Sustaining a network by controlling a fraction of nodes

Supplementary information

May 26, 2022

Contents

1 Free system states

We consider a class of systems captured by Barzel-Barabási [1–3]

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = M_0(x_i) + \lambda \sum_{j=1}^{N} A_{ij} M_1(x_i) M_2(x_j), \tag{1.1}$$

where $x_i(t)$ is node i's dynamic *activity* ($i = 1, \dots, N$) and the nonlinear functions $M_0(x)$, $M_1(x)$, $M_2(x)$ describe the system's intrinsic dynamics, *i.e.* its self-dynamics (M_0) and its interaction mechanisms (M_1, M_2). The patterns of connectivity between the nodes are captured by the network A_{ij}, a binary $N \times N$ adjacency matrix, which we assume to follow the configuration model framework, namely a random network with an arbitrary degree distribution, p_k. The strength of interactions is governed by the uniform positive parameter λ. We focus on constructive interactions, in which nodes positively impact each other's activity, or attractive interactions, in which a node attracts its neighbors towards its state. We do not consider competitive interactions.

The steady-states $\mathbf{x}_\alpha = (x_{\alpha,1}, \dots, x_{\alpha,N})^\top$ of Eq. (1.1) are obtained by setting its derivative on the left hand side to zero and satisfying linear stability. We focus here on systems with at least two steady-states, *i.e.* $\alpha = 0, 1$, one of which is desirable ($\alpha = 1$) and the other - undesirable ($\alpha = 0$). These states provide an N-dimensional descriptions of the system, capturing the stable activity $x_{\alpha,i}$ of all nodes $i = 1, \dots, N$.

To obtain a macroscopic map of the different dynamic states of the system we use the degree-based mean-field approach described in [4–8]. Assuming nodes with same degrees behave alike, according to this mean-field approximation, we replace the binary value of A_{ij} by its mean value which is the probability of being a connection between i and j given their degrees. In the configuration-model networks which we discuss, this probability equals to $k_i k_j / (N\langle k \rangle)$. Thus,

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = M_0(x_i) + \lambda M_1(x_i) \sum_{j=1}^{N} \frac{k_i k_j}{N\langle k \rangle} M_2(x_j), \tag{1.2}$$

In a relaxation, it turns to

$$0 = M_0(x_i^*) + \lambda M_1(x_i^*) k_i \sum_{j=1}^{N} \frac{k_j}{N\langle k \rangle} M_2(x_j^*), \tag{1.3}$$

where x_i^* is $x_i(t \to \infty)$, the activity of node i at steady state. Here, we define the order parameter Θ as the sum on the right hand side of (1.3), which represents the average impact that a node gets from its neighbor,

$$\Theta = \sum_{j=1}^{N} \frac{k_j}{N\langle k \rangle} M_2(x_j^*). \tag{1.4}$$

Hence, by substituting (1.4) in (1.3), we obtain

$$0 = M_0(x_i^*) + \lambda k_i M_1(x_i^*) \Theta, \tag{1.5}$$

or

$$R(x_i^*) = \lambda k_i \Theta, \tag{1.6}$$

where

$$R(x) = -\frac{M_0(x)}{M_1(x)}. \tag{1.7}$$

For invertible R, we substitute (1.6) in (1.4), to get a self-consistent equation for the order parameter Θ,

$$\Theta = \sum_{j=1}^{N} \frac{k_j}{N\langle k\rangle} M_2(R^{-1}(\lambda k_j \Theta)). \tag{1.8}$$

This equation is solvable for a given formed network which has k_i for $i = 1, 2, ..., N$, but it is not solvable theoretically for a given degree distribution p_k. However, since the term in the summation depends only on k we move to averaging over k,

$$\Theta = \sum_{k} \frac{k p_k}{\langle k\rangle} M_2(R^{-1}(\lambda k \Theta)). \tag{1.9}$$

We can solve this equation numerically for any degree distribution, p_k, and get the solutions for Θ. The solutions of Θ depend on the connectivity of the network represented by the interactions strength, λ, and the full degree distribution, p_k.

Next, to get more simple and universal equation we use another approximation. When k is not widely distributed, and/or the functions M_2 and R are close to be linear or close to be constants, we insert the average into the functions [9], *i.e.* $\overline{M_2(\mathbf{x})} = M_2(\bar{\mathbf{x}})$ and $\overline{R(\mathbf{x})} = R(\bar{\mathbf{x}})$. Doing this in Eq. (1.4), we obtain

$$\Theta = M_2\left(\sum_{j=1}^{N} \frac{k_j}{N\langle k\rangle} x_j^*\right). \tag{1.10}$$

Thus, we define the average activity as

$$\bar{\mathbf{x}} = \sum_{i=1}^{N} \frac{k_i}{N\langle k\rangle} x_i^*, \tag{1.11}$$

to get

$$\Theta = M_2(\bar{\mathbf{x}}). \tag{1.12}$$

By applying the average of $\bar{\mathbf{x}}$ on Eq. (1.6), we obtain

$$\sum_{i=1}^{N} \frac{k_i}{N\langle k\rangle} R(x_i^*) = \lambda \sum_{i=1}^{N} \frac{k_i}{N\langle k\rangle} k_i \Theta = \lambda \frac{\langle k^2\rangle}{\langle k\rangle} \Theta. \tag{1.13}$$

Using again the approximation of inserting the average into the function R, we get

$$R(\bar{\mathbf{x}}) = \beta M_2(\bar{\mathbf{x}}), \tag{1.14}$$

where

$$\beta = \lambda \kappa \tag{1.15}$$

is the parameter of the connectivity comprising of λ, the interaction strength, and

$$\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle}, \tag{1.16}$$

the average degree of a neighbor. If $M_2(\bar{\mathbf{x}})$ is not zero, then we obtain

$$\beta = \frac{R(\bar{\mathbf{x}})}{M_2(\bar{\mathbf{x}})}, \tag{1.17}$$

a simple relation between the average activity in the fixed points and the parameter β representing the network connectivity.

Box I. Example. Cellular dynamics. As an example we consider gene-regulatory dynamics, captured by the Michaelis-Menten model [10], for which (1.1) takes the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -Bx_i^a + \lambda \sum_{j=1}^{N} A_{ij} \frac{x_j^h}{1 + x_j^h}. \tag{1.18}$$

This dynamical equation maps into (1.1) through $M_0(x) = -Bx^a$, $M_1(x) = 1$ and $M_2(x) = x^h/(1+x^h)$, thus, $R(x) = Bx^a$, Eq. (1.7). Using the mean-field approximation of (1.14), we write

$$B\bar{\mathbf{x}}^a = \beta \frac{\bar{\mathbf{x}}^h}{1 + \bar{\mathbf{x}}^h}. \tag{1.19}$$

Setting $B = 1,\ a = 1$ and $h = 2$ (see Ref. [9] for a more general treatment), we extract $\bar{\mathbf{x}}$ from (1.19), predicting three fixed-points as shown in Fig. S1a: The inactive (undesirable, red)

$$\bar{\mathbf{x}}_0 = 0, \tag{1.20}$$

the active (desirable, green)

$$\bar{\mathbf{x}}_1 = \beta/2 + \sqrt{(\beta/2)^2 - 1}, \tag{1.21}$$

and the intermediate (gray dashed-line)

$$\bar{\mathbf{x}}_2 = \beta/2 - \sqrt{(\beta/2)^2 - 1}. \tag{1.22}$$

From this, we can directly obtain the system's state phase-diagram. The inactive state, $\bar{\mathbf{x}}_0 = 0$, is unconditionally stable. The active state $\bar{\mathbf{x}}_1$ is stable and exists for $\beta \geq 2$. The mid-solution $\bar{\mathbf{x}}_2$ is unstable, and hence it does not represent a potential state of the system (Fig. S1a). Therefore $\beta < 2$ forces a collapse on to $\mathbf{x}_0$, while $\beta \geq 2$ represents a bi-stable phase, where the system can reside in both $\mathbf{x}_0$ and $\mathbf{x}_1$, depending on initial conditions. As shown in Fig. S1 the transition is located where the curve of $\beta(\bar{\mathbf{x}})$ is twisted. Hence, for finding the critical point of the transition between states, we just have to demand on Eq. (1.19),

$$\left.\frac{\partial \beta}{\partial \bar{\mathbf{x}}}\right|_{\beta_c} = 0. \tag{1.23}$$

providing $\beta_c = 2$ and $\bar{\mathbf{x}}_c = 1$ (for $B = 1,\ a = 1,\ h = 2$).

In **Box I** we demonstrate this analysis on cellular dynamics, obtaining the potential states and transitions of the gene-regulation, a specific system within our general Eq. (1.1). Cellular dynamics, we find, shows a stable active state only if the adjacency matrix A is sufficiently dense (large κ) or if the interaction strength is sufficiently strong (large λ), together satisfying

$$\beta \geq 2. \tag{1.24}$$

More broadly, this example illustrates the fundamental premise of our present analysis • The potential states of the system and the critical transitions between them, *i.e.* the system's phase diagram, are determined by its intrinsic dynamics $M_0(x), M_1(x)$ and $M_2(x)$ • The specific state of a given system, however, *i.e.* where it resides along that phase diagram, is predicted by the weighted network topology via A_{ij}, as encapsulated within κ (1.16), and via λ • A bi-stable regime implying that even when a system resides in the active state, it is not completely safe since there exists also the potential inactive state, therefore we define such a state as unsustainable, as we discuss below.

1.1 Sustainable and unsustainable states

The structure of the phase diagram yielded by Eq. (1.17) (Fig. S1a) indicates that for cellular dynamics the inactive state $\mathbf{x}_0$ is steady regardless of the value of β. Hence, for large $\beta \geq 2$ where the active state $\mathbf{x}_1$ exists, even if the system resides at $\mathbf{x}_1$, it is *unsustainable*, since the inactive state is steady as well (see demonstrations in Fig. S1b,c), and thus the system is not in a safe state and might move to $\mathbf{x}_0$ under some perturbations in its activity. We define as *sustainable* a system residing at an active state for which a potential inactive state does not exist for the given network structure (A) and interaction strength (λ). System sustainability, therefore, mandates dynamic intervention or *sustaining*, as we derive in Sec. 2. A bi-stable regime where systems have both states for the same network, arise quite commonly in dynamics within the form (1.1). We therefore, seek to characterize our ability to dynamically sustain an unsustainable system via parsimonious dynamic interventions.

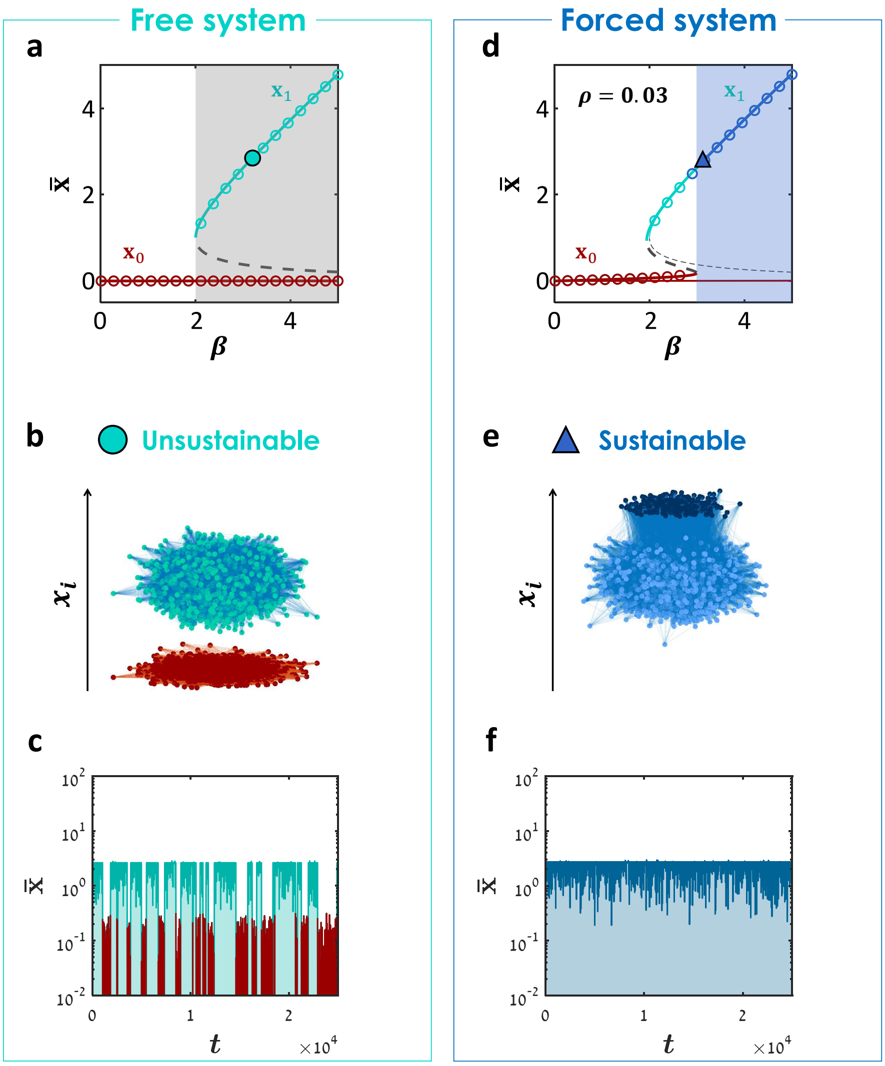

Figure S1: Sustainability of a cellular network. (a) Simulations (symbols) and theory (lines, Eq. (1.19) with $\rho = 0$) results for a free system with ER structure with $N = 10^4$ and $\kappa = 40$. The system has a bi-stable region (gray shade). (b) The activities of a system with $\beta = 3.1$ for $\rho = 0$. Both states, active and inactive, are stable, therefore the system is unsustainable. (c) Demonstration of the unsustainable character of the system in (b) for $\beta = 3.1$, and $\rho = 0$. Adding a noise to the activities x_i causes the system to transit randomly between both states, the active (green) and the inactive (red), since both are stable. Thus, this system can be regarded as unsustainable. (d) For a forced system by a fraction $\rho = 0.03$ of random nodes held with activity $\Delta = 5$, Eqs. (2.30) or (5.8) provide the phase diagram (thick curve), exhibiting an s-shape curve which has now also a regime with only a single active state (blue shade). This regime is the sustainable phase. Note that simulations (symbols) are in good agreement with the theory. The network is the same as in (a). (e) Activities for $\beta = 3.1$ as in (b) but with control of $\rho = 0.03$ and $\Delta = 5$. The dark blue nodes are the forced nodes, and the light blue nodes represent the sustainable $\mathbf{x}_1$, since the inactive state $\mathbf{x}_0$ disappears due to the intervention. (f) Demonstration of the sustainable system in (e). We added the same noise as in (c), but here the system does not collapses into a dysfunctional state since there is no such a state due to the control.

