Supplementary Material

We start this Supplementary Section with some notions and basic definitions of Category Theory. After that, we provide all the proofs from Section 3.

Category Theory

In this section we recall some basic notions and definitions of Category Theory. A category is a collection of objects $A, B, C, \cdots$ and morphisms $\eta, \gamma, \mu \cdots$ between two objects which satisfies the following axioms:

i) For each arrow η, there are given objects $dom(\eta)$ and $cod(\eta)$ called the *domain* and *codomain* of η. In this case, we write $\eta : A \longrightarrow B$ to indicate that $A = dom(\eta)$ and $B = cod(\eta)$.

ii) Given morphisms $\eta : A \longrightarrow B$ and $\gamma : B \longrightarrow C$ there is an arrow

$$\gamma \circ \eta : A \longrightarrow C$$

called the *composite* of η and γ. Observe that $cod(\eta) = dom(\gamma)$.

iii) For each object A, we have an *identity* arrow $Id_A : A \longrightarrow A$.

iv) The composition is associative:

$$\mu \circ (\gamma \circ \eta) = (\mu \circ \gamma) \circ \eta$$

for all $\eta : A \longrightarrow B, \gamma : B \longrightarrow C, \mu : C \longrightarrow D$.

v) For all $\eta : A \longrightarrow B$ we have

$$\eta \circ Id_A = \eta = Id_B \circ \eta.$$

A *functor* between the categories $\mathcal{C}$ and $\mathcal{D}$ is a map $F : \mathcal{C} \longrightarrow \mathcal{D}$ that assigns, to each object $C \in \mathcal{C}$ an object $F(C) \in \mathcal{D}$, and to each arrow $f : C \longrightarrow C'$ in $\mathcal{C}$ an arrow $F(f) : F(C) \longrightarrow F(C')$ in $\mathcal{D}$, such that the following conditions hold:

i) $F(Id_C) = Id_{F(C)}$ for every $C \in \mathcal{C}$.

ii) $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $C \xrightarrow{f} D$ and $D \xrightarrow{g} E$ in $\mathcal{C}$.

Broadly, a *subcategory* of a category $\mathcal{C}$ is a category by itself which is contained in $\mathcal{C}$. In other words, a subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is defined by a subcollection of objects and a subcollection of morphisms of $\mathcal{C}$ subject to the requirements making $\mathcal{D}$ a category by itself. For instance, the category of finite sets and functions **FSets** is a subcategory of **Sets**; the category of prime graphs and prime graph morphisms **PGraph** is a subcategory of **UndGraph**, the category of undirected graphs and graph morphisms.

Prime Graph category

Proposition Any prime labeled graph G_u has no self loops.

Proof. We proceed the proof by contradiction. Suppose that G_u is a graph having a self loop (v_0, v_0). Let φ a prime labeling on G_u. Then, if $\varphi(v_0) = j$ for some $j \in I$, we would have that v_0 has a neighbor with label j (v_0 is its own neighbor), contradicting (i) of Definition 3.1. Similarly, if $\varphi(v_0) = j'$ we would contradict (ii) of Definition 3.1. Thus, G_u can not have self loops.

Proposition 3.3 Let G_d be a directed graph in **DGraph**. Then, G_d induces a prime graph $G_u \in$ **PGraph**.

Proof. Let $G_d = (V_d, E_d)$ be a finite directed graph in **DGraph**. Without loss of generality, let us suppose that V_d is represented by the set $\{v_1, v_2, \cdots, v_n\}$. Thus, we define the undirected graph G_u as follows: the vertex set V_u is given by

$$V(G_u) :\, = \{v_1, v_2, \cdots, v_n, v_1', v_2', \cdots, v_n'\}$$

For notation's simplicity, we will denote $v_{i'} = v_i'$ for each $i' \in \{1', \cdots, n'\}$. Now, to define the edge set E_u, we will consider the following two cases:

(i) for each $i \in \{1, \ldots, n\}$, the tuple (v_i, v_i') defines an edge in G_u;

(ii) for $i \neq j$ in $\{1, \cdots, n\}$, we have that (v_i, v_j') defines an edge in G_u if, and only if, there exists a directed edge $(v_i, v_j) \in E_d$.

