Supplementary Materials of manuscript: Sequential decision making for a class of hidden Markov processes, application to medical treatment optimisation

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1 Dynamics of the continuous and discrete time Markov processes

1.1 Local characteristics of the controlled continuous-time PDMP

Recall that we consider a continuous-time Markov process with jumps, in the class of PDMPs which has both discrete and Euclidean components. The discrete component or mode $m \in \{0, 1, 2, 3\}$ corresponds to the overall state of the patient ($m = 0$: remission, $m = 1$: disease 1, $m = 2$: disease 2, $m = 3$: death of the patient) and the three Euclidean variables (ζ, u, w) corresponding to

- some marker of the disease $\zeta \in [\zeta_0, D]$ that can be measured and is the main variable of interest together with the mode,
- the time since the last jump of the process $u \in [0, H]$,
- the time since the beginning $w \in [0, H]$,

where $H > 0$ is the time horizon of the study, $\zeta_0 < D$ are real numbers.
The hybrid state space of the process is $E = \cup_{m=0}^{3} E_m$ where $E_0 = \{0\} \times \{\zeta_0\} \times [0, H]^2$, $E_m = \{m\} \times (\zeta_0, D) \times [0, H]^2$ for $m \in \{1, 2\}$, $E_3 = \{3\} \times \{D\} \times [0, H]^2$. Denote also and $E_{0:2} = \cup_{m=0}^{2} E_m$, $\mathscr{B}(E)$ the Borel σ-field of E, $\overline{E}$ the closure of E and ∂E its boundary: $\partial E = \{1, 2\} \times \{\zeta_0, D\} \times [0, H]^2$. For any $x = (m, \zeta, u, w) \in E$, we will also denote x_i the coordinates of vector x: $x_1 = m$, $x_2 = \zeta$, $x_3 = u$, $x_4 = w$.

The controlled transition kernels are denoted as $P : \mathscr{B}(E) \times \overline{E} \times \big(\{\emptyset, a, b\} \times \mathbb{T}\big) \to [0, 1]$: for any Borel subset A of the state space E, any $x \in \overline{E}$ and $d = (\ell, r) \in \{\emptyset, a, b\} \times \mathbb{T}$, one has

$$P\big(A|x, d = (\ell, r)\big) = \mathbb{P}\big(X_r \in A | X_0 = x, d = (\ell, r)\big).$$

and are explixited in Section 1.2.

The minimal requirements on the local characteristics of the PDMP are as follows, for any $d = (\ell, r) \in \{\emptyset, a, b\} \times \mathbb{T}$.

$\star$ Concerning the flow, one has

- in any mode m and for any action d, the flow $\mathbf{\Phi}^d : \{0, 1, 2, 3\} \times [\zeta_0, D] \times \mathbb{R}_+^2 \times \mathbb{R}_+ \to \{0, 1, 2, 3\} \times [\zeta_0, D] \times \mathbb{R}_+^2$ has the special form given for any $t \geq 0$ by $\mathbf{\Phi}^d\big((m, \zeta, u, w), t\big) = (m, \mathbf{\Phi}_m^d(m, \zeta, u, w, t)) = (m, \Phi_m^d(\zeta, t), u + t, w + t)$; it is continuous and $t \to \mathbf{\Phi}(x, t)$ has the semi-group property ;
- in mode $m = 0$ the flow is constant for the marker for any decision d: $\Phi_0^d(\zeta_0, t) = \zeta_0$ for all $t \geq 0$;
- in modes $m = 1$ and $m = 2$, for any decision $d = (\ell, r)$, and any $\zeta \in (\zeta_0, D)$, the application $t \mapsto \Phi_m^d(\zeta, t)$ is
  - non-increasing if $(m, \ell) \in \{(1, a), (2, b)\}$,
  - non-decreasing if $(m, \ell) \notin \{(1, a), (2, b)\}$.

Table 1: Markov kernel of the controlled continuous time PDMP

	$\ell=\emptyset$			
	$m'=0$	$m'=1$	$m'=2$	$m'=3$
$m=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)\in(0,1)$	$Q^d_m(m'\|\zeta,u,w)$ $=1-Q^d_m(1\|\zeta,u,w)\in(0,1)$	$Q^d_m(m'\|\zeta,u,w)=0$
$m=1$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=\mathbb{1}_{\zeta=D}$
$m=2$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=\mathbb{1}_{\zeta=D}$
$m=3$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$
	$\ell=a$			
	$m'=0$	$m'=1$	$m'=2$	$m'=3$
$m=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\zeta,u,w)=1$	$Q^d_m(m'\|\zeta,u,w)=0$
$m=1$	$Q^d_m(m'\|\zeta,u,w)=\mathbb{1}_{\zeta=\zeta_0}$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\zeta,u,w)=\mathbb{1}_{\zeta\neq\zeta_0}$	$Q^d_m(m'\|\zeta,u,w)=0$
$m=2$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=\mathbb{1}_{\zeta=D}$
$m=3$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$
	$\ell=b$			
	$m'=0$	$m'=1$	$m'=2$	$m'=3$
$m=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=1$	$Q^d_m(m'\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$
$m=1$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=\mathbb{1}_{\zeta=D}$
$m=2$	$Q^d_m(m'\|\zeta,u,w)=\mathbb{1}_{\zeta=\zeta_0}$	$Q^d_m(m'\|\zeta,u,w)=\mathbb{1}_{\zeta\neq\zeta_0}$	$Q^d_m(m'\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$
$m=3$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$	$Q^d_m(m'\zeta,u,w)=0$	$Q^d_m(m'\|\zeta,u,w)=0$

- in mode $m=3$ the flow is constant for the marker for any decision d: $\Phi^d_3(D,t)=D$ for all $t\geq 0$;

⋆ Concerning the jump intensity, one has

- the jump intensity $\lambda^d:\{0,1,2,3\}\times[\zeta_0,D]\times\mathbb{R}^2_+\to\mathbb{R}_+$ is denoted by $\lambda^d(m,\zeta,u,w)=\lambda^d_m(\zeta,u,w)$ and satisfies for all $(m,\zeta,u,w)\in E$
$$\int_0^\epsilon \lambda^d_m(\Phi^d_m(\zeta,s),u+s,w+s)ds<+\infty,$$
for some positive ϵ;
- if $(m,d)\in\{(1,\emptyset),(2,\emptyset),(1,b),(2,a)\}$, then $\lambda^d_m=0$. This is a technical assumption to limit the number of jumps in a given time interval;
- if $m=3$, then $\lambda^d_3=0$ for all d: no random jump is possible after death.

⋆ Concerning the jump kernel, one has

- the jump kernel $\mathbf{Q}^d$ on $\overline{E}\times\mathscr{B}(\overline{E})$ is denoted by $\mathbf{Q}^d(A|x=(m,\zeta,u,w))=Q^d_m(A|\zeta,u,w)$ for any Borel subset A of $\overline{E}$ and satisfies $\mathbf{Q}^d(E|x)=1$ for all $x\in\overline{E}$ and for $\zeta\in[\zeta_0,D]$, $u\in[0,H]$ and $w\in[0,H]$
$$Q^d_m(\{(m',\zeta,0,w)\}|\zeta,u,w)=Q^d_m(m'|\zeta,u,w),$$
for all $d=(\ell,r)$, where the values of $Q^d_m(\zeta,u,w)$ are given in Table 1.

For all $x=(m,\zeta,u,w)\in E$ and $d=(\ell,r)\in\{\emptyset,a,b\}\times\mathbb{T}$, set
$$\mathbf{\Lambda}^d(x,t)=\Lambda^d_m(\zeta,u,w,t)=\int_0^{t\wedge t^{*d}_m(\zeta)}\lambda^d_m(\Phi^d_m(\zeta,s),u+s,w+s)ds,$$
the cumulated jump intensity up to time t (or reaching the boundary), starting from (m,ζ,u,w) and following action d.

Assumption 2.1 of the main manuscript corresponds to two mathematical conditions: if $d=(\ell,r)$ with $(m,\ell)\in\{(1,\emptyset),(2,\emptyset),(1,b),(2,a)\}$ and $r\in\mathbb{T}$, one has

- $t^{*d}_m(\zeta_0)>r$ (a sound patient cannot die before the next visit) ,
- $\lambda^d_m=0$ (a patient cannot switch diseases back and forth in between two visits).

1.2 Skeleton kernels of the controlled continuous-time PDMP

The generic form of the transition kernels P of the skeleton chains with time span r is formally given below. As there are at most 2 jumps (at most one random jump and 1 boundary jump), the generic form of P for all $x = (m, \zeta, u, w) \in E$, $d = (\ell, r) \in \{\emptyset, a, b\} \times \mathbb{T}$ such that $w + r \leq H$ and any bounded measurable function h on E is

$$
\begin{aligned}
&Ph(x, d) \\
=&\mathbb{E}[h(X_r)|X_0 = x, d = (\ell, r)] \\
=&h(\mathbf{\Phi}^d(x, r))e^{-\mathbf{\Lambda}^d(x,r)}\mathbb{1}_{\mathbf{t}^{*d}(x)>r} &(1)\\
&+\int_0^r \int_E h(\mathbf{\Phi}^d(x', r - s))\lambda^d(\mathbf{\Phi}^d(x, s))e^{-\mathbf{\Lambda}^d(x,s)}\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x, s))\mathbb{1}_{\mathbf{t}^{*d}(x)>s}\mathbb{1}_{\mathbf{t}^{*d}(x')>r-s}ds &(2)\\
&+\int_E h(\mathbf{\Phi}^d(x', r - \mathbf{t}^{*d}(x)))e^{-\mathbf{\Lambda}^d(x,\mathbf{t}^{*d}(x))}e^{-\mathbf{\Lambda}^d(x',r-\mathbf{t}^{*d}(x))} \\
&\qquad\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x, \mathbf{t}^{*d}(x)))\mathbb{1}_{\mathbf{t}^{*d}(x)\leq r}\mathbb{1}_{\mathbf{t}^{*d}(x')>r-\mathbf{t}^{*d}(x)} &(3)\\
&+\int_0^{r-\mathbf{t}^{*d}(x)} \int_{E^2} h(\mathbf{\Phi}^d(x'', r - \mathbf{t}^{*d}(x) - s))e^{-\mathbf{\Lambda}^d(x,\mathbf{t}^{*d}(x))}\lambda^d(\mathbf{\Phi}^d(x', s))e^{-\mathbf{\Lambda}^d(x',s)} \\
&\qquad\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x, \mathbf{t}^{*d}(x)))\mathbf{Q}^d(dx''|\mathbf{\Phi}^d(x', s)) \\
&\qquad\mathbb{1}_{\mathbf{t}^{*d}(x)\leq r}\mathbb{1}_{\mathbf{t}^{*d}(x')>r-\mathbf{t}^{*d}(x)-s}\mathbb{1}_{\mathbf{t}^{*d}(x'')>r-\mathbf{t}^{*d}(x)-s}ds &(4)\\
&+\int_0^r \int_{E^2} h(\mathbf{\Phi}^d(x'', r - \mathbf{t}^{*d}(x') - s))\lambda^d(\mathbf{\Phi}^d(x, s))e^{-\mathbf{\Lambda}^d(x,s)} \\
&\qquad\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x, s))\mathbf{Q}^d(dx''|\mathbf{\Phi}^d(x', \mathbf{t}^{*d}(x')))\mathbb{1}_{\mathbf{t}^{*d}(x)>s}\mathbb{1}_{\mathbf{t}^{*d}(x')\leq r-s}ds. &(5)
\end{aligned}
$$

- Line (1) corresponds to the events where no jump has occurred and the process just followed the deterministic flow.
- Lines (2) and (3) correspond to the events where a single jump occurs. This jump can be
  - either a random jump (Line (2))
  - or a boundary jump at either ζ_0 or D (Line (3)).
- Lines (4) and (5) correspond to the events where two jumps occur, which can happen in either of two ways:
  - a boundary jump at ζ_0 followed by a random jump (Line (4)).
  - a random jump followed by a boundary jump at D (Line (5)).

Depending on the values of x and d, some terms may have zero value in the formula above.

1.3 Formal definition of the partially observed and fully observed MDPs

1.3.1 Partially observed MDP

The formal definition of our MDP is as follows:

- the state space is $\mathbb{X} = \{\xi = (x, y, z, w) \in (E \times \mathbb{O}); z = \mathbb{1}_{\{x_1=3\}}, x_4 = w\} \cup \{\Delta\}$, with $\mathbb{O} = \{\gamma \in I \times \{0, 1\} \times \delta^{1:N}\}$ the observation space, and gathers the values of the hidden PDMP at dates t_n as well as the observation processes Y_n, Z_n and W_n up to time H and a cemetery state Δ where the process is sent after the horizon H or the death of the patient is reached;
- the action space is $\mathbb{A} = (\{\emptyset, a, b\} \times \mathbb{T}) \times \{\check{d}\}$, where $\check{d}$ is an empty decision that is taken when the horizon H is reached or when the patient has died. This purely technical decision sends the process to the cemetery state Δ;

- the constraints set $\mathbb{K} \subset \mathbb{X} \times \mathbb{A}$ is such that its sections $\mathbb{K}(\xi) = \{d \in \mathbb{A}; (\xi, d) \in \mathbb{K}\}$ satisfy $\mathbb{K}(\Delta) = \check{d}$ and $\mathbb{K}(x, y, z, w) = \mathbb{K}(z, w)$ for $(x, y, z, w) \in \mathbb{X} - \{\Delta\}$ as decisions are taken in view of the information from the observations only. In addition, one has
  - $\mathbb{K}(0, w) = \{\emptyset, a, b\} \times (\mathbb{T} \cap \{\delta, 2\delta, \dots, N\delta - w\})$ if $w \leq \delta(N-1)$, as the patient has to come back for a last visit at horizon time H, unless the patient is dead ($z = 1$) and no treatment is possible,
  - $\mathbb{K}(0, N\delta) = \mathbb{K}(1, w) = \check{d}$.

  Note that as $\delta \in \mathbb{T}$, the set $\mathbb{T} \cap \{\delta, 2\delta, \dots, N\delta - w\}$ is never empty;

- the terminal cost function $C : \mathbb{X} \to \mathbb{R}_+$ satisfies $C(\Delta) = 0$ and $C(x, y, z, w) = C(x)$ for $(x, y, z, w) \in \mathbb{X} - \{\Delta\}$ with $C(x) = M$ if $x \in E_3$ and

$$0 < \sup_{x \in E_{0:2}} C(x) < M < +\infty,$$

  where M is a penalty for the death of the patient. Note that this cost is only paid when horizon H is reached, and does not depend on the number of observations acquired to reach H;

- the non-negative cost-per-stage function $c : \mathbb{K} \times \mathbb{X} \to \mathbb{R}_+$ satisfies

$$\begin{aligned} c(\Delta, \check{d}, \cdot) &= 0, \\ c(\xi, \check{d}, \cdot) &= C(\xi), \\ c(\xi, d, \xi') &= c(x, d, x'), \end{aligned}$$

  for $\xi = (x, \gamma) \in \mathbb{X} - \{\Delta\}$ with $x \in E_{0:2}$, $d \neq \check{d}$, $\xi' = (x', \gamma') \in \mathbb{X} - \{\Delta\}$, with $c(\xi, d, \xi') = M$ if $x' \in E_3$ and

$$0 < \sup_{(x,d,x') \in E_{0:2} \times (\mathbb{A} - \{\check{d}\}) \times E_{0:2}} c(x, d, x') < M < +\infty;$$

- the optimisation horizon is a fixed number N corresponding to date $N\delta = H$;

- the controlled transition kernels R are defined as follows: for any bounded measurable function g on $\mathbb{X}$, any $\xi \in \mathbb{X}$ and $d \in \mathbb{K}(\xi)$, one has

$$\begin{aligned} Rg(\xi, d) &= g(\Delta), \quad \text{if } d = \check{d}, \\ Rg(\xi, d) &= \int_I \int_{E_{0:2}} g(x', y', 0, w + r) f(y' - F(x')) P(dx'|x, d) dy' \\ &\quad + \int_{E_3} g(x', 0, 1, w + r) P(dx'|x, d), \quad \text{if } \xi = (x, y, 0, w),\ d = (\ell, r) \neq \check{d}, \\ Rg(\xi, d) &= 0, \quad \text{otherwise.} \end{aligned}$$

The sets of observable histories are defined recursively by $H_0 = \mathbb{O} \cup \{\Delta\}$ and $H_n = H_{n-1} \times \mathbb{A} \times (\mathbb{O} \cup \{\Delta\})$. An element of H_n is called an observable history up to observation time n. A decision rule at time n is a measurable mapping $g_n : H_n \to \mathbb{A}$ such that $g_n(h_n) \in \mathbb{K}(\gamma_n)$ for all histories $h_n = (\gamma_0, a_0, \gamma_1, a_1, \dots, \gamma_n)$. A sequence $\pi = (g_0, \dots, g_n, \dots, g_{N-1})$ where g_k is a decision rule at time k is called an admissible policy. Let Π_N denote the set of all admissible policies.

The controlled trajectory of the (partially observed) MDP following policy $\pi = (g_n)_{0:N-1} = (g_0, \dots, g_n, \dots, g_{N-1}) \in \Pi_N$ is defined recursively by

- $\Xi_0 \in \mathbb{X}$,

- for $0 \leq n \leq N - 1$,
  - $A_n = g_n(\Xi_n)$,

– $\Xi_{n+1} \sim R(\cdot|\Xi_n, A_n)$.

