On the throughput of the common target area for robotic swarm strategies – Online Appendix

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1 Proof of Lemma 1

Let $l_\theta(t) = \sqrt{(x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2}$ be the distance between the two robots. The robots must maintain their minimum distance d at all time:

$$\forall t \in \mathbb{R}, l_\theta(t) \geq d. \tag{A1}$$

To avoid a collision, we have $\theta \neq \pi$, which corresponds to the case where robots face each other exactly. As a result, $\cos(\theta) \neq -1$. For ease of calculation, we define $X = \tau v$, that is, the distance between Robot 1 and Robot 2 when Robot 1 reaches the target. We also define $P_\theta(t) = l_\theta(t)^2 - d^2$, so the constraint in (A1) for minimum distance between them is expressed by

$$\forall t \in \mathbb{R}, l_\theta(t) \geq d \Leftrightarrow \forall t \in \mathbb{R}, P_\theta(t) \geq 0.$$

In addition, we have

$$\begin{aligned}
P_\theta(t) = &(vt\cos(\theta_1) - v(t-\tau)\cos(\theta_2))^2 + (vt\sin(\theta_1) - v(t-\tau)\sin(\theta_2))^2 \\
&- d^2 \\
=& (vt)^2 - 2vt(vt-X)(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)) \\
&+ (vt-X)^2 - d^2 \\
=& (vt)^2 - 2vt(vt-X)\cos(\theta_2-\theta_1) + (vt-X)^2 - d^2 \\
=& (vt)^2 - 2vt(vt-X)\cos(\theta) + (vt-X)^2 - d^2 \\
=& 2(1-\cos(\theta))v^2t^2 - 2X(1-\cos(\theta))vt + X^2 - d^2,
\end{aligned}$$

where we used $\cos(\theta) = \cos(\theta_2 - \theta_1) = \cos(\theta_2)\cos(\theta_1) + \sin(\theta_2)\sin(\theta_1)$.

We identify two cases:

1. Case 1: $\cos(\theta) \neq 1$. Then $P_\theta(t)$ is a second-degree polynomial in t. It is of the form $at^2 + bt + c$ with $a = 2(1-\cos(\theta))v^2$, $b = -2X(1-\cos(\theta))v$ and $c = X^2 - d^2$. We know that $P_\theta(t)$ has a with positive sign for all t, because $(1-\cos(\theta)) > 0$ when $\cos(\theta) \neq 1$. Thus, as $a > 0$, by second-degree polynomial inequalities properties, $P_\theta(t) \geq 0$ for all t if and only if its discriminant $\Delta = b^2 - 4ac$ is negative, that is,

$$\forall t \in \mathbb{R}, P_\theta(t) \geq 0 \Leftrightarrow b^2 - 4ac \leq 0.$$

Thus:

$$\begin{aligned}
&2^2X^2(1-\cos(\theta))^2v^2 - 4\cdot 2(1-\cos(\theta))v^2(X^2-d^2) \leq 0 \Leftrightarrow \\
&(4(1-\cos(\theta))v^2)(X^2(1-\cos(\theta)) - 2(X^2-d^2)) \leq 0 \Rightarrow \\
&X^2(1-\cos(\theta)) - 2(X^2-d^2) \leq 0 \Leftrightarrow 2d^2 - X^2(1+\cos(\theta)) \leq 0 \Leftrightarrow \\
&\frac{2d^2}{1+\cos(\theta)} \leq X^2 \Rightarrow
\end{aligned}$$

$$X \geq d\sqrt{\frac{2}{1+\cos(\theta)}} \tag{A2}$$

2. Case 2: $\cos(\theta) = 1$. Then $P_\theta(t) = X^2 - d^2$. In this case, $P_\theta(t) \geq 0$ for all t when $X^2 - d^2 \geq 0 \Rightarrow X \geq d$. This is the same as using $\cos(\theta) = 1$ in (A2).

Hence, (A2) gives, for the robots to respect the minimum distance d for every time t, a relation between the minimum distance, the angle between the lanes and the distance between Robot 1 and Robot 2 when Robot 1 reaches the target. The final result is obtained noticing that (A2) is equivalent to

$$\tau \geq \frac{d}{v}\sqrt{\frac{2}{1+\cos(\theta)}}.$$

2 Proof of Proposition 1

We show by induction on N, which is the number of robots moving towards the target. We define θ_N as the angle between the trajectories of Robot $N-1$ and Robot N; τ_N, the minimum delay between the arrival of Robot $N-1$ and Robot N; and Δ_N, the minimum delay between the arrival of Robot 1 and Robot N. We want to show the following predicate: for all $N \geq 2$, $\Delta_N = (N-1)d/v$ for $\theta_2 = \theta_3 = \ldots = \theta_N = 0$.

Base case ($N = 2$): Let τ_2 be the delay between the arrival of Robot 1 and Robot 2. From Lemma 1 we have that the minimum delay between Robot 1 and Robot 2 is equal to $\frac{d}{v}\sqrt{\frac{2}{1+\cos(\theta_2)}}$, which is minimised by $\theta_2 = 0$. Then, the minimum delay between the two robots is $\tau_2 = d/v = \Delta_2$.

Inductive step: We suppose the predicate is true for a given $N-1 \geq 2$. We will show that it implies the predicate is true for N robots. As in the previous case, we conclude from Lemma 1 that the minimum delay between Robot $N-1$ and Robot N is equal to $\frac{d}{v}\sqrt{\frac{2}{1+\cos(\theta_N)}}$, which is minimised for $\theta_N = 0$. Then, the minimum delay between the two robots is $\tau_N = d/v$. We have

$$\Delta_N = \Delta_{N-1} + \tau_N = (N-2)\frac{d}{v} + \frac{d}{v} = (N-1)\frac{d}{v}.$$

Consequently, the minimum delay between Robot 1 and Robot N is $\Delta_N = \sum_{i=2}^{N} \tau_i = (N-1)\frac{d}{v}$ and the time of arrival of Robot N, for all N, is minimised for $\theta_2 = \theta_3 = \ldots = \theta_N = 0$. Finally, by Definition 2, the throughput is $f = \frac{N-1}{\Delta_N} = \frac{v}{d}$.