In other words,

$$E(G_u) = \begin{cases} (v_i, v_j') & \text{if } i \neq j \text{ and } (v_i, v_j) \in E(G_d) \\ (v_i, v_i') & \text{for each } i \in \{1, \cdots, n\}. \end{cases}$$

We now claim that G_u admits an I-labeling. Indeed, let $I = \{1, 2, \cdots, n\}$ and $I' = \{1', 2', \cdots, n'\}$ be the non-prime and prime sets, respectively. We thus define the labeling function $\varphi : V(G_u) \longrightarrow I \cup I'$ as follows: for each $v \in V(G_u)$

$$\varphi(v) = \begin{cases} i & \text{if } v = v_i \text{ with } v_i \in \{v_1, v_2, \cdots, v_n\} \\ i' & \text{if } v = v_i' \text{ with } v_i' \in \{v_1', v_2', \cdots, v_n'\} \end{cases}$$

Clearly, φ is a bijective function. Moreover, by how G_u is defined, condition (iii) of Definition 3.1 is satisfied. Thus, it suffices to show that G_u and φ satisfy conditions (i) and (ii). Now, if $v \in V_u$ is such that $\varphi(v) = i$ for $i \in I$, then $v = v_i$. Hence, as the incident edges to vertex $v_i \in G_u$ are either of the form (v_i, v_j') (this is, when $(v_i, v_j) \in E_d$) or (v_i, v_i'), it follows that the neighbors of v_i are either v_i' or v_j', for some $j \neq i$. Thus, in any case one has that $\varphi(v_j') = j'$ for any possible neighbor v_j' of v. Likewise, if $\varphi(v) = i'$ for some $i' \in I'$, then $v = v_i'$. Again, by how the undirected graph G_u is defined, one has that the incident edges to vertex v_i' in G_u are either of the form (v_j, v_i') (namely, when $(v_j, v_i) \in E_d$) or (v_i, v_i'). Thus, the neighbors of v_i' can be either v_j (for some $j \neq i$) or v_i, which in turn implies that $\varphi(v_j) = j$ for any possible neighbor v_j of v. Therefore, φ defines a prime labeling function on G_u. $\square$

Proposition 3.4 Let G_{d_1} and G_{d_2} be two directed graphs in **DGraph**, and let $V(G_{d_1}) \xrightarrow{f} V(G_{d_2})$ be a directed graph morphism. Then, f induces a prime graph morphism $\bar{f}$ between the prime graphs G_{u_1} and G_{u_2}.

Proof. Let $G_{d_1}, G_{d_2} \in$ **DGraph**, and let us take $f : V(G_{d_1}) \longrightarrow V(G_{d_2})$ a directed graph morphism. Without loss of generality, assume that the vertex sets are $V(G_{d_1}) = \{v_1, v_2, \cdots, v_n\}$ and $V(G_{d_2}) = \{w_1, w_2, \cdots, w_m\}$, respectively. Then, by Proposition 3.3, the vertex set of the corresponding prime graphs G_{u_1} and G_{u_2} are given by

$$V(G_{u_1}) = \{v_1, v_2, \cdots, v_n, v'_1, v'_2, \cdots, v'_n\}$$

and

$$V(G_{u_2}) = \{w_1, w_2, \cdots, w_m, w'_1, w'_2, \cdots, w'_n\},$$

respectively. With this in mind, we define $\bar{f} : V(G_{u_1}) \longrightarrow V(G_{u_2})$ by

$$\bar{f} = \begin{cases} \bar{f}(v_i) = f(v_i) & \text{for } v_i \in \{v_1, v_2, \cdots, v_n\} \\ \bar{f}(v'_i) = f(v_i)' & \text{for } v'_i \in \{v'_1, v'_2, \cdots, v'_n\}, \end{cases}$$

where $f(v_i)'$ denotes the prime labeled vertex in $V(G_{u_2})$ such that, if $f(v_i) = w_k$, then $f(v_i)' = w'_k$. We claim that $\bar{f}$ is a prime graph morphism. Indeed, we first show that $\bar{f}$ preserves adjacencies. To do so, recall that an edge $(v_i, v'_j) \in (G_{u_1})$, with $i \neq j$, corresponds to a directed edge (v_i, v_j) in $E(G_{d_1})$. Thus, as f preserves adjacencies -in the directed case-, we have that $(f(v_i), f(v_j)) \in E(G_{d_2})$ which in turn implies that $(f(v_i), f(v_j)') \in E(G_{u_2})$. Considering this, one has that

$$\bar{f}(v_i, v'_j) = \left(\bar{f}(v_i), \bar{f}(v'_j)\right) = \left(f(v_i), f(v_j)'\right) \in E(G_{u_2}).$$

