Supplementary Material for: Recycling nonlocality in a quantum network

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1 Derivation of Eq. (4) in main text

Recall that for a choice of m parties, s, the quantum probabilities are given by

$$p(a, b_s|x, y_s) = \mathrm{Tr}\left[\left(\bigotimes_{k=1}^{m} B^{(k,s_k)}_{b_{k,s_k}|y_{k,s_k}} \otimes A_{a|x}\right)\chi_s\right], \tag{1}$$

up to the ordering of the tensor product. The recycled total state, χ_s, takes the form $\chi_s = \bigotimes_{k=1}^{m} \rho_k^{(s_k-1)}$, where $\rho_k^0 = \rho_k$. Here, $\rho_k^{(j-1)}$ is the average recycled state that arrives to $B_{k,j}$. These states can be obtained recursively from the Lüders rule,

$$\rho_k^{(j)} = \frac{1}{2}\sum_{b_{k,j},y_{k,j}} \left(\sqrt{B^{(k,j)}_{b_{k,j}|y_{k,j}}}\otimes\mathbb{I}\right)\rho_k^{(j-1)}\left(\sqrt{B^{(k,j)}_{b_{k,j}|y_{k,j}}}\otimes\mathbb{I}\right), \tag{2}$$

In the quantum protocol, we let each source distribute the maximally entangled state, $|\phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$. All outer parties perform unsharp measurements in the Pauli Z and X bases respectively. The j'th one in the k'th branch has observables $\eta^Z_{k,j}\sigma_Z (y_{k,j}=0)$ and $\eta^X_{k,j}\sigma_X (y_{k,j}=1)$. Let A perform measurements $A_{a|x} = \sum_{a_1\oplus\ldots\oplus a_m = a}\bigotimes_{k=1}^{m}\Pi^{(k)}_{a_k|x}$, where $\Pi^{(k)}_{a_k|x} = \frac{1}{2}\left(\mathbb{I} + (-1)^{a_k}\left(\cos\theta\sigma_Z + (-1)^x\sin\theta\sigma_X\right.\right.$

We now recursively compute the recycled states.

$$\begin{aligned}
\rho_k^{(j)} &= \frac{1}{2}\sum_{b_{k,j},y_{k,j}} \left(\sqrt{B^{(k,j)}_{b_{k,j}|y_{k,j}}}\otimes\mathbb{I}\right)\rho_k^{(j-1)}\left(\sqrt{B^{(k,j)}_{b_{k,j}|y_{k,j}}}\otimes\mathbb{I}\right)\\
&= \frac{1}{2}\left[\sqrt{\frac{1}{2}(\mathbb{I}+\eta^Z_{k,j}\sigma_Z)}\otimes\mathbb{I}\right]\rho_k^{(j-1)}\left[\sqrt{\frac{1}{2}(\mathbb{I}+\eta^Z_{k,j}\sigma_Z)}\otimes\mathbb{I}\right]\\
&+ \frac{1}{2}\left[\sqrt{\frac{1}{2}(\mathbb{I}-\eta^Z_{k,j}\sigma_Z)}\otimes\mathbb{I}\right]\rho_k^{(j-1)}\left[\sqrt{\frac{1}{2}(\mathbb{I}-\eta^Z_{k,j}\sigma_Z)}\otimes\mathbb{I}\right]\\
&+ \frac{1}{2}\left[\sqrt{\frac{1}{2}(\mathbb{I}+\eta^X_{k,j}\sigma_X)}\otimes\mathbb{I}\right]\rho_k^{(j-1)}\left[\sqrt{\frac{1}{2}(\mathbb{I}+\eta^X_{k,j}\sigma_X)}\otimes\mathbb{I}\right]\\
&+ \frac{1}{2}\left[\sqrt{\frac{1}{2}(\mathbb{I}-\eta^X_{k,j}\sigma_X)}\otimes\mathbb{I}\right]\rho_k^{(j-1)}\left[\sqrt{\frac{1}{2}(\mathbb{I}-\eta^X_{k,j}\sigma_X)}\otimes\mathbb{I}\right]\\
&= \frac{1}{4}\left[2+\sqrt{1-\left(\eta^Z_{k,j}\right)^2}+\sqrt{1-\left(\eta^X_{k,j}\right)^2}\right]\rho_k^{(j-1)} + \frac{1}{4}\left[1-\sqrt{1-\left(\eta^Z_{k,j}\right)^2}\right](\sigma_Z\otimes\mathbb{I})\rho_k^{(j-1)}(\sigma_Z\otimes\mathbb{I})\\
&+ \frac{1}{4}\left[1-\sqrt{1-\left(\eta^X_{k,j}\right)^2}\right](\sigma_X\otimes\mathbb{I})\rho_k^{(j-1)}(\sigma_X\otimes\mathbb{I}).
\end{aligned} \tag{3}$$

