Impact of health education on controlling and preventing the transmission of cystic echinococcosis

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Additional file 1

The expression of E_0 and E^*

Let all the right hand sizes of model (1) equal to zero, one can calculate that model (1) always has a unique disease-free equilibrium $E_0 = (S_h^0, 0, 0, S_l^0, V_l^0, 0, 0, S_d^0, 0, 0)$, where

$$S_h^0 = \frac{\Lambda_h}{d_h},\ S_l^0 = \frac{\epsilon\Lambda_l}{\kappa + d_l},\ V_l^0 = \frac{(1-\epsilon)\Lambda_l}{\kappa + d_l},\ S_d^0 = \frac{\Lambda_d}{d},\ \text{and } \epsilon = \frac{\sigma + \kappa + d_l}{\nu + \sigma + \kappa + d_l}.$$

And when $R_0 > 1$, there exists an endemic equilibrium $E^* = (S_h^*, E_h^*, I_h^*, S_l^*, V_l^*, I_l^*, D_l^*, S_d^*, I_d^*, x^*)$, where

$$\begin{aligned}
S_h^* &= \frac{\Lambda_h}{d_h} - (d_h + d_h\frac{\gamma_h + d_h}{\delta})I_h^*,\ E_h^* = \frac{\gamma_h + d_h}{\delta}I_h^*,\\
I_h^* &= \frac{(1-p)\beta_h\Lambda_h m\delta I_d^*}{(1-p)m\beta_h(d_h(\delta + d_h) + \gamma_h d_h)I_d^* + (c + d_x)(\delta + d_h)(\gamma_h + d_h)d_h},\\
S_l^* &= \frac{\epsilon\Lambda_l}{\kappa + d_l} - \frac{\epsilon(\kappa + d_l)}{\kappa + d_l}I_l^*,\ V_l^* = \frac{\nu}{\sigma + \kappa + d_l}S_l^*,\ I_l^* = \frac{bd(c + d_x)(\gamma_d + d)(\kappa + d_l)(R_0^3 - 1)}{(1-q)\beta_d(\kappa + d_l)(d(c + d_x)(\kappa + dl) + m\epsilon\beta_l\Lambda_d},\\
D_l^* &= \frac{\kappa + d_l}{b}I_l^*,\ S_d^* = \frac{\Lambda_d}{d} - I_d^*,\ I_d^* = \frac{b(c + d_x)(\gamma_d + d)(\kappa + d_l)(R_0^3 - 1)}{m\epsilon\beta_l((1-q)\beta_d\Lambda_l + b(\gamma_d + d))},\ x^* = \frac{m}{c + d_x}I_d^*.
\end{aligned}$$

The global asymptotically stability of E_0

Theorem 1. *If $R_0 < 1$, then the disease-free equilibrium E_0 of model (1) is globally asymptotically stable in Γ.*

Proof. The Jacobian matrix at E_0 is

$$J(E_0) = \begin{bmatrix} J_1 & J_2 \\ 0 & J_3 \end{bmatrix},$$

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where

$$J_1 = \begin{bmatrix} -d_h & 0 & \gamma_h \\ 0 & -(\delta + d_h) & 0 \\ 0 & \delta & -(\gamma_h + d_h) \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -(1-p)\beta_h \frac{\Lambda_h}{d_h} \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-p)\beta_h \frac{\Lambda_h}{d_h} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$J_3 = \begin{bmatrix} -(\nu+\kappa+d_l) & \sigma & 0 & 0 & 0 & 0 & -\beta_l \frac{\epsilon\Lambda_l}{\kappa+d_l} \\ \nu & -(\sigma+\kappa+d_l) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\kappa+d_l) & 0 & 0 & 0 & \beta_l \frac{\epsilon\Lambda_l}{\kappa+d_l} \\ 0 & 0 & \kappa+d_l & -b & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-q)\beta_d \frac{\Lambda_d}{d} & -d & \gamma_d & 0 \\ 0 & 0 & 0 & (1-q)\beta_d \frac{\Lambda_d}{d} & 0 & -(\gamma_d+d) & 0 \\ 0 & 0 & 0 & 0 & 0 & m & -(c+d_x) \end{bmatrix}.$$