Moreover, for edges of the form $(v_i, v'_i) \in E(G_{u_1})$, one gets

$$\bar{f}(v_i, v'_i) = \left(\bar{f}(v_i), \bar{f}(v'_i)\right) = \left(f(v_i), f(v_i)'\right) \in E(G_{u_2}),$$

thus showing that $\bar{f}$ is an undirected graph morphism. Now, by how $\bar{f}$ is defined, it is clear that $\bar{f}$ is compatible with the labeling, as preserves the prime and non-prime labelings satisfies $\left(\bar{f}(v_i)\right)' = f(v_i)' = \bar{f}(v'_i)$. Therefore, $\bar{f}$ is a prime graph morphism. $\square$

Theorem 3.5 The map $\mathcal{L}$: **DGraph** $\longrightarrow$ **PGraph** that assigns to each object $G_d \in$ **DGraph** the object $G_u \in$ **PGraph**, and to each morphism $f \in$ **DGraph** the morphism $\bar{f} \in$ **PGraph** defines a functor from **DGraph** to **PGraph**.

Proof. By Proposition 3.3 and Proposition 3.4 it suffices to show that the map $\mathcal{L}$: **DGraph** $\longrightarrow$ **PGraph** preserves identity morphisms and composition of morphisms. The former follows from the fact that $\bar{I}d(v_i) = Id(v_i)$ for non-prime labeled vertices, whereas for prime labeled vertices one has that $\bar{I}d(v'_i) = Id(v_i)' = Id(v'_i) = v'_i$. For the latter, we will prove that given the directed graph morphisms $f \colon V(G_{d_1}) \longrightarrow V(G_{d_2})$ and $g \colon V(G_{d_2}) \longrightarrow V(G_{d_3})$, one has that $\overline{g \circ f} = \bar{g} \circ \bar{f}$. To this end, we must show that $\overline{g \circ f}$ and $\bar{g} \circ \bar{f}$ have the same correspondence rule, as both already have the same domain and codomain sets.

Following the notation so far, we will denote the vertex set of the directed graph G_{d_1} by $V(G_{d_1}) = \{v_1, v_2, \cdots, v_n\}$, the vertex set of the directed graph G_{d_2} by $V(G_{d_2}) = \{w_1, w_2, \cdots, w_m\}$, and the vertex set of the directed graph G_{d_3} by $V(G_{d_3}) = \{z_1, z_2, \cdots, z_l\}$. Then, by Proposition 3.3, the vertex sets of their corresponding prime graphs are given by

$$V(G_{u_1}) = \{v_1, v_2, \cdots, v_n, v_1', v_2', \cdots, v_n'\},$$

$$V(G_{u_2}) = \{w_1, w_2, \cdots, w_m, w_1', w_2', \cdots, w_m'\},$$

$$\text{and } V(G_{u_3}) = \{z_1, z_2, \cdots, z_l, z_1', z_2', \cdots, z_l'\},$$

respectively. Now, on the one hand, the image under functor $\mathcal{L}$ of the directed graph morphism defined by the composition $(g \circ f)$ is the prime graph morphism $\overline{(g \circ f)} : V(G_{u_1}) \longrightarrow V(G_{u_3})$, whose correspondence rule is given by

$$\overline{(g \circ f)} = \begin{cases} \overline{(g \circ f)}(v_i) = (g \circ f)(v_i) & \text{for } v_i \in \{v_1, v_2, \cdots, v_n\} \\ \overline{(g \circ f)}(v_i') = \big((g \circ f)(v_i)\big)' & \text{for } v_i' \in \{v_1', v_2', \cdots, v_n'\}. \end{cases}$$

Here, $\big((g \circ f)(v_i)\big)'$ denotes the prime labeled vertex on $V(G_{d_3})$ such that, if $(g \circ f)(v_i) = z_s$ then $\big((g \circ f)(v_i)\big)' = z_r'$.