To compute the network Bell inequality, we must evaluate the quantities

$$I_s = \frac{1}{2^m} \sum_{y_s} \sum_{a,b_s} (-1)^{a+\sum_k b_{k,s_k}} p(a, b_s|0, y_s), \tag{4}$$

$$J_s = \frac{1}{2^m} \sum_{y_s} \sum_{a,b_s} (-1)^{a+\sum_k (b_{k,s_k}+y_{k,s_k})} p(a, b_s|1, y_s). \tag{5}$$

We now compute these quantities in the quantum model.

$$\begin{aligned}
I_s &= \frac{1}{2^m} \sum_{y_s} \sum_{a,b_s} (-1)^{a+\sum_k b_{k,s_k}} p(a, b_s|0, y_s) \\
&= \frac{1}{2^m} \sum_{y_s} \sum_{a,b_s} (-1)^{a+\sum_k b_{k,s_k}} \operatorname{Tr}\left[\left(\bigotimes_{k=1}^{m} B^{(k,s_k)}_{b_{k,s_k}|y_{k,s_k}} \otimes A_{a|0}\right)\chi_s\right] \\
&= \frac{1}{2^m} \sum_{y_s} \sum_{a,b_s} (-1)^{a+\sum_k b_{k,s_k}} \operatorname{Tr}\left[\left(\bigotimes_{k=1}^{m} B^{(k,s_k)}_{b_{k,s_k}|y_{k,s_k}} \otimes \sum_{a_1\oplus\ldots\oplus a_m=a} \bigotimes_{k=1}^{m} \Pi^{(k)}_{a_k|0}\right) \bigotimes_{k=1}^{m} \rho_k^{(s_k-1)}\right] \\
&= \frac{1}{2^m} \sum_{y_s} \sum_{a,b_s} \sum_{a_1\oplus\ldots\oplus a_m=a} (-1)^{a+\sum_k b_{k,s_k}} \operatorname{Tr}\left[\bigotimes_{k=1}^{m} \left(B^{(k,s_k)}_{b_{k,s_k}|y_{k,s_k}} \otimes \Pi^{(k)}_{a_k|0}\rho_k^{(s_k-1)}\right)\right] \\
&= \frac{1}{2^m} \sum_{y_s} \sum_{a,b_s} \sum_{a_1\oplus\ldots\oplus a_m=a} (-1)^{a+\sum_k b_{k,s_k}} \prod_{k=1}^{m} \left[\operatorname{Tr}\left(B^{(k,s_k)}_{b_{k,s_k}|y_{k,s_k}} \otimes \Pi^{(k)}_{a_k|0}\rho_k^{(s_k-1)}\right)\right] \\
&= \frac{1}{2^m} \prod_{k=1}^{m} \operatorname{Tr}\left\{\rho_k^{(s_k-1)} \left[(B^{(k,s_k)}_{0|0} - B^{(k,s_k)}_{1|0}) \otimes \Pi^0_{A_m} + (B^{(k,s_k)}_{0|1} - B^{(k,s_k)}_{1|1}) \otimes \Pi^0_{A_m}\right]\right\} \\
&= \frac{1}{2^m} \prod_{k=1}^{m} \left\{\eta^Z_{k,s_k} \operatorname{Tr}\left[\rho_k^{(s_k-1)}(\cos\theta\sigma_Z \otimes \sigma_Z + \sin\theta\sigma_Z \otimes \sigma_X)\right] + \right. \\
&\left.\eta^X_{k,s_k} \operatorname{Tr}\left[\rho_k^{(s_k-1)}(\cos\theta\sigma_X \otimes \sigma_Z + \sin\theta\sigma_X \otimes \sigma_X)\right]\right\} \\
&= \frac{1}{2^m} \prod_{k=1}^{m} \left[\eta^Z_{k,s_k} \cos\theta \operatorname{Tr}\left(\rho_k^{(s_k-1)}\sigma_Z \otimes \sigma_Z\right) + \eta^Z_{k,s_k} \sin\theta \operatorname{Tr}\left(\rho_k^{(s_k-1)}\sigma_Z \otimes \sigma_X\right)\right. \\
&\left. +\eta^X_{k,s_k} \cos\theta \operatorname{Tr}\left(\rho_k^{(s_k-1)}\sigma_X \otimes \sigma_Z\right) + \eta^X_{k,s_k} \sin\theta \operatorname{Tr}\left(\rho_k^{(s_k-1)}\sigma_X \otimes \sigma_X\right)\right]
\end{aligned} \tag{6}$$