The corresponding characteristic equation is

$$\begin{aligned} \Phi(\lambda) := {} & (\lambda + d_h)(\lambda + \gamma_h + d_h)(\lambda + \delta + d_h)(\lambda + d)(\lambda + \kappa + d_l) \\ & \times(\lambda + \nu + \sigma + \kappa + d_l)(\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4) = 0, \end{aligned}$$

where

$$\begin{aligned} a_1 &= \kappa + d_l + b + \gamma_d + d + c + d_x > 0, \\ a_2 &= (\kappa + d_l)(b + \gamma_d + d + c + d_x) + b(\gamma_d + d + c + d_x) + (\gamma_d + d)(c + d_x) > 0, \\ a_3 &= (\kappa + d_l)b(\gamma_d + d + c + d_x) + (b + \kappa + d_l)(\gamma_d + d)(c + d_x) > 0, \\ a_4 &= (\kappa + d_l)(\gamma_d + d)(c + d_x)b(1 - R_0^3). \end{aligned}$$

When $R_0 < 1$, direct calculation derives

$$\begin{aligned} H_2 &:= a_1a_2 - a_3 \\ &= (b + \kappa + d_l)(b + \gamma_d + d + c + d_x)(\gamma_d + d + c + d_x + \kappa + d_l) \\ &\quad +(\gamma_d + d + c + d_x)(\gamma_d + d)(c + d_x) > 0, \\ H_3 &:= a_3(a_1a_2 - a_3) - a_1^2a_4 \\ &> (\kappa + d_l + b)(\gamma_d + d + c + d_x)(\gamma_d + d)^2(c + d_x)^2 > 0. \end{aligned}$$

Meanwhile, $H_1 := a_1 > 0$ and $H_4 := a_4H_3 > 0$. Therefore, by Routh-Hurwitz criteria, all roots of $\Phi(\lambda)$ have negative real parts, and hence E_0 is locally stable.

Notice that the first three equations of model (1) are independent of the rest equations, we split it up into two subsystems:

$$\begin{cases} \dfrac{dS_h}{dt} = \Lambda_h - (1-p)\beta_h x S_h + \gamma_h I_h - d_h S_h, \\ \dfrac{dE_h}{dt} = (1-p)\beta_h x S_h - (\delta + d_h)E_h, \\ \dfrac{dI_h}{dt} = \delta E_h - (\gamma_h + d_h)I_h, \end{cases} \tag{1}$$

and

$$
\begin{cases}
\dfrac{dS_l}{dt} = \Lambda_l - \beta_l x S_l + \sigma V_l - (\nu + \kappa + d_l) S_l, \\
\dfrac{dV_l}{dt} = \nu S_l - (\sigma + \kappa + d_l) V_l, \\
\dfrac{dI_l}{dt} = \beta_l x S_l - (\kappa + d_l) I_l, \\
\dfrac{dD_l}{dt} = (\kappa + d_l) I_l - b D_l, \\
\dfrac{dS_d}{dt} = \Lambda_d - (1-q)\beta_d S_d D_l + \gamma_d I_d - d S_d, \\
\dfrac{dI_d}{dt} = (1-q)\beta_d S_d D_l - (\gamma_d + d) I_d, \\
\dfrac{dx}{dt} = m I_d - (c + d_x) x.
\end{cases}
\tag{2}
$$

To proof the global attractivity of E_0, we first consider subsystem (2). Let $g(x) = x - 1 - \ln x$, $x > 0$. One can easily verify that $g(x) \geq 0$ with $g(x) = 0$ if and only if $x = 1$ and $(x-1)(y-1) = g(x) + g(y) - g(xy)$ for any $x, y \in \mathbb{R}_+$. Define

$$
V_1(t) = S_l^0 g\Big(\frac{S_l}{S_l^0}\Big) + V_l^0 g\Big(\frac{V_l}{V_l^0}\Big).
$$

Differentiating V_1 along subsystem (2) and using S_l^0 and V_l^0 is the solution of subsystem (2), simple calculation implies that

$$
\begin{aligned}
\frac{dV_1(t)}{dt}\Big|_{(2)} &= \Big(1 - \frac{S_l^0}{S_l}\Big) S_l' + \Big(1 - \frac{V_l^0}{V_l}\Big) V_l' \\
&= \Big(1 - \frac{S_l^0}{S_l}\Big)\Big[- \beta_l x S_l + \sigma V_l^0 \Big(\frac{V_l}{V_l^0} - 1\Big) - (\nu + \kappa + d_l) S_l^0 \Big(\frac{S_l}{S_l^0} - 1\Big)\Big] \\
&\quad + \Big(1 - \frac{V_l^0}{V_l}\Big)\Big[\nu S_l^0 \Big(\frac{S_l}{S_l^0} - 1\Big) - (\sigma + \kappa + d_l) V_l^0 \Big(\frac{V_l}{V_l^0} - 1\Big)\Big] \\
&= -\beta_l x S_l + \beta_l x S_l^0 - (\kappa + d_l) g\Big(\frac{S_l}{S_l^0}\Big) - (\kappa + d_l) g\Big(\frac{V_l}{V_l^0}\Big) \\
&\quad - \Lambda_l g\Big(\frac{S_l^0}{S_l}\Big) - \sigma V_l^0 g\Big(\frac{S_l^0 V_l}{S_l V_l^0}\Big) - \nu S_l^0 g\Big(\frac{S_l V_l^0}{S_l^0 V_l}\Big).
\end{aligned}
\tag{3}
$$