On the other hand, the image of f under functor $\mathcal{L}$ is the prime graph morphism $\bar{f} : V(G_{u_1}) \longrightarrow V(G_{u_2})$ given by

$$\bar{f} = \begin{cases} \bar{f}(v_i) = f(v_i) & \text{for } v_i \in \{v_1, v_2, \cdots, v_n\} \\ \bar{f}(v_i') = f(v_i)' & \text{for } v_i' \in \{v_1', v_2', \cdots, v_n'\}, \end{cases}$$

while the image of g under functor $\mathcal{L}$ is the prime graph morphism $\bar{g} : V(G_{u_2}) \longrightarrow V(G_{u_3})$ given by

$$\bar{g} = \begin{cases} \bar{g}(w_i) = f(w_i) & \text{for } w_i \in \{w_1, w_2, \cdots, w_m\} \\ \bar{g}(w_i') = f(w_i)' & \text{for } w_i' \in \{w_1', w_2', \cdots, w_m'\}. \end{cases}$$

Note that, as prime graph morphisms preserve the prime vertex and non-prime vertex labelings, then one has that $\bar{f}(v_i) \in \{w_1, \cdots, w_m\}$ whereas $\bar{f}(v_i') \in \{w_1', \cdots, w_m'\}$. This way, when considering the composite morphism $\bar{g} \circ \bar{f} : V(G_{u_1}) \longrightarrow V(G_{u_3})$ one has that, for each $v_i \in \{v_1, v_2, \cdots, v_n\}$,

$$(\bar{g} \circ \bar{f})(v_i) = \bar{g}(\bar{f}(v_i)) = \bar{g}(f(v_i)) = g(f(v_i)) = (g \circ f)(v_i) = \overline{(g \circ f)}(v_i),$$

whereas for each $v_i' \in \{v_1', v_2', \cdots, v_n'\}$,

$$(\bar{g} \circ \bar{f})(v_i') = \bar{g}(\bar{f}(v_i')) = \bar{g}(f(v_i)') = \big(g(f(v_i))\big)' = \big((g \circ f)(v_i)\big)' = \overline{(g \circ f)}(v_i').$$

Hence, $\overline{(g \circ f)}$ and $(\bar{g} \circ \bar{f})$ have the same correspondence rule, and therefore, $\overline{(g \circ f)} = (\bar{g} \circ \bar{f})$. Consequently,

$$\mathcal{L}(g \circ f) = \overline{(g \circ f)} = \bar{g} \circ \bar{f} = \mathcal{L}(g) \circ \mathcal{L}(f),$$

this is, $\mathcal{L}$ preserves compositions of morphisms. $\square$

Proposition 3.6 Let G_u be a prime labeled graph. Then, G_u induces a simple directed graph G_d.

Proof. Let G_u be a prime graph and let $\varphi_I : V(G_u) \longrightarrow I \cup I'$ be its labeling function, with $I = \{1, \cdots, n\}$. Following the notation, we can denote the vertex set of G_u as

$$V(G_u) = \{v_1, v_2, \cdots, v_n, v_1', v_2', \cdots, v_n'\}.$$

Also, recall that by the definition of prime graphs, the vertices v_i and v_i' in G_u are always adjacent. With this in mind, we define the vertex set of the directed graph G_d, induced by G_u, as

$$V(G_d) = \{v_1, v_2, \cdots, v_n\}.$$

Now, to define the edge set $E(G_d)$, we must consider all incident edges to the set of prime label vertices v_i' in $V(G_u)$. For instance, if v_j' is adjacent to v_i in G_u (with $j \neq i$), then we will get a directed edge in G_d from vertex v_i towards vertex v_j. In case that v_i' is the only adjacent vertex to v_i in G_u, then the corresponding vertex v_i in G_d will not have incoming directed edges. Thus, considering the above, we define $E(G_d)$ by the set

$$E(G_d) = \big\{(v_i, v_j) \mid \text{ whenever } i \neq j \text{ and } (v_i, v_j') \in E(G_u)\big\}$$

$\square$

Proposition 3.7 Let $G_{u_1}, G_{u_2} \in \mathbf{PGraph}$ and let $g : V(G_{u_1}) \longrightarrow V(G_{u_2})$ be a prime graph morphism. Then, f induces a directed graph morphism $\tilde{g}$ between the corresponding directed graphs G_{d_1} and G_{d_2}.