Below we compute quantities $\operatorname{Tr}\left(\rho_k^{(s_k-1)}\sigma_Z \otimes \sigma_Z\right)$, $\operatorname{Tr}\left(\rho_k^{(s_k-1)}\sigma_Z \otimes \sigma_X\right)$, $\operatorname{Tr}\left(\rho_k^{(s_k-1)}\sigma_X \otimes \sigma_Z\right)$, and $\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_X\otimes\sigma_X\Big)$,

$$
\begin{aligned}
&\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_Z\otimes\sigma_Z\Big)\\
&=\frac{1}{4}(2+\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2}+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\Big(\rho_k^{(s_k-2)}\sigma_Z\otimes\sigma_Z\Big)\\
&+\frac{1}{4}(1-\sqrt{1-\big(\eta^Z_{k,j}\big)^2})\,\mathrm{Tr}\,\rho_k^{(s_k-2)}(\sigma_Z\sigma_Z\sigma_Z\otimes\sigma_Z)+\frac{1}{4}(1-\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\,\rho_k^{(s_k-2)}(\sigma_X\sigma_Z\sigma_X\otimes\sigma_Z)\\
&=\frac{1}{4}(2+\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2}+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\Big(\rho_k^{(s_k-2)}\sigma_Z\otimes\sigma_Z\Big)\\
&+\frac{1}{4}(1-\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2})Tr\rho_k^{(s_k-2)}(\sigma_Z\otimes\sigma_Z)-\frac{1}{4}(1-\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\,\rho_k^{(s_k-2)}(\sigma_Z\otimes\sigma_Z)\\
&=\Bigg[\frac{1}{4}(2+\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2}+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})+\frac{1}{4}(1-\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2})\\
&-\frac{1}{4}(1-\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\Bigg]\,\mathrm{Tr}\Big(\rho_k^{(s_k-2)}\sigma_Z\otimes\sigma_Z\Big)\\
&=\frac{1}{2}(1+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\Big(\rho_k^{(s_k-2)}\sigma_Z\otimes\sigma_Z\Big).
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_Z\otimes\sigma_X\Big)\\
&=\frac{1}{4}(2+\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2}+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\Big(\rho_k^{(s_k-2)}\sigma_Z\otimes\sigma_X\Big)\\
&+\frac{1}{4}(1-\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2})\,\mathrm{Tr}\,\rho_k^{(s_k-2)}(\sigma_Z\sigma_Z\sigma_Z\otimes\sigma_X)\\
&+\frac{1}{4}(1-\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\,\rho_k^{(s_k-2)}(\sigma_X\sigma_Z\sigma_X\otimes\sigma_X)\\
&=\frac{1}{4}(2+\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2}+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\Big(\rho_k^{(s_k-2)}\sigma_Z\otimes\sigma_X\Big)\\
&+\frac{1}{4}(1-\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2})\,\mathrm{Tr}\,\rho_k^{(s_k-2)}(\sigma_Z\otimes\sigma_X)-\frac{1}{4}(1-\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\,\rho_k^{(s_k-2)}(\sigma_Z\otimes\sigma_X)\\
&=\Bigg[\frac{1}{4}(2+\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2}+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})+\frac{1}{4}(1-\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2})\\
&-\frac{1}{4}(1-\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\Bigg]\,\mathrm{Tr}\Big(\rho_k^{(s_k-2)}\sigma_Z\otimes\sigma_X\Big)\\
