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# Supplementary material; Propagation characteristics of an elastic bar coupled with a discrete snap-through element

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## 1 Introduction

The Finite Difference Method (FDM) is a powerful numerical tool used by engineers and mathematicians for solving a set of differential equations that can be linear/ nonlinear and ordinary/ partial and are otherwise analytically unsolvable, by discretizing them into a system of linear equations with the time and space derivatives approximated as finite differences. Both the spatial domain and time interval are discretized, or broken into a finite number of steps, and the value of the solution at these discrete points is approximated by solving algebraic equations containing finite differences and values from nearby points. There are various ways for formulating a finite difference algorithm such as the Forward in Time Centred in Space (FTCS) scheme, the Backward in Time Centred in Space (BTCS) scheme and the Centred in Time Centred in Space (CTCS) scheme also popularly known as the Crank-Nicolson method. The Crank-Nicolson method has been found to be used quite extensively for the finite difference analysis of dynamical systems [1, 2], because it being an implicit scheme in time is the most stable and accurate for small time steps.

Since the Crank-Nicolson method is implicit, it is usually very difficult to predict the future when the governing differential equations of motion are nonlinear. It is possible implicitly to find a solution by using iterative techniques such as Newton-Raphson or Runge-Kutta, but often these methods fail to converge for high-dimensional systems. In this supplementary material, we shall discuss a modification of the classical CTCS method by defining a map, for which the Crank-Nicolson prediction is an approximated stable point of our system and use a novel, explicit iterative approach to converge to a valid/ stable solution. Use of alternating Crank-Nicolson methods [3] or invariantization of the Crank-Nicolson method [4] is found to exist significantly in the literature. The explicit scheme developed here is numerically stable for small steps and also numerically less intensive than the implicit one.

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## 2 Exact non-dimensional equations of motion and boundary conditions

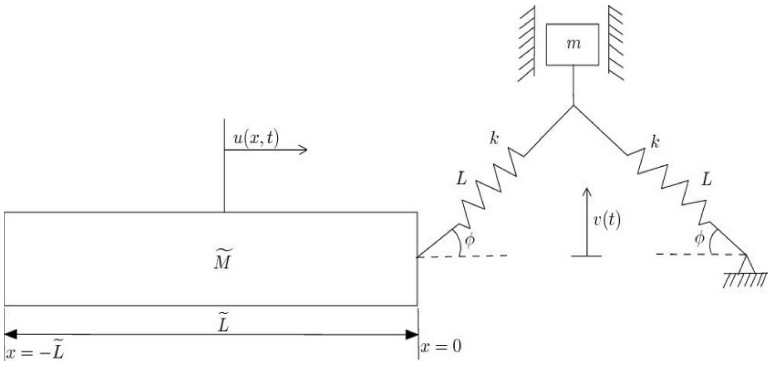


**Fig. 1** Schematic of an elastic bar coupled with a snap-through truss

The model along with the relevant dimensions and displacements is shown in Fig. 1. S. I. units are considered. The left/ free end of the bar shown in the diagram is subjected to sinusoidal excitations of the form $Q(t) = \bar{A}\sin(\Omega t)$, where $t$ is the dimensional time in seconds, $\Omega$ is the dimensional excitation frequency in radians/second and $\bar{A}$ is the dimensional excitation amplitude in meters. The characteristic length for non-dimensionalization is chosen to be $L_{ch} = \tilde{L}/l = L$, which is also the natural length of the springs in the initial configuration. We define $l$ as our length ratio or the non-dimensional length of the bar. The characteristic mass is taken as $M_{ch} = \tilde{M}/l = M = \rho AL$ The non-dimensional parameters are then obtained as

$$\tau = \frac{ct}{L}, \omega = \frac{L\Omega}{c}$$

$$l = \frac{\tilde{L}}{L}, \quad z = \frac{v}{L}, \quad w = \frac{u}{L}, \quad y = \frac{x}{L}, \quad A_m = \frac{\bar{A}}{L}, \quad P(\tau) = \frac{Q(t)}{L}$$

$$\alpha = \frac{kL}{EA}, r = \frac{m}{M}$$

The parameters are respectively, the non-dimensional time, the non-dimensional excitation frequency, length ratio or the non-dimensional length of the bar, the non-dimensional displacement of the oscillator, the non-dimensional displacement of the bar, the non-dimensional longitudinal spatial co-ordinate, the non-dimensional amplitude of the excitation pulse, the excitation pulse in non-dimensional form, the non-dimensional linear stiffness parameter, and the mass ratio. $c = \sqrt{E/\rho}$ is the longitudinal wave speed in the bar and $E$ and $\rho$ are the Modulus of Elasticity and the material density of the bar respectively.