3 Proof of Proposition 2

Consider Figure 3. The distance between the lanes is $2s$, and the minimum distance between two robots is d. Thus, $d_p = \sqrt{d^2 - (2s)^2}$. Hence, the distance between two robots in the same lane is $d_e = 2d_p = 2\sqrt{d^2 - (2s)^2}$.

The distance between two robots in the same lane must not be less than d, so $d_e \geq d$. This is true, because as we have that $0 < s \leq \frac{\sqrt{3}}{4}d$,

$$d_e = 2\sqrt{d^2 - (2s)^2} \geq 2\sqrt{d^2 - \left(2\frac{\sqrt{3}}{4}d\right)^2} = d.$$

Without loss of generality, assume the first robot to reach the target area being at the top lane in Figure 3. The number of robots on any lane is the integer division of the size of the lane by the offset between the robots plus one (because we are including the first robot in this counting). Therefore, the number of robots for a given time T in the top lane is $N_1(T) = \left\lfloor \frac{vT}{d_e} + 1 \right\rfloor$ and in the bottom lane is $N_2(T) = \left\lfloor \frac{vT - d_p}{d_e} + 1 \right\rfloor = \left\lfloor \frac{vT}{d_e} + \frac{1}{2} \right\rfloor$. By Definition 2,

$$f(T) = \frac{N_1(T) + N_2(T) - 1}{T} = \frac{1}{T}\left(\left\lfloor \frac{vT}{2\sqrt{d^2 - (2s)^2}} \right\rfloor + \left\lfloor \frac{vT}{2\sqrt{d^2 - (2s)^2}} + \frac{1}{2} \right\rfloor \right).$$

By the definition of the floor function, $\lfloor x \rfloor = x - frac(x)$, where $frac$ is the function that returns the fractional part of the number x and $0 \le frac(x) < 1$ (Graham et al, 1994). Thus,

$$\begin{aligned} \lim_{T\to\infty} f(T) &= \lim_{T\to\infty} \frac{1}{T}\left(\frac{vT}{d_e} - frac\left(\frac{vT}{d_e}\right) + \frac{vT}{d_e} + \frac{1}{2} - frac\left(\frac{vT}{d_e} + \frac{1}{2}\right)\right) \\ &= \frac{2v}{d_e} = \frac{v}{d\sqrt{1 - (2s/d)^2}}, \end{aligned}$$

as $\lim_{T\to\infty} \dfrac{frac(x)}{T} = 0$, for any x.

4 Proof of Proposition 3

As the distance between the robots must be at least d and $\frac{\sqrt{3}}{4}d < s < \frac{d}{2}$, we can assign $d_p = d/2$ in Figure 3. By doing so, two robots side by side in one lane and a robot in the other lane form an equilateral triangle with side measuring d, whose height has size $\frac{\sqrt{3}}{2}d$. Hence, the minimum diameter of the circular target region must be this value, and the hypothesis says so.

Also, the radius of the target area is less than $d/2$, implying that the three robots in Figure 3 must stay in the equilateral triangle formation because the two lanes cannot be far by d units of distance.

Thus, the throughput for a given time T is calculated similarly as in Proposition 2 resulting

$$f(T) = \frac{1}{T}\left(\left\lfloor \frac{vT}{d} \right\rfloor + \left\lfloor \frac{vT}{d} + \frac{1}{2} \right\rfloor \right) \text{ and } f = \lim_{T\to\infty} f(T) = \frac{2v}{d}.$$

5 Proof of Proposition 4

When robots move in straight lines in a single lane, we know from Proposition 1 that the optimal throughput is $\frac{v}{d}$. Since $s \ge \frac{d}{2}$, we can have multiple straight line lanes that are parallel to each other (Figure A1).

As the robots are going to a circular target region, robots next to the centre reach the region in shorter time than the others. The first robot of each lane must run an additional distance d_i from the beginning of its lane, which is related to its y-coordinate. Figure A2 illustrates this distance for a robot in Lane i. The right triangle ABC has hypotenuse $\overline{AC}$ measuring s, so the horizontal cathetus $\overline{AB}$ measures $\sqrt{s^2 - (s - (i-1)d)^2}$. Consequently, the

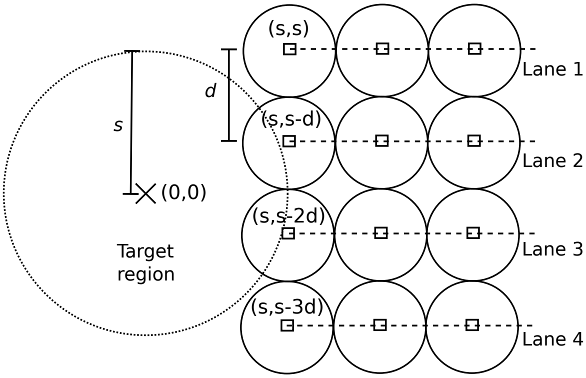


Fig. A1: Example of the parallel lanes strategy. Robots and parallel lanes are distant by d. The first lane, Lane 1, is at the top. The first robot of each lane is located at $(s, s-(i-1)d)$ for the Lane i.

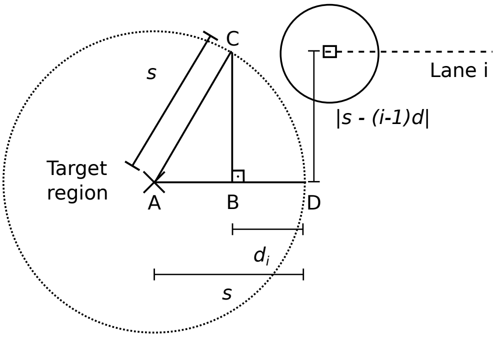


Fig. A2: The distance from the target region to a robot at the beginning of the Lane i is equal to d_i (represented by $\overline{BD}$).

robot in the Lane i needs to walk an additional distance represented by $\overline{BD}$, which has $d_i = |\overline{BD}| = s - |\overline{AB}| = s - \sqrt{s^2 - (s-(i-1)d)^2}$ units of length.