&=\frac{1}{2}(1+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2})\,\mathrm{Tr}\Big(\rho_k^{(s_k-2)}\sigma_Z\otimes\sigma_X\Big)
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&\operatorname{Tr}\Big(\rho_k^{(s_k-1)}\sigma_X\otimes\sigma_Z\Big)\\
&=\frac{1}{4}(2+\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2}+\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})\operatorname{Tr}\Big(\rho_k^{(s_k-2)}\sigma_X\otimes\sigma_Z\Big)\\
&+\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2})\operatorname{Tr}\rho_k^{(s_k-2)}(\sigma_Z\sigma_X\sigma_Z\otimes\sigma_Z)+\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})\operatorname{Tr}\rho_k^{(s_k-2)}(\sigma_X\sigma_X\sigma_X\otimes\sigma_Z)\\
&=\frac{1}{4}(2+\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2}+\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})\operatorname{Tr}\Big(\rho_k^{(s_k-2)}\sigma_X\otimes\sigma_Z\Big)\\
&-\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2})\operatorname{Tr}\rho_k^{(s_k-2)}(\sigma_X\otimes\sigma_Z) \qquad +\\
&=\Big[\frac{1}{4}(2+\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2}+\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})-\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2})\\
&+\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})\Big]\operatorname{Tr}\Big(\rho_k^{(s_k-2)}\sigma_X\otimes\sigma_Z\Big)\\
&=\frac{1}{2}(1+\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2})\operatorname{Tr}\Big(\rho_k^{(s_k-2)}\sigma_X\otimes\sigma_Z\Big).
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\operatorname{Tr}\Big(\rho_k^{(s_k-1)}\sigma_X\otimes\sigma_X\Big)\\
&=\frac{1}{4}(2+\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2}+\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})\operatorname{Tr}\Big(\rho_k^{(s_k-2)}\sigma_X\otimes\sigma_X\Big)\\
&+\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2})\operatorname{Tr}\rho_k^{(s_k-2)}(\sigma_Z\sigma_X\sigma_Z\otimes\sigma_X)+\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})\operatorname{Tr}\rho_k^{(s_k-2)}(\sigma_X\sigma_X\sigma_X\otimes\sigma_X)\\
&=\frac{1}{4}(2+\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2}+\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})\operatorname{Tr}\Big(\rho_k^{(s_k-2)}\sigma_X\otimes\sigma_X\Big)\\
&-\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2})\operatorname{Tr}\rho_k^{(s_k-2)}(\sigma_X\otimes\sigma_X)+\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})\operatorname{Tr}\rho_k^{(s_k-2)}(\sigma_X\otimes\sigma_X)\\
&=\Big[\frac{1}{4}(2+\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2}+\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})-\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2})\\
&+\frac{1}{4}(1-\sqrt{1-\big(\eta_{k,s_k-1}^{X}\big)^2})\Big]\operatorname{Tr}\Big(\rho_k^{(s_k-2)}\sigma_X\otimes\sigma_X\Big)\\
&=\frac{1}{2}(1+\sqrt{1-\big(\eta_{k,s_k-1}^{Z}\big)^2})\operatorname{Tr}\Big(\rho_k^{(s_k-2)}\sigma_X\otimes\sigma_X\Big)
\end{aligned}
\tag{10}
$$