2 Sustaining a network

2.1 Modeling system sustaining

Consider a system of the type discussed above, characterized by two stable states - an undesirable $\mathbf{x}_0$ and a desirable $\mathbf{x}_1$, see Fig. S1. Let us further assume that the system is in the bi-stable phase, presently at the desirable $\mathbf{x}_1$. We seek dynamic interventions that will help us prevent the system from potentially moving to the undesirable state $\mathbf{x}_0$ by eliminating it. To achieve this, we assign a selected set of nodes $\mathcal{F}$ - the *forced* nodes - whose dynamics we externally control. The complementary set of $\mathcal{F}$, containing the free dynamic nodes is denoted by $\mathcal{D} = \mathcal{F}^C$. For simplicity, but without loss of generality, we consider the simplest control, that is forcing the activity of nodes in $\mathcal{F}$ to be in some uniform constant high value Δ. This control, effectively, changes the system's dynamics, Eq. (1.1), into

$$\begin{cases} x_i(t) &= \Delta & i \in \mathcal{F} \\ \frac{\mathrm{d}x_i}{\mathrm{d}t} &= M_0(x_i) + \lambda \sum_{j=1}^{N} A_{ij} M_1(x_i) M_2(x_j) & i \in \mathcal{D} \end{cases}, \tag{2.1}$$

in which nodes in $\mathcal{F}$ are forced to follow the external control, while the remaining $|\mathcal{D}| = N - |\mathcal{F}|$ nodes continue to evolve via the system's natural interaction dynamics. The fraction of controlled nodes is denoted as

$$\rho = \frac{|\mathcal{F}|}{N}. \tag{2.2}$$

To follow the impact of such a control, and testing if it makes the system sustainable, we analyze Eq. (2.1) for finding system's states while it is forced by the intervention described above.

2.2 Forced system analysis

Let us observe the dynamics of the free nodes ($i \in \mathcal{D}$) in Eq. (2.1),

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = M_0(x_i) + \lambda \sum_{j=1}^{N} A_{ij} M_1(x_i) M_2(x_j), \tag{2.3}$$

which by separating the sum between neighbors in $\mathcal{D}$ and neighbors in $\mathcal{F}$ turns to

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = M_0(x_i) + \lambda M_1(x_i) \left(\sum_{j \in \mathcal{D}} A_{ij} M_2(x_j) + \sum_{j \in \mathcal{F}} A_{ij} M_2(\Delta) \right). \tag{2.4}$$

To find the steady states of the system, we demand a relaxation, thus the derivative vanishes,

$$0 = M_0(x_i^*) + \lambda M_1(x_i^*) \left(\sum_{j \in \mathcal{D}} A_{ij} M_2(x_j^*) + k_i^{\mathcal{D} \to \mathcal{F}} M_2(\Delta) \right), \tag{2.5}$$

where x_i^* represents the relaxation activity value of node i, and $k_i^{\mathcal{D}\to\mathcal{F}}$ denotes the number of neighbors in $\mathcal{F}$ of the node i which is in $\mathcal{D}$. Arranging the terms, we obtain

$$R(x_i^*) = \lambda\left(\sum_{j\in\mathcal{D}} A_{ij} M_2(x_j^*) + k_i^{\mathcal{D}\to\mathcal{F}} M_2(\Delta)\right), \tag{2.6}$$

where $R(x) = -M_0(x)/M_1(x)$ as defined in Eq. (1.7). Next, we apply the Degree-Based Mean-Field approximation [4–8], replacing the binary value, A_{ij}, by the probability of $i \in \mathcal{D}$ and $j \in \mathcal{D}$ to be connected given their degrees, which is $k_i^{\mathcal{D}\to\mathcal{D}} k_j^{\mathcal{D}\to\mathcal{D}}/(|\mathcal{D}|\langle k_{\mathcal{D}\to\mathcal{D}}\rangle)$ because of the configuration model structure. This gives

$$R(x_i^*) = \lambda\left(\sum_{j\in\mathcal{D}} \frac{k_i^{\mathcal{D}\to\mathcal{D}} k_j^{\mathcal{D}\to\mathcal{D}}}{|\mathcal{D}|\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} M_2(x_j^*) + k_i^{\mathcal{D}\to\mathcal{F}} M_2(\Delta)\right). \tag{2.7}$$

Note that now the sum does not depend on i, hence we can write

$$R(x_i^*) = \lambda k_i^{\mathcal{D}\to\mathcal{D}} \Theta + \lambda k_i^{\mathcal{D}\to\mathcal{F}} M_2(\Delta), \tag{2.8}$$

by defining the order parameter,

$$\Theta = \frac{1}{|\mathcal{D}|\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} \sum_{j\in\mathcal{D}} k_j^{\mathcal{D}\to\mathcal{D}} M_2(x_j^*), \tag{2.9}$$

which is the average impact of a node in $\mathcal{D}$ (free node) on a node in $\mathcal{D}$. Note that this definition for Θ includes the above definition, Eq. (1.4), for a free system since $\mathcal{D}$ is all the network in a free system. Using Eqs. (2.8) and (2.9), we obtain a single self-consistent equation for the order parameter Θ,

$$\Theta = \frac{1}{|\mathcal{D}|\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} \sum_{j\in\mathcal{D}} k_j^{\mathcal{D}\to\mathcal{D}} M_2(R^{-1}(\lambda k_j^{\mathcal{D}\to\mathcal{D}} \Theta + \lambda k_j^{\mathcal{D}\to\mathcal{F}} M_2(\Delta))). \tag{2.10}$$

To solve this equation, we should replace the summation on nodes in $\mathcal{D}$ by some theoretically calculable term without need to measure nodes degree. Thus, we look for different expression where Eq. (2.10) becomes

$$\Theta = \frac{1}{\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} \left\langle k^{\mathcal{D}\to\mathcal{D}} M_2(R^{-1}(\lambda k^{\mathcal{D}\to\mathcal{D}} \Theta + \lambda k^{\mathcal{D}\to\mathcal{F}} M_2(\Delta)))\right\rangle. \tag{2.11}$$

Averaging over all the possibilities of the pair $k^{\mathcal{D}\to\mathcal{D}}$ and $k^{\mathcal{D}\to\mathcal{F}}$, it gets the form

$$\Theta = \frac{1}{\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} \sum_{k,k'} k \Pr(k_j^{\mathcal{D}\to\mathcal{D}} = k, k_j^{\mathcal{D}\to\mathcal{F}} = k') M_2(R^{-1}(\lambda k \Theta + \lambda k' M_2(\Delta))), \tag{2.12}$$

where $\Pr(k_j^{\mathcal{D}\to\mathcal{D}} = k, k_j^{\mathcal{D}\to\mathcal{F}} = k')$ is the joint probability that node $j \in \mathcal{D}$ has k neighbors in $\mathcal{D}$ (free nodes) and k' neighbors in $\mathcal{F}$ (forced nodes). This joint probability is determined by the degree distribution p_k and the way of selection of the forced nodes ($\mathcal{F}$). Next, instead of running over (k, k') (which are $k^{\mathcal{D}\to\mathcal{D}}$ and $k^{\mathcal{D}\to\mathcal{F}}$ respectively), we run over (k_0, k) where here $k_0 = k^{\mathcal{D}\to\mathcal{V}}$ is the degree of a free node, where $\mathcal{V}$ is the set of all nodes, and $k = k^{\mathcal{D}\to\mathcal{D}}$ is the number of the free neighbors of a free node. Of course for each node $k' = k_0 - k$, that is $k_i^{\mathcal{D}\to\mathcal{F}} = k_i^{\mathcal{D}\to\mathcal{V}} - k_i^{\mathcal{D}\to\mathcal{D}}$. Performing this change of indexes, we obtain

$$\Theta = \frac{1}{\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} \sum_{k_0} \sum_{k \le k_0} k \Pr(k, k_0) M_2(R^{-1}(\lambda k \Theta + \lambda(k_0 - k) M_2(\Delta))), \tag{2.13}$$

and substituting $\Pr(k, k_0) = \Pr(k_0)\Pr(k|k_0)$ yields

$$\Theta = \frac{1}{\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} \sum_{k_0} \Pr(k_0) \sum_{k \le k_0} k \Pr(k|k_0) M_2(R^{-1}(\lambda k \Theta + \lambda(k_0 - k) M_2(\Delta))). \tag{2.14}$$

The terms $\langle k_{\mathcal{D}\to\mathcal{D}}\rangle$, $\Pr(k_0)$, and $\Pr(k|k_0)$ depend on the way of selection of the forced nodes besides the degree distribution p_k. For now, we consider a completely *random selection* of controlled nodes. In this case, $\Pr(k_0) = p_{k_0}$ (the degree distribution of the network) because $\mathcal{D}$ is random and therefore has the same degree distribution p_k as the whole network $\mathcal{V}$. The probability that a neighbor is in $\mathcal{F}$ is ρ, thus $\langle k_{\mathcal{D}\to\mathcal{D}}\rangle = \langle k_{\mathcal{V}\to\mathcal{D}}\rangle = \langle k\rangle(1-\rho)$ as an average of binomial distribution, and $\Pr(k|k_0) = \binom{k_0}{k}(1-\rho)^k \rho^{k_0-k}$, binomial distributed. Substituting these, we finally obtain for a random selection of controlled nodes,

$$\Theta = \frac{1}{\langle k\rangle(1-\rho)} \sum_{k_0} p_{k_0} \sum_{k \le k_0} k \binom{k_0}{k} (1-\rho)^k \rho^{k_0-k} M_2(R^{-1}(\lambda k \Theta + \lambda(k_0 - k) M_2(\Delta))), \tag{2.15}$$

where p_{k_0} is the degree distribution and $\langle k\rangle$ is the mean degree. This self-consistent equation is solvable theoretically based on the degree distribution. We use Eq. (2.15) to plot the theoretical lines in Fig. 4 in the main text.

2.2.1 Small fluctuations MF

To get a simpler and informative equation, we apply another Mean-Field approximation [9] that works well for small fluctuations as we discuss below. According to this MF, we insert the average in Eq. (2.9) into the function $M_2(x)$, to obtain

$$\Theta = M_2\left(\frac{1}{|\mathcal{D}|\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} \sum_{j\in\mathcal{D}} k_j^{\mathcal{D}\to\mathcal{D}} x_j^*\right) = M_2(\bar{\mathbf{x}}), \tag{2.16}$$

where the average activity of the unforced nodes, $\bar{\mathbf{x}}$, is defined by

$$\bar{\mathbf{x}} = \frac{1}{|\mathcal{D}|\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} \sum_{j\in\mathcal{D}} k_j^{\mathcal{D}\to\mathcal{D}} x_j^*. \tag{2.17}$$

This definition includes Eq. (1.11) because for free system $\mathcal{D} = \mathcal{V}$, *i.e.* all the network. Operating this average on Eq. (2.8) and using the MF approximation, we get

$$R(\bar{\mathbf{x}}) = \lambda \kappa_{\mathcal{D}\to\mathcal{D}} \Theta + \lambda \kappa_{\mathcal{D}\to\mathcal{F}} M_2(\Delta), \tag{2.18}$$

where

$$\kappa_{\mathcal{D}\to\mathcal{D}} = \frac{\langle k^2_{\mathcal{D}\to\mathcal{D}}\rangle}{\langle k_{\mathcal{D}\to\mathcal{D}}\rangle},$$
$$\kappa_{\mathcal{D}\to\mathcal{F}} = \frac{\langle k_{\mathcal{D}\to\mathcal{D}} k_{\mathcal{D}\to\mathcal{F}}\rangle}{\langle k_{\mathcal{D}\to\mathcal{D}}\rangle}, \tag{2.19}$$

are the average degrees into $\mathcal{D}$ and into $\mathcal{F}$ respectively of node approached by link within $\mathcal{D}$. Substituting Eq. (2.16) into Eq. (2.18) we obtain an equation for the activity of a forced system,

$$R(\bar{\mathbf{x}}) = \lambda\kappa_{\mathcal{D}\to\mathcal{D}} M_2(\bar{\mathbf{x}}) + \lambda\kappa_{\mathcal{D}\to\mathcal{F}} M_2(\Delta). \tag{2.20}$$

The quantities $\kappa_{\mathcal{D}\to\mathcal{D}}$ and $\kappa_{\mathcal{D}\to\mathcal{F}}$ depend of course on the way of selecting the set $\mathcal{F}$ of controlled nodes. For now, we assume for simplicity that $\mathcal{F}$ is selected completely randomly, and find these quantities for this case. Below in Section 3 we analyze the general case.