Proof. Let $G_{u_1}, G_{u_2} \in \mathbf{PGraph}$, and let $f : V(G_{u_1}) \longrightarrow V(G_{u_2})$ be a prime graph morphism. Let us assume that $V(G_{u_1}) = \{v_1, \cdots, v_n, v_1', \cdots, v_n'\}$ and that $V(G_{u_2}) = \{w_1, \cdots, w_m, w_1', \cdots, w_m'\}$. By Proposition 3.6, we can write the vertex sets $V(G_{d_1})$ and $V(G_{d_2})$ by $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_m\}$, respectively. Thus, we define $\tilde{f} : V(G_{d_1}) \longrightarrow V(G_{d_2})$ as follows: for each $v_i \in V(G_{d_1})$, we set $\tilde{f}(v_i) = f(v_i)$. We claim that $\tilde{f}$ preserves adjacencies. Indeed, given $(v_i, v_j) \in E(G_{d_1})$ we get that $(v_i, v_j') \in E(G_{u_1})$. Thus, as f is a prime graph morphism, one has that $(f(v_i), f(v_j')) \in E(G_{u_2})$. Moreover, since f is compatible with the prime and non-prime labelings, it follows that $f(v_j') = f(v_j)'$, from which we have that $(f(v_i), f(v_j)) \in E(G_{d_2})$. Hence,

$$\tilde{f}(v_i, v_j) = (\tilde{f}(v_i), \tilde{f}(v_j)) = (f(v_i), f(v_j)) \in E(G_{d_2}).$$

Therefore, $\tilde{f}$ is a directed graph morphism.

$\square$

Theorem 3.8 Let $G_{u_1}, G_{u_2} \in$ **PGraph** and let $g : V(G_{u_1}) \longrightarrow V(G_{u_2})$ be a prime graph morphism. Then, f induces a directed graph morphism $\tilde{g}$ between the corresponding directed graphs G_{d_1} and G_{d_2}.

Proof. By Proposition 3.6 and Proposition 3.7 it suffices to show that the map $\mathcal{M}$ preserves identity morphisms and composition of morphisms. The former follows from the fact that $\tilde{Id}(v_i) = v_i$, for every vertex v_i. For the latter, we will shot that given the prime graph morphisms $f : V(G_{u_1}) \longrightarrow V(G_{u_2})$ and $g : V(G_{u_2}) \longrightarrow V(G_{u_3})$, one has that $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$. Without loss of generality, let us suppose that

$$\begin{aligned} V(G_{u_1}) &= \{u_1, u_2, \cdots, u_n, u'_1, u'_2, \cdots, u'_n\}, \\ V(G_{u_2}) &= \{v_1, v_2, \cdots, v_m, v'_1, v'_2, \cdots, v'_m\} \text{ and} \\ V(G_{u_3}) &= \{w_1, w_2, \cdots, w_l, w'_1, w'_2, \cdots, w'_l\}. \end{aligned}$$

Following the notation so far, we denote the vertex sets of the directed graphs G_{d_1}, G_{d_2} and G_{d_3} by $V(G_{d_1}) = \{u_1, u_2, \cdots, u_n\}$, $V(G_{u_2}) = \{v_1, v_2, \cdots, v_m\}$ and $V(G_{u_3}) = \{w_1, w_2, \cdots, w_l\}$, respectively. Now, on the one hand, the image under $\mathcal{M}$ of the prime graph morphism obtained by composing f followed by g, is the directed graph morphism $\widetilde{(g \circ f)} : V(G_{d_1}) \longrightarrow V(G_{d_3})$, whose correspondence rule given by

$$\widetilde{(g \circ f)}(u_i) = (g \circ f)(u_i), \forall u_i \in V(G_{d_1}).$$

On the other hand, the directed graph morphism $\tilde{f} : V(G_{d_1}) \longrightarrow V(G_{d_2})$ is defined by $\tilde{f}(u_i) = f(u_i)$, for all $u_i \in V(G_{d_1})$, and similarly, the directed graph morphism $\tilde{g} : V(G_{d_2}) \longrightarrow V(G_{d_3})$ is defined by $\tilde{g}(v_j) = g(v_j)$, for all $v_j \in V(G_{d_2})$. Therefore, the composition $(\tilde{g} \circ \tilde{f}) : V(G_{d_1}) \longrightarrow V(G_{d_3})$ is a directed graph morphism such that, for each vertex $u_i \in \{u_1, u_2, \cdots, u_n\}$

$$(\tilde{g} \circ \tilde{f})(u_i) = \tilde{g}(\tilde{f}(u_i)) = \tilde{g}(f(u_i)) = g(f(u_i)) = (g \circ f)(u_i) = \widetilde{(g \circ f)}(u_i).$$