This distance is minimised when $|\overline{BD}| = 0$, that would happen if $i = \frac{s}{d} + 1$. However, i must be integer, so $|\overline{BD}|$ is minimised for an integer J that minimises d_J. If $\frac{s}{d} \notin \mathbb{Z}$, the two nearest integers are $\left\lfloor \frac{s}{d} \right\rfloor$ and $\left\lceil \frac{s}{d} \right\rceil$. Thus, if $J = \left\lfloor \frac{s}{d} \right\rfloor + 1$ then, equivalently, $d_{\left\lfloor \frac{s}{d} \right\rfloor + 1} \leq d_{\left\lceil \frac{s}{d} \right\rceil + 1} \Leftrightarrow s - \sqrt{s^2 - \left(s - \left\lfloor \frac{s}{d} \right\rfloor d\right)^2} \leq s - \sqrt{s^2 - \left(s - \left\lceil \frac{s}{d} \right\rceil d\right)^2} \Leftrightarrow \sqrt{s^2 - \left(s - \left\lceil \frac{s}{d} \right\rceil d\right)^2} \leq \sqrt{s^2 - \left(s - \left\lfloor \frac{s}{d} \right\rfloor d\right)^2} \Leftrightarrow s^2 - \left(s - \left\lceil \frac{s}{d} \right\rceil d\right)^2 \leq s^2 - \left(s - \left\lfloor \frac{s}{d} \right\rfloor d\right)^2 \Leftrightarrow \left(s - \left\lfloor \frac{s}{d} \right\rfloor d\right)^2 \leq \left(s - \left\lceil \frac{s}{d} \right\rceil d\right)^2 \Leftrightarrow \left|s - \left\lfloor \frac{s}{d} \right\rfloor d\right| \leq \left|s - \left\lceil \frac{s}{d} \right\rceil d\right|$. Thus,

$$J = \begin{cases} \left\lfloor \frac{s}{d} \right\rfloor + 1, & \text{if } \left|s - \left\lfloor \frac{s}{d} \right\rfloor d\right| \leq \left|s - \left\lceil \frac{s}{d} \right\rceil d\right|, \\ \left\lceil \frac{s}{d} \right\rceil + 1, & \text{otherwise.} \end{cases}$$

Let $N(T)$ be the number of robots that arrive at the target region until a given time T after the first robot has reached it. Thus,

$$N(T) = \sum_{i=1}^{\lfloor \frac{2s}{d} \rfloor + 1} N_i(T),$$

for the number of robots at Lane i, $N_i(T)$, that arrived at the target region by time T. As every robot has the same linear velocity and started at the same x-coordinate, when the first robot at Lane J reaches the target region, all robots have run d_J units of length. Hence, at each Lane i, instead of running an additional d_i to reach the target region, they need to run $d_i - d_J$. Consequently,

$$N_i(T) = \begin{cases} \left\lfloor \frac{vT-(d_i-d_J)}{d} + 1 \right\rfloor, & \text{if } T \geq \frac{d_i - d_J}{v}, \\ 0, & \text{otherwise,} \end{cases}$$

and, by Definition 2,

$$f_p(T) = \frac{N(T) - 1}{T} = \frac{1}{T} \left(\sum_{i=1}^{\lfloor \frac{2s}{d} \rfloor + 1} N_i(T) \right) - \frac{1}{T}.$$

Also,

$$\begin{aligned}
f_p &= \lim_{T \to \infty} f_p(T) = \lim_{T \to \infty} \left(\frac{1}{T} \left(\sum_{i=1}^{\lfloor \frac{2s}{d} \rfloor + 1} N_i(T) \right) - \frac{1}{T} \right) \\
&= \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{\lfloor \frac{2s}{d} \rfloor + 1} N_i(T) = \lim_{T \to \infty} \sum_{i=1}^{\lfloor \frac{2s}{d} \rfloor + 1} \frac{N_i(T)}{T} \\
&= \lim_{T \to \infty} \sum_{i=1}^{\lfloor \frac{2s}{d} \rfloor + 1} \frac{1}{T} \left\lfloor \frac{vT - d_i + d_J}{d} + 1 \right\rfloor \qquad \left[\text{as } T \to \infty \Rightarrow T \geq \frac{d_i - d_J}{v} \right] \\
&= \lim_{T \to \infty} \sum_{i=1}^{\lfloor \frac{2s}{d} \rfloor + 1} \frac{1}{T} \left(\frac{vT - d_i + d_J}{d} + 1 - frac\left(\frac{vT - d_i + d_J}{d} + 1 \right) \right) \\
&= \lim_{T \to \infty} \sum_{i=1}^{\lfloor \frac{2s}{d} \rfloor + 1} \left(\frac{v}{d} - \frac{d_i - d_J}{dT} + \frac{1}{T} \right) - \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{\lfloor \frac{2s}{d} \rfloor + 1} frac\left(\frac{vT - d_i + d_J}{d} + 1 \right) \\
&= \left\lfloor \frac{2s}{d} + 1 \right\rfloor \frac{v}{d},
\end{aligned}$$

as $frac$ and d_i are bounded for every i due to $0 \leq d_i \leq s$ and $0 \leq frac(x) < 1$ for any x.