Note that the cemetery state Δ ensures that all trajectories have the same length N, even if they don't have the same number of actual observation dates t_n. Then can define the cost of policy $\pi \in \Pi_N$ starting at $\xi_0 \in \mathbb{X}$ as

$$J_\pi(\xi_0) = \mathbb{E}^\pi_{\xi_0}\left[\sum_{n=0}^{N-1} c(\Xi_n, A_n, \Xi_{n+1}) + C(\Xi_N)\right],$$

and our control problem corresponds to the optimisation problem

$$V(\xi_0) = \inf_{\pi\in\Pi_N} J_\pi(\xi_0).$$

1.3.2 Fully observed MDP

Recall that Θ_n is such that for any Borel subset A of E, and $\Xi_n = (X_n, Y_n, Z_n, W_n)$ in $\mathbb{X} - \{\Delta\}$,

$$\Theta_n(A) \quad = \quad \mathbb{P}(X_n \in A|\mathscr{F}^O_n) = \mathbb{P}(\Xi_n \in A \times \{(Y_n, Z_n, W_n)\}|\mathscr{F}^O_n),$$

Proposition 1.1 *For any $n \geq 0$, one has:*

1. *Conditionally on $(Y_{n+1}, Z_{n+1}, W_{n+1}) = (y', 0, w')$, $d = (\ell, r) \in (\{\emptyset, a, b\} \times \mathbb{T})$ and $\Theta_n = \theta$, one has $\Theta_{n+1}(E_{0:2}) = 1$ and more specifically $\Theta_{n+1} = \Psi(\theta, y', 0, w', d)$ with*

$$\Psi(\theta, y', 0, w', d)(A) = \frac{\int_{E_{0:2}} \mathbb{1}_{w'-r}(x_4) \int_{E_{0:2}} f(y' - F(x'))\mathbb{1}_A(x')P(dx'|x, d)\theta(dx)}{\int_{E_{0:2}} \mathbb{1}_{w'-r}(x_4) \int_{E_{0:2}} f(y' - F(x'))P(dx'|x, d)\theta(dx)},$$

   for any Borel subset A of $E_{0:2}$.

2. *Conditionally on $(Y_{n+1}, Z_{n+1}, W_{n+1}) = (y', 1, w')$, $d = (\ell, r) \in (\{\emptyset, a, b\} \times \mathbb{T})$ and $\Theta_n = \theta$, one has $\Theta_{n+1}(E_3) = 1$, and more specifically $\Theta_{n+1} = \Psi(\theta, 0, 1, w', d)$ with*

$$\Psi(\theta, 0, 1, w', d)(A) = \frac{\int_{E_{0:2}} \mathbb{1}_{w'-r}(x_4)\big(\int_{E_3} \mathbb{1}_A(x')P(dx'|x, d)\big)\theta(dx)}{\int_{E_{0:2}} \mathbb{1}_{w'-r}(x_4)\big(\int_{E_3} P(dx'|x, d)\big)\theta(dx)},$$

   for any Borel subset A of E_3.

Let $\mathscr{P}(E)$ denote the set of probability measures on E, $\mathscr{P}(E_{0:2})$ the set of probability measures on $E_{0:2}$, and $\mathscr{P}(E_3)$ the set of probability measures on E_3. The new MDP is defined as follows:

- the state space is a subset $\mathbb{X}'$ of $(P(E) \times \mathbb{O}) \cup \{\Delta\}$ satisfying the following constraints: all $\xi = (\theta, y, z, w) \in \mathbb{X}'$ satisfy
  - $\theta(\{x \in E; x_4 = w\}) = 1$;
  - $\theta(E_{0:2}) = 1$ or $\theta(E_3) = 1$ and if $\theta(E_3) = 1$, then $y = 0$ and $z = 1$;
  - if $\theta(E_{0:2}) = 1$, then $\theta(E_0) \geq \big(\underline{f}\,\overline{f}^{-1}\big)^{\frac{w}{\delta}\vee 1} e^{-w\|\lambda\|}$, where the constants $\underline{f}$, $\overline{f}$ and $\|\lambda\|$ are defined in Assumptions .1 and .3 in the Appendix.

  It is necessary to restrict the state space to ensure the regularity of our operators. The first constraint imposes adequation between the observed elapsed time w and the process' spent time. The second constraint comes from the fact that the death of the patient is observed. As the filter (Θ_n) is adapted to the filtration $(\mathscr{F}^O_n)$, it charges E_3 accordingly. The last constraint is more technical, and is guaranteed as soon as the process starts in mode 0.

- the action space is still $\mathbb{A} = (\{\emptyset, a, b\} \times \mathbb{T}) \times \{\check{d}\}$;

- the constraints set $\mathbb{K}' \subset \mathbb{X}' \times \mathbb{A}$ is such that its sections $\mathbb{K}'(\xi) = \{d \in \mathbb{A}; (\xi, d) \in \mathbb{K}'\}$ satisfy $\mathbb{K}'(\Delta) = \check{d}$ and $\mathbb{K}'(\theta, y, z, w) = \mathbb{K}'(z, w) = \mathbb{K}(z, w)$ for $(\theta, y, z, w) \in \mathscr{P}(E) \times \mathbb{O}$

- the non-negative cost-per-stage function $c' : \mathbb{K}' \to \mathbb{R}_+$ and the terminal cost function $C' : \mathbb{X}' \to \mathbb{R}_+$ are defined by $C'(\Delta) = c'(\Delta, \check{d}, \cdot) = 0$ and for $\xi = (\theta, \gamma) \in \mathscr{P}(E) \times \mathbb{O}$ and $d \in \mathbb{K}'(\xi)$,

$$\begin{aligned} c'(\xi, d) = c'(\theta, \gamma, d) &= \int_{E^2} c(x, d, x')P(dx'|x, d)\theta(dx), \\ C'(\xi) = C'(\theta, \gamma) &= \int_E C(x)\theta(dx); \end{aligned}$$

- the optimisation horizon is still N;
- the controlled transition kernels R' are defined as follows: for any bounded measurable function g on $\mathbb{X}'$, any $\xi \in \mathbb{X}'$ and $d \in \mathbb{K}'(\xi)$, one has

$$\begin{aligned} &R'g(\xi, d) = g(\Delta), \quad \text{if } d = \check{d}, \\ &R'g(\xi, d) = \int_{E_{0:2}} \int_I \int_{E_{0:2}} g\big(\Psi(\theta, y', 0, w + r, d), y', 0, w + r\big) f(y' - F(x'))P(dx'|x, d)dy'\theta(dx) \\ &\qquad + \int_{E_{0:2}} \int_{E_3} g(\Psi(\theta, 0, 1, w + r, d), 0, 1, w + r)P(dx'|x, d)\theta(dx), \\ &\qquad \text{if } \xi = (\theta, y, 0, w), d = (\ell, r), \\ &R'g(\xi, d) = 0, \quad \text{otherwise.} \end{aligned}$$

  See Section 1.4 for the proof that R' maps $\mathbb{X}'$ onto itself.

Denote (Ξ'_n) a trajectory of the fully observed MDP. The cost of a strategy π in Π_N is

$$J'(\pi, \xi'_0) = \mathbb{E}^{\pi}_{\xi'_0}\left[\sum_{n=0}^{N-1} c'(\Xi'_n, A_n) + C'(\Xi'_N)\right],$$

and the value function of the fully observed problem is,

$$V'(\xi'_0) = \inf_{\pi \in \Pi_N} J'(\pi, \xi'_0).$$

1.4 Kernel R' maps $\mathbb{X}'$ onto itself

Set $\xi \in \mathbb{X}'$ and $d \in \mathbb{K}'(\xi)$, and denote Ξ a random variable with distribution $R'(\cdot|\xi, d)$.

If $d = \check{d}$ then $\Xi = \Delta \in \mathbb{X}'$.

Otherwise, by definition one has $\xi = (\theta, y, z, w)$ such that

- $\theta(\{x \in E; x_4 = w\}) = 1$;
- $\theta(E_{0:2}) = 1$ or $\theta(E_3) = 1$;
- if $\theta(E_{0:2}) = 1$, then $\theta(E_0) \geq \left(\frac{\underline{f}}{\overline{f}}\right)^{\frac{w}{\delta} \vee 1} e^{-w\|\lambda\|}$;
- $\theta(E_3) = 1$, then $y = 0$ and $z = 1$,

and $d = (\ell, r) \in \mathbb{K}'(\xi)$.

If $\theta(E_3) = 1$, then $y = 0$ and $z = 1$ so that $d = \check{d}$ and this case has already been treated.

Finally, if $\theta(E_{0:2}) = 1$, then either $\Xi = (\Theta, Y, Z, W)$ with $Y = 0$, $Z = 1$, $W = w + r$ and therefore $\Theta(E_3) = 1$, or $\Xi = (\Theta, Y, Z, W)$ with $Z = 0$, $W = w + r$ and conditionally to Y,

$$\Theta = \Psi(\theta, Y, 0, w + r, d).$$

Again, the definition of Ψ directly yields $\Theta(E_{0:2}) = 1$, $\Theta(\{x \in E; x_4 = w + r\}) = 1$. To obtain the lower bound on $\Theta(E_0)$, note that

$$\Theta(E_0) = \frac{\int_{E_{0:2}} \int_{E_{0:2}} f(Y - F(x'))\mathbb{1}_{E_0}(x')P(dx'|x, d)\theta(dx)}{\int_{E_{0:2}} \int_{E_{0:2}} f(Y - F(x'))P(dx'|x, d)\theta(dx)}.$$

By definition of kernel P, if $x \in E_0$ and $d = (\ell, r)$, one has

$$P(E_0|x, d) = e^{-\mathbf{\Lambda}^d(x,r)} \geq e^{-r\|\lambda\|}, \tag{6}$$

thanks to Assumption A.1. Assumption A.3 then yields

$$\Theta(E_0) \geq \frac{\underline{f} e^{-r\|\lambda\|}\theta(E_0)}{\bar{f}} \geq \left(\frac{\underline{f}}{\bar{f}}\right)^{\frac{w+r}{\delta}\vee 1} e^{-(w+r)\|\lambda\|},$$

as $\frac{\underline{f}}{\bar{f}} \leq 1$ and $r \geq \delta$.

1.5 Kernel $\bar{R}'$ maps $\bar{\mathbb{X}}'$ onto itself

Set $\xi \in \bar{\mathbb{X}}'$ and $d \in \mathbb{K}'(\xi)$, and denote Ξ a random variable with distribution $\bar{R}'(\cdot|\xi, d)$.

If $d = \check{d}$ then $\Xi = \Delta \in \bar{\mathbb{X}}'$.

Otherwise, by definition one has $\xi = (\theta, y, z, w)$ such that

- $\theta(\{x \in \Omega; x_4 = w\}) = 1$;
- $\theta(\Omega_{0:2}) = 1$ or $\theta(\Omega_3) = 1$;
- if $\theta(\Omega_{0:2}) = 1$, then $\theta(\Omega_0) \geq \left(\frac{\underline{f}}{\bar{f}}\right)^{\frac{w}{\delta}\vee 1} e^{-w\|\lambda\|}$,
- $\theta(\Omega_3) = 1$, then $y = 0$ and $z = 1$,

and $d = (\ell, r) \in \mathbb{K}'(\xi)$.

If $\theta(\Omega_3) = 1$, as above, $d = \check{d}$ and this case has already been treated.

If $\theta(\Omega_{0:2}) = 1$, then either $\Xi = (\Theta, Y, Z, W)$ with $Y = 0$, $Z = 0$, $W = w + r$ and $\Xi(\Omega_3) = 1$ or $\Xi = (\Theta, Y, Z, W)$ with $Z = 0$, $W = w + r$ and conditionally to Y,

$$\Theta = \bar{\Psi}(\theta, Y, 0, w + r, d).$$

The definition of $\bar{\Psi}$ directly yields $\Theta(\Omega_{0:2}) = 1$, $\Theta(\{x \in E; x_4 = w + r\}) = 1$. To obtain the lower bound on $\Theta(\Omega_0)$, for all $\omega^i \in \Omega_{0:2}$, one has

$$\Theta(E_0)(\omega^j) = \frac{\sum_{\omega^i \in \Omega_{0:2}} f(Y - F(\omega^j))\mathbb{1}_{\Omega_{0:2}}(\omega^j)\bar{P}(\omega^j|\omega^i, d)\bar{\theta}(\omega^i)}{\sum_{\omega^i \in \Omega_{0:2}} \sum_{\omega^k \in \Omega_{0:2}} f(Y - F(\omega^k))\bar{P}(\omega^k|\omega^i, d)\bar{\theta}(\omega^i)}$$

By definition of kernel P, if $\omega^i \in E_0$ and $d = (\ell, r)$, one has

$$\bar{P}(\Omega_0|\omega^i, d) = P(E_0|x, d) = e^{-\mathbf{\Lambda}^d(x,r)} \geq e^{-r\|\lambda\|}, \tag{7}$$

thanks to Assumption A.1, and because the Voronoi cells form a partition of E that preserves the mode. Assumption A.3 then yields

$$\begin{aligned}\Theta(E_0) &\geq \frac{\underline{f} e^{-r\|\lambda\|}\theta(E_0)}{\bar{f}} \\ &\geq \left(\frac{\underline{f}}{\bar{f}}\right)^{\frac{w+r}{\delta}\vee 1} e^{-(w+r)\|\lambda\|},\end{aligned}$$

as $\frac{\underline{f}}{\bar{f}} \leq 1$ and $r \geq \delta$.

2 Error bounds for the value functions

2.1 First discretization

The proof of Theorem 3.3 of the main manuscript is based on Lipschitz regularity properties of the operators involved. Because of the boundaries of the state space at ζ_0 and D, the operators are not regular on the whole state space, but only locally regular in some sub-areas. More specifically, we consider the sets $B_i = \{\zeta \in [0, D], \exists d = (\ell, r) \in \mathbb{A} - \{\check{d}\}, t_i^{*d}(\zeta) = r\} \cup \{\zeta_0, D\}$ for $i = 1$ or 2 and $b_i^{(1)} < b_i^{(2)} < \cdots < b_i^{(T)}$ its $T = 3|\mathbb{T}| + 2$ ordered elements. We then consider $(F_j)_{1 \le j \le 2T}$ the following partition of E:

$$\begin{aligned}
F_1 &= E_0,\\
F_j &= \{1\} \times (b_1^{(j-1)}, b_1^{(j)}) \times [0, H]^2, \text{ for } 2 \le j \le T,\\
F_j &= \{2\} \times (b_2^{(j-T)}, b_2^{(j-T+1)}) \times [0, H]^2, \text{ for } T+1 \le j \le 2T-1,\\
F_{2T} &= E_3.
\end{aligned}$$

The splitting points satisfying $t_m^{*d}(\zeta) = r$ for $m \in \{1, 2\}$ and $d = (\ell, r) \in \mathbb{A} - \{\check{d}\}$ separate values of ζ for which the probability of reaching the cemetery in one observation time is strictly positive from those with null probability.

Let us now introduce some additional notation and function spaces compatible with this partition. Let $BL(E)$ be the set of Borel functions from $\mathbb{X}$ onto $\mathbb{R}$ for which there exist finite constants $\|\varphi\|_E$ and $[\varphi]_E$ such that for all (x, γ) and (x', γ) in $\mathbb{X} - \{\Delta\}$, one has

- $|\varphi(\Delta)| \le \|\varphi\|_E$, $|\varphi(x, \gamma)| \le \|\varphi\|_E$,
- $|\varphi(x, \gamma) - \varphi(x', \gamma)| \le [\varphi]_E \, \|x - x'\|$, if x and x' belong to the same subset F_j,
- φ is constant on $(E_3 \times \mathbb{O}) \cap \mathbb{X}$.

Denote also the unit ball of $BL(E)$ by

$$BL_1(E) \quad = \quad \big\{\varphi \in BL(E) : \|\varphi\|_E + [\varphi]_E \le 1\big\}.$$

For θ and $\bar{\theta}$ two probability measures in $\mathscr{P}(E)$, define the distance $d_E(\theta, \bar{\theta})$ by

$$d_E(\theta, \bar{\theta}) = \sup_{g \in BL_1(E)} \sup_{\gamma \in \mathbb{O}} \left| \int g(x, \gamma) \mathbb{1}_{\mathbb{X}}(x, \gamma) \theta(dx) - \int g(x, \gamma) \mathbb{1}_{\mathbb{X}}(x, \gamma) \bar{\theta}(dx) \right|.$$

Let $BLP(E)$ be the set of Borel functions from $\mathbb{X}'$ onto $\mathbb{R}$ for which there exist finite constants $\|\varphi\|_{E,\mathscr{P}}$ and $[\varphi]_{E,\mathscr{P}}$ such that for all (θ, γ) and $(\bar{\theta}, \gamma)$ in $\mathbb{X}'$ such that $\theta(E_{0:2}) = \bar{\theta}(E_{0:2}) = 1$ or $\theta(E_3) = \bar{\theta}(E_3) = 1$ one has

- $|\varphi(\Delta)| \le \|\varphi\|_{E,\mathscr{P}}$,
- $|\varphi(\theta, \gamma)| \le \|\varphi\|_{E,\mathscr{P}}$,
- $|\varphi(\theta, \gamma) - \varphi(\bar{\theta}, \gamma)| \le [\varphi]_{E,\mathscr{P}} d_E(\theta, \bar{\theta})$,
- φ is constant on the set $\{(\theta, \gamma) \in \mathbb{X}' - \{\Delta\} \ ; \ \theta(E_3) = 1\}$.