Let

$$
(S_l(t), V_l(t), I_l(t), D_l(t), S_d(t), I_d(t), x_(t))
$$

be any solution of system (2) in Γ, then for any $t \geq 0$, we have $S_d(t) \leq \Lambda_d/d = S_d^0$. Define

$$
V_2(t) = I_l + \frac{m\beta_l S_l^0 (1-q)\beta_d S_d^0}{b(\gamma_d + d)(c + d_x)} D_l + \frac{m\beta_l S_l^0}{(\gamma_d + d)(c + d_x)} I_d + \frac{\beta_l S_l^0}{c + d_x} x.
$$

Differentiating $V_2(t)$ along subsystem (2), one can derive

$$\begin{aligned}\frac{dV_2(t)}{dt}\Big|_{(2)} &= \beta_l x S_l - (\kappa+d_l)I_l + \frac{m\beta_l S_l^0(1-q)\beta_d S_d^0}{b(\gamma_d+d)(c+d_x)}[(\kappa+d_l)I_l - bD_l] \\ &\quad + \frac{m\beta_l S_l^0}{(\gamma_d+d)(c+d_x)}[(1-q)\beta_d S_d D_l - (\gamma_d+d)I_d] + \frac{\beta_l S_l^0}{c+d_x}[mI_d - (c+d_x)x] \\ &\le \beta_l x S_l - (\kappa+d_l)I_l + \frac{m\beta_l S_l^0(1-q)\beta_d S_d^0}{b(\gamma_d+d)(c+d_x)}[(\kappa+d_l)I_l - bD_l] \qquad (4)\\ &\quad + \frac{m\beta_l S_l^0}{(\gamma_d+d)(c+d_x)}[(1-q)\beta_d S_d^0 D_l - (\gamma_d+d)I_d] + \frac{\beta_l S_l^0}{c+d_x}[mI_d - (c+d_x)x] \\ &= \beta_l x S_l - (\kappa+d_l)I_l + \frac{m\beta_l S_l^0(1-q)\beta_d S_d^0}{b(\gamma_d+d)(c+d_x)}(\kappa+d_l)I_l - \beta_l S_l^0 x.\end{aligned}$$

Consider the following Lyapunov candidate function

$$V(t) = V_1(t) + V_2(t).$$

Then combing equations (3) and (4) together and using the expression of R_0, we have

$$\begin{aligned}\frac{dV}{dt}\Big|_{(2)} &= -\Lambda_l g\Big(\frac{S_l^0}{S_l}\Big) - (\kappa+d_l)\Big[g\Big(\frac{S_l}{S_l^0}\Big) + g\Big(\frac{V_l}{V_l^0}\Big)\Big] - \sigma V_l^0 g\Big(\frac{S_l^0 V_l}{S_l V_l^0}\Big) \\ &\quad -\nu S_l^0 g\Big(\frac{S_l V_l^0}{S_l^0 V_l}\Big) - \Big(1-R_0\Big)(\kappa+d_l)I_l.\end{aligned} \qquad (5)$$

Therefore, if $R_0 \le 1$, then $dV/dt \le 0$ and $dV/dt = 0$ if and only if $S_l = S_l^0, V_l = V_l^0, I_l = 0$. It is not difficult to verify that $(S_l^0, V_l^0, 0, 0, S_d^0, 0, 0)$ is the only invariant set of subsystem (2). By LaSalle's Invariant Principle, $(S_l^0, V_l^0, 0, 0, S_d^0, 0, 0)$ is globally asymptotically stable.