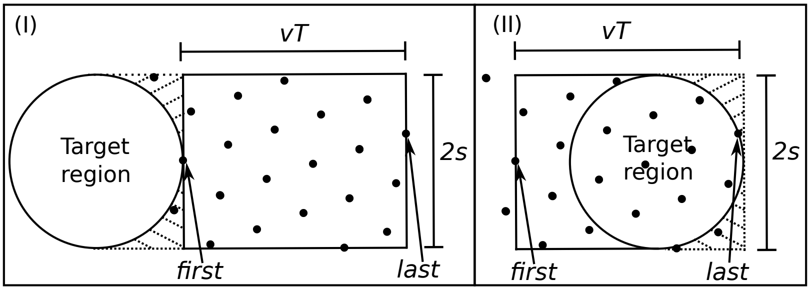


Fig. A3: Robots are represented by black dots and they are in hexagonal formation and distant by d. In (I), only the first robot reached the target and, in (II), all robots that are not in the dashed area arrived in the target region before the last robot. The robots on the right dashed area should not be counted because their arrival time is greater than the arrival time of the last robot, hence they arrive after the considered time frame T. That is, all robots on the dashed area in (I) should be counted as part of the number of robots that reached the target region in the time between the first and the last robot, while the robots on the dashed area in (II) should not. As these dashed areas have the same value, we consider in this proof the number of robots inside a rectangle $vT \times 2s$. Then, the dashed area used for counting in (II) replaces the unconsidered robots in the dashed area in (I). As we are concerned with $T \to \infty$, any possible difference between the number of robots on the dashed areas of either side due to the configuration of the hexagonal packing is negligible.

6 Proof of Proposition 5

Without loss of generality, we consider the target at the origin of a coordinate system and the robots are moving parallel to the x-axis. By Definition 2, the throughput considers the number of robots that cross the target during a unit of time, after the first robot has reached it. We evaluate that number of robots, N_T, during a time T. As a result, computing the maximum throughput is reduced to finding the maximum number of robots (their centre of mass) that can fit in a rectangle of width $w(T) = vT$ and height $h = 2s$ (Figure A3).

Since we have the constraint that robots must be at a minimum distance d from each other, we consider discs of radius $d/2$ as reserved areas for each robot, and any two reserved areas must not intersect. Therefore, the problem is equivalent to finding the optimal arrangement of circles of radius $d/2$ in a rectangle of width $W(T) = w(T)+2\frac{d}{2}$ and height $H = h+2\frac{d}{2}$. This formulation is a variant of the *circle packing* problem, which is already well studied[1]. The term $2\frac{d}{2}$ was added because the circle packing problem deals with full circles, not their centres.

[1] See http://packomania.com/ and http://hydra.nat.uni-magdeburg.de/packing/crc_var/crc.html for an informal introduction.

The optimal surface occupied by the circles divided by the rectangle area was proven to be $\pi\sqrt{3}/6$ in the case of hexagonal packing over an infinite area (Chang and Wang, 2010). Thus, the total area occupied by the circles representing the reserved areas of the robots is given by $\left(\pi\sqrt{3}/6\right) HW(T)$. Hence, the maximum number of robots N_T that can fit inside the $HW(T)$ area is bounded by $N_{opt}(T) \geq N_T$, for

$$N_{opt}(T) = \left\lfloor \frac{\left(\pi\sqrt{3}/6\right) HW(T)}{\pi d^2/4} \right\rfloor = \left\lfloor \frac{2HW(T)}{\sqrt{3}d^2} \right\rfloor .$$

By Definition 2, the maximum throughput is

$$f_h^{max}(T) = \frac{N_{opt}(T) - 1}{T} = \frac{\left\lfloor \frac{2HW(T)}{\sqrt{3}d^2} \right\rfloor - 1}{T}.$$

As for any x, $\lfloor x \rfloor = x - frac(x)$ and $0 \leq frac(x) < 1$, the upper bound of the asymptotic throughput is

$$\begin{aligned}
f_h^{max} &= \lim_{T\to\infty} f_h^{max}(T) = \lim_{T\to\infty} \frac{\left\lfloor \frac{2HW(T)}{\sqrt{3}d^2} \right\rfloor - 1}{T} = \lim_{T\to\infty} \frac{2HW(T)}{\sqrt{3}d^2 T} \\
&= \lim_{T\to\infty} \frac{2(2s+d)(vT+d)}{\sqrt{3}d^2 T} = \lim_{T\to\infty} \frac{2}{\sqrt{3}} \frac{2s+d}{d^2} \frac{vT+d}{T} \\
&= \lim_{T\to\infty} \frac{2}{\sqrt{3}} \left(\frac{2s}{d^2} + \frac{1}{d}\right)\left(v + \frac{d}{T}\right) = \frac{2}{\sqrt{3}} \left(\frac{2s}{d^2} + \frac{1}{d}\right) v = \frac{2}{\sqrt{3}} \left(\frac{2s}{d} + 1\right) \frac{v}{d}.
\end{aligned}$$

7 Proof of Proposition 6

We are concerned about the throughput of the target region for a given time and hexagonal packing angle θ, $f_h(T, \theta) = \frac{N(T,\theta)-1}{T}$, where $N(T, \theta)$ denotes the number of robots which arrived at the target region. Figure A4 illustrates the arrival of the robots on the target region. As this region has a circular shape, not all robots at the distance vT arrive at target region by the time T. Thence, the number of robots in hexagonal packing are divided into the number of robots located inside a rectangle, N_R, and of robots inside a semicircle N_S (Figure A4 (III)). That is, $N(T, \theta) = N_S(T, \theta) + N_R(T, \theta)$ and $N_R = 0$ whenever $vT \leq s$.

This proof is divided in lemmas for helping the construction of the equation to compute $N_R(T, \theta)$ and $N_S(T, \theta)$ as well for calculating $\lim_{T\to\infty} f_h(T, \theta)$. Before presenting them, we discuss a coordinate space transformation which will be used to count the robots for N_R and N_S. This transformation was inspired by (Red Blob Games, 2021).