Denote also the unit ball of $BLP(E)$ by

$$BLP_1(E) \quad = \quad \big\{\varphi \in BLP(E) : \|\varphi\|_{E,\mathscr{P}} + [\varphi]_{E,\mathscr{P}} \le 1\big\}.$$

Finally, for any function $h \in BL(E)$, $\xi = (x, \gamma) \in \mathbb{X} - \{\Delta\}$ and $d \in \mathbb{K}(\xi)$, denote with a slight abuse of notation

$$Ph(\xi, d) = \int_E h(x', \gamma') P(dx'|x, d),$$

with $\gamma' = (y \mathbb{1}_{(x_1' \ge 3)}, \mathbb{1}_{(x_1' = 3)}, w + r)$ if $\gamma = (y, z, w)$ and $d = (\ell, r)$.

2.1.1 Regularity of operator P

Proposition 2.1 *Under assumptions A.1 and A.2, there exists a positive constant C_P such that for all $h \in BL_1(E)$, for all x^1 and x^2 belonging to the same subset F_j of E for some (i,j) and such that $x^1_4 = x^2_4 = w$, for all $(y,z) \in I \times \{0,1\}$ and $d \in \mathbb{K}(z,w) - \{\check{d}\}$, one has*

$$\big|Ph(x^1,y,z,w,d) - Ph(x^2,y,z,w,d)\big| \leq C_P \|x^1 - x^2\|, \tag{8}$$

and P maps $BL(E)$ onto itself and for $g \in BL(E)$, one has $\|Pg\|_E \leq \|g\|_E$ and $[Pg]_E \leq C_P(\|g\|_E + [g]_E)$.

Proof: First note that if x^1 and x^2 have different modes, the result holds true as the right-hand side equals infinity. Second, note that P maps E_3 onto itself, as no jump is possible in mode 3. Therefore, if x^1 and x^2 belong to E_3, then $Ph(x^1,y,z,w,d) = Ph(x^2,y,z,w,d)$ as h is constant on $E_3 \times \mathbb{O}$. Hence, Ph is also constant on $(E_3 \times \mathbb{O}) \cap \mathbb{X}$.

Second, note that if $z = 1$, then $\mathbb{K}(z,w) = \{\check{d}\}$ and there is nothing to prove.

Suppose now that x^1 and x^2 share same mode $x^1_1 = x^2_1 = m$, and and same time since the beginning $x^1_4 = x^2_4 = w$. Select $(y,z) \in I \times \{0\}$, $d \in \mathbb{K}(z,w)$ and a function $h \in BL_1(E)$ and denote $\gamma = (y,z,w)$ and $\gamma' = (y,0,w+r)\mathbb{1}_{x'_1 \neq 3} + (0,1,w+r)\mathbb{1}_{x'_1=3}$. As operator P involves indicator functions, we split the computation of the difference $\big|Ph(x^1,y,z,w,d) - Ph(x^2,y,z,w,d)\big|$ into 3 cases depending on the values of $\mathbf{t}^{*d}(x^1)$ and $\mathbf{t}^{*d}(x^2)$ compared to r.

• *First case:* $\mathbf{t}^{*d}(x^1) > r$ and $\mathbf{t}^{*d}(x^2) > r$. In this case, the explicit formula in Section 1.2 becomes

$$\begin{aligned}
&\big|Ph(x^1,y,z,w,d) - Ph(x^2,y,z,w,d)\big| \\
&\leq \Big|h(\mathbf{\Phi}^d(x^1,r),\gamma')e^{-\mathbf{\Lambda}^d(x^1,r)} - h(\mathbf{\Phi}^d(x^2,r),\gamma')e^{-\mathbf{\Lambda}^d(x^2,r)}\Big| \\
&\quad + \Big|\int_0^r \int_E h(\mathbf{\Phi}^d(x',r-s),\gamma')\lambda^d(\mathbf{\Phi}^d(x^1,s))e^{-\mathbf{\Lambda}^d(x^1,s)}\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x^1,s))\mathbb{1}_{\mathbf{t}^{*d}(x')>r-s}ds \\
&\qquad - \int_0^r \int_E h(\mathbf{\Phi}^d(x',r-s),\gamma')\lambda^d(\mathbf{\Phi}^d(x^2,s))e^{-\mathbf{\Lambda}^d(x^2,s)}\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x^2,s))\mathbb{1}_{\mathbf{t}^{*d}(x')>r-s}ds\Big| \\
&\quad + \Big|\int_0^r \int_{E^2} h(\mathbf{\Phi}^d(x'',r-\mathbf{t}^{*d}(x')-s),\gamma')\lambda^d(\mathbf{\Phi}^d(x^1,s))e^{-\mathbf{\Lambda}^d(x^1,s)} \\
&\qquad\qquad \mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x^1,s))\mathbf{Q}^d(dx''|\mathbf{\Phi}^d(x',\mathbf{t}^{*d}(x')))\mathbb{1}_{\mathbf{t}^{*d}(x')\leq r-s}ds \\
&\qquad - \int_0^r \int_{E^2} h(\mathbf{\Phi}^d(x'',r-\mathbf{t}^{*d}(x')-s),\gamma')\lambda^d(\mathbf{\Phi}^d(x^2,s))e^{-\mathbf{\Lambda}^d(x^2,s)} \\
&\qquad\qquad \mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x^2,s))\mathbf{Q}^d(dx''|\mathbf{\Phi}^d(x',\mathbf{t}^{*d}(x')))\mathbb{1}_{\mathbf{t}^{*d}(x')\leq r-s}ds\Big| \\
&= A_1 + B_1 + C_1.
\end{aligned}$$

We split the expression into a sum of 3 terms that we study separately.

$\star$*Term A_1:* This term corresponds to no jump occurring. As h is in $BL_1(E)$ on obtains from Assumptions A.1 and A.2 that

$$\begin{aligned}
A_1 &\leq \big|h(\mathbf{\Phi}^d(x^1,r),\gamma') - h(\mathbf{\Phi}^d(x^2,r),\gamma')\big|\,|e^{-\mathbf{\Lambda}^d(x^1,r)}| \\
&\quad + \big|h(\mathbf{\Phi}^d(x^2,r),\gamma')\big|\,|e^{-\mathbf{\Lambda}^d(x^1,r)} - e^{-\mathbf{\Lambda}^d(x^2,r)}| \\
&\leq [h][\Phi]\|x^1 - x^2\| + \|h\| r[\lambda][\Phi]\|x^1 - x^2\| \\
&\leq [\Phi](1 + \bar{\delta}[\lambda])\|x^1 - x^2\|,
\end{aligned}$$

where $\bar{\delta} = \max \mathbb{T}$.

$\star$*Term B_1:* This term corresponds to one random jump, which can only lead to modes 1 or 2.

We therefore split this term in 2 parts that can be controlled identically:

$$
\begin{aligned}
B_1 \leq \sum_{i=1}^{2} \Big| &\int_0^r \int_{E_i} h(\mathbf{\Phi}^d(x', r-s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^1, s))e^{-\mathbf{\Lambda}^d(x^1,s)}\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x^1, s))\mathbb{1}_{\mathbf{t}^{*d}(x')>r-s}ds \\
&- \int_0^r \int_{E_i} h(\mathbf{\Phi}^d(x', r-s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^2, s))e^{-\mathbf{\Lambda}^d(x^2,s)}\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x^2, s))\mathbb{1}_{\mathbf{t}^{*d}(x')>r-s}ds\Big| \\
\leq \sum_{i=1}^{2} \Big| &\int_0^r h(i, \mathbf{\Phi}_i^d(m, \mathbf{\Phi}_m^d(x^1, s), r-s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^1, s))e^{-\mathbf{\Lambda}^d(x^1,s)}\mathbb{1}_{t_i^{*d}(\Phi_m^d(x_2^1,s))>r-s}\ ds \\
&- \int_0^r h(i, \mathbf{\Phi}_i^d(m, \mathbf{\Phi}_m^d(m, x^2, s), r-s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^2, s))e^{-\mathbf{\Lambda}^d(x^2,s)}\mathbb{1}_{t_i^{*d}(\Phi_m^d(x_2^2,s))>r-s}\ ds\Big| \\
= B_{11} + B_{12}&
\end{aligned}
$$

Moreover, this jump can only happen if coming initially from mode $m = 0$, in which case $\Phi_m^d(x^j, s)$ is constant, or from the opposite disease mode with its appropriate treatment, in which case $\Phi_m^d(x^j, s)$ is non-increasing. In both cases, the current (lack of) treatment in inappropriate for the mode i after the jump, in which case $\mathbf{t}_i^{*d}(x')$ is non-increasing. To deal with the indicator functions in terms B_{11} and B_{12}, we now consider 3 different subcases:

▲ *first sub-case:* $t_i^{*d}(x_2^1) > r$ and $t_i^{*d}(x_2^2) > r$ (hence for all $0 \leq s \leq r$, $t_i^{*d}(\Phi_m^d(x_2^j, s)) > r - s$). One has

$$
\begin{aligned}
B_{1i} \leq &\int_0^r \big|h(i, \mathbf{\Phi}_i^d(m, \mathbf{\Phi}_m^d(x^1, s), r-s), \gamma') - h(i, \mathbf{\Phi}_i^d(m, \mathbf{\Phi}_m^d(x^2, s), r-s), \gamma')\big| \\
&\qquad \lambda^d(\mathbf{\Phi}^d(x^2, s))e^{-\mathbf{\Lambda}^d(x^2,s)}\ ds \\
&+ \int_0^r \big|h(i, \mathbf{\Phi}_i^d(m, \mathbf{\Phi}_m^d(x^1, s), r-s), \gamma')\big|\,\big|\lambda^d(\mathbf{\Phi}^d(x^1, s)) - \lambda^d(\mathbf{\Phi}^d(x^2, s))\big|\, e^{-\mathbf{\Lambda}^d(x^2,s)}\ ds \\
&+ \int_0^r \big|h(i, \mathbf{\Phi}_i^d(m, \mathbf{\Phi}_m^d(x^1, s), r-s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^1, s))\big|\,\Big|e^{-\mathbf{\Lambda}^d(x^2,s)} - e^{-\mathbf{\Lambda}^d(x^2,s)}\Big|\ ds \\
\leq\ & [h][\Phi]^2\|x^1 - x^2\| + r\|h\|[\lambda][\Phi]\|x^1 - x^2\| + \|h\|\|\lambda\|r^2[\lambda][\Phi]\|x^1 - x^2\| \\
\leq\ & \big([\Phi]^2 + \bar{\delta}[\lambda][\Phi] + \|\lambda\|\bar{\delta}^2[\lambda][\Phi]\big)\|x^1 - x^2\|
\end{aligned}
$$

▲ *second sub-case:* $t_i^{*d}(x_2^1) < r$ and $t_i^{*d}(x_2^2) < r$. Hence there exists $0 < s_i^1 < r$ and $0 < s_i^2 < r$ such that for all $s \leq s_i^j$, one has $t_i^{*d}(\Phi_m^d(x_2^j, s)) \leq r - s$, and for all $s \leq s_i^j$, one has $t_i^{*d}(\Phi_m^d(x_2^j, s)) > r - s$. Under Assumption A.2, one also has $t_i^{*d}(\Phi_m^d(x_2^j, s_i^j)) = r - s_i^j$. Suppose, without loss of generality that $s_i^1 \leq s_i^2$. Similar computations as in the previous sub-case lead to

$$
\begin{aligned}
B_{1i} \leq &\left|\int_{s_i^1}^{s_i^2} h(i, \mathbf{\Phi}_i^d(m, \mathbf{\Phi}_m^d(x^1, s), r-s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^1, s))e^{-\mathbf{\Lambda}^d(x^1,s)}ds\right| \\
&+ \Big|\int_{s_i^2}^r h(i, \mathbf{\Phi}_i^d(m, \mathbf{\Phi}_m^d(x^1, s), r-s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^1, s))e^{-\mathbf{\Lambda}^d(x^1,s)}ds \\
&- \int_{s_i^2}^r h(i, \mathbf{\Phi}_i^d(m, \mathbf{\Phi}_m^d(m, x^2, s), r-s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^2, s))e^{-\mathbf{\Lambda}^d(x^2,s)}ds\Big| \\
B_{1i} \leq &\big([\Phi]^2 + \bar{\delta}[\lambda][\Phi] + \|\lambda\|\bar{\delta}^2[\lambda][\Phi]\big)\|x^1 - x^2\| + |s_i^1 - s_i^2| \\
\leq &\big([\Phi]^2 + \bar{\delta}[\lambda][\Phi] + \|\lambda\|\bar{\delta}^2[\lambda][\Phi] + [S]\big)\|x^1 - x^2\|.
\end{aligned}
$$

▲ *other sub-cases:* If $t_i^{*d}(x_2^1)$ and $t_i^{*d}(x_2^2)$ are on different sides of r, then the difference $\big|Ph(x^1, \gamma, d) - Ph(x^2, \gamma, d)\big|$ cannot be made arbitrarily small as $\|x^1 - x^2\|$ is small. However, this case is impossible as x^1 and x^2 are in the same subset F_j.

$\star$*Term C_1:* Recall that this term comes from a first random jump followed by a boundary jump. Under our assumptions, this is only possible if $m \in \{1, 2\}$, the first random jump yields to mode

$i \in \{1,2\} \neq m$, the treatment is appropriate thus Φ_m^d is non increasing, and the boundary jump is at D. In a similar manner as for B_1 we split the term as

$$
\begin{aligned}
C_1 \leq \sum_{i=1}^{2} \Big| & \int_0^r h(3, D, r - t_i^{*d}(\Phi_m^d(x_2^1, s)) - s, w + r, \gamma')\lambda^d(\mathbf{\Phi}^d(x^1, s))e^{-\mathbf{\Lambda}^d(x^1,s)} \\
& \qquad \mathbb{1}_{t_i^{*d}(\Phi_m^d(x_2^1,s)) \leq r-s} ds \\
& - \int_0^r h(3, D, r - t_i^{*d}(\Phi_m^d(x_2^2, s)) - s, w + r, \gamma'')\lambda^d(\mathbf{\Phi}^d(x^2, s))e^{-\mathbf{\Lambda}^d(x^2,s)} \\
& \qquad \mathbb{1}_{t_i^{*d}(\Phi_m^d(x_2^2,s)) \leq r-s} ds \Big| \\
\leq C_{11} + C_{12}&.
\end{aligned}
$$

Note that only one term in the sum above is non-zero as $i \neq m$.
▲ *first sub-case:* $t_i^{*d}(x_2^1) > r$ and $t_i^{*d}(x_2^2) > r$. Hence, as above, for all $0 \leq s \leq r$, $t_i^{*d}(\Phi_m^d(x_2^j, s)) > r - s$ and C_1 has zero value.
▲ *second sub-case:* $t_i^{*d}(x_2^1) < r$ and $t_i^{*d}(x_2^2) < r$. Then as above, there exists $0 < s_i^1 < r$ and $0 < s_i^2 < r$ such that for all $s \leq s_i^j$, one has $t_i^{*d}(\Phi_m^d(x_2^j, s)) \leq r - s$, and for all $s \leq s_i^j$, one has $t_i^{*d}(\Phi_m^d(x_2^j, s)) > r - s$. Under Assumption .2, one also has $t_i^{*d}(\Phi_m^d(x_2^j, s_i^j)) = r - s_i^j$. Suppose, without loss of generality that $s_i^1 \leq s_i^2$. One has

$$
\begin{aligned}
C_{1i} \leq & \int_{s_i^1}^{s_i^2} \left|\lambda^d(\mathbf{\Phi}^d(x^1, s))e^{-\mathbf{\Lambda}^d(x^1,s)}\right| ds \\
& + \int_{s_i^2}^{r} \big|h(3, D, r - t_i^{*d}(\Phi_m^d(x_2^1, s)) - s, w + r, \gamma)\lambda^d(\mathbf{\Phi}^d(x^1, s))e^{-\mathbf{\Lambda}^d(x^1,s)} \\
& - h(3, D, r - t_i^{*d}(\Phi_m^d(x_2^2, s)) - s, w + r, \gamma')\lambda^d(\mathbf{\Phi}^d(x^2, s))e^{-\mathbf{\Lambda}^d(x^2,s)}\big| ds \\
\leq & ([S] + [t^*][\Phi] + [\lambda][\Phi]\bar{\delta} + \|\lambda\|[\lambda][\Phi]\bar{\delta}^2)\|x^1 - x^2\|.
\end{aligned}
$$

▲ *other sub-cases:* We once again exclude other sub-cases by choosing x^1 and x^2 in the same subset F_j of E.

• *Second case:* $\mathbf{t}^{*d}(x^1) \leq r$ and $\mathbf{t}^{*d}(x^2) \leq r$. This implies again that $m \in \{1,2\}$. We suppose, without loss of generality, that $\mathbf{t}^{*d}(x^1) \leq \mathbf{t}^{*d}(x^2)$. Then $\left|Ph(x^1, \gamma, d) - Ph(x^2, \gamma, d)\right|$ can be split in 4 terms A_2, B_2, C_2 and D_2 corresponding to lines (2) to (5) in Appendix 1.2. depending on the jumps of the process.