The initial conditions for all the displacements and the corresponding velocities have been taken to be zero. The non-dimensional equations governing the motion of the original un-approximated system are given by

$$\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial \tau^2} = 0 \tag{1}$$

$$rz'' + \alpha z\left(2 - \frac{1}{\sqrt{(\cos\phi - w(0,\tau))^2 + z^2}} - \frac{1}{\sqrt{\cos^2\phi + z^2}}\right) = 0 \tag{2}$$

The non-dimensional boundary conditions are given by

$$\frac{\partial w}{\partial y}(0,\tau) = \alpha\left(1 - \frac{1}{\sqrt{(\cos\phi - w(0,\tau))^2 + z^2}}\right)(\cos\phi - w(0,\tau))) \tag{3}$$

$$w(-l,\tau) = P(\tau) = A_m \sin\omega\tau \tag{4}$$

## 3 Discretization of the exact equations of motion and boundary conditions

The discretization used for the Finite Difference Analysis is as follows:

- $N$ is defined as the number of time steps, $S$ is defined as the number of spatial grids.
- $T_{span}$ is taken as the total time window for the numerical simulation, $y_{position}$ is taken as the desired position or the point in the bar at which the numerically obtained displacement $w(y,\tau)$ is plotted.
- The time step is then obtained as $\Delta t = T_{span}/N$. The length of each small element into which the bar is discretized is obtained as $\Delta y = \tilde{L}/S$.
- The displacement matrix of the bar is of size $(S+1, N+1)$ and is denoted by $\mathbf{W}$ and the displacement vector of the oscillator from its initial position is of size $(1, N+1)$ and is denoted by $\mathbf{Z}$.
- $i$ is chosen to be the iterative index for the spatial discretization and $j$ is chosen to be the iterative index for the time discretization.

The displacement of the snap-through oscillator $z$ is measured from its initial position as done in the paper and hence $z = \sin\phi + \eta$ is substituted in equations (2) and (3) while formulating the algorithm. The rows of the displacement matrix $\mathbf{W}$ denote the spatial positions of the elements into which the bar is divided and the columns denote the time instants obtained as a result of the chosen discretization.

## 4 The pseudo-coded algorithm

### 4.1 Assigning the initial conditions and the essential boundary condition of the bar

The initial displacement and velocity have been assumed to be zero for both the bar and the oscillator. The time-derivatives of the displacements, i.e. the velocities are written in discretized form using the central difference approach as follows.

- For $i = 1 : S+1$, $\mathbf{W}(i,1) = 0$ which comes from the initial displacement of the bar that has been taken to be zero at all points.
- The initial displacement of the oscillator from its position at $z = \sin\phi$ has been taken to be zero and hence $\mathbf{Z}(1,1) = 0$.
- The initial velocity of the bar is written in discretized form using the central difference at time $\tau(j)$ as $\frac{\partial w}{\partial \tau}(y,0) = \frac{\mathbf{W}(i,j+1) - \mathbf{W}(i,j-1)}{2\Delta\tau}$, for $i = 1 : S+1$ and $j = 1$. Since the initial velocity of the bar has been taken to be zero, it is obtained that $\frac{\partial w}{\partial \tau}(y,0) = 0 \implies \mathbf{W}(i,j+1) = \mathbf{W}(i,j-1)$, for $i = 1 : S+1$ and $j = 1$.
- Similarly, the initial velocity of the snap-through oscillator is written in central difference form at time $\tau(j)$. The initial velocity is obtained in discretized form as $\frac{d\eta}{d\tau}(0) = \frac{\mathbf{Z}(1,j+1) - \mathbf{Z}(1,j-1)}{2\Delta\tau}$, for $j = 1$. Since the initial velocity of the oscillator has also been taken to be zero, it is obtained that $\frac{d\eta}{d\tau}(0) = 0 \implies \mathbf{Z}(1,j+1) = \mathbf{Z}(1,j-1)$, for $j = 1$.
- The time vector is formulated as $\tau(j) = (j-1)\Delta\tau$, for $j = 1 : N+1$.
- The essential boundary condition at the left end of the bar is then obtained from (4) in discretized form as $\mathbf{W}(1,j) = A_m \sin(\omega\tau(j))$, for $j = 1 : N+1$.