Recursively, we have

$$\begin{aligned}
\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_Z\otimes\sigma_Z\Big) &= 2^{1-s_k}\,\mathrm{Tr}\big(\rho_k^0\sigma_Z\otimes\sigma_Z\big)\prod_{j=0}^{s_k-2}(1+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2}),\\
\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_Z\otimes\sigma_X\Big) &= 2^{1-s_k}\,\mathrm{Tr}\big(\rho_k^0\sigma_Z\otimes\sigma_X\big)\prod_{j=0}^{s_k-2}(1+\sqrt{1-\big(\eta^X_{k,s_k-1}\big)^2}),\\
\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_X\otimes\sigma_Z\Big) &= 2^{1-s_k}\,\mathrm{Tr}\big(\rho_k^0\sigma_X\otimes\sigma_Z\big)\prod_{j=0}^{s_k-2}(1+\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2}),\\
\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_X\otimes\sigma_X\Big) &= 2^{1-s_k}\,\mathrm{Tr}\big(\rho_k^0\sigma_X\otimes\sigma_X\big)\prod_{j=0}^{s_k-2}(1+\sqrt{1-\big(\eta^Z_{k,s_k-1}\big)^2}).
\end{aligned}\tag{11}$$

Inserting these results into Equation (6) and noting that $\mathrm{Tr}(\rho_k^0\sigma_Z\otimes\sigma_Z)=\mathrm{Tr}(\rho_k^0\sigma_X\otimes\sigma_X)=1$ and $\mathrm{Tr}(\rho_k^0\sigma_X\otimes\sigma_Z)=\mathrm{Tr}(\rho_k^0\sigma_Z\otimes\sigma_X)=0$ we obtain

$$\begin{aligned}
I_s &= \frac{1}{2^m}\prod_{k=1}^{m}\Big[\eta^Z_{k,s_k}\cos\theta\,\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_Z\otimes\sigma_Z\Big)+\eta^Z_{k,s_k}\sin\theta\,\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_Z\otimes\sigma_X\Big)\\
&+\eta^X_{k,s_k}\cos\theta\,\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_X\otimes\sigma_Z\Big)+\eta^X_{k,s_k}\sin\theta\,\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_X\otimes\sigma_X\Big)\Big]\\
&= \frac{1}{2^m}\prod_{k=1}^{m}2^{1-s_k}\left[\eta^Z_{k,s_k}\cos\theta\prod_{j=1}^{s_k-1}(1+\sqrt{1-\big(\eta^X_{k,j}\big)^2})+\eta^X_{k,s_k}\sin\theta\prod_{j=1}^{s_k-1}(1+\sqrt{1-\big(\eta^Z_{k,j}\big)^2})\right]
\end{aligned}\tag{12}$$

Similarly we can write J_s explicitly as,

$$\begin{aligned}
J_s &= \frac{1}{2^m}\sum_{y_s}\sum_{a,b_s}(-1)^{a+\sum_k(b_{k,s_k}+y_{k,s_k})}p(a,b_s|1,y_s)\\
&= \frac{1}{2^m}\prod_{k=1}^{m}\Big[\eta^Z_{k,s_k}\cos\theta\,\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_Z\otimes\sigma_Z\Big)-\eta^Z_{k,s_k}\sin\theta\,\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_Z\otimes\sigma_X\Big)\\
&-\eta^X_{k,s_k}\cos\theta\,\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_X\otimes\sigma_Z\Big)+\eta^X_{k,s_k}\sin\theta\,\mathrm{Tr}\Big(\rho_k^{(s_k-1)}\sigma_X\otimes\sigma_X\Big)\Big]\\
&= \frac{1}{2^m}\prod_{k=1}^{m}2^{1-s_k}\left[\eta^Z_{k,s_k}\cos\theta\prod_{j=1}^{s_k-1}(1+\sqrt{1-\big(\eta^X_{k,j}\big)^2})+\eta^X_{k,s_k}\sin\theta\prod_{j=1}^{s_k-1}(1+\sqrt{1-\big(\eta^Z_{k,j}\big)^2})\right]
\end{aligned}\tag{13}$$