2.2.2 The approximation $\overline{M_2(\mathbf{x})} = M_2(\bar{\mathbf{x}})$ and $\overline{R(\mathbf{x})} = R(\bar{\mathbf{x}})$

The approximation we used above is exact if at least one of the following two conditions applies: (i) $M_2(x)$ and $R(x)$ are linear; (ii) $x_i(t)$ are uniform. Clearly, these conditions are not guaranteed, however, under many practical scenarios, they represent a sufficient approximation, designed to detect the macro-scale behavior of the system - as fully corroborated by our numerical examination. Indeed, while Eq. (1.1) is, generally, nonlinear, $M_2(x), R(x)$, in many useful models, are often sub-linear, linear or weakly super-linear, *i.e.* involving powers that are not much higher than unity. This satisfies, approximately, condition (i). In other cases we may observe strong nonlinearities in $M_2(x)$ and $R(x)$, but in such cases, we often have bounded activities $x_i(t)$. This ensures a narrow distribution of $x_i(t)$, roughly satisfying condition (ii). We further elaborate on the relevance of these conditions in the appropriate sections, where we analyze each of our specific dynamic systems, Sec. 5.

2.2.3 Random selection

Selecting the set of controlled nodes $\mathcal{F}$ randomly determines that the degree distribution of the controlled nodes, $\mathcal{F}$, the free nodes $\mathcal{D}$, and all of the nodes $\mathcal{V}$, are the same, p_k. This allows us to analyze the quantities in Eq. (2.19), appearing in Eq. (2.20), since we can replace the average over subset of the network ($\mathcal{F}$ or $\mathcal{D}$) by an average over all the network $\mathcal{V}$. Considering that an arbitrary node in $\mathcal{V}$ belongs to $\mathcal{F}$ with likelihood ρ, we obtain, using Wald's identity,

$$\langle k_{\mathcal{D}\to\mathcal{D}}\rangle = \langle k_{\mathcal{V}\to\mathcal{D}}\rangle = \langle k\rangle(1-\rho). \tag{2.21}$$

For the second moment $\langle k^2_{\mathcal{D}\to\mathcal{D}}\rangle$, we first change the average to be on $\mathcal{V}$, $\langle k^2_{\mathcal{V}\to\mathcal{D}}\rangle$. Then we define a variable y_j which is an indicator of whether neighbor number j of a given node belongs to $\mathcal{D}$, such that $y_j = 1$ if neighbor j belongs to $\mathcal{D}$ and $y_j = 0$ otherwise. Thus, $k_{\mathcal{V}\to\mathcal{D}} = \sum_{j=1}^{k} y_j$ where the index j runs over neighbors of an arbitrary node. The random variables y_j are independent, and have the mean $1-\rho$, therefore,

$$\begin{aligned}\langle k^2_{\mathcal{D}\to\mathcal{D}}\rangle = \langle k^2_{\mathcal{V}\to\mathcal{D}}\rangle &= \Big\langle \Big(\sum_{j=1}^{k} y_j\Big)^2 \Big\rangle = \Big\langle \sum_{j=1}^{k} y_j^2 + 2\sum_{1\le i<j\le k} y_i y_j \Big\rangle \\ &= \langle k\rangle(1-\rho) + \langle k^2 - k\rangle(1-\rho)^2.\end{aligned} \tag{2.22}$$

Substituting Eqs. (2.21) and (2.22) in Eq. (2.19) provides

$$\kappa_{\mathcal{D}\to\mathcal{D}} = \frac{\langle k^2_{\mathcal{D}\to\mathcal{D}}\rangle}{\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} = 1 + (\kappa-1)(1-\rho). \tag{2.23}$$

This result is very intuitive because this quantity represents the expected number of free neighbors of a free node approached via free node as well. Hence, it has certainly one free neighbor, and since the average degree of an arbitrary neighbor is κ, it has $\kappa-1$ more neighbors but a fraction ρ of them is expected to be controlled.

Next, we move to evaluate the second term in Eq. (2.19).

$$\begin{aligned}\langle k_{\mathcal{D}\to\mathcal{D}} k_{\mathcal{D}\to\mathcal{F}}\rangle &= \langle k_{\mathcal{V}\to\mathcal{D}} k_{\mathcal{V}\to\mathcal{F}}\rangle \\ &= \langle k_{\mathcal{V}\to\mathcal{D}}(k - k_{\mathcal{V}\to\mathcal{D}})\rangle \\ &= \langle k_{\mathcal{V}\to\mathcal{D}} k\rangle - \langle k^2_{\mathcal{V}\to\mathcal{D}}\rangle.\end{aligned} \tag{2.24}$$

We already found the second term in Eq. (2.22). The first term is

$$\begin{aligned}\langle k_{\mathcal{V}\to\mathcal{D}} k\rangle &= \Big\langle \Big(\sum_{j=1}^{k} y_j\Big)\Big(\sum_{i=1}^{k} 1\Big) \Big\rangle \\ &= \Big\langle \sum_{i=1}^{k}\sum_{j=1}^{k} y_j \Big\rangle = \langle k^2\rangle(1-\rho).\end{aligned} \tag{2.25}$$

Plugging Eqs. (2.22) and (2.25) into Eq. (2.24) gives

$$\begin{aligned}\langle k_{\mathcal{D}\to\mathcal{D}} k_{\mathcal{D}\to\mathcal{F}}\rangle &= \langle k_{\mathcal{V}\to\mathcal{D}} k\rangle - \langle k^2_{\mathcal{V}\to\mathcal{D}}\rangle \\ &= \langle k^2\rangle(1-\rho) - (\langle k\rangle(1-\rho) + \langle k^2 - k\rangle(1-\rho)^2) \\ &= (\langle k^2\rangle - \langle k\rangle)(1-\rho)\rho.\end{aligned} \tag{2.26}$$

Substituting Eqs. (2.21) and (2.26) into Eq. (2.19) we get

$$\kappa_{\mathcal{D}\to\mathcal{F}} = \frac{\langle k_{\mathcal{D}\to\mathcal{D}} k_{\mathcal{D}\to\mathcal{F}}\rangle}{\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} = (\kappa-1)\rho. \tag{2.27}$$

This formula is intuitive as well, since it represents the number of forced neighbors of a free node approached via another free node. κ is the expected degree, and among the remain $\kappa-1$ neighbors, a fraction ρ is expected to be forced.

Substituting Eqs. (2.23) and (2.27) into Eq. (2.20) we get the Mean-Field equation for a forced system for a random selection of controlled nodes,

$$R(\bar{\mathbf{x}}) = \lambda(1 + (\kappa - 1)(1 - \rho))M_2(\bar{\mathbf{x}}) + \lambda(\kappa - 1)\rho M_2(\Delta). \tag{2.28}$$

Taking the limit $\kappa \gg 1$ and recalling $\beta = \lambda\kappa$, this equation takes the form

$$R(\bar{\mathbf{x}}) = \beta(1 - \rho)M_2(\bar{\mathbf{x}}) + \beta\rho M_2(\Delta). \tag{2.29}$$

Isolating β, we finally obtain Eq. (8) in the main text,

$$\beta = \frac{R(\bar{\mathbf{x}})}{(1 - \rho)M_2(\bar{\mathbf{x}}) + \rho M_2(\Delta)}, \tag{2.30}$$

which yields a relation between $\bar{\mathbf{x}}$ and β, determining the average activity of the forced system for each topology captured by β. Also the system states depend on the intervention characteristics, ρ and Δ. Note that substituting $\rho = 0$ in Eq. (2.30) recovers Eq. (1.17) of a free system. Recognizing the stable states of forced system compared to those of free system, we can answer our main question of sustaining a system by holding a fraction of nodes. In the following Section we find a region where unsustainable network becomes sustainable due to the external control. We call this region the sustainability window.

2.3 Sustainability window

From Eq. (2.30) we get a new phase diagram for a forced given ρ and Δ, which is different from the free system's phase diagram. When there is a range of β-values in which the free system is unsustainable, *i.e.* there exist both the active and the inactive states ($\mathbf{x}_1$ and $\mathbf{x}_0$), and in the same region of β-values the forced system is sustainable, *i.e.* there exists only $\mathbf{x}_1$, then this range is regarded as a sustainability window. This is because a system residing in this region becomes sustainable by controlling a fraction ρ with force Δ.

In Fig. S1 we demonstrate this control effect on gene regulation (elaborately analyzed below in Section 5). Fig. S1d shows the phase diagram of a forced system derived from Eq. (2.30). The thin light lines in Fig. S1d as well as in Fig. S1a represent the states of a free system. One can see that above some value of $\beta = \beta_c$ the forced system shows only the active state, while a free system has both states. Thus, this region, $\beta > \beta_c$, is the sustainability window (blue shade). Fig. S1e demonstrates the elimination of the inactive state $\mathbf{x}_0$ due to the external control. Fig. S1f demonstrates the implication of the system's being sustainable. When adding large noise to the activities x_i, the system never collapses but always goes up into the stable active state. This is in contrary to a free system with same β, Fig. S1c, which makes transitions between both states due to the random noise.

As seen in Fig. S1d, the value of β_c, above which the network is sustainable, can be obtained by finding the local maximum of $\beta(\bar{\mathbf{x}})$. Therefore it is found using Eq. (2.30) by

$$\left.\frac{\partial\beta}{\partial\bar{\mathbf{x}}}\right|_{\beta_c} = 0. \tag{2.31}$$

Solving Eq. (2.31), we find the transition point $(\beta_c, \bar{\mathbf{x}}_c)$. From Eqs. (2.30) and (2.31) it can be seen that β_c depends on the fraction ρ, and the force of control, Δ. Having the relation $\beta_c(\rho, \Delta)$, we derive from it the inverse relation $\rho_c(\beta, \Delta)$, *i.e.* the minimal fraction of nodes required to be controlled, given the connectivity β and the force Δ, for sustaining the network.

Increasing ρ more unveils another critical value of $\rho = \rho_0$ as explained below, the tricritical point.

2.4 Tricritical point

As shown in Figs. 3d,e in the main text, increasing ρ, shortens the unsustainable interval (the bi-stable regime). At some critical point $\rho = \rho_0$, the unsustainable phase completely vanishes and the transition between the active state, $\mathbf{x}_1$, and the inactive state, $\mathbf{x}_0$, becomes continuous. One can see that at this point the maximum and the minimum of $\beta(\bar{\mathbf{x}})$ merge. This consideration yields the additional condition besides Eq. (2.31) for the tricritical point,

$$\left.\frac{\partial^2 \beta}{\partial \bar{\mathbf{x}}^2}\right|_{\beta_c} = 0. \tag{2.32}$$

These two conditions, Eqs. (2.31) and (2.32), together determine a tricritical point (β_0, ρ_0) in (β, ρ)-space where the three phases (sustainable, unsustainable and inactive) meet, and beyond which the system does not experience an abrupt transition at all. See Section 5 for explicit calculations for particular dynamics.

3 Methods of selecting the controlled nodes

In this section we analyze several different methods of selecting nodes to sustain the system via controlling them, besides the simple case of random selection that we analyzed above in Sec. 2.2.3. All selection methods we shall consider, obey the character of random spread throughout the network. However, they differ from each other in the degree distribution of the controlled node. Therefore, we explore the general case of degree distribution of the set $\mathcal{F}$, and then analyze particular cases.

3.1 Arbitrary degree distribution of controlled nodes

Let the degree distribution of the forced nodes be f_k, while the degree distribution of the whole network is p_k. The forced nodes are distributed randomly over the network, excluding *e.g.* a localized selection. We aim to modify the above equations predicting the state a forced system, which were obtained for a random selection, for a general degree distribution. Hence, we wish to update both Eq. (2.15), used for large fluctuations, and Eq. (2.30), used for small fluctuations.

To this end, we have to step back to the stages before we assumed that the selection is random. For large fluctuations, we return to Eq. (2.14) and we recalculate the quantities $\langle k_{\mathcal{D}\to\mathcal{D}}\rangle$, $\Pr(k_0)$ and $\Pr(k|k_0)$. For small fluctuations, we go back to Eq. (2.20), and we recalculate $\kappa_{\mathcal{D}\to\mathcal{D}}$ and $\kappa_{\mathcal{D}\to\mathcal{F}}$ defined in Eq. (2.19). We will follow step by step until we obtain all these quantities, then we rewrite the equations for a forced system for a selection of $\mathcal{F}$ with arbitrary degree distribution f_k.