Figure A5 shows the coordinate spaces used in this proof. Let $\psi = \pi/3 - \theta$ (because the angle of the equilateral triangle formed by neighbours is $\pi/3$, as

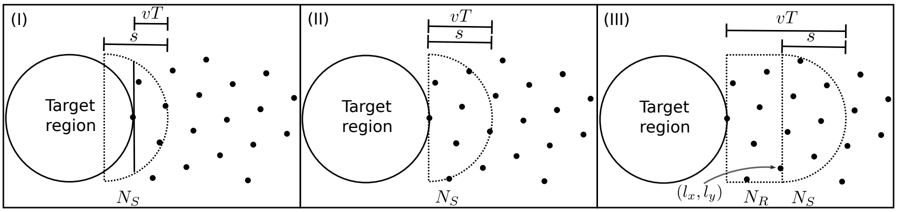


Fig. A4: (I) When the robots – here represented by black dots – in hexagonal packing begin to arrive at the target region, only the robots inside a part of the semicircle are counted. (II) We consider the first robot to reach the target region being at (x_0, y_0) at time 0. As T grows, this continues until $vT = s$. (III) When $vT > s$, the robots are counted on two regions: a rectangular, N_R, and a semicircular, N_S. When $vT > s$, the semicircular region counting starts after the last robot on the rectangular region located at (l_x, l_y).

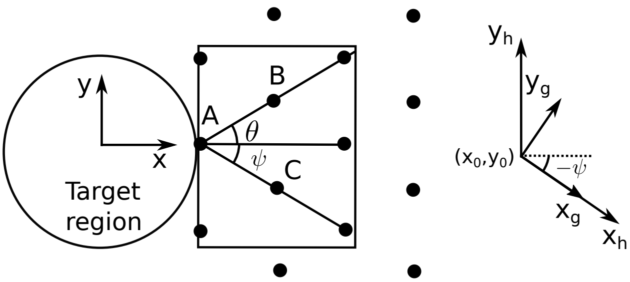


Fig. A5: The reference frames used in this proof: the usual Euclidean space (x, y) in relation to the target region and the rectangle region formed by robots in hexagonal packing going to it; the coordinate space (x_g, y_g), formed by the usual space after a translation to the first robot to reach the target region at (x_0, y_0), followed by a rotation by $-\psi$; the coordinate space (x_h, y_h), a hexagonal grid coordinate space made after this transformation and a linear transformation H. Robots are represented by the black dots and they are on hexagonal formation. Each neighbour of a robot is distant by d, so $\triangle ABC$ is equilateral. Thus, $\theta + \psi = \pi/3$.

explained in Figure A5). Accordingly, $\psi \in [0, \pi/3)$ too. The usual Euclidean coordinate space which represents the location of all robots is denoted here by (x, y) coordinates. The next coordinate space is denoted by (x_g, y_g), and it is the result of a translation of the usual Euclidean coordinate space by the position of the first robot to reach the target region at (x_0, y_0), then a rotation of $-\psi$, that is,

$$\begin{bmatrix} x_g \\ y_g \end{bmatrix} = \begin{bmatrix} \cos(-\psi) & -\sin(-\psi) \\ \sin(-\psi) & \cos(-\psi) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.$$

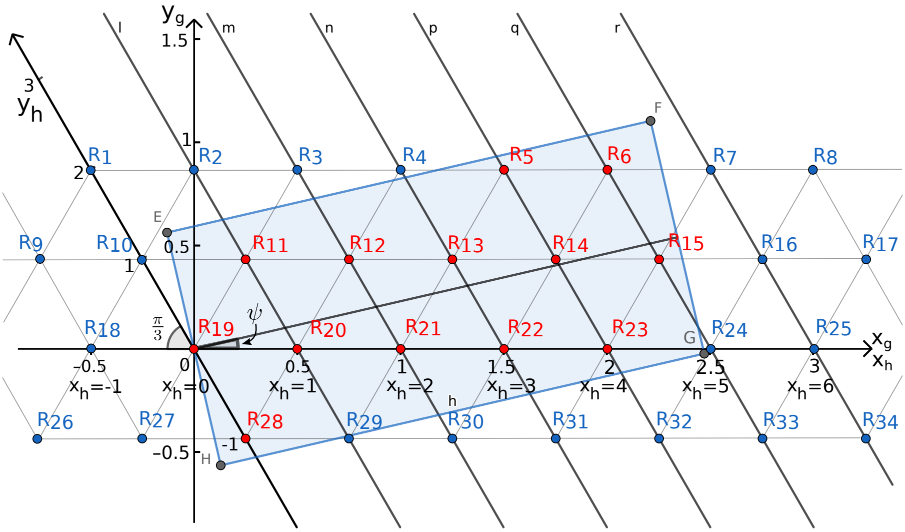


Fig. A6: Robots in hexagonal packing formation, and the corresponding rectangular corridor which will reach the target region. Robots are located in the l, m, n, p, q and r lines, which are parallel to the y_h-axis. In this example, $\psi = 0.227$ and the distance between all robots is $d = 0.5$. The distance between those parallel-to-y_h lines is $\sqrt{3}d/2$. The robots inside the rectangle EFGH are counted and are indicated by red points, while blue points are robots outside the rectangle. Although the x_h-axis coincides with the x_g-axis, x_h is scaled by d.

The last coordinate space is denoted by (x_h, y_h) and it is intended to represent a hexagonal grid such that the position of each robot is an integer pair.

Figure A6 shows the location of robots with respect to that hexagonal grid. Let $(x_h, y_h) \in \mathbb{Z}^2$ be the hexagonal coordinates of a robot in this hexagonal grid space. In this figure, there is an integer grid in grey – the horizontal lines correspond to fixed integer y_h values and the inclined ones, x_h values. For example, in Figure A6 robots R_{10}, R_{11} and R_{20} respectively are at $(0, 1)$, $(1, 1)$ and $(1, 0)$ at (x_h, y_h) coordinate system, which is equivalent to $\left(-1/4, \sqrt{3}/4\right)$, $\left(1/4, \sqrt{3}/4\right)$ and $(1/2, 0)$ on the usual two dimensional coordinate system with origin at (x_0, y_0).