Then we obtain Eq. (4) in the main text,

$$S_s = \left[\prod_{k=1}^{m} 2^{1-s_k}\left(\eta^Z_{k,s_k}\cos\theta\prod_{j=1}^{s_k-1} f^X_{k,j} + \eta^X_{k,s_k}\sin\theta\prod_{j=1}^{s_k-1} f^Z_{k,j}\right)\right]^{1/m}. \tag{14}$$

2 Unsharp measurement

The positive-operator-valued-measures (POVMs) of a qubit system is given as

$$\Pi_{0|\vec{r}} = \frac{1}{2}(\mathbb{I} + \eta\sigma_{\vec{r}}), \quad \Pi_{1|\vec{r}} = \mathbb{I} - \Pi_{0|\vec{r}}, \tag{15}$$

where $\mathbb{I}$ is the identity operator, $\sigma_{\vec{r}} = \vec{r}\cdot\vec{\sigma}$, $\vec{r}$ is the Bloch vector with $|\vec{r}| = 1$, $\vec{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z)$ are Pauli matrices, and $\eta \in [0, 1]$.

A quantum circuit to implement the unsharp measurement is given in Fig. S1(a), in which the meter qubit $|\Phi_\mathrm{m}\rangle = S_\mathrm{m}(\theta)\,|0_\mathrm{m}\rangle = \cos\theta|0_\mathrm{m}\rangle + \sin\theta|1_\mathrm{m}\rangle$ couples to the system qubit ρ_s via a Controlled-Not (C-NOT) gate,

$$U^{\text{C-NOT}}_{\mathrm{sm}} = |0_\mathrm{s}\rangle\langle 0_\mathrm{s}| \otimes \mathbb{I}_m + |1_\mathrm{s}\rangle\langle 1_\mathrm{s}| \otimes \sigma_{X,\mathrm{m}}. \tag{16}$$

The POVMs of the system qubit are given as

$$\begin{aligned} E_0 &= M_0^\dagger M_0 = \frac{1}{2}(\mathbb{I} + \cos 2\theta\sigma_Z) = \Pi_{0|\vec{r}}, \\ E_1 &= M_1^\dagger M_1 = \frac{1}{2}(\mathbb{I} - \cos 2\theta\sigma_Z) = \Pi_{1|\vec{r}}, \end{aligned} \tag{17}$$

with

$$\begin{aligned} M_0 &= \langle 0_\mathrm{m}|\, U^{\text{C-NOT}}_{\mathrm{sm}}(\mathbb{I}_s \otimes S_\mathrm{m}(\theta))\,|0_\mathrm{m}\rangle = \cos\theta\,|0_\mathrm{s}\rangle\langle 0_\mathrm{s}| + \sin\theta\,|1_\mathrm{s}\rangle\langle 1_\mathrm{s}|, \\ M_1 &= \langle 1_\mathrm{m}|\, U^{\text{C-NOT}}_{\mathrm{sm}}(\mathbb{I}_s \otimes S_\mathrm{m}(\theta))\,|0_\mathrm{m}\rangle = \sin\theta\,|0_\mathrm{s}\rangle\langle 0_\mathrm{s}| + \sin\theta\,|1_\mathrm{s}\rangle\langle 1_\mathrm{s}|, \end{aligned} \tag{18}$$

where the unsharp measurement strength is $\eta = \cos 2\theta$.