First, we find the probability that a random node approached via a free neighbor is forced. We assume that there are no correlations between the degrees according to the configuration model. Using Bayes' theorem for the likelihood of an arbitrary node to be forced and to have degree k, we get

$$\Pr(k)\Pr(\mathcal{F}|k) = \Pr(\mathcal{F})\Pr(k|\mathcal{F}), \tag{3.1}$$

where $\Pr(\mathcal{F}|k)$ is the probability that an arbitrary node belongs to $\mathcal{F}$ given its degree is k, and $\Pr(k|\mathcal{F})$ is the opposite. Three of the terms in the last equation are already known,

$$p_k \Pr(\mathcal{F}|k) = \rho f_k. \tag{3.2}$$

Thus, we obtain the probability of a random node with degree k to belong to $\mathcal{F}$, *i.e.* to be forced,

$$\Pr(\mathcal{F}|k) = \rho f_k/p_k. \tag{3.3}$$

Now, when we approach a node via a neighbor, using the degree distribution of a neighbor $kp_k/\langle k\rangle$, we know that the chance it is forced, is obtained by

$$\Pr(\mathcal{F}|\text{neighbor}) = \sum_{k=0}^{\infty}\frac{kp_k}{\langle k\rangle}\Pr(\mathcal{F}|k) = \sum_{k=0}^{\infty}\frac{kp_k}{\langle k\rangle}\rho f_k/p_k = \rho\langle k\rangle_{\mathcal{F}}/\langle k\rangle. \tag{3.4}$$

This quantity is useful since the nodes that appear in the interaction terms in Eq. (1.1) are approached via their neighbors, hence we denote this probability, $\Pr(\mathcal{F}|\text{neighbor})$, by ρ^*, and get

$$\rho^* = \rho\langle k\rangle_{\mathcal{F}}/\langle k\rangle. \tag{3.5}$$

For random selection $\rho^* = \rho$, since $\langle k\rangle_{\mathcal{F}} = \langle k\rangle$. This is reasonable because there is no correlation between the degree and the identity of the node, *i.e.* whether the node is free or forced. However, in degree-dependent selection, since a neighbor has on average larger degree, it is more likely to be forced compared to random node if $\langle k\rangle_{\mathcal{F}} > \langle k\rangle$ and vice versa.

Next, we move to calculate the degree distribution of the subset $\mathcal{D}$ of free nodes. Let us observe the number of nodes with degree k in each set from our three sets (free, forced and all) that fulfill the disjoint union $\mathcal{D} \sqcup \mathcal{F} = \mathcal{V}$,

$$|\mathcal{F}| \Pr(k|\mathcal{F}) + |\mathcal{D}| \Pr(k|\mathcal{D}) = N \Pr(k). \tag{3.6}$$

Plugging into this equation what we already know, we get

$$N\rho f_k + N(1-\rho) \Pr(k|\mathcal{D}) = N p_k, \tag{3.7}$$

resulting in

$$\Pr(k|\mathcal{D}) = (p_k - \rho f_k)/(1-\rho). \tag{3.8}$$

This is the distribution that we denoted above as $\Pr(k_0)$. Knowing this distribution, we can easily calculate its moments,

$$\begin{aligned} \langle k\rangle_{\mathcal{D}} &= \left(\langle k\rangle - \rho\langle k\rangle_{\mathcal{F}}\right)/(1-\rho), \\ \langle k^2\rangle_{\mathcal{D}} &= \left(\langle k^2\rangle - \rho\langle k^2\rangle_{\mathcal{F}}\right)/(1-\rho). \end{aligned} \tag{3.9}$$

Hence, we obtain the expected degree of a neighbor in $\mathcal{D}$,

$$\kappa_{\mathcal{D}} = \frac{\langle k^2\rangle_{\mathcal{D}}}{\langle k\rangle_{\mathcal{D}}} = \frac{\langle k^2\rangle - \rho\langle k^2\rangle_{\mathcal{F}}}{\langle k\rangle - \rho\langle k\rangle_{\mathcal{F}}}. \tag{3.10}$$

Next, we progress to calculate the components of $\kappa_{\mathcal{D}\to\mathcal{D}}$ and $\kappa_{\mathcal{D}\to\mathcal{F}}$ in Eq. (2.19). As above for a random selection, Eqs. (2.21), (2.22) and (2.24), we define the variable y_j which equals 1 if neighbor number j of a given node belongs to $\mathcal{D}$ and 0 otherwise. Thus, according to Wald's identity,

$$\langle k_{\mathcal{D}\to\mathcal{D}}\rangle = \left\langle \sum_{j=1}^{k_{\mathcal{D}\to\mathcal{V}}} y_j \right\rangle = \langle k_{\mathcal{D}\to\mathcal{V}}\rangle(1-\rho^*) = \langle k\rangle_{\mathcal{D}}(1-\rho^*). \tag{3.11}$$

The quantity ρ^* appears instead of ρ in the random case, since the probability that a neighbor is forced is ρ^* as obtained above, Eq. (3.5). For the second moment $\langle k^2_{\mathcal{D}\to\mathcal{D}}\rangle$,

$$\begin{aligned} \langle k^2_{\mathcal{D}\to\mathcal{D}}\rangle &= \left\langle \Big(\sum_{j=1}^{k_{\mathcal{D}\to\mathcal{V}}} y_j \Big)^2 \right\rangle = \left\langle \sum_{j=1}^{k_{\mathcal{D}\to\mathcal{V}}} y_j^2 + 2 \sum_{1\le i<j\le k_{\mathcal{D}\to\mathcal{V}}} y_i y_j \right\rangle \\ &= \langle k_{\mathcal{D}\to\mathcal{V}}\rangle(1-\rho^*) + \langle k^2_{\mathcal{D}\to\mathcal{V}} - k_{\mathcal{D}\to\mathcal{V}}\rangle(1-\rho^*)^2 \\ &= \langle k\rangle_{\mathcal{D}}(1-\rho^*) + \langle k^2 - k\rangle_{\mathcal{D}}(1-\rho^*)^2. \end{aligned} \tag{3.12}$$

For $\langle k_{\mathcal{D}\to\mathcal{D}} k_{\mathcal{D}\to\mathcal{F}} \rangle$, we obtain,

$$\begin{aligned}
\langle k_{\mathcal{D}\to\mathcal{D}} k_{\mathcal{D}\to\mathcal{F}} \rangle &= \langle k_{\mathcal{D}\to\mathcal{D}} (k_{\mathcal{D}\to\mathcal{V}} - k_{\mathcal{D}\to\mathcal{D}}) \rangle \\
&= \langle k_{\mathcal{D}\to\mathcal{D}} k_{\mathcal{D}\to\mathcal{V}} \rangle - \langle k^2_{\mathcal{D}\to\mathcal{D}} \rangle \\
&= \left\langle \Big(\sum_{j=1}^{k_{\mathcal{D}\to\mathcal{V}}} y_j \Big) \Big(\sum_{i=1}^{k_{\mathcal{D}\to\mathcal{V}}} 1 \Big) \right\rangle - \langle k^2_{\mathcal{D}\to\mathcal{D}} \rangle \\
&= \left\langle \sum_{i=1}^{k_{\mathcal{D}\to\mathcal{V}}} \sum_{j=1}^{k_{\mathcal{D}\to\mathcal{V}}} y_j \right\rangle - \langle k^2_{\mathcal{D}\to\mathcal{D}} \rangle \\
&= \langle k^2_{\mathcal{D}\to\mathcal{V}} \rangle (1-\rho^*) - \langle k^2_{\mathcal{D}\to\mathcal{D}} \rangle .
\end{aligned} \tag{3.13}$$

Substituting Eq. (3.12) into Eq. (3.13) yields

$$\begin{aligned}
\langle k_{\mathcal{D}\to\mathcal{D}} k_{\mathcal{D}\to\mathcal{F}} \rangle &= \langle k^2 \rangle_{\mathcal{D}} (1-\rho^*) - \big(\langle k \rangle_{\mathcal{D}} (1-\rho^*) + \langle k^2 - k \rangle_{\mathcal{D}} (1-\rho^*)^2 \big) \\
&= \big(\langle k^2 \rangle_{\mathcal{D}} - \langle k \rangle_{\mathcal{D}} \big) (1-\rho^*) \rho^* .
\end{aligned} \tag{3.14}$$

Using Eqs. (3.11), (3.12) and (3.14), we get the version of Eq. (2.19) for an arbitrary degree distribution of $\mathcal{F}$,

$$\begin{aligned}
\kappa_{\mathcal{D}\to\mathcal{D}} &= 1 + (\kappa_{\mathcal{D}} - 1)(1-\rho^*), \\
\kappa_{\mathcal{D}\to\mathcal{F}} &= (\kappa_{\mathcal{D}} - 1)\, \rho^* .
\end{aligned} \tag{3.15}$$

This result is again intuitive since our equation deals with an average free node which was approached by neighbor in $\mathcal{D}$. Therefore its expected degree is $\kappa_{\mathcal{D}}$. One of its neighbors is certainly free because we came from a free neighbor. Every other neighbor is forced with probability ρ^*. Thus, the expected number of forced is $(\kappa_{\mathcal{D}} - 1)\rho^*$, and the rest are free. Substituting Eq. (3.15) into Eq. (2.20) we get the Mean-Field equation for the states of a forced system,

$$R(\bar{\mathbf{x}}) = \lambda(1 + (\kappa_{\mathcal{D}} - 1)(1-\rho^*)) M_2(\bar{\mathbf{x}}) + \lambda(\kappa_{\mathcal{D}} - 1)\rho^* M_2(\Delta). \tag{3.16}$$

Considering $\kappa_{\mathcal{D}} \gg 1$ and denoting

$$\beta_{\mathcal{D}} = \lambda \kappa_{\mathcal{D}}, \tag{3.17}$$

we get a similar equation to Eq. (2.29) but for a general degree distribution of the controlled nodes,

$$R(\bar{\mathbf{x}}) = \beta_{\mathcal{D}} (1-\rho^*) M_2(\bar{\mathbf{x}}) + \beta_{\mathcal{D}} \rho^* M_2(\Delta). \tag{3.18}$$

This equation allows us to predict the states of the forced system and thus to find the window of sustainability as shown above.
Yet, there is a significant difference between Eq. (3.18), and the one of random selection, (2.29), because here $\beta_{\mathcal{D}}$ depends on ρ and the moments of f_k (degree distribution of $\mathcal{F}$), Eqs. (3.17) and (3.10). Also, ρ^* depends on the moments of p_k and f_k, Eq. (3.5), unlike for a random selection. Thus, the separation of parameters of the topology, β, and the dynamic intervention, ρ and Δ, does not work in this general selection case.

Finally, we find the quantities of Eq. (2.14), to obtain the self-consistent equation for large fluctuations,

$$\begin{aligned}
\Pr(k_0) &= \Pr(k_0|\mathcal{D}) = (p_{k_0} - \rho f_{k_0})/(1-\rho), \\
\Pr(k|k_0) &= \tbinom{k_0}{k}(1-\rho^*)^k \rho^{*k_0-k}, \\
\langle k_{\mathcal{D}\to\mathcal{D}} \rangle &= \langle k \rangle_{\mathcal{D}}(1-\rho^*).
\end{aligned} \tag{3.19}$$

Plugging these three terms into Eq. (2.14), we obtain,

$$\Theta = \frac{1}{\langle k \rangle_{\mathcal{D}}(1-\rho^*)} \times \\ \sum_{k_0} \frac{p_{k_0} - \rho f_{k_0}}{1-\rho} \sum_{k \le k_0} k \binom{k_0}{k} (1-\rho^*)^k \rho^{*k_0-k} M_2\left(R^{-1}\left(\lambda k \Theta + \lambda (k_0 - k) M_2(\Delta)\right)\right), \tag{3.20}$$

which provides, for given degree distributions of the network and of the forced nodes, the order parameter Θ as a function of λ, ρ, and Δ. Eq. (3.20), for an arbitrary degree distribution of controlled nodes, replaces Eq. (2.15), which stands for a random selection.

3.2 Sustainability window

From Eq. (3.18), we can find the conditions under which the system is sustainable, and the limits of this sustainable region, *i.e.* the transitions point. Since Eq. (3.18) has the same shape as Eq. (2.29), the same condition on β, ρ, Δ for a random selection, which is obtained by Eq. (2.31), applies also to $\beta_{\mathcal{D}}, \rho^*, \Delta$ for a general selection. Thus, at the transition we get a relation between those three quantities $\beta_{\mathcal{D}}, \rho^*, \Delta$, and *e.g.* for given $\beta_{\mathcal{D}}$ and Δ, if ρ^* is larger than the value obtained from the transition relation, the system is sustainable. The same holds for $\beta_{\mathcal{D}}$ or Δ. If they are larger than their value at criticality the system is sustainable, see Fig. S5 for results for cellular dynamics.

However, it is more complicated than the random case, because $\beta_{\mathcal{D}}$ and ρ^* are not independent variables and also not directly controllable, thus we cannot keep one of them fixed, and then change the other until it gets the critical value. For example, increasing ρ enlarges ρ^* but in turn might decrease $\kappa_{\mathcal{D}}$ and as a result also $\beta_{\mathcal{D}}$. Nevertheless, for given p_k and all the properties of the control, ρ, f_k, Δ, we can find the critical interaction strength, λ_c, because it appears in a simple way only in $\beta_{\mathcal{D}} = \lambda \kappa_{\mathcal{D}}$.

In specific ways of selecting the set $\mathcal{F}$, we can find simplified expressions for $\beta_{\mathcal{D}}$ and ρ^* obtain an explicit condition for sustainability. We shall consider one example of non-random selection, which is selecting a fraction ρ of the nodes with the highest degrees to be controlled.

3.3 High degree nodes selection

In this Section, to explore the impact of targeted control aimed to the most influential nodes, we select as the forced set, $\mathcal{F}$, a fraction ρ of the network, comprising of the nodes with the highest degrees. We aim to find explicit expressions for the terms in Eq. (3.18), $\beta_{\mathcal{D}}$ and ρ^*. We do this for power-law degree distribution which characterizes scale-free networks.

3.3.1 Scale-free networks

Here we consider scale-free (SF) networks whose degree distribution allows an extensive analysis. Another importance of SF is that the variation of its degrees is large, thus we expect that different selections of the forced nodes would impact significantly. However, these networks challenge the MF of small fluctuations represented by Eq. (3.18), and the MF might be inaccurate in some cases. Thus, all the following results for SF are useful only where MF is valid also for SF networks. Yet, as we show in Fig. S4c, the scaling relation can be predicted quite good although quantitatively the theory does not recover well the simulations, see Fig. 4b in the main text.