Now, we consider the subsystem (1). Since $x \to 0$ as $t \to +\infty$, it is not difficult to show that $S_h(t) \to \Lambda_h/d_h, E_h(t) \to 0$ and $I_h(t) \to 0$ as $t \to +\infty$. Hence, $(\Lambda_h/d, 0, 0)$ is attractive with respect to subsystem (1). Thus, according to the theory of asymptotic autonomous systems [1], the disease-free equilibrium E_0 of system (1) is globally asymptotically stable when $R_0 \le 1$.

The global asymptotically stability of E^*

Theorem 2. *If $R_0 > 1$, then endemic equilibrium E^* of model (1) is globally asymptotically stable in Γ.*

Proof. First, we investigate the globally asymptotic stability of the endemic equilibrium of subsystem (2). Let $g(x) = x - 1 - \ln x$ and

$$V_\sharp = \sharp^* g\Big(\frac{\sharp}{\sharp^*}\Big),$$

here $\sharp$ represents $S_l, V_l, I_l, D_l, S_d, I_d$ and x. Note that $(x-1)(1-y) = g(x) + g(y) - g(xy)$ for all $x, y \in \mathbb{R}_+$. Then, using the equilibrium equation $\Lambda_l - \beta_l x^* S_l^* + \sigma V_l^* - (\nu+\kappa+d_l)S_l^* = 0$ and differentiating V_{S_l} along subsystem (2), one have

$$\begin{aligned}
\frac{dV_{S_l}}{dt}\Big|_{(2)} &= \Big(1-\frac{S_l^*}{S_l}\Big)S_l' \\
&= \Big(1-\frac{S_l^*}{S_l}\Big)[-\beta_l x S_l + \beta_l x^* S_l^* + \sigma(V_l - V_l^*) - (\nu+\kappa+d_l)(S_l - S_l^*)] \\
&= \beta_l x^* S_l^*\Big(1-\frac{S_l^*}{S_l}\Big)\Big(1-\frac{xS_l}{x^*S_l^*}\Big) + \sigma V_l^*\Big(1-\frac{S_l^*}{S_l}\Big)\Big(\frac{V_l}{V_l^*}-1\Big) \\
&\quad -(\nu+\kappa+d_l)S_l^*\Big(1-\frac{S_l^*}{S_l}\Big)\Big(\frac{S_l}{S_l^*}-1\Big) \\
&= \beta_l x^* S_l^* g\Big(\frac{x}{x^*}\Big) - \beta_l x^* S_l^* g\Big(\frac{S_l^*}{S_l}\Big) - \beta_l x^* S_l^* g\Big(\frac{xS_l}{x^*S_l^*}\Big) + \sigma V_l^* g\Big(\frac{V_l}{V_l^*}\Big) \\
&\quad +\sigma V_l^* g\Big(\frac{S_l^*}{S_l}\Big) - \sigma V_l^* g\Big(\frac{S_l^* V_l}{S_l V_l^*}\Big) - (\nu+\kappa+d_l)S_l^*\Big[g\Big(\frac{S_l^*}{S_l}\Big) + g\Big(\frac{S_l}{S_l^*}\Big)\Big] \\
&= \beta_l x^* S_l^* g\Big(\frac{x}{x^*}\Big) - \Lambda_l g\Big(\frac{S_l^*}{S_l}\Big) - \beta_l x^* S_l^* g\Big(\frac{xS_l}{x^*S_l^*}\Big) + \sigma V_l^* g\Big(\frac{V_l}{V_l^*}\Big) \\
&\quad -\sigma V_l^* g\Big(\frac{S_l^* V_l}{S_l V_l^*}\Big) - (\nu+\kappa+d_l)S_l^* g\Big(\frac{S_l}{S_l^*}\Big).
\end{aligned} \tag{6}$$

Using the equilibrium equation $\nu S_l^* - (\sigma+\kappa+d_l)V_l^* = 0$ and differentiating V_{V_l} along subsystem (2), one derive