The above equation shows that $(\tilde{g} \circ \tilde{f})$ and $\widetilde{(g \circ f)}$ have the same correspondence rule. Since both functions have also the same domain and codomain, it follows that $(\tilde{g} \circ \tilde{f}) = \widetilde{(g \circ f)}$. Therefore,

$$\mathcal{M}(g \circ f) = \widetilde{(g \circ f)} = \tilde{g} \circ \tilde{f} = \mathcal{M}(g) \circ \mathcal{M}(f),$$

this is, $\mathcal{M}$ preserves compositions of morphisms.

$\square$

Proposition 3.9 Let G_d be a directed graph in **DGraph**. Then

$$(\mathcal{M} \circ \mathcal{L})(G_d) = G_d.$$

Proof. Let $G_d \in$ **DGraph**. Without loss of generality, we can assume that $V(G_d) = \{v_1, \cdots, v_n\}$. Then, the prime graph $\mathcal{L}(G_d) := G_u$ has as vertex set

$$V(G_u) = \{v_1, v_2, \cdots, v_n, v'_1, v'_2, \cdots, v'_n\},$$

and as edge set

$$E(G_u) = \begin{cases} (v_i, v'_j) & \text{if } i \neq j \text{ and } (v_i, v_j) \in E(G_d) \\ (v_i, v'_i) & \text{for each } v_i \in V(G_d). \end{cases}$$

If we now consider the directed graph induced by G_u, this is, $\mathcal{M}(G_u) = G^*_d$, then its vertex set $V(G^*_d)$ is given by $\{v_1, \cdots, v_n\}$ and its edge set is given by

$$E(G^*_d) = \big\{(v_i, v_j) \mid i \neq j \text{ and } (v_i, v'_j) \in E(G_u)\big\}.$$

In this way, as $(v_i, v'_j) \in E(G_u)$ if and only if $(v_i, v_j) \in E(G_d)$, it follows that $(v_i, v_j) \in E(G^*_d)$ if and only if $(v_i, v_j) \in E(G_d)$. Moreover, since G_d and G^*_d have the same vertex set $\{v_1, v_2, \cdots, v_n\}$, we can conclude that $G_d = G^*_d$. Therefore,

$$(\mathcal{M} \circ \mathcal{L})(G_d) = G_d.$$

□

Proposition 3.10 Let $G_{d_1}, G_{d_2} \in$ **DGraph** and let $f : V(G_{d_1}) \longrightarrow V(G_{d_2})$ be a directed graph morphism. Then

$$(\mathcal{M} \circ \mathcal{L})(f) = f.$$

Proof. Let G_{d_1} and G_{d_2} be directed graphs and let $f : V(G_{d_1}) \longrightarrow V(G_{d_2})$ be a directed graph morphism. Let us suppose that $V(G_{d_1}) = \{v_1, \cdots, v_n\}$. Then, by Theorem 3.5, we obtain the prime graphs $\mathcal{L}(G_{d_1}) = G_{u_1}$ and $\mathcal{L}(G_{d_2}) = G_{u_2}$, along with the prime graph morphism $\mathcal{L}(f) = \bar{f}$ defined by

$$\bar{f} = \begin{cases} \bar{f}(v_i) = f(v_i) & \text{for } v_i \in \{v_1, v_2, \cdots, v_n\} \\ \bar{f}(v'_i) = f(v_i)' & \text{for } v'_i \in \{v'_1, v'_2, \cdots, v'_n\}. \end{cases}$$

Here, $V(G_{u_1}) = \{v_1, v_2, \cdots, v_n, v'_1, v'_2, \cdots, v'_n\}$.
Now, as

$$\mathcal{M}(G_{u_i}) = \mathcal{M}(\mathcal{L}(G_{d_i})) = (\mathcal{M} \circ \mathcal{L})(G_{d_i}) = G_{d_i}$$

for $i = 1, 2$, it suffices to show that the functions $\mathcal{M}(\bar{f})$ and f have the same correspondence rule, since both have the same domain set $V(G_{d_1})$ and the same codomain set $V(G_{d_2})$. But that follows from the fact that $\mathcal{M}(\bar{f}) = \widetilde{\bar{f}}$, and $\widetilde{\bar{f}}(v_i) = \bar{f}(v_i) = f(v_i)$ for each vertex $v_i \in \{v_1, \cdots, v_n\}$. Therefore, $\mathcal{M}(\bar{f}) = f$ and thus,