Let the degree distribution be

$$p_k = Ak^{-\gamma}, \tag{3.21}$$

for $k_0 \leq k \leq k_f$, and $p_k = 0$ otherwise. The minimal degree k_0 is set arbitrarily, while the maximal degree k_f is obtained naturally as the *natural cutoff* which is evaluated by $k_f \approx k_0 N^{1/(\gamma-1)}$ [11]. We analyze separately three regimes of the exponent γ, (i) $\gamma > 3$ which yields a finite variance of the degree, (ii) $2 < \gamma < 3$ which yields a divergent variance of the degree, (iii) $\gamma = 3$ with a divergent variance as well.

$\gamma > 3$

First we consider the interval of the exponent, $\gamma > 3$, such that the second moment $\langle k^2 \rangle$ is finite. For such distribution, the moments are

$$\begin{aligned}
\langle k \rangle &\approx \frac{\int_{k_0}^{k_f} k k^{-\gamma}\,\mathrm{d}k}{\int_{k_0}^{k_f} k^{-\gamma}\,\mathrm{d}k} = \frac{\gamma-1}{\gamma-2}\frac{k_f^{2-\gamma} - k_0^{2-\gamma}}{k_f^{1-\gamma} - k_0^{1-\gamma}} \approx \frac{\gamma-1}{\gamma-2} k_0, \\
\langle k^2 \rangle &\approx \frac{\int_{k_0}^{k_f} k^2 k^{-\gamma}\,\mathrm{d}k}{\int_{k_0}^{k_f} k^{-\gamma}\,\mathrm{d}k} = \frac{\gamma-1}{\gamma-3}\frac{k_f^{3-\gamma} - k_0^{3-\gamma}}{k_f^{1-\gamma} - k_0^{1-\gamma}} \approx \frac{\gamma-1}{\gamma-3} k_0^2, \\
\kappa &\approx \frac{\gamma-2}{\gamma-3} k_0,
\end{aligned} \tag{3.22}$$

where we assumed $k_0 \ll k_f$.

Next, we find k_s, which is the edge degree separating between the forced and free nodes. Above k_s, all degrees of nodes in $\mathcal{F}$ are located, and below, all nodes are free. Thus,

$$\rho \approx \frac{\int_{k_s}^{k_f} k^{-\gamma}\,\mathrm{d}k}{\int_{k_0}^{k_f} k^{-\gamma}\,\mathrm{d}k} = \frac{k_f^{1-\gamma} - k_s^{1-\gamma}}{k_f^{1-\gamma} - k_0^{1-\gamma}} \approx \left(\frac{k_0}{k_s}\right)^{\gamma-1}, \tag{3.23}$$

where we assumed $k_s \ll k_f$. Consequently,

$$k_s \approx k_0 \rho^{-\frac{1}{\gamma-1}}. \tag{3.24}$$

Now, over the forced nodes, the average degree is

$$\langle k\rangle_{\mathcal{F}} \approx \frac{\int_{k_s}^{k_f} k k^{-\gamma}\,\mathrm{d}k}{\int_{k_s}^{k_f} k^{-\gamma}\,\mathrm{d}k} = \frac{\gamma-1}{\gamma-2}\frac{k_f^{2-\gamma}-k_s^{2-\gamma}}{k_f^{1-\gamma}-k_s^{1-\gamma}} \approx \frac{\gamma-1}{\gamma-2}k_s. \tag{3.25}$$

Using Eqs. (3.22)-(3.25), we get

$$\langle k\rangle_{\mathcal{F}} \approx \langle k\rangle\frac{k_s}{k_0} \approx \langle k\rangle\rho^{-\frac{1}{\gamma-1}}. \tag{3.26}$$

Thus, based on Eqs. (3.5) and (3.26), we obtain

$$\rho^* = \rho\langle k\rangle_{\mathcal{F}}/\langle k\rangle \approx \rho^{\frac{\gamma-2}{\gamma-1}}. \tag{3.27}$$

Next, we move to find $\kappa_{\mathcal{D}}$ which appears in Eqs. (3.18), (3.17) and (3.10). The second moment of degree over the free nodes is

$$\langle k^2\rangle_{\mathcal{D}} \approx \frac{\int_{k_0}^{k_s} k^2 k^{-\gamma}\,\mathrm{d}k}{\int_{k_0}^{k_s} k^{-\gamma}\,\mathrm{d}k} = \frac{\gamma-1}{\gamma-3}\frac{k_s^{3-\gamma}-k_0^{3-\gamma}}{k_s^{1-\gamma}-k_0^{1-\gamma}} \approx \frac{\gamma-1}{\gamma-3}k_0^2 \approx \langle k^2\rangle, \tag{3.28}$$

where we assumed $k_0 \ll k_s$. The first moment is

$$\langle k\rangle_{\mathcal{D}} \approx \frac{\int_{k_0}^{k_s} k k^{-\gamma}\,\mathrm{d}k}{\int_{k_0}^{k_s} k^{-\gamma}\,\mathrm{d}k} = \frac{\gamma-1}{\gamma-2}\frac{k_s^{2-\gamma}-k_0^{2-\gamma}}{k_s^{1-\gamma}-k_0^{1-\gamma}} \approx \frac{\gamma-1}{\gamma-2}k_0 \approx \langle k\rangle. \tag{3.29}$$

Therefore,

$$\kappa_{\mathcal{D}} = \frac{\langle k^2\rangle_{\mathcal{D}}}{\langle k\rangle_{\mathcal{D}}} \approx \kappa, \tag{3.30}$$

and thus,

$$\beta_{\mathcal{D}} = \lambda\kappa_{\mathcal{D}} \approx \beta. \tag{3.31}$$

Substituting Eqs. (3.31) and (3.27) into Eq. (3.18), we get the equation for system states when forcing the highest-degree nodes of SF with $\gamma > 3$. Since for this case the expressions of $\beta_{\mathcal{D}}$ and ρ^* depend only on β and ρ independently, we can just use the relation for *random selection*, Section 2.3, and replace the pair β, ρ, by $\beta_{\mathcal{D}}$ and ρ^* that we obtained here.

If the relation $\rho_c(\beta)$ for random selection has a power-law scaling as in cellular dynamics, Eq. (5.16),

$$\rho_c \sim \beta^{-\theta}, \tag{3.32}$$

then, for high degrees selection, in SF network with $\gamma > 3$,

$$\rho_c^* \sim \beta_{\mathcal{D}}^{-\theta}. \tag{3.33}$$

Substituting into this Eqs. (3.27) and (3.31), we obtain

$$\rho_c^{\frac{\gamma-2}{\gamma-1}} \sim \beta^{-\theta}, \tag{3.34}$$

yielding the scaling,

$$\rho_c \sim \beta^{-\theta \frac{\gamma-1}{\gamma-2}}. \tag{3.35}$$

See Section 5.1 and Fig. S5c for cellular dynamics.

$2 < \gamma < 3$

For degree distribution $p_k = Ak^{-\gamma}$ with $2 < \gamma < 3$ for $k_0 \le k \le k_f$, the second moments $\langle k^2 \rangle$, $\langle k^2 \rangle_{\mathcal{F}}$ and $\langle k^2 \rangle_{\mathcal{D}}$ diverge, and their leading term involves k_f or k_s. The first moments, however, stay the same. Thus, ρ^* remains the same as in Eq. (3.27), while $\beta_{\mathcal{D}}$ should be recalculated. Hence, we go back to Eq. (3.22) and recalculate, for the leading term,

$$\langle k^2 \rangle \approx \frac{\int_{k_0}^{k_f} k^2 k^{-\gamma}\,\mathrm{d}k}{\int_{k_0}^{k_f} k^{-\gamma}\,\mathrm{d}k} = \frac{\gamma-1}{\gamma-3} \frac{k_f^{3-\gamma} - k_0^{3-\gamma}}{k_f^{1-\gamma} - k_0^{1-\gamma}} \approx \frac{\gamma-1}{3-\gamma} k_f^{3-\gamma} k_0^{\gamma-1}. \tag{3.36}$$

In similar way,

$$\langle k^2 \rangle_{\mathcal{D}} \approx \frac{\gamma-1}{3-\gamma} k_s^{3-\gamma} k_0^{\gamma-1}. \tag{3.37}$$

Using Eq. (3.36), Eq. (3.37) becomes

$$\langle k^2 \rangle_{\mathcal{D}} \approx \langle k^2 \rangle \left(\frac{k_s}{k_f}\right)^{3-\gamma}. \tag{3.38}$$

The first moment is still $\langle k \rangle_{\mathcal{D}} \approx \langle k \rangle$ as in Eq. (3.29). Therefore,

$$\kappa_{\mathcal{D}} = \frac{\langle k^2 \rangle_{\mathcal{D}}}{\langle k \rangle_{\mathcal{D}}} \approx \kappa \left(\frac{k_s}{k_f}\right)^{3-\gamma} \approx \kappa \left(\frac{k_s}{k_0}\right)^{3-\gamma} \left(\frac{k_0}{k_f}\right)^{3-\gamma}, \tag{3.39}$$

and using Eq. (3.23) we obtain

$$\kappa_{\mathcal{D}} \approx \kappa \left(\frac{k_0}{k_f}\right)^{3-\gamma} \rho^{-\frac{3-\gamma}{\gamma-1}}. \tag{3.40}$$

Using the natural cutoff for k_f, we know that [11]

$$k_f \approx k_0 N^{\frac{1}{\gamma-1}}. \tag{3.41}$$

Therefore Eq. (3.40) turns to

$$\kappa_{\mathcal{D}} \approx \kappa N^{-\frac{3-\gamma}{\gamma-1}} \rho^{-\frac{3-\gamma}{\gamma-1}}. \tag{3.42}$$

resulting in

$$\beta_{\mathcal{D}} = \lambda \kappa_{\mathcal{D}} \approx \beta N^{-\frac{3-\gamma}{\gamma-1}} \rho^{-\frac{3-\gamma}{\gamma-1}}. \tag{3.43}$$

Here, differently from the above case of $\gamma > 3$, $\beta_{\mathcal{D}}$ is not independent variable but depends on both β and ρ. Therefore, for a given β, increasing ρ changes both ρ^* and $\beta_{\mathcal{D}}$, thus there is no guarantee that ρ_c will be exactly at the value according to the relation $\rho_c^*(\beta_{\mathcal{D}})$ derived from the relation $\rho_c(\beta)$ of random selection.

$\gamma = 3$

Let us find the second moments and $\kappa_{\mathcal{D}}$ for this case.

$$\langle k^2 \rangle \approx \frac{\int_{k_0}^{k_f} k^2 k^{-3}\,\mathrm{d}k}{\int_{k_0}^{k_f} k^{-3}\,\mathrm{d}k} = 2\frac{\ln k_f - \ln k_0}{k_0^{-2} - k_f^{-2}} \approx 2k_0^2 \ln(k_f/k_0). \tag{3.44}$$

In a similar way,

$$\langle k^2 \rangle_{\mathcal{D}} \approx 2k_0^2 \ln(k_s/k_0). \tag{3.45}$$

Using Eqs. (3.44) and (3.23), Eq. (3.45) becomes

$$\langle k^2 \rangle_{\mathcal{D}} \approx \langle k^2 \rangle \frac{1}{\ln(k_f/k_0)} \frac{1}{2} \ln\left(\frac{1}{\rho}\right). \tag{3.46}$$

The first moment is still $\langle k \rangle_{\mathcal{D}} \approx \langle k \rangle$ as in Eq. (3.29). Therefore,

$$\kappa_{\mathcal{D}} = \frac{\langle k^2 \rangle_{\mathcal{D}}}{\langle k \rangle_{\mathcal{D}}} \approx \kappa \frac{1}{\ln(k_f/k_0)} \frac{1}{2} \ln\left(\frac{1}{\rho}\right). \tag{3.47}$$

Using the natural cutoff, $k_f \approx k_0 N^{1/2}$, we get

$$\kappa_{\mathcal{D}} \approx \kappa \frac{1}{\ln N} \ln\left(\frac{1}{\rho}\right), \tag{3.48}$$

resulting in

$$\beta_{\mathcal{D}} = \lambda \kappa_{\mathcal{D}} \approx \beta \frac{1}{\ln N} \ln\left(\frac{1}{\rho}\right). \tag{3.49}$$

Also here, similarly to $\gamma < 3$, $\beta_{\mathcal{D}}$ depends both on β and on ρ.

4 Distance-based method for the limit $\rho \to 0$

In this section we introduce a method to find the required fraction ρ_c for sustaining a network, also in the limit of $\rho \to 0$ in which our above MF theory, Eqs. (2.28) and (2.15), fails. The above MF predicts that if $\rho \to 0$, then the phase diagram of the system converges to that of a free system which is obtained by substituting $\rho = 0$. Namely, a single node which is a zero fraction cannot make a global change. However, simulation results and theory [12] show otherwise. Under certain conditions, controlling a single node makes a system sustainable. Therefore, we generalize here the method in Ref. [12] developed for controlling only a single node, and develop a method that covers both limits of a few controlled nodes and of an extensive fraction of controlled nodes.

We recognize that the above MF does not cover the case of a microscopic number of controlled nodes since the forced nodes break the uniformity and homogeneity of nodes along the network and divide the nodes to classes according to their distance from the forced nodes. Ref. [12] used a shells-based mean-field around the single controlled node. Here we do a similar assumption for many forced nodes. We perform a distance-based mean-field where we assume that nodes having the same distance from the closest controlled node behave alike.