$$\begin{aligned}
\frac{dV_{V_l}}{dt}\Big|_{(2)} &= \Big(1-\frac{V_l^*}{V_l}\Big)V_l' \\
&= \Big(1-\frac{V_l^*}{V_l}\Big)[\nu(S_l - S_l^*) - (\sigma+\kappa+d_l)(V_l - V_l^*)] \\
&= \nu S_l^*\Big(1-\frac{V_l^*}{V_l}\Big)\Big(\frac{S_l}{S_l^*}-1\Big) - (\sigma+\kappa+d_l)V_l^*\Big(1-\frac{V_l^*}{V_l}\Big)\Big(\frac{V_l}{V_l^*}-1\Big) \\
&= \nu S_l^* g\Big(\frac{S_l}{S_l^*}\Big) - \nu S_l^* g\Big(\frac{S_l V_l^*}{S_l^* V_l}\Big) + \nu S_l^* g\Big(\frac{V_l^*}{V_l}\Big) \\
&\quad -(\sigma+\kappa+d_l)V_l^* g\Big(\frac{V_l^*}{V_l}\Big) - (\sigma+\kappa+d_l)V_l^* g\Big(\frac{V_l}{V_l^*}\Big) \\
&= \nu S_l^* g\Big(\frac{S_l}{S_l^*}\Big) - \nu S_l^* g\Big(\frac{S_l V_l^*}{S_l^* V_l}\Big) - (\sigma+\kappa+d_l)V_l^* g\Big(\frac{V_l}{V_l^*}\Big).
\end{aligned} \tag{7}$$

Similarly, we have

$$
\begin{aligned}
\frac{dV_{I_l}}{dt}\Big|_{(2)} &= \beta_l x^* S_l^* g\Big(\frac{xS_l}{x^*S_l^*}\Big) - \beta_l x^* S_l^* g\Big(\frac{xS_lI_l^*}{x^*S_l^*I_l}\Big) - (\kappa+d_l)I_l^* g\Big(\frac{I_l}{I_l^*}\Big), \\
\frac{dV_{D_l}}{dt}\Big|_{(2)} &= (\alpha+\kappa+d_l)I_l^* g\Big(\frac{I_l}{I_l^*}\Big) - (\alpha+\kappa+d_l)I_l^* g\Big(\frac{I_lD_l^*}{I_l^*D_l}\Big) - bD_l^* g\Big(\frac{D_l}{D_l^*}\Big), \\
\frac{dV_{S_d}}{dt}\Big|_{(2)} &= -\Lambda_d g\Big(\frac{S_d^*}{S_d}\Big) - (1-q)\beta_d S_d^* D_l^* g\Big(\frac{S_dD_l}{S_d^*D_l^*}\Big) + (1-q)\beta_d S_d^* D_l^* g\Big(\frac{D_l}{D_l^*}\Big) \\
&\quad + \gamma_d I_d^* g\Big(\frac{I_d}{I_d^*}\Big) - \gamma_d I_d^* g\Big(\frac{S_d^*I_d}{S_dI_d^*}\Big) - dS_d^* g\Big(\frac{S_d}{S_d^*}\Big), \\
\frac{dV_{I_d}}{dt}\Big|_{(2)} &= (1-q)\beta_d S_d^* D_l^* g\Big(\frac{S_dD_l}{S_d^*D_l^*}\Big) - (1-q)\beta_d S_d^* D_l^* g\Big(\frac{S_dD_lI_d^*}{S_d^*D_l^*I_d}\Big) - (\gamma_d+d)I_d^* g\Big(\frac{I_d}{I_d^*}\Big), \\
\frac{dV_x}{dt}\Big|_{(2)} &= mI_d^* g\Big(\frac{I_d}{I_d^*}\Big) - mI_d^* g\Big(\frac{x^*I_d}{xI_d^*}\Big) - (c+d_x)x^* g\Big(\frac{x}{x^*}\Big).
\end{aligned}
\tag{8}
$$

Then, consider the following Lyapunov candidate function

$$
\begin{aligned}
\bar{V} = \quad & V_{S_l} + V_{V_l} + V_{I_l} + \frac{m\beta_l S_l^*(1-q)\beta_d S_d^*}{b(\gamma_d+d)(c+d_x)} V_{D_l} \\
& + \frac{m\beta_l S_l^*}{b(\gamma_d+d)(c+d_x)}(V_{S_d}+V_{I_d}) + \frac{\beta_l S_l^*}{c+d_x} V_x.
\end{aligned}
\tag{9}
$$

It follows from the equations $\beta_l x^* S_l^* - (\kappa+d_l)I_l^* = 0$ and $(1-q)\beta_d S_d^* D_l^* - (\gamma_d+d)I_d^* = 0$ that

$$
\frac{m\beta_l S_l^*(1-q)\beta_d S_d^*}{b(\gamma_d+d)(c+d_x)} = \frac{m\frac{(\kappa+d_l)I_l^*}{x^*}\cdot\frac{(\gamma_d+d)I_d^*}{D_l^*}}{b(\gamma_d+d)(c+d_x)}.
$$