### 4.2 Implementation of the 'centred in time centred in space' scheme

After substituting $z(\tau) = \sin\phi + \eta(\tau)$ in (2), (3) and writing the equations in discretized form using a second-order central difference for the space derivative at non-dimensional position $y(i)$ and a second-order central difference for the time derivative at non-dimensional time $\tau(j)$ gives the recurrence equations

$$\frac{\mathbf{W}(i+1,j) - 2\mathbf{W}(i,j) + \mathbf{W}(i-1,j)}{(\Delta y)^2} = \frac{\mathbf{W}(i,j+1) - 2\mathbf{W}(i,j) + \mathbf{W}(i,j-1)}{(\Delta\tau)^2} \tag{5}$$

$$r\left(\frac{\mathbf{Z}(1,j+1)-2\mathbf{Z}(1,j)+\mathbf{Z}(1,j-1)}{(\Delta\tau)^2}\right)+\alpha\Big(\mathbf{Z}(1,j)+\sin\phi\Big)\left(2-\frac{1}{\sqrt{\cos^2\phi+(\mathbf{Z}(1,j)+\sin\phi)^2}}\right.$$
$$\left.-\frac{1}{\sqrt{(\cos\phi-\mathbf{W}(S+1,j))^2+(\mathbf{Z}(1,j)+\sin\phi)^2}}\right)=0 \tag{6}$$

*4.2.1 Filling the bar-displacement matrix* $\boldsymbol{W}$

The index $i$ runs from 2 to $S$ and $j$ runs from 2 to $N$. In equations (5) and (6), the quantities $\mathbf{W}(i,j+1)$ and $\mathbf{Z}(1,j+1)$ are unknown at this stage to be found out by running an iteration. But, if $j$ starts from 2 in (5), then in order to find out $\mathbf{W}(i,3)$, we need to find out $\mathbf{W}(i,2)$ and $\mathbf{W}(i,1)$ first. From the first step of our Finite Difference formulation in section 4.1, we have $\mathbf{W}(i,1)=0$ for $i=1:S+1$. $\mathbf{W}(i,2)$ is found out by using the central difference scheme as follows:

- From the initial velocity condition of the bar as discussed in the third step of section 4.1, we have $\mathbf{W}(i,j+1)=\mathbf{W}(i,j-1)$, for $i=1:S+1$ and $j=1$.
- In (5), putting $j=1$ and using the condition recalled in the previous step gives for $i=2:S$

$$\frac{2\mathbf{W}(i,2)-2\mathbf{W}(i,1)}{(\Delta\tau)^2}=\frac{\mathbf{W}(i+1,1)-2\mathbf{W}(i,1)+\mathbf{W}(i-1,1)}{(\Delta y)^2} \tag{7}$$

- All the quantities in (7) except $\mathbf{W}$(i,2) are known which can be easily found out. If $i$ runs upto $S$, then in order to find out $\mathbf{W}(S,3)$, we need to find out $\mathbf{W}(S+1,2)$ which is still an unknown. It is found out by using the natural boundary condition of the bar (3) at the nonlinear interface ($y=0$). Writing this condition in discretized form using the first-order central difference for the space derivative at position $y(S+1)$ gives the recurrence equation

$$\frac{\mathbf{W}(S+2,j)-\mathbf{W}(S,j)}{2\alpha\Delta y}=\left(1-\frac{1}{\sqrt{(\cos\phi-\mathbf{W}(S+1,j))^2+\mathbf{Z}(1,j)^2}}\right)\Big(\cos\phi-\mathbf{W}(S+1,j))\Big) \tag{8}$$

- The only unknown quantity in (8) is the dummy variable $\mathbf{W}$(S+2,j) which exceeds our matrix dimensions for the displacement matrix of the bar. Since all the other quantities in (8) are known, the dummy variable would not be a bother and can found out by substituting the known quantities.
- Once the dummy variable is found out, we use the condition deduced in the third step of section 4.1 and re-write equation (5) for $i=S+1$ and $j=1$ to obtain

$$\frac{2\mathbf{W}(S+1,2)-2\mathbf{W}(S+1,1)}{(\Delta\tau)^2}=\frac{\mathbf{W}(S+2,1)-2\mathbf{W}(S+1,1)+\mathbf{W}(S,1)}{(\Delta y)^2} \tag{9}$$

- Thus, from equation (9), the unknown $\mathbf{W}(S+1,2)$ can be found out in terms of the other known quantities.