⋆ The first term corresponds to a single random jump. This is only possible if $i \in \{1,2\} \neq m$ and the treatment is appropriate thus Φ_m^d is non increasing. One has

$$
\begin{aligned}
A_2 \leq \sum_{i=1}^{2} \int_0^{\mathbf{t}^{*d}(x^1)} \int_{E_i} \Big| & h(\mathbf{\Phi}^d(x', r - s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^1, s))e^{-\mathbf{\Lambda}^d(x^1,s)}\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x^1, s))\mathbb{1}_{\mathbf{t}^{*d}(x') > r-s} \\
& - h(\mathbf{\Phi}^d(x', r - s), \gamma')\lambda^d(\mathbf{\Phi}^d(x^2, s))e^{-\mathbf{\Lambda}^d(x^2,s)}\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x^2, s))\mathbb{1}_{\mathbf{t}^{*d}(x') > r-s}\Big| ds \\
+ \sum_{i=1}^{2} \int_{\mathbf{t}^{*d}(x^1)}^{\mathbf{t}^{*d}(x^2)} \int_{E_i} & \left|h(\mathbf{\Phi}^d(x', r - s))\lambda^d(\mathbf{\Phi}^d(x^2, s))e^{-\mathbf{\Lambda}^d(x^2,s)}\mathbf{Q}^d(dx'|\mathbf{\Phi}^d(x^2, s))\mathbb{1}_{\mathbf{t}^{*d}(x') > r-s}\right| ds.
\end{aligned}
$$

Once again this term is studied by separating cases based on the position of $t_i^{*d}(x^1)$ and $t_i^{*d}(x^2)$ with respect to r. Then, similarly to term B_1 above, if x^1 and x^2 are in the same subset F_j of E, we obtain

$$
A_2 \leq \big([\Phi]^2 + \bar{\delta}[\lambda][\Phi] + \|\lambda\|\bar{\delta}^2[\lambda][\Phi] + [S]\big)\|x^1 - x^2\| + [t^*]\|x^1 - x^2\|.
$$

⋆ The second term corresponds to a single boundary jump at ζ_0 or at D :

$$
\begin{aligned}
B_2 \leq \int_E \Big| & h(\mathbf{\Phi}^d(x', r - \mathbf{t}^{*d}(x^1)), \gamma') e^{-\mathbf{\Lambda}^d(x^1, \mathbf{t}^{*d}(x^1))} e^{-\mathbf{\Lambda}^d(x', r - t^{*d}(x^1))} \\
& \qquad \mathbf{Q}^d(dx' | \mathbf{\Phi}^d(x^1, \mathbf{t}^{*d}(x^1))) \mathbb{1}_{\mathbf{t}^{*d}(x') > r - \mathbf{t}^{*d}(x^1)} \\
& h(\mathbf{\Phi}^d(x', r - \mathbf{t}^{*d}(x^2)), \gamma') e^{-\mathbf{\Lambda}^d(x^2, \mathbf{t}^{*d}(x^2))} e^{-\mathbf{\Lambda}^d(x', r - \mathbf{t}^{*d}(x^2))} \\
& \qquad \mathbf{Q}^d(dx' | \mathbf{\Phi}^d(x^2, \mathbf{t}^{*d}(x^2))) \mathbb{1}_{\mathbf{t}^{*d}(x') > r - \mathbf{t}^{*d}(x^2)} \Big| .
\end{aligned}
$$

Recall that since x^1 and x^2 have the same mode and the same decision d is taken, thus this jump occurs at the same boundary for each term. Moreover, in either case we have $\mathbf{t}^{*d}(x') = +\infty$. Therefore similar computations as in the first case lead to

$$
B_2 \leq ([t^*] + \bar{\delta}[\lambda][\Phi] + \|\lambda\|[t^*]) \|x^1 - x^2\|.
$$

⋆ The third term corresponds to a boundary jump at ζ_0 followed by a random jump to mode $i \in \{1, 2\} \neq m$. It is only possible when the treatment is appropriate for the initial mode m, hence the flow Φ_m^d is again non increasing.

$$
\begin{aligned}
C_2 \leq \sum_{i=1}^{2} \int_0^{r - \mathbf{t}^{*d}(x^2)} \int_{E_i} \Big| & h(\mathbf{\Phi}^d(x'', r - \mathbf{t}^{*d}(x^1) - s), \gamma'') e^{-\mathbf{\Lambda}^d(x^1, \mathbf{t}^{*d}(x^{1'}))} \lambda^d(\mathbf{\Phi}^d(x^{1'}, s)) \\
& e^{-\mathbf{\Lambda}^d(x^{1'}, s)} \mathbf{Q}^d(dx'' | \mathbf{\Phi}^d(x^{1'}, s)) \mathbb{1}_{\mathbf{t}^{*d}(x^{1'}) > r - \mathbf{t}^{*d}(x^1) - s} \mathbb{1}_{\mathbf{t}^{*d}(x'') > r - \mathbf{t}^{*d}(x^1) - s} \\
& - h(\mathbf{\Phi}^d(x'', r - \mathbf{t}^{*d}(x^2) - s), \gamma'') e^{-\mathbf{\Lambda}^d(x^2, \mathbf{t}^{*d}(x^2))} \lambda^d(\mathbf{\Phi}^d(x^{2'}, s)) \\
& e^{-\mathbf{\Lambda}^d(x^{2'}, s)} \mathbf{Q}^d(dx'' | \mathbf{\Phi}^d(x^{2'}, s)) \mathbb{1}_{\mathbf{t}^{*d}(x^{2'}) > r - \mathbf{t}^{*d}(x^2) - s} \mathbb{1}_{\mathbf{t}^{*d}(x'') > r - \mathbf{t}^{*d}(x^2) - s} \Big| ds \\
+ \sum_{i=1}^{2} \int_{r - \mathbf{t}^{*d}(x^2)}^{r - \mathbf{t}^{*d}(x^1)} \int_{E_i} \Big| & h(\mathbf{\Phi}^d(x'', r - \mathbf{t}^{*d}(x^1) - s), \gamma'') e^{-\mathbf{\Lambda}^d(x^1, \mathbf{t}^{*d}(x^1))} \lambda^d(\mathbf{\Phi}^d(x^{1'}, s)) \\
& e^{-\mathbf{\Lambda}^d(x^{1'}, s)} \mathbf{Q}^d(dx'' | \mathbf{\Phi}^d(x^{1'}, s)) \mathbb{1}_{\mathbf{t}^{*d}(x^{1'}) > r - \mathbf{t}^{*d}(x^1) - s} \mathbb{1}_{\mathbf{t}^{*d}(x'') > r - \mathbf{t}^{*d}(x^1) - s} \Big| ds
\end{aligned}
$$

with $x^{1'} = (0, \zeta_0, 0, w + \mathbf{t}^{*d}(x^1))$ and $x^{2'} = (0, \zeta_0, 0, w + \mathbf{t}^{*d}(x^2))$ hence $\mathbf{t}^{*d}(x^{1'}) = \mathbf{t}^{*d}(x^{2'}) = +\infty$. Moreover, Assumption 2.1 of the main manuscript prevents more than two jumps from happening in between two visits, and therefore $\mathbf{t}^{*d}(x'') > r - \mathbf{t}^{*d}(x^j)$. Hence one has

$$
\begin{aligned}
C_2 \leq \sum_{i=1}^{2} \int_0^{r - \mathbf{t}^{*d}(x^2)} \int_{E_i} \Big| & h(\mathbf{\Phi}^d(x'', r - \mathbf{t}^{*d}(x^1) - s), \gamma'') e^{-\mathbf{\Lambda}^d(x^1, \mathbf{t}^{*d}(x^1))} \lambda^d(\mathbf{\Phi}^d(x^{1'}, s)) \\
& e^{-\mathbf{\Lambda}^d(x^{1'}, s)} \mathbf{Q}^d(dx'' | \mathbf{\Phi}^d(x^{1'}, s)) \\
& - h(\mathbf{\Phi}^d(x'', r - \mathbf{t}^{*d}(x^2) - s), \gamma') e^{-\mathbf{\Lambda}^d(x^2, \mathbf{t}^{*d}(x^2))} \lambda^d(\mathbf{\Phi}^d(x^{2'}, s)) \\
& e^{-\mathbf{\Lambda}^d(x^{2'}, s)} \mathbf{Q}^d(dx'' | \mathbf{\Phi}^d(x^{2'}, s)) \Big| ds \\
+ \sum_{i=1}^{2} \int_{r - \mathbf{t}^{*d}(x^2)}^{r - \mathbf{t}^{*d}(x^1)} \int_{E_i} \Big| & h(\mathbf{\Phi}^d(x'', r - \mathbf{t}^{*d}(x^1) - s), \gamma'') e^{-\mathbf{\Lambda}^d(x^1, \mathbf{t}^{*d}(x^1))} \lambda^d(\mathbf{\Phi}^d(x^{1'}, s)) \\
& e^{-\mathbf{\Lambda}^d(x^{1'}, s)} \mathbf{Q}^d(dx'' | \mathbf{\Phi}^d(x^{1'}, s)) \Big| ds \\
\leq ([t^*] + [\Phi][t^*] + \bar{\delta}[\lambda][\Phi] & + [t^*] \|\lambda\| + \bar{\delta}^2 \|\lambda\| [\lambda][\Phi][t^*] + \bar{\delta} \|\lambda\| [\lambda][\Phi][t^*]) \|x^1 - x^2\|.
\end{aligned}
$$

⋆ The last term corresponds to a random jump to $i \in \{1, 2\} \neq m$ followed by a boundary jump at

D. It is treated similarly to term C_1. One has

$$\begin{aligned}
D_2 \le \int_0^{\mathbf{t}^{*d}(x^1)} \int_{E_3}\int_{E_i} \Big| & h(\boldsymbol{\Phi}^d(x'', r - t^{*d}(x^1) - s), \gamma'')\lambda^d(\boldsymbol{\Phi}^d(x^1, s))e^{-\boldsymbol{\Lambda}^d(x^1,s)} \\
& \mathbf{Q}^d(dx'|\boldsymbol{\Phi}^d(x^1, s))\mathbf{Q}^d(dx''|\boldsymbol{\Phi}^d(x', \mathbf{t}^{*d}(x')))\mathbb{1}_{\mathbf{t}^{*d}(x')\le r-s} \\
& - h(\boldsymbol{\Phi}^d(x'', r - t^{*d}(x^2) - s), \gamma'')\lambda^d(\boldsymbol{\Phi}^d(x^2, s))e^{-\boldsymbol{\Lambda}^d(x^2,s)} \\
& \mathbf{Q}^d(dx'|\boldsymbol{\Phi}^d(x^2, s))\mathbf{Q}^d(dx''|\boldsymbol{\Phi}^d(x', \mathbf{t}^{*d}(x')))\mathbb{1}_{\mathbf{t}^{*d}(x')\le r-s}\Big|\, ds \\
+ \int_{\mathbf{t}^{*d}(x^1)}^{\mathbf{t}^{*d}(x^2)} \int_{E_3}\int_{E_i} \Big| & h(\boldsymbol{\Phi}^d(x'', r - t^{*d}(x^2) - s)), \gamma''\lambda^d(\boldsymbol{\Phi}^d(x^2, s))e^{-\boldsymbol{\Lambda}^d(x^2,s)} \\
& \mathbf{Q}^d(dx'|\boldsymbol{\Phi}^d(x^2, s))\mathbf{Q}^d(dx''|\boldsymbol{\Phi}^d(x', \mathbf{t}^{*d}(x')))\Big|\, ds \\
\le{} & C_1 + [t^*]\|x^1 - x^2\| \\
\le{} & ([S] + [t^*][\Phi] + [\lambda][\Phi]\bar{\delta} + \|\lambda\|[\lambda][\Phi]\bar{\delta}^2 + [t^*]\ )\|x^1 - x^2\|.
\end{aligned}$$

• *Other cases:* We exclude all other cases by considering points x^1 and x^2 in the same subset F_j of E.

2.1.2 Regularity of operator R'

We first need to investigate the regularity of the filter operator.

Proposition 2.2 *Under Assumptions A.1, A.2 and A.3, for all* $z' \in \{0, 1\}$*,* $w' \in [0, H]$*,* $d = (\ell, r) \in \mathbb{A} - \{\check{d}\}$*, and* $\theta, \bar{\theta} \in \mathscr{P}(E_{0:2})$ *satisfying* $\theta(\{x \in E_{0:2}; x_4 = w' - r\}) = \bar{\theta}(\{x \in E_{0:2}; x_4 = w' - r\}) = 1$ *and* $\theta(E_0) > 0$*,* $\bar{\theta}(E_0) > 0$*, we have*

$$\int_I d_E(\Psi(\theta, y', z', w', d), \Psi(\bar{\theta}, y', z', w', d))dy' \le \frac{C_\Psi}{\theta(E_0)\bar{\theta}(E_0)} d_E(\theta, \bar{\theta}). \tag{9}$$

Proof: <u>For $z' = 1$ and $y' = 0$</u> we have

$$\Psi(\theta, 0, 1, w', d)(A) = \frac{\int_{E_{0:2}} \big(\int_{E_3} \mathbb{1}_{\{w'\}}(x'_4)\mathbb{1}_A(x')P(dx'|x, d)\big)\theta(dx)}{\int_{E_{0:2}} \big(\int_{E_3} \mathbb{1}_{\{w'\}}(x'_4)P(dx'|x, d)\big)\theta(dx)}.$$

Set $g \in BL_1(E)$ and recall that by assumption, $g(x, \gamma)$ is constant for $(x, \gamma) \in E_3 \times \mathbb{O}$. Say $g(x, \gamma) = \xi_C$. Hence, one has

$$\int_E g(x, \gamma)\Psi(\theta, 0, 1, w', d)(dx) = \frac{\int_{E_{0:2}} \big(\int_{E_3} \mathbb{1}_{\{w'\}}(x'_4)g(x', \gamma)P(dx'|x, d)\big)\theta(dx)}{\int_{E_{0:2}} \big(\int_{E_3} \mathbb{1}_{\{w'\}}(x'_4)P(dx'|x, d)\big)\theta(dx)} = \xi_C.$$

Hence $d_E(\Psi(\theta, 0, 1, w', d), \Psi(\bar{\theta}, 0, 1, w', d)) = 0$.

For $z'=0$: let $g\in BL_1(E)$ and $y'\in I$, $w'\in[0,H]$, and $\gamma\in\mathbb{O}$, we have

$$
\begin{aligned}
&\int_E g(x,\gamma)\mathbb{1}_{\mathbb{X}}(x,\gamma)\Psi(\theta,y',0,w',d)(dx)-\int_E g(x,\gamma)\mathbb{1}_{\mathbb{X}}(x,\gamma)\Psi(\bar\theta,y',0,w',d)(dx)\\
&=\frac{\int_{E_{0:2}}\int_{E_{0:2}} f(y'-F(x'))g(x',\gamma)\mathbb{1}_{\mathbb{X}}(x,\gamma)\mathbb{1}_{\{w'\}}(x_4')P(dx'|x,d)\theta(dx)}{\int_{E_{0:2}}\int_{E_{0:2}}\mathbb{1}_{\{w'\}}(x_4')f(y'-F(x'))P(dx'|x,d)\theta(dx)}\\
&\qquad-\frac{\int_{E_{0:2}}\int_{E_{0:2}} f(y'-F(x'))g(x',\gamma)\mathbb{1}_{\mathbb{X}}(x,\gamma)\mathbb{1}_{\{w'\}}(x_4')P(dx'|x,d)\bar\theta(dx)}{\int_{E_{0:2}}\int_{E_{0:2}}\mathbb{1}_{\{w'\}}(x_4')f(y'-F(x'))P(dx'|x,d)\bar\theta(dx)}\\
&=\Big[\int_{E_{0:2}}\int_{E_{0:2}} f(y'-F(x'))g(x',\gamma)\mathbb{1}_{\mathbb{X}}(x',\gamma)\mathbb{1}_{\{w'\}}(x_4')P(dx'|x,d)\theta(dx)\\
&\qquad-\int_{E_{0:2}}\int_{E_{0:2}} f(y'-F(x'))g(x',\gamma)\mathbb{1}_{\mathbb{X}}(x',\gamma)\mathbb{1}_{\{w'\}}(x_4')P(dx'|x,d)\bar\theta(dx)\Big]\\
&\qquad\Big[\frac{1}{\int_{E_{0:2}}\int_{E_{0:2}}\mathbb{1}_{\{w'\}}(x_4')f(y'-F(x'))P(dx'|x,d)\theta(dx)}\Big]\\
&\quad+\Big[\int_{E_{0:2}}\int_{E_{0:2}} f(y'-F(x'))g(x',\gamma)\mathbb{1}_{\mathbb{X}}(x',\gamma)\mathbb{1}_{\{w'\}}(x_4')P(dx'|x,d)\theta(dx)\Big]\\
&\qquad\Big[\frac{1}{\int_{E_{0:2}}\int_{E_{0:2}}\mathbb{1}_{\{w'\}}(x_4')f(y'-F(x'))P(dx'|x,d)\theta(dx)}\\
&\qquad\quad-\frac{1}{\int_{E_{0:2}}\int_{E_{0:2}}\mathbb{1}_{\{w'\}}(x_4')f(y'-F(x'))P(dx'|x,d)\bar\theta(dx)}\Big]\\
&=\frac{A(\gamma,y',w')}{B(y',w')}+C(\gamma,y',w')D(y',w').
\end{aligned}
$$

First, note that if g is in $BL_1(E)$, then application $\varphi:(x,\gamma)\mapsto\int_E f(y'-F(x'))\mathbb{1}_{\{w'\}}(x_4')g(x',\gamma)\mathbb{1}_{\mathbb{X}}(x',\gamma)P(dx'|x,d)$ is also in $BL(E)$ with $\|\varphi\|_E\le\overline{f}$ and $[\varphi]_E\le\big(\overline{f}+L\big)C_P$ thanks to Proposition 2.1 and Assumptions A.3, so that one has

$$|A(\gamma,y',w')|\le\big(\overline{f}(1+C_P)+LC_P\big)d_E(\theta,\bar\theta).$$

On the other hand, as $\theta(\{x\in E_{0:2};x_4=w'-r\})=1$ by assumption, the indicator function in term $B(y',w')$ equals 1 and one has

$$
\begin{aligned}
|B(y',w')|&\ge\underline{f}\int_{E_{0:2}}P(E_{0:2}|x,d)\theta(dx)\\
&\ge\underline{f}\int_{E_0}P(E_0|x,d)\theta(dx).
\end{aligned}
$$

By definition of kernel P, if $x\in E_0$ and $d=(\ell,r)$, one has

$$P(E_0|x,d)=e^{-\mathbf{\Lambda}^d(x,r)}\ge e^{-r\|\lambda\|}\ge e^{-H\|\lambda\|},\tag{10}$$

thanks to Assumption A.1. Hence, one has

$$|B(y',w')|\ge\underline{f}e^{-H\|\lambda\|}\theta(E_0).$$

Similarly, the second term can be bounded by

$$|C(\gamma,y',w')D(y',w')|\le\overline{f}\,\frac{\overline{f}(1+C_P)+LC_P}{\underline{f}^2e^{-2H\|\lambda\|}\theta(E_0)\bar\theta(E_0)}d_E(\theta,\bar\theta).$$

As I is a bounded interval, the result follows. □

Now we can turn to the regularity of operator R'.