Thus, we denote $K(l)$ as the set of all nodes in distance l from the closest controlled node,

$$K(l) = \left\{ i \;\middle|\; \min_{j \in \mathcal{F}} d(i,j) = l \right\}, \tag{4.1}$$

where $d(i,j)$ is the distance between nodes i and j, and $\mathcal{F}$ is the set of the forced nodes. Next, we denote ϕ_l as the average activity in relaxation (x_i^*) over all the nodes in distance l from the closest controlled node,

$$\phi_l = \frac{1}{|K(l)|} \sum_{i \in K(l)} x_i^*, \tag{4.2}$$

where $x_i^* = x_i(t \to \infty)$ is the relaxation activity of node i, and $|K(l)|$ refers to the size of the set $K(l)$ defined in (4.1). Next, we seek for an equation connecting between the average activities ϕ_l of different distances l's. Hence, we denote the number of neighbors in $K(l+n)$ of a node $i \in K(l)$ as $k_{n,i}$, that is,

$$k_{n,i} = \sum_{j \in K(l+n)} A_{ij}. \tag{4.3}$$

We notice that a node in $K(l)$, has neighbors only in $K(l-1)$, $K(l)$, or $K(l+1)$. Thus $k_{n,i}$ is zero for any n unless $n = -1, 0, 1$. See Fig. S2 for a demonstration. Since we seek for the average behavior in distance l from the controlled nodes, we denote the average of $k_{n,i}$ over $K(l)$ by $k_n(l)$,

$$k_n(l) = \frac{1}{|K(l)|} \sum_{i \in K(l)} k_{n,i}. \tag{4.4}$$

This quantity, $k_n(l)$, depends both on l but also on ρ, thus we denote it by $k_n(l,\rho)$. For instance, if $l = l_{\max}$, *i.e.* the maximal distance from a forced node, then $k_{+1}(l_{\max}, \rho) = 0$ since there are no neighbors in distance $l+1$. In addition, the value of $l_{\max}$ depends on the value of ρ. This implies that $k_n(l,\rho)$ depends on both l and ρ.

Suppose we have for a given network and a given set of controlled nodes $\mathcal{F}$ the quantity $k_n(l,\rho)$. Let us focus on a node in distance l from the closest forced node. The dynamics of this node, following Eq. (1.1), is

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = M_0(x_i) + \lambda M_1(x_i) \sum_{n=-1}^{+1} \sum_{j\in K(l+n)} A_{ij} M_2(x_j), \tag{4.5}$$

where $i \in K(l)$. In relaxation the derivative vanishes, and we obtain

$$R(x_i^*) = \lambda \sum_{n=-1}^{+1} \sum_{j\in K(l+n)} A_{ij} M_2(x_j^*), \tag{4.6}$$

where $R(x) = -M_0(x)/M_1(x)$. Next, we apply our mean-field approximation where we assume that inside $K(l)$ nodes are similar, thus we replace $x_j^* \in K(l)$ by the average ϕ_l defined in (4.2). Thus we obtain

$$R(x_i^*) = \lambda \sum_{n=-1}^{+1} M_2(\phi_{l+n}) \sum_{j\in K(l+n)} A_{ij}. \tag{4.7}$$

Using Eq. (4.3) it turns to

$$R(x_i^*) = \lambda \sum_{n=-1}^{+1} k_{n,i} M_2(\phi_{l+n}). \tag{4.8}$$

Averaging the nodes in $i \in K(l)$, we replace x_i^* by ϕ_l, and $k_{n,i}$ by the averages $k_n(l,\rho)$ defined in (4.4), to obtain

$$R(\phi_l) = \lambda \sum_{n=-1}^{+1} k_n(l,\rho) M_2(\phi_{l+n}). \tag{4.9}$$

To summarize, we get a second order recurrence relation for ϕ_l,

$$\begin{cases} \phi_0 = \Delta, \\ R(\phi_l) = \lambda k_{-1}(l,\rho)M_2(\phi_{l-1}) + \lambda k_0(l,\rho)M_2(\phi_l) + \lambda k_{+1}(l,\rho)M_2(\phi_{l+1}), \quad 1 \le l \le l_{\max}, \end{cases} \tag{4.10}$$

where $l_{\max}$ is the maximal distance from the controlled nodes all over the network. The value $\phi_{l_{\max}+1}$ is not defined since there is no nodes in $K(l_{\max}+1)$. However, since $k_{+1}(l_{\max}) = 0$, the variable $\phi_{l_{\max}+1}$ does not really appear in the last equation for $\phi_{l_{\max}}$, thus it is not a problem. The initial condition in Eq. (4.10) is $\phi_0 = \Delta$ since the nodes in distance 0 from the forced nodes are the forced nodes themselves, and their activity is forced to be Δ. To conclude, Eq. (4.10) provides $l_{\max}$ equations for $l_{\max}$ variables, and as such, it is solvable numerically.

Once we have the solution for ϕ_l, we have to determine whether the system has two solutions, inactive state $\mathbf{x}_0$ and active state $\mathbf{x}_1$, or only $\mathbf{x}_1$ exists, and in other words, whether $\mathbf{x}_0$ exists. If and only if $\mathbf{x}_0$ does not exist then the system is sustainable. For the variable ϕ_l two solutions are possible, convergence to $\bar{\mathbf{x}}_1$ of the free system for large l, or convergence to $\bar{\mathbf{x}}_0$ of the free system for large l. Note that indeed if we seek for a fixed point of the recurrence relation, (4.10), by demanding $\phi_{l+1} = \phi_l = \phi$, we get the equation $R(\phi) = \lambda\kappa M_2(\phi)$ as Eq. (1.17), and thus the two stable fixed points are $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_0$. However, the initial condition with high value $\phi_0 = \Delta$ might cause that the solution of a convergence to $\bar{\mathbf{x}}_0$ for large l does not exist.

We found numerically if the solution of convergence to $\bar{\mathbf{x}}_0$ exists or only the solution of convergence to $\bar{\mathbf{x}}_1$ exists, to find the critical λ_c needed for sustaining a given network for each ρ. By this, we plotted the dashed lines representing the results of this theory in Fig. 3l in the main text. Note that we used numerical calculations not only for solving the set of equations (4.10), but also to obtain the expression $k_n(l, \rho)$ defined in (4.4) that appears in (4.10). We do not have $k_n(l, \rho)$ theoretically, and thus we measure it on many constructed networks and average the results. For solving Eq. (4.10) we used the method of adding the term $\mathrm{d}\phi_l/\mathrm{d}t$ and letting the system reach a relaxation since then $\mathrm{d}\phi_l/\mathrm{d}t = 0$.

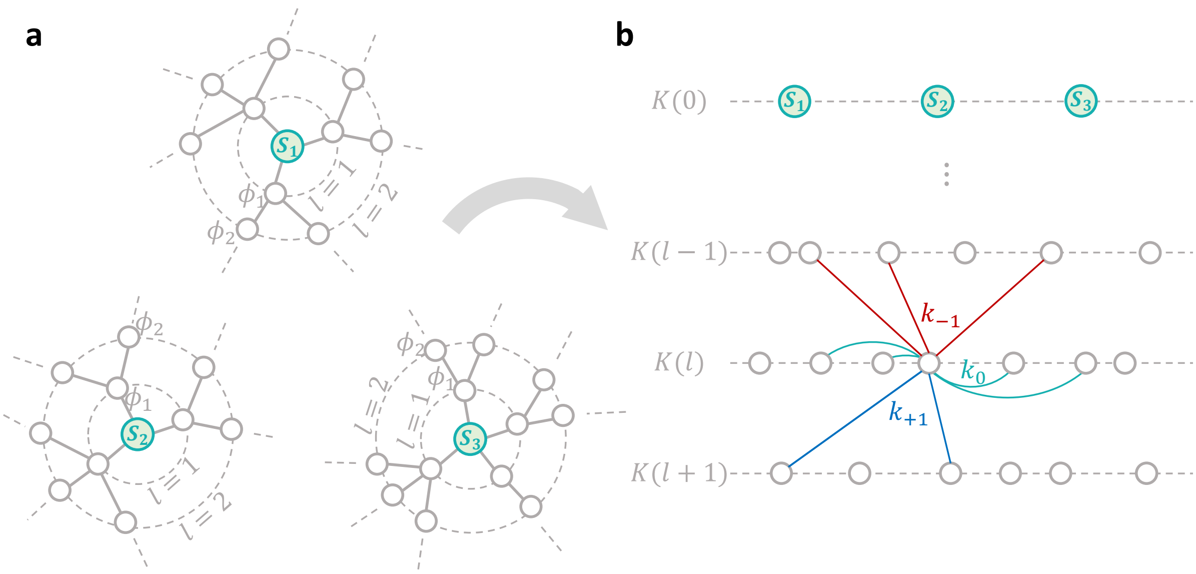

Figure S2: Distance-Based MF for covering both limits of $\rho > 0$ and $\rho \to 0$. (a) Controlling multiple nodes $s_1, s_2, s_3, \ldots \in \mathcal{F}$, separates the network into shells around each one of the controlled nodes. However, these "circles" meet at some finite l and shells become blended. Thus, we decide to separate the network according to the distance of each node from the closest forced node $s_i \in \mathcal{F}$. The average activity in distance l from the closest controlled node is denoted by ϕ_l, Eq. (4.2). (b) We denote $K(l)$, Eq. (4.1), as the set of all the nodes in distance l from the closest forced node. Hence, the set $K(0)$ contains all the forced nodes, *i.e.* the complete set $\mathcal{F}$. $K(l-1)$ contains all nodes in distance $l-1$ from their closest forced node $s_i \in \mathcal{F}$. The same are $K(l)$ and $K(l+1)$. To find a relation between consecutive distances we define the number of neighbors of a node in the same distance, k_0, in a distance lower in one, k_{-1}, and in a distance larger in one, k_{+1}, as defined in Eq. (4.3). For instance, the node presented in this demonstration has $k_{-1} = 3$, $k_0 = 4$, and $k_{+1} = 2$. Based on this separation of the network into distances, we obtain a mean field approximation, Eq. (4.10) that gives a good prediction for ρ_c for both limits of $\rho_c > 0$ and $\rho_c \to 0$ as shown in Fig. 3l in the main text.

5 Dynamical models

To demonstrate our framework, we examined the sustainability of three relevant dynamic systems. Below we detail the analytical treatment of each of these systems, starting from the *free system*, in which we examine the states of the system without forcing Δ, then treating the *forced system*, in which we control the activity of a fraction of the system.

5.1 Cellular dynamics

We consider gene-regulatory dynamics, as captured by the Michaelis-Menten model [10, 13], for which (1.1) takes the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -Bx_i^a + \lambda \sum_{j=1}^{N} A_{ij} \frac{x_j^h}{1+x_j^h}. \tag{5.1}$$

Under this framework $M_0(x_i) = -Bx_i^a$, describing degradation ($a = 1$), dimerization ($a = 2$) or a more complex bio-chemical depletion process (fractional a), occurring at a rate B [14]. For simplicity, in our simulations we set $B = 1$. The activation interaction is captured by a Hill function of the form $M_1(x_i) = 1$, $M_2(x_j) = x_j^h/(1 + x_j^h)$, a *switch-like* function that saturates to $M_2(x_j) \to 1$ for large x_j, representing j's positive, albeit bounded, contribution to i's activity $x_i(t)$.

5.1.1 Free system

First we seek the natural fixed-points of (5.1), by mapping it to the one dimensional space of $\bar{\mathbf{x}}$ via Eq. (1.14). The fixed-points follow

$$\bar{\mathbf{x}}_0 = 0, \tag{5.2}$$

the inactive state, and

$$\beta = \bar{\mathbf{x}}^a \left(1 + \bar{\mathbf{x}}^{-h}\right), \tag{5.3}$$

whose solutions provide the potentially active and intermediate states. For $a = 1,\ h = 2$, the system we examine in the main text, Eqs. (5.2) and (5.3) provide the three solutions shown in Fig. S1a in the main text (see also **Box 1**): an always stable $\bar{\mathbf{x}}_0$ (red), and a stable $\bar{\mathbf{x}}_1$ (green) for $\beta > \beta_c$. The basins of attraction of $\bar{\mathbf{x}}_0$ and $\bar{\mathbf{x}}_1$ are separated by the intermediate unstable state $\bar{\mathbf{x}}_2$ (gray dashed line).

To obtain the critical point β_c, we find the transition point where β versus $\bar{\mathbf{x}}$ in Eq. (5.3) has an extremum as demanded according to Eq. (1.23), yielding

$$a\bar{\mathbf{x}}^{a-1} \left(1 + \bar{\mathbf{x}}^{-h}\right) - h\bar{\mathbf{x}}^a\bar{\mathbf{x}}^{-h-1} = 0. \tag{5.4}$$

The solution of this equation is

$$\bar{\mathbf{x}}_c = \left(\frac{h}{a} - 1\right)^{1/h}. \tag{5.5}$$

Substituting this in Eq. (5.3) we arrive at the solution for the critical value of β,

$$\beta_c = \frac{h}{a}\left(\frac{h}{a} - 1\right)^{a/h-1}, \tag{5.6}$$

capturing the bifurcation in Fig. S1a, where the active state (green) emerges as a stable fixed-point at $\beta \geq \beta_c$. Setting $a = 1,\ h = 2$, the parameters used in the main-text simulations we obtain

$$\begin{aligned} \bar{\mathbf{x}}_c &= 1 \\ \beta_c &= 2, \end{aligned} \tag{5.7}$$

precisely the transitions observed in Fig. S1a. Therefore, the free system exhibits an inactive phase for $\beta < 2$, and a bi-stable regime (gray shaded) for $\beta \geq 2$, where both $\bar{\mathbf{x}}_0$ and $\bar{\mathbf{x}}_1$ are potentially stable. Thus, in all the range $\beta \geq 2$ the system is regarded *unsustainable* since although there exists an active state $\mathbf{x}_1$, yet the existence of inactive state $\mathbf{x}_0$ puts the system in a danger of falling into dysfunctionality.