Notice that $D_l^* = \frac{\kappa+d_l}{b} I_l^*$ and $x^* = \frac{m}{c+d_x} I_d^*$, then

$$
\frac{m\beta_l S_l^*(1-q)\beta_d S_d^*}{b(\gamma_d+d)(c+d_x)} = 1 \tag{10}
$$

Combing equations (6) (7), (8) together and using equation (10), we have

$$
\begin{aligned}
\frac{d\bar{V}}{dt}\Big|_{(2)} = \quad & -\Lambda_l g\Big(\frac{S_l^*}{S_l}\Big) - \sigma V_l^* g\Big(\frac{S_l^*V_l}{S_lV_l^*}\Big) - (\kappa+d_l)S_l^* g\Big(\frac{S_l}{S_l^*}\Big) - \nu S_l^* g\Big(\frac{S_lV_l^*}{S_l^*V_l}\Big) \\
& -(\kappa+d_l)V_l^* g\Big(\frac{V_l}{V_l^*}\Big) - \beta_l x^* S_l^* g\Big(\frac{xS_lI_l^*}{x^*S_l^*I_l}\Big) - \frac{m\beta_l S_l^* \Lambda_d}{(\gamma_d+d)(c+d_x)} g\Big(\frac{S_d^*}{S_d}\Big) \\
& -\frac{m\beta_l S_l^*(1-q)\beta_d S_d^*(\alpha+\kappa+d_l)I_l^*}{b(\gamma_d+d)(c+d_x)} g\Big(\frac{I_lD_l^*}{I_l^*D_l}\Big) - \frac{m\beta_l S_l^*(1-q)\beta_d S_d^* D_l^*}{(\gamma_d+d)(c+d_x)} g\Big(\frac{S_dD_lI_d^*}{S_d^*D_l^*I_d}\Big) \\
& -\frac{m\beta_l S_l^* \gamma_d I_d^*}{(\gamma_d+d)(c+d_x)} g\Big(\frac{S_d^*I_d}{S_dI_d^*}\Big) - \frac{md\beta_l S_l^* S_d^*}{(\gamma_d+d)(c+d_x)} g\Big(\frac{S_d}{S_d^*}\Big) - \frac{m\beta_l S_l^* I_d^*}{c+d_x} g\Big(\frac{x^*I_d}{xI_d^*}\Big).
\end{aligned}
\tag{11}
$$

Therefore, $d\bar{V}/dt \le 0$ and $d\bar{V}/dt = 0$ if and only if $S_l = S_l^*, V_l = V_l^*, I_l = I_l^*, D_l = D_l^*, S_d = S_d^*, I_d = I_d^*$ and $x = x^*$, that is, $(S_l^*, V_l^*, I_l^*, D_l^*, S_d^*, I_d^*, x^*)$ is the only invariant set of subsystem (2). It follows from LaSalle's Invariant Principle that $(S_l^*, V_l^*, I_l^*, D_l^*, S_d^*, I_d^*, x^*)$ is globally asymptotically stable.

The pervious proof implies that $x \to x^*$ as $t \to +\infty$. Then, the limiting system of subsystem (1) is

$$\begin{cases} \dfrac{dS_h}{dt} = \Lambda_h - (1-p)\beta_h x^* S_h + \gamma_h I_h - d_h S_h, \\ \dfrac{dE_h}{dt} = (1-p)\beta_h x^* S_h - (\delta + d_h)E_h, \\ \dfrac{dI_h}{dt} = \delta E_h - (\gamma_h + d_h)I_h, \end{cases} \tag{12}$$

Using the Lyapunov candidate function

$$\tilde{V} = S_h^* g\Big(\frac{S_h}{S_h^*}\Big) + E_h^* g\Big(\frac{E_h}{E_h^*}\Big) + I_h^* g\Big(\frac{I_h}{I_h^*}\Big),$$

one can easily derive that $d\tilde{V}/dt|_{(12)} \leq 0$ and $d\tilde{V}/dt|_{(12)} = 0$ if and only if $S_h = S_h^*, E_h = E_h^*, I_h = I_h^*$. Therefore, (S_h^*, E_h^*, I_h^*) is the only invariant set of (12) and is globally asymptotically stable based on the LaSalle's Invariant Principle. Hence, according to the theory of asymptotic autonomous systems [1], if $R_0 > 1$ the unique endemic equilibrium E^* of system (1) is globally asymptotically stable.

References

[1] Thieme HR. Convergence results and a Poincare-Bendixson trichotomy for asymptotically autonomous differential equations. Journal of Mathematical Biology. 1992, 30: 755-763.