Proposition 2.3 *Under Assumptions A.1, A.2 and A.3, there exists a positive constant $C_{R'}$ such that for all $g \in BLP_1(E)$, $\gamma = (y,z,w) \in \mathbb{O}$, $d \in \mathbb{K}'(\gamma)$ and for all $\theta, \bar{\theta}$ in $\mathscr{P}(E)$ such that $(\theta,\gamma) \in \mathbb{X}'$, $(\bar{\theta},\gamma) \in \mathbb{X}'$ and*

- *either $d \neq \check{d}$ and $\theta(E_{0:2}) = \bar{\theta}(E_{0:2}) = 1$,*
- *or $d = \check{d}$,*

one has

$$\big|R'g(\theta,y,z,d) - R'g(\bar{\theta},y,z,d)\big| \leq C_{R'} d_E(\theta,\bar{\theta}) \tag{11}$$

In particular, operator R' maps $BLP(E)$ onto itself and for $g \in BLP(E)$, one has $\|R'g\|_{E,\mathscr{P}} \leq \|g\|_{E,\mathscr{P}}$ and $[R'g]_{E,\mathscr{P}} \leq (\|g\|_{E,\mathscr{P}} + [g]_{E,\mathscr{P}})C_{R'}$.

Proof: If $d = \check{d}$, then $R'g(\theta,\gamma,\check{d}) = R'g(\bar{\theta},\gamma,\check{d}) = g(\Delta)$, hence $|R'g(\theta,\gamma,\check{d}) - R'g(\bar{\theta},\gamma,\check{d})| = 0$.
If $d = (\ell, r) \neq \check{d}$, then by assumption $\theta(E_{0:2}) = \bar{\theta}(E_{0:2}) = 1$ so that one has $\theta(E_0) > 0$ and $\bar{\theta}(E_0) > 0$ as $(\theta,\gamma) \in \mathbb{X}'$ and $(\bar{\theta},\gamma) \in \mathbb{X}'$. In addition, one has $z = 0$ and $w < N\delta$, such that

$$\begin{aligned}
R'g(\theta,\gamma,d) = &\int_I \int_{E_{0:2}} \int_{E_{0:2}} g\big(\Psi(\theta,y',0,w+r,d),y',0,w+r\big) f(y'-F(x'))P(dx'|x,d)\theta(dx)dy' \\
&+ \int_{E_{0:2}} \Big(\int_{E_3} g(\Psi(\theta,0,1,w+r,d),0,1,w+r)P(dx'|x,d)\Big)\theta(dx).
\end{aligned}$$

We will study the two terms separately. Let us first consider the second term. Note that by definition $\Psi(\theta,0,1,w+r,d)(E_3) = \Psi(\bar{\theta},0,1,w+r,d)(E_3) = 1$. Hence, as g is in $BLP(E)$, one has

$$g(\Psi(\theta,0,1,w+r,d),0,1,w+r) = g(\Psi(\bar{\theta},0,1,w+r,d),0,1,w+r) = \alpha,$$

for some constant $\alpha \leq \|g\|_{E,\mathscr{P}} \leq 1$, so that one has

$$\begin{aligned}
&\int_{E_{0:2}} \Big(\int_{E_3} g(\Psi(\theta,0,1,w+r,d),0,1,w+r)P(dx'|x,d)\Big)\theta(dx) \\
&\qquad - \int_{E_{0:2}} \Big(\int_{E_3} g(\Psi(\bar{\theta},0,1,w+r,d),0,1,w+r)P(dx'|x,d)\Big)\bar{\theta}(dx) \\
&= \alpha\Big(\int_E P(E_3|x,d)\theta(dx) - \int_E P(E_3|x,d)\bar{\theta}(dx)\Big).
\end{aligned}$$

Application $x \to P(E_3|x,d)$ is in $BL(E)$ thanks to Proposition 2.1 as $\mathbb{1}_{E_3}$ is clearly in $BL_1(E)$ (with $\|\mathbb{1}_{E_3}\|_E = 1$ and $[\mathbb{1}_{E_3}]_E = 0$), hence one obtains

$$\begin{aligned}
&\Big|\int_{E_{0:2}} \Big(\int_{E_3} g(\Psi(\theta,0,1,w+r,d),0,1,w+r)P(dx'|x,d)\Big)\theta(dx) \\
&\qquad - \int_{E_{0:2}} \Big(\int_{E_3} g(\Psi(\bar{\theta},0,1,w+r,d),0,1,w+r)P(dx'|x,d)\Big)\bar{\theta}(dx)\Big| \\
&\leq C_P d_E(\theta,\bar{\theta}).
\end{aligned}$$

Now let us study the first term in $R'g$. Set

$$h_\theta(x,\gamma) = \int_I \int_{E_{0:2}} g\big(\Psi(\theta,y',0,w+r,d),y',0,w+r\big) f(y'-F(x))dy',$$

if $x \in E_{0:2}$ and $\gamma = (y,0,w+r)$ and $h(\xi) = 0$ otherwise and

$$g_{\theta,d}(x,\gamma) = Ph_\theta(x,\gamma,d),$$

if $x \in E_{0:2}$ and $\gamma = (y, 0, w + r)$ and 0 otherwise. From Assumptions A.3, h_θ is in $BL(E)$ with $\|h_\theta\|_E \le B_f$ and $[h_\theta]_E \le L_f$. Proposition 2.1 thus yields that $g_{\theta,d}$ in still in $BL(E)$ with $\|g_{\theta,d}\|_E \le B_f$ and $[g_{\theta,d}]_E \le (B_f + L_f)C_P$. Therefore one has

$$\begin{aligned}
&\left|\int_{E_{0:2}} g_{\theta,d}(x,\gamma)\theta(dx) - \int_{E_{0:2}} g_{\bar\theta,d}(x,\gamma)\bar\theta(dx)\right| \\
&\le \int_{E_{0:2}} \left|g_{\theta,d}(x) - g_{\bar\theta,d}(x,\gamma)\right| \bar\theta(dx) + \left|\int_{E_{0:2}} g_{\theta,d}(x,\gamma)\theta(dx) - \int_{E_{0:2}} g_{\theta,d}(x,y)\bar\theta(dx)\right| \\
&= A_1 + A_2.
\end{aligned}$$

The second term A_2 is bounded by $([g_{\theta,d}]_E + \|g_{\theta,d}\|_E)\, d(\theta,\bar\theta)$. For A_1, we write

$$\left|g_{\theta,d}(x,\gamma) - g_{\bar\theta,d}(x,\gamma)\right| \le \int_I d_E(\Psi(\theta, y', 0, w + r, d), \Psi(\bar\theta, y', 0, w + r, d))dy'.$$

Note in particular that $(\theta,\gamma) \in \mathbb{X}'$, $(\bar\theta,\gamma) \in \mathbb{X}'$ and the other assumptions on θ and $\bar\theta$ guarantee that the assumptions of Proposition 2.2 are satisfied. Hence we conclude using Proposition 2.2 and the minoration of $\theta(E_0)$ and $\bar\theta(E_0)$. □

2.1.3 Projection error for operators P and $\bar P$

We need some assumptions on the discretisation grid Ω in order to be able to control the discretisation error. This is achieved by designing the Voronoi cells such that they are compatible with the partition $(F_j)_{1\le j\le 2T}$.

Assumption 2.4 *The grid Ω and its Voronoi cells $(C_k)_{1\le k\le K}$ are such that for all $1 \le k \le K$, there exists some $1 \le j \le 2T$ such that*

$$C_k \subset F_j.$$

In other words, the hyperplanes with equation $t_m^{*d}(\zeta) = r$ for $m \in \{1, 2\}$ and $d = (\ell, r) \in \mathbb{A} - \{\check d\}$ are included in the boundary of the Vorono cells of the grid Ω, which means that the closest points to these hyperplanes in the grid are symmetric with respect to these hyperplanes, see Section 4 of the main manuscript for details about the grid construction.

For $\omega = (m, \zeta, u, w) \in \Omega$, we denote ω_i the coordinates of vector ω: $\omega_1 = m$, $\omega_2 = \zeta$, $\omega_3 = u$, $\omega_4 = w$. Denote also $\Omega_i = \Omega \cap E_i$ for $i \in \{0, 1, 2, 3, 0 : 2\}$.

For any function $h \in BL(E)$, $\xi = (x,\gamma) \in \Omega \times \mathbb{O}$ and $d \in \mathbb{K}(\xi)$, denote

$$\bar P h(\xi, d) = \int_E h(x',\gamma)\bar P(dx'|x, d).$$

Proposition 2.5 *Under Assumption 2.4, for all $i \in \{1, \dots, K\}$, $\gamma \in \mathbb{O}$, $d \in \mathbb{A} - \{\check d\}$ and all $g \in BL_1(E)$, we have*

$$\left|Pg(\omega^i,\gamma, d) - \bar P g(\omega^i,\gamma, d)\right| \le \sup_{j\in\{1,\dots,K\}} \mathscr{D}_j. \tag{12}$$

Proof: For $x \in E$, denote by C_x the Voronoi cell of x. For $g \in BL_1(E)$, $i \in \{1, \dots, K\}$, $\gamma = (y, z, w) \in \mathbb{O}$, $d \in \mathbb{A} - \{\check d\}$, one has

$$\begin{aligned}
\left|Pg(\omega^i,\gamma, d) - \bar P g(\omega^i,\gamma, d)\right| &\le \Big|\int_E g(x',\gamma')P(dx'|\omega^i, d) - \sum_{j=1}^K g(\omega^j,\gamma')P(C_j|\omega^i, d)\Big| \\
&\le \sum_{j=1}^K \int_{C_j} \left|g(x',\gamma') - g(\omega^j,\gamma')\right| P(dx'|\omega^i, d) \\
&\le \sum_{j=1}^K \int_{C_j} |x' - \omega^j| P(dx'|\omega^i, d) \\
&\le \sup_{j\in\{1,\dots,K\}} \mathscr{D}_j
\end{aligned}$$

as g is Lipschitz-continuous on each of the cells C_j thanks to assumption 2.4. $\square$

2.1.4 Projection error for operators R' and $\bar{R}'$

Proposition 2.6 *Under assumptions A.3 and 2.4, for all $\gamma' = (y', z', w') \in \mathbb{O}$, $d = (\ell, r) \in \mathbb{A} - \{\check{d}\}$ and all $\bar{\theta} \in \mathscr{P}(\Omega)$ satisfying $\bar{\theta}(\{\omega \in \Omega_{0:2}; \omega_4 = w' - r\}) = 1$ and $\bar{\theta}(\Omega_0) > 0$, we have*

$$\int_I d_E(\Psi(\bar{\theta}, \gamma', d), \bar{\Psi}(\bar{\theta}, \gamma', d))dy' \leq \frac{\bar{C}_\Psi}{\bar{\theta}(\Omega_0)} \sup_{j \in \{1, \dots, \ell\}} \mathscr{D}_j. \tag{13}$$

Proof: Set $\gamma \in \mathbb{O}$, $\gamma' = (y', z', w') \in \mathbb{O}$, $d = (\ell, r) \in \mathbb{A} - \{\check{d}\}$ and $\bar{\theta} \in \mathscr{P}(\Omega)$ satisfying $\bar{\theta}(\{\omega \in \Omega_{0:2}; \omega_4 = w' - r\}) = 1$ and $\bar{\theta}(\Omega_0) > 0$, and $g \in BL_1(E)$.
<u>If $z' = 1$ and $y' = 0$</u>, we have

$$\int_E g(x,\gamma)\Psi(\bar{\theta}, 0, 1, w', d)(dx) = \frac{\sum_{\omega^i \in \Omega_{0:2}} \left(\int_{E_3} \mathbb{1}_{\{w'\}}(x'_4) g(x', \gamma) P(dx'|\omega^i, d) \right) \bar{\theta}(\omega^i)}{\sum_{\omega^i \in \Omega_{0:2}} \left(\int_{E_3} \mathbb{1}_{\{w'\}}(x'_4) P(dx'|\omega^i, d) \right) \bar{\theta}(\omega^i)} = \xi_C,$$

$$\int_E g(x,\gamma)\bar{\Psi}(\bar{\theta}, 0, 1, w', d)(dx) = \frac{\sum_{\omega^i \in \Omega_{0:2}} \sum_{\omega^j \in \Omega_3} \mathbb{1}_{\{w'\}}(\omega^j_4) g(\omega^j, \gamma) \bar{P}(\omega^j|\omega^i, d) \bar{\theta}(\omega^i)}{\sum_{\omega^i \in \Omega_{0:2}} \sum_{\omega^k \in \Omega_3} \mathbb{1}_{\{w'\}}(\omega^k_4) \bar{P}(\omega^k|\omega^i, d) \bar{\theta}(\omega^i)} = \xi_C.$$

as $g(x, \gamma) = \xi_0$ is constant for $(x, \gamma) \in E_3 \times \mathbb{O}$. Hence in this cas one has $d_E(\Psi(\bar{\theta}, \gamma, d), \bar{\Psi}(\bar{\theta}, \gamma, d)) = 0$.