5.1.2 Forced system

To examine the behavior of our cellular dynamics (5.1) under sustaining we seek to construct the forced system states according to Eq. (2.30),

$$\beta = \frac{\bar{\mathbf{x}}^a}{(1-\rho)/(1+\bar{\mathbf{x}}^{-h}) + \rho/(1+\Delta^{-h})}. \tag{5.8}$$

This formula generates a new phase diagram for a forced system shown in Fig. S1d (thick curve) for $\rho = 0.03$ and $\Delta = 5$, exhibiting an s-shape diagram which has now also a new active regime (blue shade), in which only $\mathbf{x}_1$ exists. This regime is regarded as the *sustainable phase*. Note that the simulation results (symbols) agree well with theory. The simulations are for ER network with $\kappa = 40$ and $N = 10^4$. The value of λ varies.

5.1.3 Sustaining a network

For finding β_c, above which the forced system is sustainable, according to Eq. (2.31), we take the derivative of β in Eq. (5.8) with respect to $\bar{\mathbf{x}}$ to be zero as Eq. (2.31) demands, yielding

$$a\bar{\mathbf{x}}^{a-1}\left(\frac{1-\rho}{1+\bar{\mathbf{x}}^{-h}} + \frac{\rho}{1+\Delta^{-h}}\right) + \bar{\mathbf{x}}^a\frac{(1-\rho)(-h)\bar{\mathbf{x}}^{-h-1}}{(1+\bar{\mathbf{x}}^{-h})^2} = 0. \tag{5.9}$$

Denoting $u = 1 + \bar{\mathbf{x}}^{-h}$, we obtain,

$$\frac{a(1-\rho)}{u} + \frac{a\rho}{1+\Delta^{-h}} - \frac{h(1-\rho)(u-1)}{u^2} = 0. \tag{5.10}$$

Arranging terms gives a solvable quadratic equation,

$$\frac{a\rho}{1+\Delta^{-h}}u^2-(h-a)(1-\rho)u+h(1-\rho)=0, \tag{5.11}$$

whose solutions are given by

$$u=\frac{(h-a)(1-\rho)\pm\sqrt{(h-a)^2(1-\rho)^2-4ah\rho(1-\rho)/(1+\Delta^{-h})}}{2a\rho/(1+\Delta^{-h})}. \tag{5.12}$$

Taking the solution which gives the smaller $\bar{\mathbf{x}}_c$ for the transition to the sustainable regime (see Fig. S1d), we get

$$\bar{\mathbf{x}}_c^{-h}=-1+\frac{(h-a)(1-\rho)-\sqrt{(h-a)^2(1-\rho)^2-4ah\rho(1-\rho)/(1+\Delta^{-h})}}{2a\rho/(1+\Delta^{-h})}. \tag{5.13}$$

Substituting this in Eq. (5.8) provides $\beta_c(\rho,\Delta)$, yields a complicated expression. However, let us expand this in the limit of small ρ and as a result small $\bar{\mathbf{x}}_c$ as can be seen in Fig. S1d, to get the scaling of β_c and ρ in the limit of small ρ. To this end, we go back to Eq. (5.9) to obtain for the lead order

$$\bar{\mathbf{x}}_c^h\approx\frac{\rho}{1+\Delta^{-h}}\frac{1}{(h-a)(1-\rho)}\approx\frac{\rho}{1+\Delta^{-h}}\frac{1}{(h-a)}\sim\rho, \tag{5.14}$$

and substituting this in Eq. (5.8), we finally obtain

$$\beta_c\approx\frac{\bar{\mathbf{x}}_c^a}{\bar{\mathbf{x}}_c^h+\rho/(1+\Delta^{-h})}\sim\rho^{-\frac{h-a}{h}}, \tag{5.15}$$

which provides the scaling between the connectivity β and the fraction of control ρ at the transition from the unsustainable phase to the sustainable phase for small ρ in cellular dynamics. The inverse relation gives for a given β the critical required fraction ρ_c for sustaining a network,

$$\rho_c\sim\beta^{-\frac{h}{h-a}}. \tag{5.16}$$

In Fig. 3k in the main text we present the results for $a=1,\ h=2$, thus we observe the scaling $\rho_c\sim\beta^{-2}=(\lambda\kappa)^{-2}$. In Fig. S3 we show results also for different values of a,h resulting in different exponents and a good agreement between simulations and theory.

Notice that as we discussed above in Section 4, that the limits of small ρ and small $\langle k\rangle$ challenge our mean-field that provides the scaling of (5.16) which is valid only for small ρ. Thus, even though the simulation results in Fig. 3k in the main text show the predicted exponent because of the large degrees, in contrary, the results in Fig. 3l show a deviation from the predicted scaling because the MF assumption is broken due to the small ρ and small average degrees.

5.1.4 Tricritical point

Beyond some point (β_0,ρ_0) the unsustainable phase vanishes, and the transition between the high-active state and the low-active state becomes continuous, see Fig. 3d and Fig. 3g,h in the main text. This happens when the two transition points on the edges of the

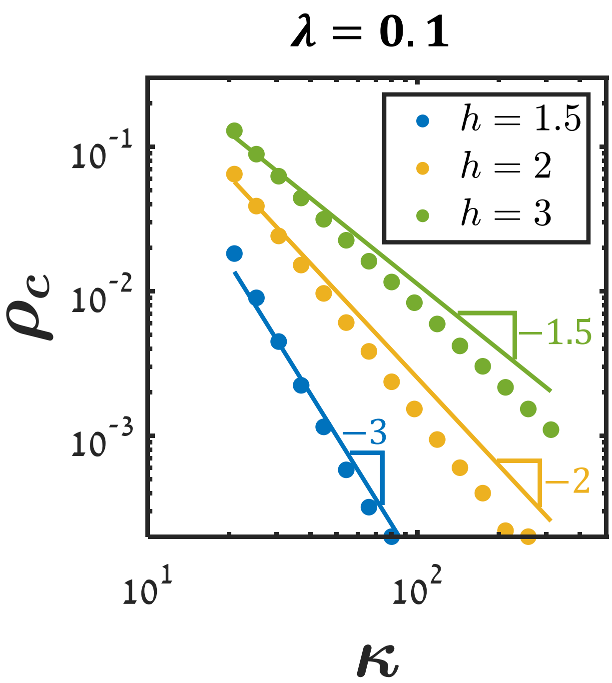

Figure S3: The scaling of ρ_c and κ for different exponents of cellular dynamics. For random selection of controlled nodes, we show how the exponents of dynamics impact the critical fraction for sustaining the network. We set $a = 1$ and change h. Symbols represent simulations, continuous lines are the slopes obtained from Eq. (5.16). The network is ER with size of $N = 10^4$ and the results were averaged over 10 realizations. The interaction strength was set to be $\lambda = 0.1$.

unsustainable region merge. In order for this to be fulfilled, there should be only a single solution to Eq. (5.10), therefore the term inside the square root has to equal zero. Thus we demand

$$(h-a)^2(1-\rho)^2 - \frac{4ah\rho(1-\rho)}{1+\Delta^{-h}} = 0, \tag{5.17}$$

yielding

$$\rho_0 = \frac{1}{1 + \frac{4ah}{(h-a)^2}\frac{1}{1+\Delta^{-h}}}, \tag{5.18}$$

which is the critical value of ρ above which the system is fully sustainable because the bi-stable regime disappears. Plugging this in Eqs. (5.13) and (5.8) provides the value β_c of the critical point.

In our simulations in Fig. 3 in the main text we set $a = 1$, $h = 2$, thus we get $\rho_c = 1/(1 + 8/(1 + \Delta^{-h}))$, and setting $\Delta = 5$ gives $\rho_c \approx 1/9$, while using $\Delta = 1$ yields $\rho_c = 1/5$. In general, for high Δ, we get $\rho_c \approx (h-a)^2/(h+a)^2$.

5.1.5 Sustainability for high-degree nodes selection

As obtained above, Eq. (3.18), when the forced nodes have an arbitrary degree distribution, and they are spread randomly across the network, then $\beta_{\mathcal{D}}$ (Eq. (3.17)) just replaces β while

ρ^* (Eq. (3.5)) replaces ρ.

In cellular dynamics as shown in Eq. (5.16) there exists a power-law relation $\rho_c \sim \beta^{-\theta}$ where $\theta = h/(h-a)$. Thus, for high degrees selection of the forced nodes, we use Eq. (3.35) to get

$$\rho_c \sim \beta^{-\frac{h}{h-a}\frac{\gamma-1}{\gamma-2}}. \tag{5.19}$$

For $a = 1$, $h = 2$ and $\gamma = 3.5$, we get the scaling $\rho_c \sim \beta^{\frac{10}{3}}$ for controlled nodes with high degrees, and $\rho_c \sim \beta^{-2}$ for random selection as shown in Fig. S5c with qualitatively agreement. We do not expect a perfect agreement since our MF of small fluctuations has deviations from the simulations in SF networks as shown in Fig. 4a,b in the main text.

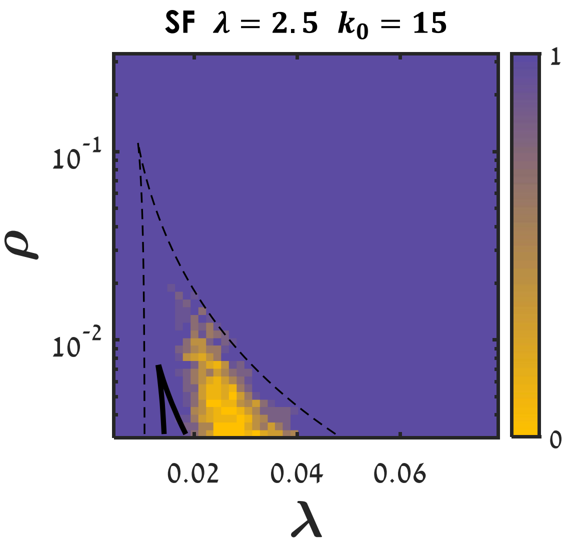

Figure S4: Results for scale-free networks with small exponent γ. In contrast to Fig. 4a,b in the main text for SF network with $\gamma = 3.5$, in which the theory of Eq. (2.15) is excellent while the theory of small fluctuations, Eq. (2.30) has a deviation from simulations, here, for SF with $\gamma = 2.5$ which has larger degree variation, the theory of Eq. (2.15) fails. However, the theory of Eq. (2.30) still works quite well. However, it might be coincidentally since Eq. (2.30) relies upon Eq. (2.15) and based on further assumption, so by definition it is not capable of providing better results.

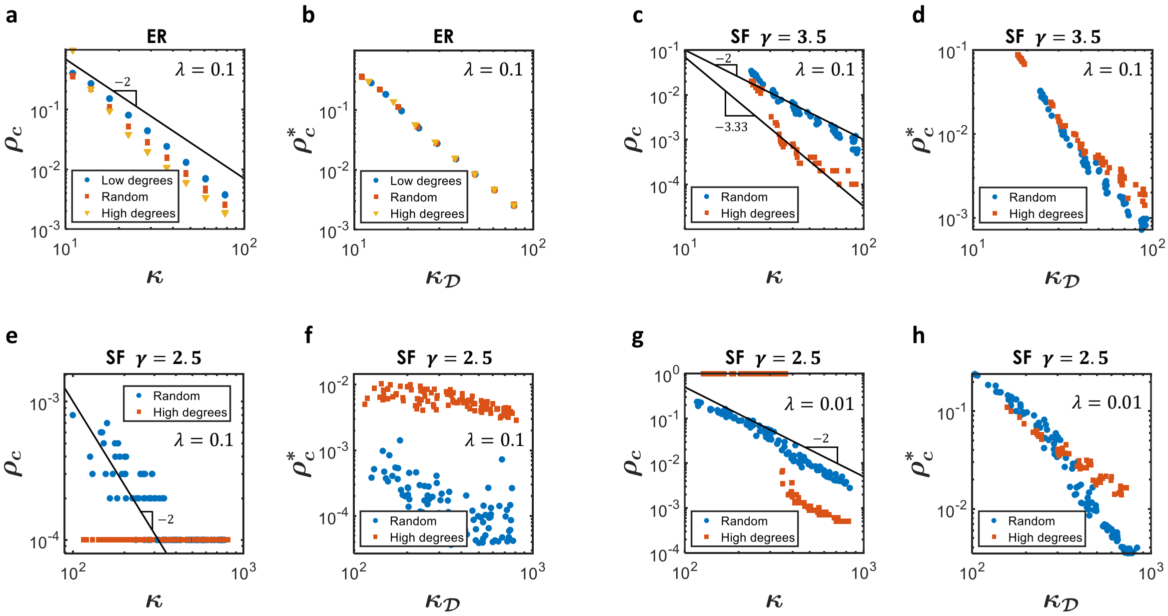

Figure S5: Selection methods of the controlled nodes. (a) For cellular dynamics on ER network, we show the results for three different ways to select the controlled nodes, highest degree nodes, lowest degree nodes and randomly. The scaling of the minimal fraction for sustaining, ρ_c, as κ is similar, but the value of ρ_c changes remarkably. The interaction strength is $\lambda = 0.1$. ρ_c was calculated for 10 ER networks of $N = 10^4$ for each mean degree $\langle k \rangle$. The parameters of the cellular dynamics were set to $a = 1$, $h = 2$. (b) According to Eq. (3.18), for arbitrary degree distribution of the controlled nodes, the relevant parameters are $\beta_{\mathcal{D}}$ and ρ^* instead of β and ρ. Indeed, in these variables the three cases yield a single curve. The data points are the same as in (a), shown with respect to the new variables $\beta_{\mathcal{D}}$ and ρ^*. (c)-(d) Results for scale-free networks with $\gamma = 3.5$ and varying minimal degree k_0. Here the scaling of ρ_c with β changes between high degrees and random selections, as predicted by Eq. (5.19) for high degrees selection and Eq. (5.16) for a random selection. (d) The two selections are close to be on a single curve, but it is much less significant compared to ER in (b) since the results for SF are less accurate because of the larger fluctuations. (e)-(f) Results for SF with exponent $\gamma = 2.5$. This network is more challenging for the mean-field theory since the variation of node degrees is larger. The exponent -2 for random selection still shows a good agreement with simulations. For high degree selection we see a single node sustaining, since the network size is 10^4. (f) Our theory does not cover this challenging case, particularly because we reach the limit of microscopic sustaining. Thus, we decreased λ to be 0.01. (g)-(h) The same as (e)-(g) with smaller interaction strength $\lambda = 0.01$. Here we do not see anymore a single node sustaining. The exponent -2 for random selection still works. However, our theory does not capture the behavior for high degrees selection.