We get the linear transformation H from a point (x_h, y_h) to (x_g, y_g) basis by knowing the result of this transformation for the standard vectors $(1, 0)$ and $(0, 1)$. Observing Figure A6 and having that the angle between the x-axis and y_h-axis is by definition $2\pi/3$, we get the following mappings $(x_h, y_h) \mapsto (x_g, y_g)$: $(1, 0) \mapsto (d, 0)$ and $(0, 1) \mapsto (d \cos(2\pi/3), d \sin(2\pi/3)) = (-\frac{d}{2}, \frac{\sqrt{3}d}{2})$ (in Figure A6 these two mappings are represented by robots R_{20} and R_{10}, respectively,

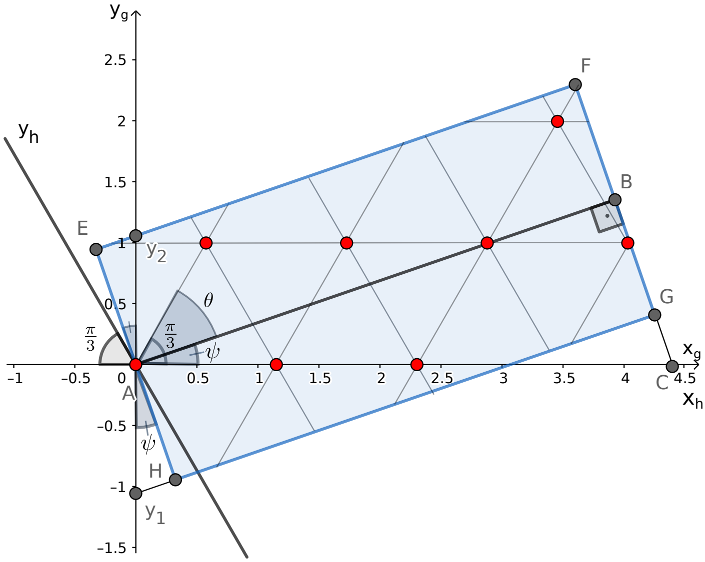


Fig. A7: The problem involves the rectangle EFGH in a hexagonal grid (grey lines inside the rectangle) of robots (the red dots). The x_h-axis is horizontal and coincides with the x-axis. The y_h-axis forms a $2\pi/3$ angle with it. $\overline{EH}$ and $\overline{AB}$ have length $2s$ and $vT - s$, respectively. In this example, $\psi = 19\pi/180$ and $s = 1$. The angles marked with a line are equal to ψ, because the angle formed by $\overrightarrow{y_2 A}$ and the x-axis is right, as well as $\widehat{EAB}$. Accordingly, $\widehat{y_2AB} = \pi/2 - \psi$ implies that $\widehat{EAy_2} = \psi$.

with $d = 0.5$). Then,

$$\begin{bmatrix} x_g \\ y_g \end{bmatrix} = \left[H\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \; H\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] \begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} d & -\frac{d}{2} \\ 0 & \frac{\sqrt{3}d}{2} \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix}. \tag{A3}$$

Counting the robots inside the rectangle is the same as counting the number of integer hexagonal coordinate points lying inside it. Figure A7 shows the rectangular part with some robots in hexagonal packing, where the robots are the red dots and the hexagonal packing is guided by the grey lines inside the rectangle, based on the value of the angle ψ. The rectangle is of width $vT - s$ and of height $2s$. The reference frame of the hexagonal grid is rotated in relation to the target region (Figure A5). From Figure A7, we have

$$2s = (y_2 - y_1)\cos(\psi), \;\; y_2 = \frac{s}{\cos(\psi)} \text{ and } y_1 = -\frac{s}{\cos(\psi)}. \tag{A4}$$

We consider a robot with coordinates (x_g, y_g). The four sides of the rectangle EFGH, $\overline{HG}$, $\overline{EF}$, $\overline{EH}$ and $\overline{FG}$, have the following equations of line: $y_g = y_1 + \tan(\psi)x_g$, $y_g = y_2 + x_g\tan(\psi)$, $y_g = \tan\left(\psi + \frac{\pi}{2}\right)x_g$ and

$y_g = \tan(\psi+\frac{\pi}{2})\left(x_g - \frac{vT-s}{\cos(\psi)}\right)$, respectively. The term $\frac{vT-s}{\cos(\psi)}$ in the last equation arises because of the length of $\overline{AC}$, which is the hypotenuse of $\triangle ABC$ whose side $\overline{AB}$ measures vT. Knowing that $\tan\left(\psi+\frac{\pi}{2}\right) = -\cot(\psi)$, the equations below are all true for a robot at (x_g, y_g) to be inside or on the boundary of the previously defined rectangle,

$$\begin{aligned} &y_g \geq y_1 + x_g\tan(\psi),\ y_g \leq y_2 + x_g\tan(\psi),\ -x_g \leq \tan(\psi)y_g,\ \text{and} \\ &-\left(x_g - \frac{vT-s}{\cos\psi}\right) \geq \tan(\psi)y_g. \end{aligned} \tag{A5}$$

Now we take the minimum and maximum y_h value for each parallel-to-y_h line depending on the x_h value. Using (A3) for converting (A5) to x_h and y_h coordinate system, i.e., hexagonal coordinates, we have for $\overline{HG}$ and $\overline{EF}$

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\tan(\psi)\right)y_h - \tan(\psi)x_h \geq \frac{y_1}{d} \text{ and}$$

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\tan(\psi)\right)y_h - \tan(\psi)x_h \leq \frac{y_2}{d}.$$

Hence,

$$\begin{aligned} \frac{y_1}{d} \leq \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\tan\psi\right)y_h - \tan(\psi)x_h \leq \frac{y_2}{d} \Leftrightarrow \\ \frac{\frac{2y_1}{d} + 2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)} \leq y_h \leq \frac{\frac{2y_2}{d} + 2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)}. \end{aligned} \tag{A6}$$