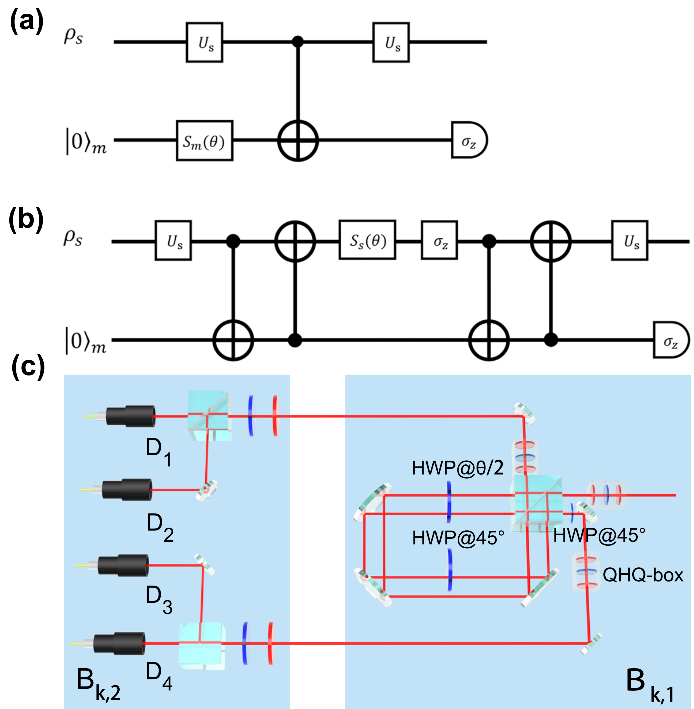

Figure 1: (a) The quantum circuit to realize unsharp measurement Eq. (15). (b) The quantum circuit to realize unsharp measurement with a Sagnac interferometer and (c) its optical realization.

Measuring system qubit in other directions $\sigma_{\vec{r}} = \vec{r} \cdot \vec{\sigma}$ are implemented by applying a unitary rotation U_{s} to the system qubit accordingly, for example, U_{s} is Hadamard gate for measurement σ_X,

$$U_{\mathrm{s}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{19}$$

and U_s for σ_Y is

$$U_{\mathrm{s}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{20}$$

In this experiment, $B_{k,1}$ encodes system qubit to the polarization state and meter qubit to the path state of photon, with the quantum circuit shown in Fig. S1 (b) and experimental realization shown in Fig. S1 (c). One can show that the quantum circuit of Fig. S1 (b) is equivalent to that of Fig. S1 (a). We set meter qubit to state $|0\rangle_{\mathrm{m}}$ ($|1\rangle_{\mathrm{m}}$) as photon propagates (counter-) clockwise in the Sagnac interferometer. Below we describe the photon traveling through the Sagnac interferometer with a unitary,

$$U_{\text{sm}}^{\text{Sagnac}} = U_{\text{ms}}^{\mathrm{C-NOT}} U_{\text{sm}}^{\mathrm{C-NOT}} \left[(S_{\text{s}}(\theta) * \sigma_Z) \otimes \mathbb{I}_{\text{m}} \right] U_{\text{ms}}^{\mathrm{C-NOT}} U_{\text{sm}}^{\mathrm{C-NOT}} \tag{21}$$

C-NOT gate $U_{\text{sm}}^{\mathrm{C-NOT}}$ with meter qubit as target is implemented by the PBS in the Sagnac interferometer. The photon passes the PBS twice and experiences $U_{\text{sm}}^{\mathrm{C-NOT}}$ twice.

C-NOT gate $U_{\text{ms}}^{\mathrm{C-NOT}}$ with system qubit as target is implemented by passing photon in state $|0\rangle_{\mathrm{m}}$ through a HWP oriented at 45° (HWP@45°). The HWP@45° inside (outside) the Sagnac interferometer is to implement the first (second) $U_{\text{ms}}^{\mathrm{C-NOT}}$.