5.2 Neuronal dynamics

We consider a modified Wilson-Cowan model [15, 16] for excitation in neuronal networks,

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -x_i + \lambda \sum_{j=1}^{N} A_{ij} \frac{1}{1+e^{\mu-\delta x_j}}, \tag{5.20}$$

which we examine for $\mu = 5$ and $\delta = 1$. In this version of the model, adapted to the form of Eq. (1.1), the summation is extracted outside of the exponential function, namely we write $\lambda \sum_{j=1}^{N} A_{ij}(1 + e^{\mu-\delta x_j})^{-1}$ instead of $(1 + e^{\mu-\delta\lambda \sum_{j=1}^{N} A_{ij}x_j})^{-1}$. As a result, each node receives accumulating inputs from all its neighbors, and hence higher in-degree nodes gain a stronger overall activation signal from their surrounding neighbors. This is in contrast to the standard version of the model, with the summation appears inside the exponent, and hence the effect of the multiple incoming signals quickly reaches saturation. In the present context, where we wish to observe the role of degree heterogeneity on reigniting, *e.g.*, through κ, the form of Eq. (5.20) provides a more relevant testing ground.

5.2.1 Free system

To obtain the fixed-points of the system we use Eq. (1.17) to obtain the relation,

$$\beta = \bar{\mathbf{x}}(1 + e^{\mu-\delta\bar{\mathbf{x}}}). \tag{5.21}$$

Plotting $\bar{\mathbf{x}}$ vs. β (Fig. 5b in the main text) we obtain the three dynamic phases of the free system. The *inactive* state $\mathbf{x}_0$ (red), in which all activities are suppressed, is obtained when the network is extremely sparse, *i.e.* small β. The *active* $\mathbf{x}_1$ (green), in which x_i are relatively high, is observed when β is large. In between these two extremes, the system features a *bi-stable* phase, in which both $\mathbf{x}_0$ and $\mathbf{x}_1$ are potentially stable. These phases are separated by two critical points $\beta_{c,1} < \beta_{c,2}$, between which the system is bi-stable and thus unsustainable. However, above $\beta_{c,2}$ the system is naturally sustainable without any external intervention. To find $\beta_{c,12}$ we look for the extremum points of β vs. $\bar{\mathbf{x}}$ in Eq. (5.21), getting

$$1 + e^{\mu-\delta\bar{\mathbf{x}}} + \bar{\mathbf{x}}(-\delta e^{\mu-\delta\bar{\mathbf{x}}}) = 0, \tag{5.22}$$

that turns to

$$e^{\delta\bar{\mathbf{x}}-\mu} + 1 = \delta\bar{\mathbf{x}}. \tag{5.23}$$

This equation is transcendental but solvable numerically with two solutions $\bar{\mathbf{x}}_{c,1}$ and $\bar{\mathbf{x}}_{c,2}$, providing through Eq. (5.21) the borders of the three phases of a free neuronal system, inactive ($\beta < \beta_{c,1}$), unsustainable ($\beta_{c,1} \le \beta \le \beta_{c,2}$) and naturally sustainable ($\beta > \beta_{c,2}$).

5.2.2 Forced system

To obtain the states of the system when a random fraction of nodes is controlled with high activity Δ, we use Eqs. (2.30) and (5.20) to get

$$\beta = \frac{\bar{\mathbf{x}}}{\dfrac{1-\rho}{1+e^{\mu-\delta\bar{\mathbf{x}}}} + \dfrac{\rho}{1+e^{\mu-\delta\Delta}}}. \tag{5.24}$$

In Fig. 5b in the main text we present this result which agrees well with simulations results, and shows a new phase diagram for the forced system. Even though both forced and free system for brain dynamics show an s-shape diagram, the diagram of the forced system is biased to the left, and thus has a significantly smaller $\beta_{c,2}$. This creates a window (blue shade) of sustainability where the control drives the system to become sustainable.

5.2.3 Sustaining a network

For finding β_c above which a forced system becomes sustainable, we again find the local maximum of β vs. $\bar{\mathbf{x}}$ in Eq. (5.24), yielding

$$\frac{1-\rho}{1+e^{\mu-\delta\bar{\mathbf{x}}}} + \frac{\rho}{1+e^{\mu-\delta\Delta}} - \bar{\mathbf{x}}\frac{1-\rho}{\left(1+e^{\mu-\delta\bar{\mathbf{x}}}\right)^2}\delta e^{\mu-\delta\bar{\mathbf{x}}} = 0. \tag{5.25}$$

Solving numerically this transcendental equation, we get $\bar{\mathbf{x}}_{c,1}$ and $\bar{\mathbf{x}}_{c,2}$, and using Eq. (5.24) we obtain $\beta_{c,1}$ and $\beta_{c,2}$ for the edges of the unsustainable range of a forced system. The higher value among them is the critical $\beta_c(\rho, \Delta)$ required for sustaining the network. This result is presented in Fig. 5c,d in the main text, showing a good agreement with simulations.

5.2.4 Tricritical point

For finding ρ_0, above which there is no unsustainable region, we have to apply the condition of Eq. (2.32) on Eq. (5.24). Together with the condition of Eq. (2.31) on Eq. (5.24), and Eq. (5.24) itself, we obtain numerically the tricritical point (β_0, ρ_0), beyond which there is no unsustainable region as can be seen in Fig. 5d in the main text.

5.3 Spin dynamics

Here we explore the last example, spin dynamics. We consider a kinetic model based on Ising-Glauber model [17] to follow the state of spins connected by a network of ferromagnetic interactions.

5.3.1 Free system

The dynamics of the spins according to this model [17] is captured by the set of equations which are not included in Eq. (1.1),

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -x_i + \tanh\left(\lambda \sum_{j=1}^{N} A_{ij} x_j\right). \tag{5.26}$$

Even though this form is beyond our general framework in (1.1), still a similar analysis can be performed as follows. First, we look on a relaxed system, *i.e.* $\mathrm{d}x_i/\,\mathrm{d}t = 0$, giving

$$x_i^* = \tanh\left(\lambda \sum_{j=1}^{N} A_{ij} x_j^*\right). \tag{5.27}$$

The next step is the DBMF approximation [4–8] that we used above, Eq. (1.2), according which we substitute A_{ij} by $k_i k_j/(N\langle k\rangle)$,

$$x_i^* = \tanh\left(\lambda k_i \sum_{j=1}^{N} \frac{k_j}{N\langle k\rangle} x_j^*\right). \tag{5.28}$$

As above, Eq. (1.11), we use the average $\bar{\mathbf{x}}$ to write

$$x_i^* = \tanh\left(\lambda k_i \bar{\mathbf{x}}\right). \tag{5.29}$$

Next, we insert the last equation into the definition of $\bar{\mathbf{x}}$ to obtain a self-consistent equation for $\bar{\mathbf{x}}$,

$$\bar{\mathbf{x}} = \sum_{i=1}^{N} \frac{k_i}{N\langle k\rangle} \tanh\left(\lambda k_i \bar{\mathbf{x}}\right). \tag{5.30}$$

Replacing the summation on nodes i by summation on the degrees k, we obtain

$$\bar{\mathbf{x}} = \sum_{k} \frac{k p_k}{\langle k\rangle} \tanh\left(\lambda k \bar{\mathbf{x}}\right) \tag{5.31}$$

which is a single solvable theoretically equation given the degree distribution. As above, Eq. (1.10), as in Ref. [9] and explained above in Section 2.2.2, we insert the average into the function for small fluctuations and/or if $\lambda k \bar{\mathbf{x}}$ values are close to zero were tanh is close to linear, or $\lambda k \bar{\mathbf{x}}$ values are large where tanh is close to constant. Using this approximation, we obtain

$$\bar{\mathbf{x}} = \tanh(\beta \bar{\mathbf{x}}), \tag{5.32}$$

which predicts the states of a free system of spin dynamics. The phase diagram constructed from this equation is shown in Fig. 5f (thin light lines) in the main text. Isolating β in Eq. (5.32) yields a simpler equation,

$$\beta = \frac{\tanh^{-1}(\bar{\mathbf{x}})}{\bar{\mathbf{x}}} = \frac{1}{2\bar{\mathbf{x}}} \ln\left(\frac{1+\bar{\mathbf{x}}}{1-\bar{\mathbf{x}}}\right). \tag{5.33}$$

5.3.2 Forced system

Here we follow the steps we have done above in Section 2.2. When controlling the set $\mathcal{F}$ to have an activity Δ, we have to separate the sum in Eq. (5.26) between neighbors in $\mathcal{D}$ in $\mathcal{F}$. Thus, the equation for node $i \in \mathcal{D}$ becomes

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -x_i + \tanh\left(\lambda \sum_{j\in\mathcal{D}} A_{ij}x_j + \lambda \sum_{j\in\mathcal{F}} A_{ij}\Delta\right). \tag{5.34}$$

In relaxation,

$$x_i^* = \tanh\left(\lambda \sum_{j\in\mathcal{D}} A_{ij}x_j^* + \lambda k_i^{\mathcal{D}\to\mathcal{F}}\Delta\right). \tag{5.35}$$

where $k_i^{\mathcal{D}\to\mathcal{F}} = \sum_{j\in\mathcal{F}} A_{ij}$ is the number of neighbors in $\mathcal{F}$ of the node i belonging to $\mathcal{D}$. Next, we perform again the DBMF approximation [4–8], replacing A_{ij} by the probability of being a link (i, j) given their relevant degrees, which here are degrees inside $\mathcal{D}$. Hence,

$$x_i^* = \tanh\left(\lambda k_i^{\mathcal{D}\to\mathcal{D}} \sum_{j\in\mathcal{D}} \frac{k_j^{\mathcal{D}\to\mathcal{D}}}{|\mathcal{D}|\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} x_j^* + \lambda k_i^{\mathcal{D}\to\mathcal{F}}\Delta\right). \tag{5.36}$$

Using the same definition as above of $\bar{\mathbf{x}}$, Eq. (2.17), we obtain

$$x_i^* = \tanh\left(\lambda k_i^{\mathcal{D}\to\mathcal{D}}\bar{\mathbf{x}} + \lambda k_i^{\mathcal{D}\to\mathcal{F}}\Delta\right). \tag{5.37}$$

Plugging the last equation into the definition of $\bar{\mathbf{x}}$, we get a self-consistent equation for $\bar{\mathbf{x}}$,

$$\bar{\mathbf{x}} = \sum_{j\in\mathcal{D}} \frac{k_j^{\mathcal{D}\to\mathcal{D}}}{|\mathcal{D}|\langle k_{\mathcal{D}\to\mathcal{D}}\rangle} \tanh\left(\lambda k_j^{\mathcal{D}\to\mathcal{D}}\bar{\mathbf{x}} + \lambda k_j^{\mathcal{D}\to\mathcal{F}}\Delta\right). \tag{5.38}$$

Next, we use again the mean-field approximation [9] of moving the averaging into the function tanh, as above Eq. (2.16), Section 2.2.2 and Eq. (5.32), to obtain

$$\bar{\mathbf{x}} = \tanh\left(\lambda\kappa_{\mathcal{D}\to\mathcal{D}}\bar{\mathbf{x}} + \lambda\kappa_{\mathcal{D}\to\mathcal{F}}\Delta\right), \tag{5.39}$$

where $\kappa_{\mathcal{D}\to\mathcal{D}}$ and $\kappa_{\mathcal{D}\to\mathcal{F}}$ are defined in Eq. (2.19) and found for a random selection in Eqs. (2.23) and (2.27). Substituting their values for a random selection of the forced nodes, yields

$$\bar{\mathbf{x}} = \tanh\left(\lambda(1 + (\kappa - 1)(1 - \rho))\bar{\mathbf{x}} + \lambda(\kappa - 1)\rho\Delta\right), \tag{5.40}$$

which for large κ becomes

$$\bar{\mathbf{x}} = \tanh\left(\beta(1 - \rho)\bar{\mathbf{x}} + \beta\rho\Delta\right). \tag{5.41}$$

This equation predicts the states of the forced system as shown in Fig. 5f in the main text. A simpler relation can be obtained by isolating β,

$$\beta = \frac{\tanh^{-1}(\bar{\mathbf{x}})}{(1 - \rho)\bar{\mathbf{x}} + \rho\Delta} = \frac{1}{2}\frac{\ln((1 + \bar{\mathbf{x}})/(1 - \bar{\mathbf{x}}))}{(1 - \rho)\bar{\mathbf{x}} + \rho\Delta}. \tag{5.42}$$

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