Analogously, but considering $\overline{EH}$ and $\overline{FG}$,

$$-x_h \leq \left(\tan(\psi)\frac{\sqrt{3}}{2} - \frac{1}{2}\right)y_h \text{ and } \left(\tan(\psi)\frac{\sqrt{3}}{2} - \frac{1}{2}\right)y_h \leq \frac{vT-s}{d\cos(\psi)} - x_h. \tag{A7}$$

Based on the sign of $\left(\tan(\psi)\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ and excluding the null case (when $\psi = \pi/6$), we have two different inequalities over y_h. Assuming $\psi \in [0, \pi/3)$, we have $\left(\tan(\psi)\frac{\sqrt{3}}{2} - \frac{1}{2}\right) > 0 \Leftrightarrow \tan(\psi)\frac{\sqrt{3}}{2} > \frac{1}{2} \Leftrightarrow \tan(\psi) > \frac{1}{\sqrt{3}} \Leftrightarrow \psi > \pi/6$. Thus, from (A7),

$$\begin{aligned} \frac{-2x_h}{\sqrt{3}\tan(\psi)-1} \leq y_h \leq \frac{\frac{2(vT-s)}{d\cos(\psi)} - 2x_h}{\sqrt{3}\tan(\psi)-1}, \text{ if } \psi > \pi/6, \\ \frac{\frac{2(vT-s)}{d\cos(\psi)} - 2x_h}{\sqrt{3}\tan(\psi)-1} \leq y_h \leq \frac{-2x_h}{\sqrt{3}\tan(\psi)-1}, \text{ if } \psi < \pi/6. \end{aligned} \tag{A8}$$

We have that (A6) and (A8) restrict the value of y_h depending on the value of x_h by the relation

$$\begin{aligned}
&\max\left(\frac{\frac{2y_1}{d}+2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)}, \frac{-2x_h}{\sqrt{3}\tan(\psi)-1}\right)\le y_h \\
&\le \min\left(\frac{\frac{2y_2}{d}+2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)}, \frac{\frac{2(vT-s)}{d\cos(\psi)}-2x_h}{\sqrt{3}\tan(\psi)-1}\right), \text{ if } \psi>\pi/6, \\
&\max\left(\frac{\frac{2y_1}{d}+2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)}, \frac{\frac{2(vT-s)}{d\cos(\psi)}-2x_h}{\sqrt{3}\tan(\psi)-1}\right)\le y_h \\
&\le \min\left(\frac{\frac{2y_2}{d}+2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)}, \frac{-2x_h}{\sqrt{3}\tan(\psi)-1}\right), \text{ if } \psi<\pi/6.
\end{aligned} \tag{A9}$$

Using hexagonal coordinates the position of each robot is represented by a pair of integers. Then, assuming x_h and y_h integers, (A9) becomes $\lceil Y_1^R(x_h)\rceil \le y_h \le \lfloor Y_2^R(x_h)\rfloor$, for

$$\begin{aligned}
Y_1^R(x_h) &= \begin{cases}
\max\left(\frac{\frac{2y_1}{d}+2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)}, \frac{-2x_h}{\sqrt{3}\tan(\psi)-1}\right), & \text{if } \psi>\pi/6, \\
\max\left(\frac{\frac{2y_1}{d}+2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)}, \frac{\frac{2(vT-s)}{d\cos(\psi)}-2x_h}{\sqrt{3}\tan(\psi)-1}\right), & \text{if } \psi<\pi/6, \\
\frac{\sqrt{3}y_1+dx_h}{2d}, & \text{if } \psi=\pi/6,
\end{cases} \\
&= \begin{cases}
\max\left(\frac{\sin(\psi)x_h-\frac{s}{d}}{\cos\left(\frac{\pi}{6}-\psi\right)}, \frac{-\cos(\psi)x_h}{\sin\left(\psi-\frac{\pi}{6}\right)}\right), & \text{if } \psi>\pi/6, \\
\max\left(\frac{\sin(\psi)x_h-\frac{s}{d}}{\cos\left(\frac{\pi}{6}-\psi\right)}, \frac{\frac{vT-s}{d}-\cos(\psi)x_h}{\sin\left(\psi-\frac{\pi}{6}\right)}\right), & \text{if } \psi<\pi/6, \\
\frac{x_h}{2}-\frac{s}{d}, & \text{if } \psi=\pi/6,
\end{cases}
\end{aligned} \tag{A10}$$

$$\begin{aligned}
Y_2^R(x_h) &= \begin{cases}
\min\left(\frac{\frac{2y_2}{d}+2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)}, \frac{\frac{2(vT-s)}{d\cos(\psi)}-2x_h}{\sqrt{3}\tan(\psi)-1}\right), & \text{if } \psi>\pi/6, \\
\min\left(\frac{\frac{2y_2}{d}+2\tan(\psi)x_h}{\sqrt{3}+\tan(\psi)}, \frac{-2x_h}{\sqrt{3}\tan(\psi)-1}\right), & \text{if } \psi<\pi/6, \\
\frac{\sqrt{3}y_2+dx_h}{2d}, & \text{if } \psi=\pi/6,
\end{cases} \\
&= \begin{cases}
\min\left(\frac{\sin(\psi)x_h+\frac{s}{d}}{\cos\left(\frac{\pi}{6}-\psi\right)}, \frac{\frac{vT-s}{d}-\cos(\psi)x_h}{\sin\left(\psi-\frac{\pi}{6}\right)}\right), & \text{if } \psi>\pi/6, \\
\min\left(\frac{\sin(\psi)x_h+\frac{s}{d}}{\cos\left(\frac{\pi}{6}-\psi\right)}, \frac{-\cos(\psi)x_h}{\sin\left(\psi-\frac{\pi}{6}\right)}\right), & \text{if } \psi<\pi/6, \\
\frac{x_h}{2}+\frac{s}{d}, & \text{if } \psi=\pi/6.
\end{cases}
\end{aligned} \tag{A11}$$

We simplified above using (A4), $\cos\left(\frac{\pi}{6}-\psi\right) = \frac{\sqrt{3}}{2}\cos(\psi)+\frac{1}{2}\sin(\psi)$ and $\sin\left(\psi-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\sin(\psi) - \frac{1}{2}\cos(\psi)$.