<u>If $z' = 0$</u>, we have

$$\begin{aligned}
&\int_E g(x,\gamma)\Psi(\bar{\theta}, y', 0, w', d)(dx) - \int_E g(x,\gamma)\bar{\Psi}(\bar{\theta}, y', 0, w', d)(dx) \\
&= \frac{\sum\limits_{\omega^i \in \Omega_{0:2}} \int_{E_{0:2}} f(y' - F(x')) g(x', \gamma) \mathbb{1}_{\{w'\}}(x'_4) P(dx'|\omega^i, d) \bar{\theta}(\omega^i)}{\sum\limits_{\omega^i \in \Omega_{0:2}} \int_{E_{0:2}} \mathbb{1}_{\{w'\}}(x'_4) f(y' - F(x')) P(dx'|\omega^i, d) \bar{\theta}(\omega^i)} \\
&\qquad - \frac{\sum\limits_{\omega^i \in \Omega_{0:2}} \sum\limits_{\omega^j \in \Omega_{0:2}} f(y' - F(\omega^j)) \mathbb{1}_{\{w'\}}(\omega^j_4) g(\omega^j, \gamma) \bar{P}(\omega^j|\omega^i, d) \bar{\theta}(\omega^i)}{\sum\limits_{\omega^i \in \Omega_{0:2}} \sum\limits_{\omega^k \in \Omega_{0:2}} f(y' - F(\omega^k)) \mathbb{1}_{\{w'\}}(\omega^k_4) \bar{P}(\omega^k|\omega^i, d) \bar{\theta}(\omega^i)} \\
&= \left[\left(\sum_{\omega^i \in \Omega_{0:2}} \int_{E_{0:2}} f(y' - F(x')) g(x', \gamma) \mathbb{1}_{\{w'\}}(x'_4) P(dx'|\omega^i, d) \right.\right. \\
&\qquad \left.\left. - \sum_{\omega^j \in \Omega_{0:2}} f(y' - F(\omega^j)) \mathbb{1}_{\{w'\}}(\omega^j_4) g(\omega^j, \gamma) \bar{P}(\omega^j|\omega^i, d) \right) \bar{\theta}(\omega^i) \right] \\
&\qquad \left[\frac{1}{\sum\limits_{\omega^i \in \Omega_{0:2}} \int_{E_{0:2}} \mathbb{1}_{\{w'\}}(x'_4) f(y' - F(x')) P(dx'|\omega^i, d) \bar{\theta}(\omega^i)} \right] \\
&+ \left[\sum_{\omega^i \in \Omega_{0:2}} \sum_{\omega^j \in \Omega_{0:2}} f(y' - F(\omega^j)) \mathbb{1}_{\{w'\}}(\omega^j_4) g(\omega^j, \gamma) \bar{P}(\omega^j|\omega^i, d) \bar{\theta}(\omega^i) \right] \\
&\qquad \left[\frac{1}{\sum\limits_{\omega^i \in \Omega_{0:2}} \int_{E_{0:2}} \mathbb{1}_{\{w'\}}(x'_4) f(y' - F(x')) P(dx'|\omega^i, d) \bar{\theta}(\omega^i)} \right. \\
&\qquad \left. - \frac{1}{\sum\limits_{\omega^i \in \Omega_{0:2}} \sum\limits_{\omega^k \in \Omega_{0:2}} f(y' - F(\omega^k)) \mathbb{1}_{\{w'\}}(\omega^k_4) \bar{P}(\omega^k|\omega^i, d) \bar{\theta}(\omega^i)} \right] \\
&= \frac{A(\gamma, y', w')}{B(y', w')} + C(\gamma, y', w') D(y', w').
\end{aligned}$$

Let us study the different terms separately. First, note that as g is in $BL_1(E)$, then application $\varphi : (x', \gamma) \mapsto f(y' - F(x'))\mathbb{1}_{\{w'\}}(x'_4)g(x', \gamma)$ is also in $BL(E)$ with $\|\varphi\|_E \leq \overline{f}$ and $[\varphi]_E \leq (\overline{f} + L)$ thanks to Assumptions A.3. Hence, Proposition 2.5 yields

$$|A(\gamma, y', w')| \leq (2\overline{f} + L) \sup_{j \in \{1,\dots,\ell\}} \mathscr{D}_j.$$

Similar arguments as in the proof of Proposition 2.2 directly yield

$$\begin{aligned}
|B(y', w')| &\geq \underline{f} e^{-H\|\lambda\|} \bar{\theta}(\Omega_0), \\
|C(\gamma, y', w')| &\leq \overline{f}, \\
|D(y', w')| &\leq (2\overline{f} + L) \sup_{j \in \{1,\dots,\ell\}} \mathscr{D}_j \frac{1}{\underline{f}^2 e^{-2H\|\lambda\|} \bar{\theta}(\Omega_0)}
\end{aligned}$$

Hence the result by integration on y'. $\square$

Proposition 2.7 *Under assumptions A.2 and 2.4, for all $g \in BLP_1(E)$, $(\theta, \gamma) \in \bar{\mathbb{X}}'$ and $d \in \mathbb{K}'(\gamma)$, one has*

$$\left| R'g(\bar{\theta}, \gamma, d) - \bar{R}'g(\bar{\theta}, \gamma, d) \right| \leq \bar{C}_{R'} \sup_{j \in \{1,\dots,\ell\}} \mathscr{D}_j.$$

Proof: This is a direct consequence of Propositions 2.5 and 2.6 with $\bar{C}_{R'} = L_f + 2B_f + L_f \bar{C}_\Psi$, as $\bar{\theta}(\Omega_0) > 0$. $\square$

2.1.5 Regularity and approximation error for the cost functions

The last preliminary result we need in order to prove Theorem 3.3 of the main manuscript is to ensure that the cost functions c', C' belong to the appropriate function spaces and the error between c' and $\bar{c}'$ is controlled.

Lemma 2.8 *Under assumptions A.4, the cost function C' is in $BLP(E)$.*

Proof: Under Assumptions A.4, C is clearly in $BL(E)$ with $\|C\|_E \leq B_C$ and $[C]_E \leq L_C$ (recall that $C = M$ is constant on E_3). Thus, C' is still bounded by B_C and for all (θ, γ) and $(\bar{\theta}, \gamma)$ in $\mathbb{X}'$ one has

$$|C'(\theta, \gamma) - C'(\bar{\theta}, \gamma)| = \Big| \int_E C(x)\theta(dx) - \int_E C(x)\bar{\theta}(dx) \Big| \leq (B_C + L_C) d_E(\theta, \bar{\theta}).$$

As $C = M$ is constant on E_3, C' is also constant (equal to M) on the set $\{(\theta, \gamma) \in \mathbb{X}' \; ; \; \theta(E_3) = 1\}$ Hence C' is in $BLP(E)$ with $\|C'\|_{E,\mathscr{P}} \leq B_C$ and $[C']_{E,\mathscr{P}} \leq B_C + L_C$. $\square$

Lemma 2.9 *Under assumptions A.1, A.2 and A.4, for all $d \in \mathbb{A}$ the function $c'_d : \xi \mapsto c'(\xi, d)$ is in $BLP(E)$.*

Proof: Under assumptions A.4, c'_d is bounded by B_c, and as $c = M$ is constant on E_3, c'_d is also constant on the set $\{(\theta, \gamma) \in \mathbb{X}' \; ; \; \theta(E_3) = 1\}$. Let us study the application $Pc : x \mapsto \int_E c(x, d, x')P(dx'|x, d)$ for fixed $d \in \mathbb{A} - \{\tilde{d}\}$. It is bounded by B_c, constant on E_3 and for x^1, x^2 in E, one has

$$\begin{aligned}
|Pc(x^1) - Pc(x^2)| &= \Big| \int_E c(x^1, d, x')P(dx'|x^1, d) - \int_E c(x^2, d, x')P(dx'|x^2, d) \Big| \\
&\leq \int_E |c(x^1, d, x') - c(x^2, d, x')| P(dx'|x^1, d) \\
&\quad + \Big| \int_E c(x^2, d, x')P(dx'|x^1, d) - \int_E c(x^2, d, x')P(dx'|x^2, d) \Big| \\
&\leq L_c\|x^1 - x^2\| + C_P(B_c + L_c)\|x^1 - x^2\|
\end{aligned}$$

by applying Proposition 2.1 to $x \mapsto c(x^2, d, x)$ that is clearly in $BL(E)$ under assumptions A.4. Hence application Pc is still in $BL(E)$. Then, for all (θ, γ) and $(\bar{\theta}, \gamma)$ in $\mathbb{X}'$ one obtains

$$\begin{aligned}|c'_d(\theta,\gamma) - c'_d(\bar{\theta},\gamma)| &= \Big| \int_{E^2} c(x,d,x')P(dx'|x,d)\theta(dx) - \int_{E^2} c(x,d,x')P(dx'|x,d)\bar{\theta}(dx)\Big| \\ &\leq (B_c + L_c)(C_P + 1)d_E(\theta,\bar{\theta}).\end{aligned}$$

Hence the result. $\square$

Lemma 2.10 *Under assumptions A.4 and 2.4, for all $\xi \in \bar{\mathbb{X}}'$ and $d \in \mathbb{K}'(\xi)$, one has*

$$|c'(\xi,d) - \bar{c}'(\xi,d)| \leq (B_c + L_c) \sup_{j\in\{1,\ldots,\ell\}} \mathscr{D}_j.$$

Proof: For all $\xi = (\bar{\theta}, \gamma) \in \bar{\mathbb{X}}'$ and $d \in \mathbb{K}'(\xi)$, one has

$$|c'(\xi,d) - \bar{c}'(\xi,d)| = \sum_{\omega^i\in\Omega} \Big|\Big(\int_E c(\omega^i,d,x')P(x'|\omega^i,d) - \sum_{\omega^j\in\Omega} c(\omega^i,d,\omega^j)\bar{P}(\omega^j|\omega^i,d)\Big)\Big|\bar{\theta}(\omega^i).$$

The result the follows directly form Proposition 2.5 and the fact that $x \to c(\omega^i, d, x)$ is in $BL(E)$. $\square$

2.1.6 Proof of Theorem 3.3

We establish the result by (backward) induction on n, with the additional statement that v'_n is in $BLP(E)$ for all n.

• For $n = N$, $v'_N = C'$ is in $BLP(E)$ by Lemma 2.8. In addition, set ξ in $\bar{\mathbb{X}}'$. Then by definition one has

$$|v'_N(\xi) - \bar{v}'_N(\xi)| = |C'(\xi) - C'(\xi)| = 0.$$

• Suppose the result holds true for some $n + 1 \leq N$. By induction, v'_{n+1} is in $BL(E)$, thus $R'v'_{n+1}$ is also in $BL(E)$ by Proposition 2.3. For all $d \in \mathbb{A}$, c'_d is in $BLP(E)$ by Lemma 2.9. As $BLP(E)$ is clearly stable by finite maximum, it follows that v'_n is also in $BLP(E)$. Set ξ in $\bar{\mathbb{X}}'$. If $\xi = \Delta$, recall that $\mathbb{K}'(\Delta) = \{\check{d}\}$, so that on the one hand, one has

$$v'_n(\xi) = c'(\xi,\check{d}) + R'v'_{n+1}(\xi,\check{d}) = 0 + v'_{n+1}(\Delta),$$

and on the other hand, one has

$$\bar{v}'_n(\xi) = \bar{c}'(\xi,\check{d}) + \bar{R}'\bar{v}'_{n+1}(\xi,\check{d}) = 0 + \bar{v}'_{n+1}(\Delta).$$

Hence, one has

$$|v'_n(\Delta) - \bar{v}'_n(\Delta)| = |v'_{n+1}(\Delta) - \bar{v}'_{n+1}(\Delta)| = 0,$$

by induction.

If $\xi = (\bar{\theta}, \gamma) \in \bar{\mathbb{X}}' - \{\Delta\}$, with $\gamma = (y, z, w)$, then by definition one has $\bar{\theta}(\{\omega \in \Omega; \omega_4 = w\}) = 1$ and $\theta(\Omega_0) \geq \big(\underline{f}\overline{f}^{-1}\big)^{\frac{w}{\delta}\vee 1} e^{-w\|\lambda\|}$. The dynamic programming equations yield

$$\begin{aligned}|v'_n(\bar{\theta},\gamma) - \bar{v}'_n(\bar{\theta},\gamma)| &\leq \max_{d\in\mathbb{K}'(\xi)} |c'(\xi,d) - \bar{c}'(\xi,d)| + \max_{d\in\mathbb{K}'(\xi)} |R'v'_{n+1}(\xi,d) - \bar{R}'\bar{v}'_{n+1}(\xi,d)| \\ &\leq \max_{d\in\mathbb{K}'(\xi)} |c'(\xi,d) - \bar{c}'(\xi,d)| + \max_{d\in\mathbb{K}'(\xi)} |R'v'_{n+1}(\xi,d) - \bar{R}'v'_{n+1}(\xi,d)| \\ &\qquad + \max_{d\in\mathbb{K}'(\xi)} |\bar{R}'v'_{n+1}(\xi,d) - \bar{R}'\bar{v}'_{n+1}(\xi,d)| \\ &= A_1 + A_2 + A_3.\end{aligned}$$

∘ The fist term A_1 is bounded thanks to Lemma 2.10 by $(B_c + L_c) \sup_{j\in\{1,\dots,\ell\}} \mathscr{D}_j$.

∘ The second term A_2 is bounded by Proposition 2.7 as v'_n is in$BLP(E)$:

$$\begin{aligned} A_2 &\leq (\|v'_{n+1}\|_{E,\mathscr{P}} + [v'_{n+1}]_{E,\mathscr{P}})\bar{C}_{R'} \left(\underline{f}^{-1}\overline{f}\right)^{\frac{w}{\delta}\vee 1} e^{w\|\lambda\|} \sup_{j\in\{1,\dots,\ell\}} \mathscr{D}_j \\ &\leq (\|v'_{n+1}\|_{E,\mathscr{P}} + [v'_{n+1}]_{E,\mathscr{P}})\bar{C}_{R'} \left(\underline{f}^{-1}\overline{f}\right)^{N\vee 1} e^{N\delta\|\lambda\|} \sup_{j\in\{1,\dots,\ell\}} \mathscr{D}_j. \end{aligned}$$

∘ The last term A_3 is bounded by the induction hypothesis and using the fact that $\bar{R}'$ is a Markov kernel

$$A_3 \leq \sup_{\xi'\in\bar{\mathbb{X}}'} |v'_{n+1}(\xi') - \bar{v}'_{n+1}(\xi')| \leq C_{v'_{n+1}} \sup_{j\in\{1,\dots,\ell\}} \mathscr{D}_j.$$

hence the result.

2.2 Second discretization

The proof of Theorem 3.4 follows similar lines as that of Theorem 3.3. Again, it is based on Lipschitz regularity properties of the operators involved.

We first introduce additional function spaces. Let $BLP(\Omega)$ be the set of Borel functions from $\bar{\mathbb{X}}'$ onto $\mathbb{R}$ for which there exist finite constants $\|\varphi\|_{\Omega,\mathscr{P}}$ and $[\varphi]_{\Omega,\mathscr{P}}$ such that for all (θ,γ) and $(\bar{\theta},\gamma)$ in $\bar{\mathbb{X}}'$ such that $\theta(\Omega_{0:2}) = \bar{\theta}(\Omega_{0:2}) = 1$ or $\theta(\Omega_3) = \bar{\theta}(\Omega_3) = 1$ one has

- $|\varphi(\Delta)| \leq \|\varphi\|_{\Omega,\mathscr{P}}$,
- $|\varphi(\theta,\gamma)| \leq \|\varphi\|_{\Omega,\mathscr{P}}$,
- $|\varphi(\theta,\gamma) - \varphi(\bar{\theta},\gamma)| \leq [\varphi]_{\Omega,\mathscr{P}} d_\Omega(\theta,\bar{\theta})$,
- φ is constant on the set $\{(\theta,\gamma) \in \bar{\mathbb{X}}' - \{\Delta\}\ ;\ \theta(\Omega_3) = 1\}$.

Denote also the unit ball of $BLP(\Omega)$ by

$$BLP_1(\Omega) \quad = \quad \big\{\varphi \in BLP(\Omega) : \|\varphi\|_{\Omega,\mathscr{P}} + [\varphi]_{\Omega,\mathscr{P}} \leq 1\big\}.$$

2.2.1 Regularity of operator $\bar{R}'$

We first need to investigate the regularity of the approximate filter operator.

Proposition 2.11 *Under assumptions A.3, for all $\gamma' = (y', z', w') \in \mathbb{O}$, $d = (\ell, r) \in \mathbb{A} - \{\check{d}\}$ and all $\theta, \bar{\theta} \in \mathscr{P}(\Omega)$ satisfying $\theta(\{\omega \in \Omega_{0:2}; \omega_4 = w' - r\}) = \bar{\theta}(\{\omega \in \Omega_{0:2}; \omega_4 = w' - r\}) = 1$ and $\theta(\Omega_0) > 0$, $\bar{\theta}(\Omega_0) > 0$, we have*

$$\int_I d_\Omega(\bar{\Psi}(\theta, y', z', d), \bar{\Psi}(\bar{\theta}, y', z', d))dy' \leq \frac{C_{\bar{\Psi}}}{\theta(\Omega_0)\bar{\theta}(\Omega_0)} d_\Omega(\theta, \bar{\theta}). \tag{14}$$

Proof: Set $\omega^j \in \Omega$. $\underline{\text{If } z' = 1 \text{ and } y' = 0}$, one has

$$
\begin{aligned}
&|\bar{\Psi}(\theta, 0, 1, w', d)(\omega^j) - \bar{\Psi}(\bar{\theta}, 0, 1, w', d)(\omega^j)| \\
&= \Bigg| \frac{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \mathbb{1}_{\Omega_3}(\omega^j) \bar{P}(\omega^j|\omega^i, d)\theta(\omega^i)}{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \sum_{\omega^k \in \Omega_3} \bar{P}(\omega^k|\omega^i, d)\theta(\omega^i)} \\
&\qquad\qquad - \frac{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \mathbb{1}_{\Omega_3}(\omega^j) \bar{P}(\omega^j|\omega^i, d)\bar{\theta}(\omega^i)}{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \sum_{\omega^k \in \Omega_3} \bar{P}(\omega^k|\omega^i, d)\bar{\theta}(\omega^i)} \Bigg| \\
&\le \frac{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \mathbb{1}_{\Omega_3}(\omega^j) \bar{P}(\omega^j|\omega^i, d)|\theta(\omega^i) - \bar{\theta}(\omega^i)|}{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \sum_{\omega^k \in \Omega_3} \bar{P}(\omega^k|\omega^i, d)\theta(\omega^i)} \\
&+ \left[\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \mathbb{1}_{\Omega_3}(\omega^j) \bar{P}(\omega^j|\omega^i, d)\bar{\theta}(\omega^i) \right] \\
&\qquad\qquad \Bigg| \frac{1}{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \sum_{\omega^k \in \Omega_3} \bar{P}(\omega^k|\omega^i, d)\theta(\omega^i)} \\
&\qquad\qquad\quad - \frac{1}{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \sum_{\omega^k \in \Omega_3} \bar{P}(\omega^k|\omega^i, d)\bar{\theta}(\omega^i)} \Bigg| \\
&\le \frac{1}{\theta(\Omega_0)} d_\Omega(\theta, \bar{\theta}) + \frac{1}{\theta(\Omega_0)\bar{\theta}(\Omega_0)} d_\Omega(\theta, \bar{\theta}).
\end{aligned}
$$

$\underline{\text{If } z' = 0}$, similarly one has

$$
\begin{aligned}
&|\bar{\Psi}(\theta, 0, 1, w', d)(\omega^j) - \bar{\Psi}(\bar{\theta}, 0, 1, w', d)(\omega^j)| \\
&= \Bigg| \frac{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) f(y' - F(\omega^j)) \mathbb{1}_{\Omega_{0:2}}(\omega^j) \bar{P}(\omega^j|\omega^i, d)\theta(\omega^i)}{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \sum_{\omega^k \in \Omega_{0:2}} f(y' - F(\omega^k)) \bar{P}(\omega^k|\omega^i, d)\theta(\omega^i)} \\
&- \frac{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) f(y' - F(\omega^j)) \mathbb{1}_{\Omega_{0:2}}(\omega^j) \bar{P}(\omega^j|\omega^i, d)\bar{\theta}(\omega^i)}{\sum_{\omega^i \in \Omega_{0:2}} \mathbb{1}_{\{w'-r\}}(\omega_4^i) \sum_{\omega^k \in \Omega_{0:2}} f(y' - F(\omega^k)) \bar{P}(\omega^k|\omega^i, d)\bar{\theta}(\omega^i)} \Bigg| \\
&\le \frac{\overline{f}}{\underline{f}\theta(\Omega_0)} d_\Omega(\theta, \bar{\theta}) + \frac{\overline{f}^2}{\underline{f}^2 \theta(\Omega_0)\bar{\theta}(\Omega_0)} d_\Omega(\theta, \bar{\theta}),
\end{aligned}
$$

hence the result by integration on y' □

Now we can turn to the regularity of operator $\bar{R}'$.