Unitary rotation $S(\theta)$ of system qubit is implemented by passing photon through a HWP oriented at θ (HWP@$\theta/2$) in the Sagnac interferometer.

σ_Z is implemented as horizontally polarized photons gain phase π upon reflection by a mirror.

We then have

$$
\begin{aligned}
U_{\mathrm{sm}}^{\mathrm{Sagnac}} =& (\cos\theta \left|0_{\mathrm{s}}\right\rangle \left\langle 0_{\mathrm{s}}\right| + \sin\theta \left|1_{\mathrm{s}}\right\rangle \left\langle 1_{\mathrm{s}}\right|) \otimes \left|0_{\mathrm{m}}\right\rangle \left\langle 0_{\mathrm{m}}\right| + (-\sin\theta \left|0_{\mathrm{s}}\right\rangle \left\langle 1_{\mathrm{s}}\right| + \cos\theta \left|1_{\mathrm{s}}\right\rangle \left\langle 0_{\mathrm{s}}\right|) \otimes \left|0_{\mathrm{m}}\right\rangle \left\langle 1_{\mathrm{m}}\right| \\
&+ (\sin\theta \left|0_{\mathrm{s}}\right\rangle \left\langle 0_{\mathrm{s}}\right| + \cos\theta \left|1_{\mathrm{s}}\right\rangle \left\langle 1_{\mathrm{s}}\right|) \otimes \left|1_{\mathrm{m}}\right\rangle \left\langle 0_{\mathrm{m}}\right| + (\cos\theta \left|0_{\mathrm{s}}\right\rangle \left\langle 1_{\mathrm{s}}\right| - \sin\theta \left|1_{\mathrm{s}}\right\rangle \left\langle 0_{\mathrm{s}}\right|) \otimes \left|1_{\mathrm{m}}\right\rangle \left\langle 1_{\mathrm{m}}\right|
\end{aligned}
\tag{22}
$$

We then reproduce Eq. (18),

$$
\begin{aligned}
M_0 &= \left\langle 0_{\mathrm{m}}\right| U_{\mathrm{sm}}^{\mathrm{Sagnac}} \left|0_{\mathrm{m}}\right\rangle = \cos\theta \left|0_{\mathrm{s}}\right\rangle \left\langle 0_{\mathrm{s}}\right| + \sin\theta \left|1_{\mathrm{s}}\right\rangle \left\langle 1_{\mathrm{s}}\right|, \\
M_1 &= \left\langle 1_{\mathrm{m}}\right| U_{\mathrm{sm}}^{\mathrm{Sagnac}} \left|0_{\mathrm{m}}\right\rangle = \sin\theta \left|0_{\mathrm{s}}\right\rangle \left\langle 0_{\mathrm{s}}\right| + \sin\theta \left|1_{\mathrm{s}}\right\rangle \left\langle 1_{\mathrm{s}}\right|.
\end{aligned}
\tag{23}
$$

To implement arbitrary unitary U_s, we pass photons sequentially through HWP, QWP, and HWP (QHQ-box), as shown in Fig. 1(c).

From Eq. (23), we can read that photon exiting the Saganc interferometer into the upper or lower path of the setup in Fig. 1(c) corresponds to output $b_{k,1} = 0$ or $b_{k,1} = 1$ of party $B_{k,1}$, respectively.

Finally, we perform projective polarization measurements by passing photons sequentially through HWP, QWP and PBS, with outcomes $b_{k,2} = 0$ if the photon transmits the PBS and $b_{k,2} = 1$ if the photon is reflected by the PBS. The single photon detection at detectors D_1, D_2, D_3, or D_4 corresponds to one of the four outcome combination $\{b_{k,1}, b_{k,2}\}$ as shown in Tab. 1.

[htbp!]

Table 1: Outcome combination of $\{b_{k,1}, b_{k,2}\}$

Detector in Fig. 1(c)	$b_{k,1}$	$b_{k,2}$
D_1	0	0
D_2	0	1
D_3	1	1
D_4	1	0