If $j$ runs upto $N$ and $i$ runs upto $S$ in the iterative equation (5), then in order to find out the only remaining term in the bar-displacement matrix, $\mathbf{W}(S+1,N+1)$, we first need to find out the dummy variable $\mathbf{W}(S+2,N)$ which can be done by putting $j=N$ in equation (8).

- Once the dummy variable is found out, we use the condition deduced in the third step of section 4.1 and re-write equation (5) for $i=S+1$ and $j=N$ to obtain

$$\frac{\mathbf{W}(S+1,N+1)-2\mathbf{W}(S+1,N)+\mathbf{W}(S+1,N-1)}{(\Delta\tau)^2}$$
$$=\frac{\mathbf{W}(S+2,N)-2\mathbf{W}(S+1,N)+\mathbf{W}(S,N)}{(\Delta y)^2} \tag{10}$$

- The only unknown quantity in (10) is our previously stated $(S+1, N+1)$th term in the bar-displacement matrix, $\mathbf{W}(S+1, N+1)$ which can be found out now by substituting the other known quantities.
- After finding these unknowns, all the $\mathbf{W}(i, j+1)$ for $i = 2 : S$ and $j = 2 : N$ can be easily found out by running the iterative scheme as described in (5).
- Once the displacement matrix $\mathbf{W}$ is formulated/ filled, the bar displacement at a desired location $(y = y_{position})$ is extracted as follows:

$$w(y, \tau)|y = y_{position} = \mathbf{W}\left(\frac{y_{position}}{\Delta y} + 1, :\right) \tag{11}$$

*4.2.2 Filling the oscillator-displacement vector* $\boldsymbol{Z}$

Similarly, if $j$ starts from 2 in (6), then in order to find $\mathbf{Z}(1,3)$, first we need to find out $\mathbf{Z}(1,2)$, $\mathbf{Z}(1,1)$ and $\mathbf{W}(S+1,2)$. From the second step of our Finite Difference formulation in section 4.1, we have $\mathbf{Z}(1,1) = 0$. $\mathbf{W}(S+1,2)$ has already been found out using equation (9). $\mathbf{Z}(1,2)$ is found out by using the central difference scheme as follows:

- For $j = 1$, we have respectively from the third and fourth steps of the discretization in section 4.1, $\mathbf{W}(i, j+1) = \mathbf{W}(i, j-1)$ for $i = 1 : S+1$ and $\mathbf{Z}(1, j+1) = \mathbf{Z}(1, j-1)$.
- In (6), putting $j = 1$ and using the conditions recalled in the previous step gives

$$r\left(\frac{2\mathbf{Z}(1,2) - 2\mathbf{Z}(1,1)}{(\Delta\tau)^2}\right) + \alpha\Big(\mathbf{Z}(1,1) + \sin\phi\Big)\left(2 - \frac{1}{\sqrt{\cos^2\phi + (\mathbf{Z}(1,1) + \sin\phi)^2}} - \frac{1}{\sqrt{(\cos\phi - \mathbf{W}(S+1,1))^2 + (\mathbf{Z}(1,1) + \sin\phi)^2}}\right) = 0 \tag{12}$$

- $\mathbf{W}(S+1,1) = 0$ and $\mathbf{Z}(1,1) = 0$ from the first and second steps of the discretization in section 4.1. So, the only unknown in equation (12) is $\mathbf{Z}(1,2)$ which can be found out from the other known quantities.
- After finding these unknowns, all the $\mathbf{Z}(1, j+1)$ for $j = 2 : N$ can be easily found out by running the iterative scheme as described in (6).

## 5 Nature of the iterative scheme used

It is important to mention that the scheme developed for the numerical simulation of our system is an explicit one, that is at each step, the unknown quantity is found out directly/ explicitly in terms of the known quantities and not by using any implicit solver such as Newton-Raphson or Runge-Kutta. In the conventional CTCS method, which is the basis of our algorithm and otherwise implicit, the errors are quadratic over both the time step and the space step $\Delta u = O(k^2) + O(h^2)$, where $\Delta u$ is the error and $k$, $h$ are the time and space steps. So, the time and space steps have been both chosen to be adequately small for the explicit scheme discussed here. The iterative algorithm has been executed computationally in a MATLAB platform.

**Contents**



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