Proposition 2.12 *Under Assumptions A.3, there exists a positive constant $C_{\bar{R}'}$ such that for all $g \in BLP_1(\Omega)$, for all $\theta, \bar{\theta}$ in $\mathscr{P}(\Omega)$, $(y, \bar{y}, z, w) \in I^2 \times \{0, 1\} \times [0, H]$ and $d = (\ell, r) \in \mathbb{A} - \{\check{d}\}$, such that $(\theta, y, z, w) \in \bar{X}'$ and $(\bar{\theta}, \bar{y}, z, w) \in \bar{X}'$, and*

- *either $d \neq \check{d}$ and $\theta(\Omega_{0:2}) = \bar{\theta}(\Omega_{0:2}) = 1$,*
- *or $d = \check{d}$,*

one has

$$
\left|\bar{R}'g(\theta, y, z, d) - \bar{R}'g(\bar{\theta}, \bar{y}, z, d)\right| \le C_{\bar{R}'} d_\Omega(\theta, \bar{\theta}). \tag{15}
$$

In particular, operator $\bar{R}'$ maps $BLP(\Omega)$ onto itself and for $g \in BLP(\Omega)$, one has $\|\bar{R}'g\|_\Omega \le \|g\|_\Omega$ and $[\bar{R}'g]_\Omega \le (\|g\|_\Omega + [g]_\Omega)C_{\bar{R}'}$.

Proof: Set g in $BLP_1(\Omega)$, $\gamma = (y, z, w)$ and $\bar{\gamma} = (\bar{y}, z, w)$.
If $d = \check{d}$, then $\bar{R}'g(\theta, \gamma, \check{d}) = \bar{R}'g(\bar{\theta}, \bar{\gamma}, \check{d}) = g(\Delta)$, hence $|\bar{R}'g(\theta, \gamma, \check{d}) - \bar{R}'g(\bar{\theta}, \bar{\gamma}, \check{d})| = 0$.
If $d = (\ell, r) \neq \check{d}$, then by assumption $\theta(\Omega_{0:2}) = \bar{\theta}(\Omega_{0:2}) = 1$ so that one has $\theta(\Omega_0) > 0$ and

$\bar{\theta}(\Omega_0) > 0$ as $(\theta, \gamma) \in \bar{\mathbb{X}}'$ and $(\bar{\theta}, \gamma) \in \bar{\mathbb{X}}'$. In addition, one has $z = 0$ and $w < N\delta$, such that

$$\begin{aligned}\bar{R}' g(\theta, \gamma, d) = \int_I \sum_{\omega^i \in \Omega_{0:2}} & g\big(\bar{\Psi}(\theta, y', 0, w + r, d), y', 0, w + r\big) \\ & \sum_{\omega^k \in \Omega_{0:2}} f(y' - F(\omega^k))\bar{P}(\omega^k | \omega^i, d)\theta(\omega^i) dy' \\ + \sum_{\omega^i \in \Omega_{0:2}} & g\big(\bar{\Psi}(\theta, 0, 1, w + r, d), y', 0, w + r\big) \\ & \sum_{\omega^k \in \Omega_3} f(y' - F(\omega^k))\bar{P}(\omega^k | \omega^i, d)\theta(\omega^i) dy'.\end{aligned}$$

Set

$$\begin{aligned} g_{\theta,d,0:2}(\omega^i) &= \int_I g\big(\bar{\Psi}(\theta, y', 0, w + r, d), y', 0, w + r\big) \sum_{\omega^k \in \Omega_{0:2}} f(y' - F(\omega^k))\bar{P}(\omega^k | \omega^i, d)\theta(\omega^i) dy', \\ g_{\theta,d,3}(\omega^i) &= \int_I g\big(\bar{\Psi}(\theta, 0, 1, w + r, d), y', 0, w + r\big) \sum_{\omega^k \in \Omega_3} f(y' - F(\omega^k))\bar{P}(\omega^k | \omega^i, d)\theta(\omega^i) dy'. \end{aligned}$$

In particular, one has $g_{\theta,d,0:2}(\omega^i)| \leq B_f$ and $|g_{\theta,d,3}(\omega^i)| \leq B_f$ for all $1 \leq i \leq K$. In addition, one has

$$\begin{aligned} &\big|\bar{R}' g(\theta, y, z, d) - \bar{R}' g(\bar{\theta}, y', z, d)\big| \\ &\leq \left| \sum_{\omega^i \in \Omega_{0:2}} g_{\theta,d,0:2}(\omega^i)\theta(\omega^i) - \sum_{\omega^i \in \Omega_{0:2}} g_{\bar{\theta},d,0:2}(\omega^i)\bar{\theta}(\omega^i) \right| \\ &\qquad + \left| \sum_{\omega^i \in \Omega_{0:2}} g_{\theta,d,3}(\omega^i)\theta(\omega^i) - \sum_{\omega^i \in \Omega_{0:2}} g_{\bar{\theta},d,3}(\omega^i)\bar{\theta}(\omega^i) \right| \\ &\leq \sum_{\omega^i \in \Omega_{0:2}} (|g_{\theta,d,0:2}(\omega^i)| + |g_{\theta,d,3}(\omega^i)|)|\theta(\omega^i) - \bar{\theta}(\omega^i)| \\ &\qquad + \sum_{\omega^i \in \Omega_{0:2}} (|g_{\theta,d,0:2}(\omega^i) - g_{\bar{\theta},d,0:2}(\omega^i)| + |g_{\theta,d,3}(\omega^i) - g_{\bar{\theta},d,3}(\omega^i)|)\bar{\theta}(\omega^i) \\ &\leq A_1 + A_2. \end{aligned}$$

The first term A_1 is directly controlled by $2B_f d_\Omega(\theta, \bar{\theta})$. For the second term, we have

$$\big|g_{\theta,d,u}(\omega^i) - g_{\bar{\theta},d,u}(\omega^i)\big| \leq \overline{f} \int_I d_\Omega(\bar{\Psi}(\theta, y', 0, w + r, d), \bar{\Psi}(\bar{\theta}, y', 0, w + r, d))dy',$$

for $u \in \{0 : 2, 3\}$, and we conclude using Proposition 2.12 to obtain $C_{\bar{R}'} \leq B_f + \overline{f} C_{\bar{\Psi}} \left(\frac{\underline{f}}{\overline{f}}\right)^{-2} e^{2H\|\lambda\|}$.
□

2.2.2 Projection error for operators $\bar{R}'$ and $\hat{R}'$

Proposition 2.13 *For all $g \in BLP_1(\Omega)$ and $1 \leq j \leq L$, one has*

$$\left|\bar{R}' g(\rho^j, d) - \hat{R}' g(\rho^j, d)\right| \leq \sup_{\{1 \leq j \leq L\}} \mathscr{D}_j.$$

Proof: One has

$$\begin{aligned}\left|\bar{R}'g(\rho^j,d)-\hat{R}'g(\rho^j,d)\right| &= \left|\int_{\bar{\mathbb{X}}'} g\big(\xi\big)\,\bar{R}'(d\xi|\rho^j,d)-\sum_{k=1}^{L} g\big(\rho^k\big)\,\bar{R}'(\bar{C}_k|\rho^j,d)\right|\\ &\le \sum_{k=1}^{L}\int_{\bar{C}_k}\big|g(\xi)-g(\rho^k)\big|\bar{R}'(d\xi|\rho^j,d)\\ &\le \sum_{k=1}^{L}\int_{\bar{C}_k} d(\xi,\rho^k)\bar{R}'(d\xi|\rho^j,d)\\ &\le \sup_{\{1\le k\le L\}}\bar{\mathscr{D}}_k,\end{aligned}$$

hence the result. □

2.2.3 Regularity and approximation error for the cost functions

The last preliminary result we need in order to prove Theorem 3.4 is to ensure that the cost functions c', C' also belong to $BLP(\Omega)$.

Lemma 2.14 *Let g be a function from $\mathbb{X}'$ onto $\mathbb{R}$ belonging to $BLP(E)$, then the restriction of g to $\bar{X}'$ is in $BLP(\Omega)$.*

Proof: First, if g is bounded by $\|g\|_{E,\mathscr{P}}$ on $\mathbb{X}'$, it is also bounded by $\|g\|_{E,\mathscr{P}}$ on $\bar{\mathbb{X}}'$. Second, if g is constant on E_3 it is also constant on Ω_3. Last, for all (θ,γ) and $(\bar{\theta},\gamma)$ in $\bar{\mathbb{X}}'$ one has

$$\begin{aligned}&|g(\theta,\gamma)-g(\bar{\theta},\gamma)|\le [g]_{E,\mathbb{P}}d_E(\theta,\bar{\theta})\\ &\le [g]_{E,\mathbb{P}}\sup_{h\in BL_1(E)}\sup_{\gamma\in\mathbb{O}}\left|\int h(x,\gamma)\mathbb{1}_{\mathbb{X}}(x,\gamma)\theta(dx)-\int h(x,\gamma)\mathbb{1}_{\mathbb{X}}(x,\gamma)\bar{\theta}(dx)\right|\\ &\le [g]_{E,\mathbb{P}}\sup_{h\in BL_1(E)}\sup_{\gamma\in\mathbb{O}}\left|\sum_{i=1}^{K}\Big(h(\omega^i,\gamma)\mathbb{1}_{\mathbb{X}}(\omega^i,\gamma)\theta(\omega^i,)-h(\omega^i,\gamma)\mathbb{1}_{\mathbb{X}}(\omega^i,\gamma)\bar{\theta}(\omega^i)\Big)\right|\\ &\le [g]_{E,\mathbb{P}}d_\Omega(\theta,\bar{\theta}),\end{aligned}$$

hence the result. □

In particular, the restrictions of C' and $\bar{c}'$ to $\bar{X}'$ belong to $BLP(\Omega)$.

2.2.4 Proof of Theorem 3.4

We establish the result by (backward) induction on k, with the additional statement that $\bar{v}'_k$ is in $BLP(\Omega)$ for all k.

- For $k=N$ and $1\le j\le L$, one has

$$\left|\bar{v}'_N(\rho^j)-\hat{v}'(\rho^j)\right|=\left|C'(\rho^j)-C'(\rho^j)\right|=0,$$

by definition. In addition, $\bar{v}'=C'$ is in $BLP(\Omega)$.

- Suppose the result holds true for some $k+1\le N$. By induction, $\bar{v}'_{n+1}$ is in $BLP(\Omega)$, thus $\bar{R}'v'_{n+1}$ is also in $BLP(E)$ by Proposition 2.12. For all $d\in\mathbb{A}$, $\bar{c}'_d$ is in $BLP(\Omega)$. As $BLP(\Omega)$ is clearly stable by finite maximum, it follows that $\bar{v}'_n$ is also in $BLP(\Omega)$. Now, the dynamic programming equations yield

$$\begin{aligned}\left|\bar{v}'_k(\rho^j)-\hat{v}'_k(\rho^j)\right| &\le \max_{d\in\mathbb{K}'(\rho^j)}\left|\bar{R}'\bar{v}'_{k+1}(\rho^j,d)-\hat{R}'\hat{v}'_{k+1}(\rho^j,d)\right|\\ &\le \max_{d\in\mathbb{K}'(\rho^j)}\left|\bar{R}'\bar{v}'_{k+1}(\rho^j,d)-\hat{R}'\bar{v}'_{k+1}(\rho^j,d)\right|\\ &\qquad+\max_{d\in\mathbb{K}'(\rho^j)}\left|\hat{R}'\bar{v}'_{k+1}(\rho^j,d)-\hat{R}'\hat{v}'_{k+1}(\rho^j,d)\right|.\end{aligned}$$

The last term on the right-hand side is smaller than $\max_{1\leq j\leq L}|\bar{v}'_{k+1}(\rho^j)-\hat{v}'_{k+1}(\rho^j)|$ as $\hat{R}'$ is a Markov kernel. The first term on the right-hand side is bounded by Proposition 2.13 as $\bar{v}'_n$ is in $BLP(\Omega)$:

$$\max_{d\in\mathbb{K}'(\rho^j)}|\bar{R}'\bar{v}'_{k+1}(\rho^j,d)-\hat{R}'\bar{v}'_{k+1}(\rho^j,d)|\leq C_{\bar{R}'}(\|\bar{v}'_{k+1}\|_\Omega+[\bar{v}'_{k+1}]_\Omega)\sup_{j\in\{1,\dots,L\}}\bar{\mathscr{D}}_j.$$

which concludes the proof.

3 Dynamic programming

In this section, we show that in practice the points of grid Ω can be constructed without keeping track of the time since the beginning of the observation, w. For such purpose, we consider the observed space E.

By definition, Ω is such that each element ω is a set (m,ζ,u,w). Because observations can only be obtained at times corresponding to a subset of $\{\delta,2\delta,\dots,N\delta\}$, Ω needs only to discretise E at these time-points. Hence we denote Ω_k the set of all elements of Ω for which the fourth coordinate is equal to $k\delta$, that is $\Omega_k=\{\omega=(m,\zeta,u,w)\in\Omega|w = k\delta\}$, and $\Omega_k^{|(1,2,3)}$ its restriction to the first three coordinates, that is $\Omega_k^{|(1,2,3)}=\{\bar{\omega}=(m,\zeta,u)|\exists\omega=(m,\zeta,u,k\delta)\in\Omega_k\}$.
We choose to construct the grid such that $\forall k,j\in\{1,\dots,N\}$, $\Omega_k^{|(1,2,3)}=\Omega_j^{|(1,2,3)}$, ie if $\omega=(m,\zeta,u,k\delta)\in\Omega_k$, then for all $j\in\{1,\dots,N\}$, $\omega'=(m,\zeta,u,j\delta)\in\Omega_j$. Let $\Omega^{|(1,2,3)}=\Omega_1^{|(1,2,3)}$.