$$(\mathcal{M} \circ \mathcal{L})(f) = f.$$

□

Proposition 3.12 Let G_u be a prime labeled graph. Then

$$(\mathcal{L} \circ \mathcal{M})(G_u) = G_u.$$

Proof. Let $G_u \in$ **PGraph**. Let us assume that the vertex set $V(G_u)$ is defined by the set $\{v_1, v_2, \cdots, v_n, v_1', v_2', \cdots, v_n'\}$. Then $\mathcal{M}(G_u) := G_d$ is the directed graph with vertex set $V(G_d) = \{v_1, v_2, \cdots, v_n\}$ and edge set

$$E(G_d) = \big\{(v_i, v_j) \mid i \neq j \text{ and } (v_i, v_j') \in E(G_u)\big\}.$$

Now, if we denote by G_u^* the prime graph induced by G_d, this is, $\mathcal{L}(G_d) = G_u^*$, then $V(G_d^*) = \{v_1, \cdots, v_n, v_1', \cdots, v_n'\}$, and $E(G_u^*)$ is given by

$$E(G_u^*) = \begin{cases} (v_i, v_j') & \text{if } i \neq j \text{ and } (v_i, v_j) \in E(G_d) \\ (v_i, v_i') & \text{for each } v_i \in V(G_d). \end{cases}$$

Notice that $(v_i, v_j') \in E(G_u^*)$ if and only if $(v_i, v_j) \in E(G_d)$, which holds if and only if $(v_i, v_j') \in E(G_u)$. Since both G_u and G_u^* have as vertex set the set $\{v_1, v_2, \cdots, v_n, v_1', \cdots, v_n'\}$, it follows that $G_u = G_u^*$. Therefore,

$$(\mathcal{L} \circ \mathcal{M})(G_u) = G_u.$$

□

Proposition 3.13 Let $f : V(G_{u_1}) \longrightarrow V(G_{u_2})$ be a prime graph morphism. Then

$$(\mathcal{L} \circ \mathcal{M})(f) = f.$$

Proof. Let $f \colon V(G_{u_1}) \longrightarrow V(G_{u_2})$ be a prime graph morphism, and let us suppose that $V(G_{u_1}) = \{v_1, \cdots, v_n, v_1', \cdots, v_n'\}$. By Theorem 3.8, we have the directed graphs $\mathcal{M}(G_{u_1}) = G_{d_1}$ and $\mathcal{M}(G_{u_2}) = G_{d_2}$, along with the directed graph morphism $\mathcal{M}(f) = \tilde{f} \colon V(G_{d_1}) \longrightarrow V(G_{d_2})$ defined by $\tilde{f}(v_i) = f(v_i)$, for each $i = 1, \cdots, n$. Now, as

$$\mathcal{L}(G_{d_i}) = \mathcal{L}(\mathcal{M}(G_{u_i})) = (\mathcal{L} \circ \mathcal{M})(G_{u_i}) = G_{u_i}$$

for $i = 1, 2$, then it suffices to show that $\mathcal{L}(\tilde{f}) = \bar{\tilde{f}} \colon V(G_{u_1}) \longrightarrow V(G_{u_2})$ and $f \colon V(G_{u_1}) \longrightarrow V(G_{u_2})$ have the same correspondence rule, as these have the same domain and codomain. But this last follows by considering that $\mathcal{L}(\tilde{f}) = \bar{\tilde{f}}$ is such that

$$\bar{\tilde{f}} = \begin{cases} \bar{\tilde{f}}(v_i) = \tilde{f}(v_i) = f(v_i) & \text{for } v_i \in \{v_1, v_2, \cdots, v_n\} \\ \bar{\tilde{f}}(v_i') = \tilde{f}(v_i)' = f(v_i)' & \text{for } v_i' \in \{v_1', v_2', \cdots, v_n'\}. \end{cases}$$

Hence, $\mathcal{L}(\tilde{f})$ and f have the same correspondence rule, which in turn implies that $\mathcal{L}(\tilde{f}) = f$. Therefore,

$$(\mathcal{L} \circ \mathcal{M})(f) = f.$$

□