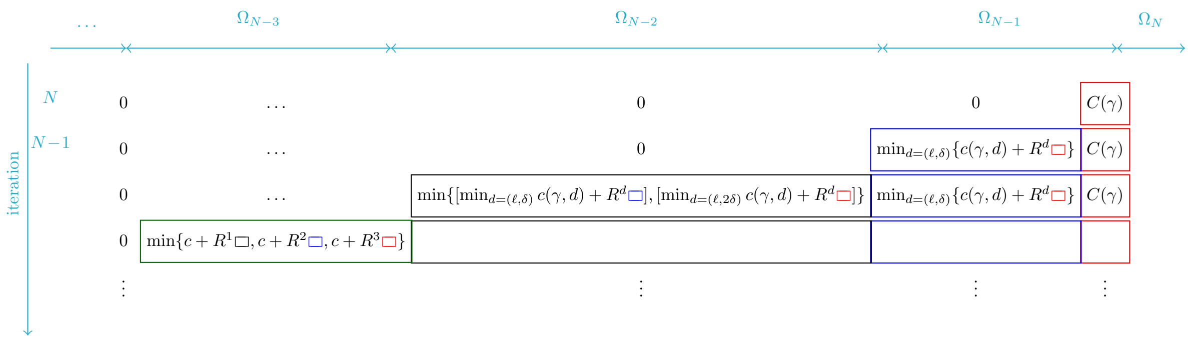
To show that $\Omega=\Omega^{|(1,2,3)}$ is a sufficient discretisation of E, first one can notice that if $d\neq\check{d}$, $c(m,\zeta,u,k\delta,d)=c(m,\zeta,u,j\delta,d)$ for all (k,j) and for both our choice of cost functions.
Then, we can show by backward induction that the number of iterations n to reach $k\delta$ does not affect v_n, that is, $v_n(m,\zeta,u,k\delta)=v_{n'}(m,\zeta,u,k\delta)$ for all n,n' such that $k\delta$ can be reached. Indeed, it is true for $k=N$ as then the only admissible decision is $\check{d}$ for which $Rv_n(m,\zeta,u,N\delta,\check{d})=0$ for all n. Then if the assumption is true for all $k+1,\dots,N$, we have

$$\begin{aligned}
&v_n(m,\zeta,u,k\delta)\\
&=\min_{r\in\mathbb{T}}\min_{d=(\ell,r)\in\mathbb{K}(m,\zeta,u,k\delta)}\{c(m,\zeta,u,k\delta,d)+Rv_{n+1}(m,\zeta,u,k\delta,d)\}\\
&=\min_{r\in\mathbb{T}}\min_{d=(\ell,r)\in\mathbb{K}(m,\zeta,u,k\delta)}\{c(m,\zeta,u,k\delta,d)+\\
&\qquad\int_I\int_\Omega v_{n+1}(m',\zeta',u',k'\delta)f(y'-F(\zeta))\bar{P}(m',\zeta',u',k'\delta|m,\zeta,u,k\delta,d)\Big\}\\
&=\min_{r\in\mathbb{T}}\min_{d=(\ell,r)\in\mathbb{K}(m,\zeta,u,k\delta)}\{c(m,\zeta,u,k\delta,d)+\\
&\qquad\int_I\int_{\Omega_{k+r}} v_{n+1}(m',\zeta',u',(k+r)\delta)f(y'-F(\zeta))\bar{P}(m',\zeta',u',(k+r)\delta|m,\zeta,u,k\delta,d)\Big\}\\
&=\min_{r\in\mathbb{T}}\min_{d=(\ell,r)\in\mathbb{K}(m,\zeta,u,k\delta)}\{c(m,\zeta,u,k\delta,d)+\\
&\qquad\int_I\int_{\Omega_{k+r}} v_{n'+1}(m',\zeta',u',(k+r)\delta)f(y'-F(\zeta))\bar{P}(m',\zeta',u',(k+r)\delta|m,\zeta,u,k\delta,d)\Big\}\\
&=\min_{r\in\mathbb{T}}\min_{d=(\ell,r)\in\mathbb{K}(m,\zeta,u,k\delta)}\{c(m,\zeta,u,k\delta,d)+Rv_{n'+1}(m,\zeta,u,k\delta,d)\}\\
&=v_{n'}(m,\zeta,u,k\delta)
\end{aligned}$$

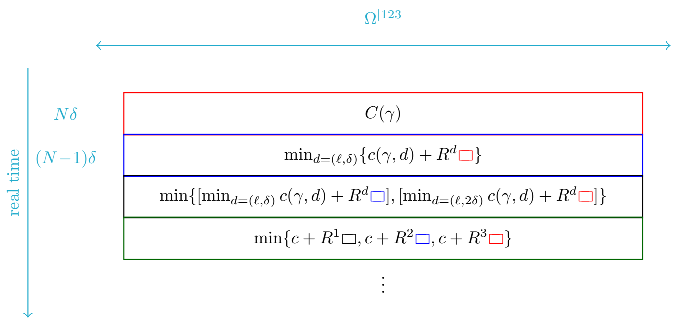
Let us therefore denote $\tilde{v}_k(m,\zeta,u)=v_n(m,\zeta,u,k\delta)$ for all compatible n. If we consider the dynamic programming matrix $(DP_{n,k})$ for which $DP_{n,k}$ is the vector $v_n(\{\omega\in\Omega_k\})$, then the elements of a given column are either 0 if iteration n prevents reaching time $k\delta$, or take one unique value, it can concatenated into the dynamic programming matrix $(\tilde{DP}_k)$ for which $\tilde{DP}_k$ is the vector $\tilde{v}_k(\Omega^{|(1,2,3)})$.

Hence the two following dynamic programming operators are equivalent, where $R\square$ is the integration of the vector $\square$ with respect to R:

$$(DP_{n,k}) =$$



$$(\tilde{DP}_k) =$$



where in the first case one should read the rows as dynamic programming iterations, and refer to the columns to identify the optimal decisions at a given time, while in the second case one should refer to the rows of the matrix to read the optimal decision at a given time.

4 Details on Simulation study

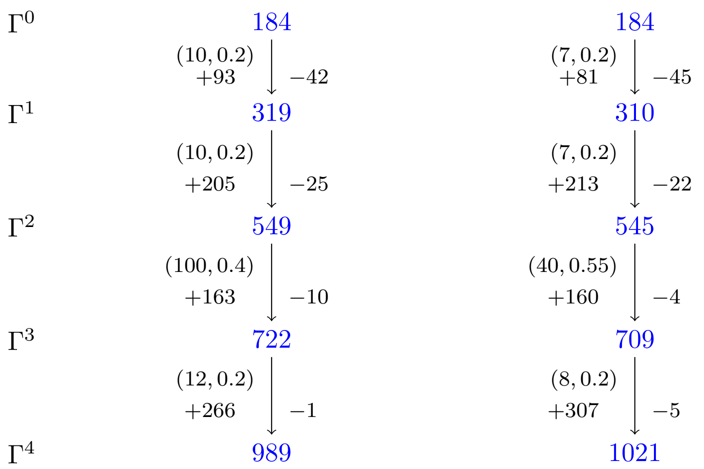
4.1 Grids construction

As explained in the main manuscript, starting from an initial grid (with $184 = n_\Omega$ points chosen to emphasise strong beliefs on each atom of grid Ω), grids were extended iteratively by simulating a number n_{sim} of trajectories using optimal strategies obtained by dynamic programming on the previous grids, and including all estimated filters with distance to their projection larger than a fixed threshold s.
We varied the couples $((n_{sim}, s)$ for both distances (L_2 and L_m) at each iteration. Extensive simulations (work not shown) indicates that for a similar number of points in the resulting grids, the results are better when using a small number of simulations with a stringent threshold than a larger number of simulations with a larger threshold.
At each iteration, we also removed from the current grid all points whose density did not reach a given threshold. To do so, we simulated 10000 trajectories using the current optimal strategy keeping track of all visited points in the grid. Then we removed all points with density smaller than $0.001/n_\Gamma$ (note that if all points were used equally, the density would be $1/n_\Gamma$). Due to the large amount of points removed at the first iteration, the second grid was computed without additional points.
The final process is illustrated in Figure 1.

4.2 Distance impact on trajectory

Setting the same seed, we illustrate the impact of the grid and distance choice on a trajectory in the following Figures. The first row shows the (true) value of X_2, with X_1 indicated by circles for

Figure 1: Iterative grids for distance L_2 (left) and L_m (right)



$X_1 = 0$, triangles for $X_2 = 1$ and pluses for $X_2 = 3$. The second row shows the observed process Y, with colours indicating treatments: black for $\emptyset$, green for a and red for b. The third row shows the mass probability of each mode of the estimated filter, and the fourth for its projected counterpart. Finally, the fifth row shows the distance between estimated and projected filters, in black for the L_2 distance, and in blue for the L_m distance.

Figure 2 shows that though the distance between the filter and its projection significantly decreases between the two grids, the projection does not preserve the mode, hence leads to decision which do not always seem appropriate seeing the data, in particular regarding the next visit date. On the contrary, Figure 3 shows that distance L_m decreases the distance meanwhile maintaining the mass distribution between modes.

4.3 Choice of $\hat{\psi}$ or $\bar{\psi}$ in practice

The discretisation strategy lead to a Markov kernel $\hat{R}'$ on $\bar{\mathbb{X}}'$ to which can be associated a Markov chain $\hat{\psi}$. In theory, the dynamic programming algorithm operates on this Markov chain, and error bounds from the main theorem are computed accordingly. This should imply that at iteration k, filter $\bar{\psi}_k$ is computed from the new observation and the current filter $\hat{\psi}_{k-1}$, then projected on Γ to identify the optimal decision. Then this projection $\hat{\psi}_k = p_{|\Gamma}(\bar{\psi}_k)$ is saved as current filter for the next observation.

In practice, this implies that the projection error $\hat{\psi}_k - \bar{\psi}_k$ is propagated through the dynamic programming recursion. We propose instead to save $\bar{\psi}_k$ as current filter for the next iteration, *i.e.* the filter at iteration $k+1$ will be computed using Y_{k+1} and $\bar{\psi}_k$. We do not propose any error bound on this practical strategy, and there is no guarantee that this should lead to better results, as projection errors on Γ may sometimes compensate previous errors. However in our simulation studies we have observed a significant difference between the strategies, as illustrated in Table 2 (grid 1021 with Lm distance).

One can note that the estimated cost is identical in the visit choice framework between filter $\bar{\psi}$ and $\hat{\psi}$ as they correspond to how grids were calibrated, but in practice the real cost is lower with $\bar{\psi}$ as the latter is closer to the true data.

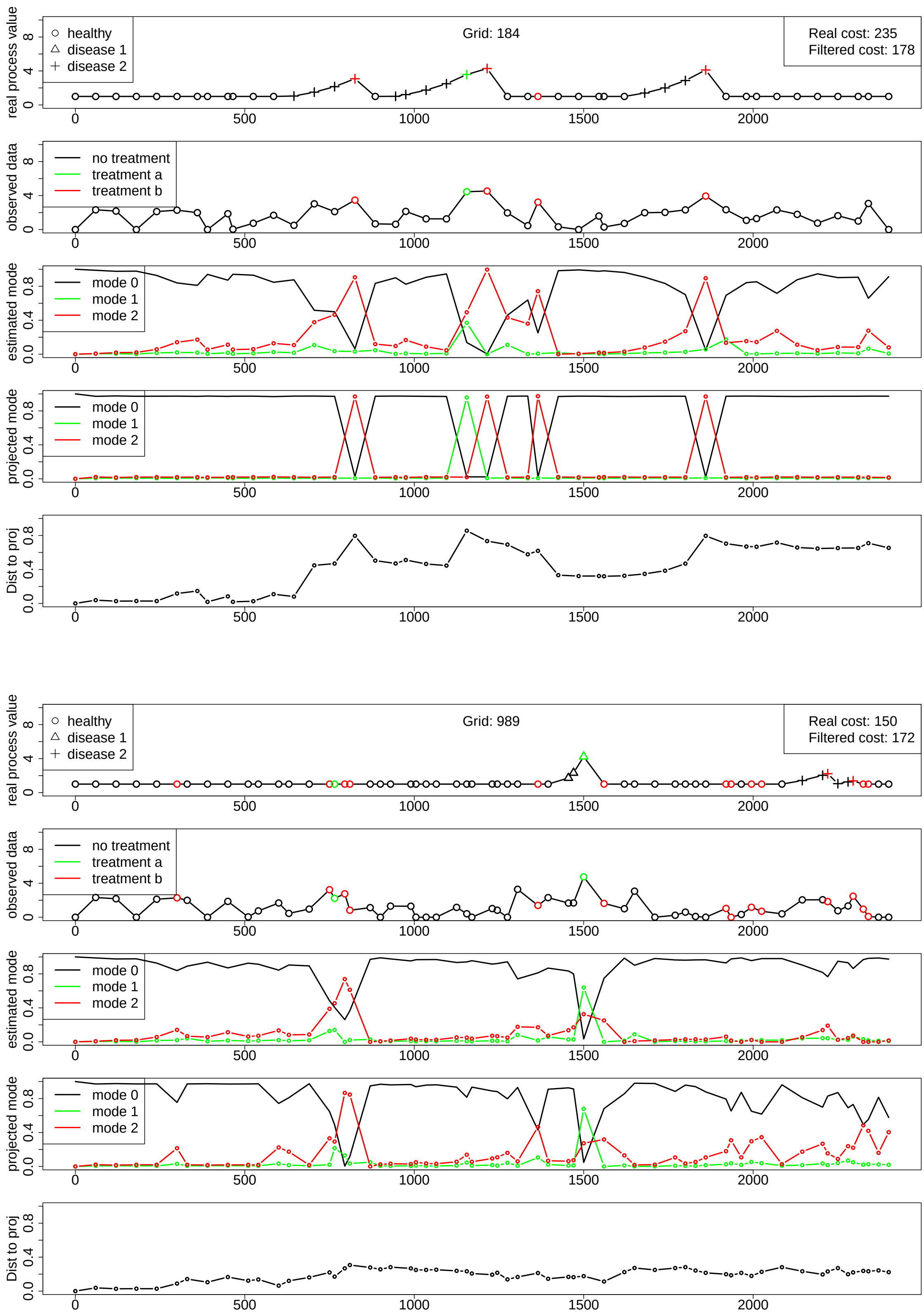


Figure 2: **Trajectories for L_2 distance.**

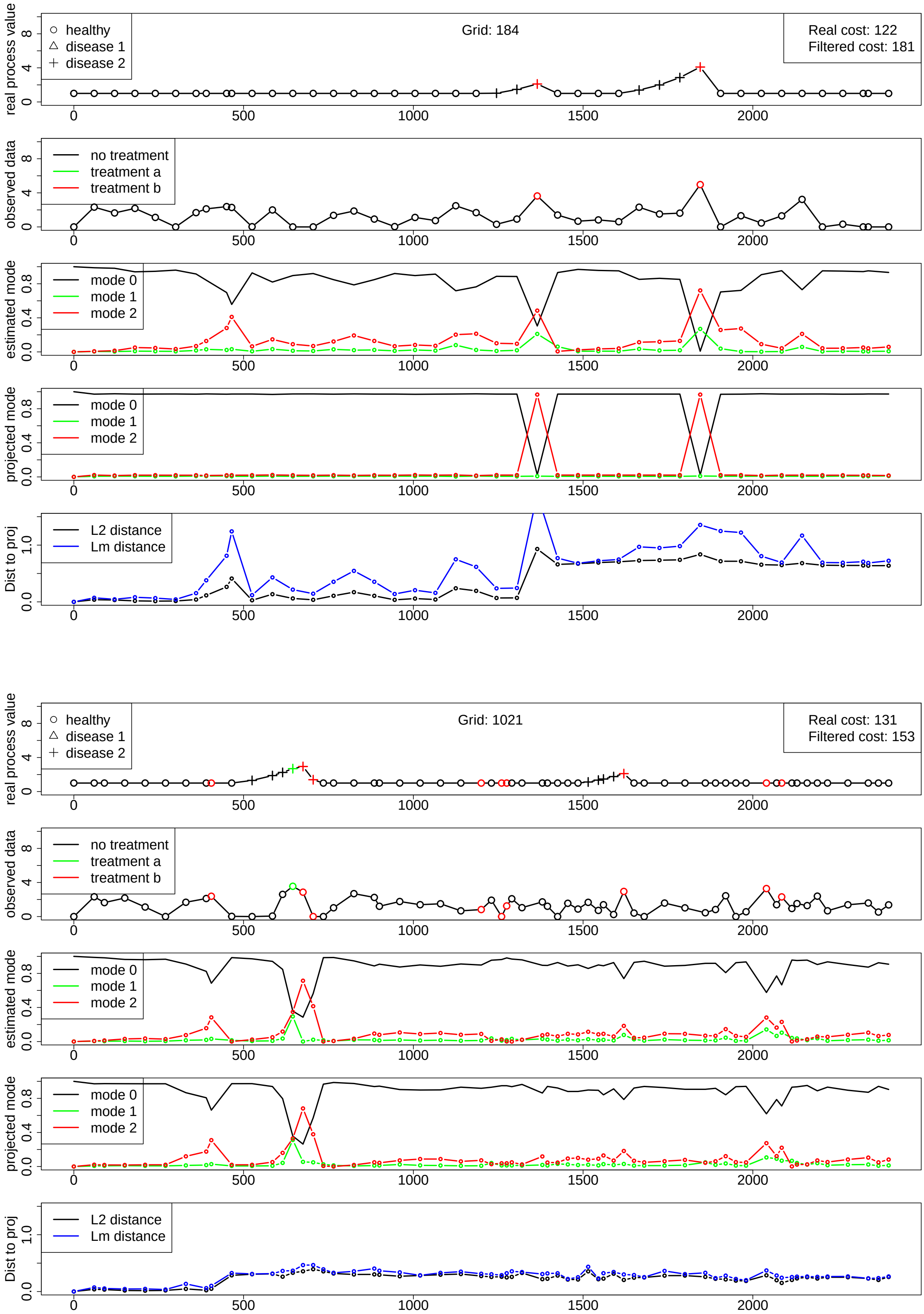


Figure 3: **Trajectories for L_m distance.**

Table 2: Choice of $\hat{\psi}$ or $\bar{\psi}$ as current filter

		$\bar{\psi}$			$\hat{\psi}$	
	Visits	$\hat{v}_0$	real cost (sd)	est. cost (sd)	real cost (sd)	est. cost (sd)
	OS	132.43	136.96 (3.97)	134.8 (0.75)	161.38 (7.2)	134.82 (0.96)
Grid	FD-15	253.28	214.29 (1.67)	215.83 (0.78)	227.84 (3.13)	264.19 (2.31)
1231	FD-60	163.57	143.02 (5.29)	141.8 (1.05)	163.19 (7.29)	165.22 (1.05)