Now we get the possible integer values for the x_h-axis which are inside the rectangle EFGH, that is, we count the number of lines parallel with the y_h-axis that intersect the rectangle for x_h integer values. Let n_l be the number of such parallel lines. We consider $n_l = n_l^- + n_l^+$, such that n_l^- is the number of lines parallel to the y_h-axis whose intersection with the x_h-axis is a point $(i, 0)$ for $i < 0$ and $i \in \mathbb{Z}$, and n_l^+ is similar but for non-negative integer i. For example, in Figure A6 we have $n_l^- = 0$ and $n_l^+ = 6$ (we marked below the values of the points over the x-axis, the equivalent over x_h-axis in order to aid enumerating them). Note that the point $(i, 0)$ may be outside of the rectangle, but it will still be counted if there are integer (i, y_h) coordinates inside the rectangle. The next lemma shows how to compute n_l^+ and n_l^- to aid in this proof development.

Lemma A1. *On the (x_h, y_h) coordinate system, the integer values for x_h robot coordinates inside the rectangle EFGH are in the set $\{-n_l^-, \ldots, n_l^+ - 1\}$ with*

$$n_l^+ = \left\lfloor \frac{2(vT-s)\cos(\psi-\pi/6) + 2s\sin(|\psi-\pi/6|)}{\sqrt{3}d} + 1 \right\rfloor, \tag{A12}$$

and

$$n_l^- = \left\lfloor \frac{2s\sin\left(|\psi-\pi/6|\right)}{\sqrt{3}d} \right\rfloor. \tag{A13}$$

Proof For getting n_l^+, we count how many parallel-to-y_h lines when projected over the x-axis are distant from each other by d on this axis and are inside the rectangle. These lines must intersect the diagonal $\overline{HF}$ of the rectangle, but commencing from the intersection between the y_h-axis and the diagonal (i.e., from B_0 in the Figure A8). Let $\phi = \arctan\left(\frac{2s}{vT-s}\right)$ be the angle of the diagonal in relation to the rectangle base. We have two cases depending on the value of ψ.

- Case $\psi \leq \frac{\pi}{6}$: from Figure A8, every line parallel to y_h is distant by d on the projection onto the x-axis. The triangles AD_iC_i for any $i \in \{1, \ldots, n_l^+ - 1\}$ and ADC are similar, $|\overline{AD_1}| = d$ and $|\overline{AC_1}| = e$, whose value is unknown for the moment. $\triangle ADC$ has angles $\widehat{CAD} = \psi + \phi$, $\widehat{ADC} = \pi/3$ and $\widehat{ACD} = \pi - \widehat{CAD} - \widehat{ADC} = 2\pi/3 - \psi - \phi$. As for every i, $\triangle AD_iC_i \sim \triangle ADC$, we have that
$$\frac{|\overline{AC}|}{|\overline{AC_1}|} = \frac{|\overline{AD}|}{|\overline{AD_1}|} \Leftrightarrow \frac{|\overline{AC}|}{e} = \frac{|\overline{AD}|}{d}. \tag{A14}$$
As AHFI is a parallelogram, $|\overline{HF}| = |\overline{AI}|$ and $|\overline{FI}| = |\overline{AH}| = s$, then $|\overline{BI}| = 2s$. Thus, $|\overline{AI}| = \sqrt{(2s)^2 + (vT-s)^2}$, because $\triangle ABI$ is right-angled.

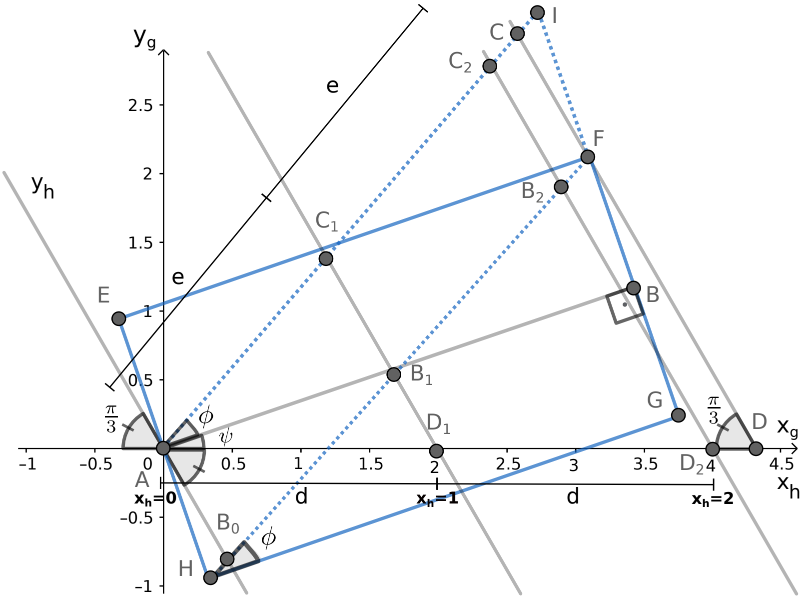


Fig. A8: The goal is to count how many points named B_i lie in the diagonal $\overline{HF}$. B_i is the intersection of a parallel-to-y_h line on a x_h integer coordinate and the diagonal $\overline{HF}$. The triangles AD_iC_i for any $i \in \{1, 2\}$ and ADC are similar. $\overline{AD_i}$ and $\overline{AC_i}$ have distance $i \cdot d$ and $i \cdot e$, respectively. In this example, $d = 2$ and there are three points lying over $\overline{HF}$.

Also, by the law of sines, we get $\frac{|\overline{AD}|}{\sin(\widehat{ACD})} = \frac{|\overline{AC}|}{\sin(\widehat{ADC})} \Leftrightarrow$

$$\begin{aligned} |\overline{AD}| = |\overline{AC}| \frac{\sin(\widehat{ACD})}{\sin(\widehat{ADC})} &= \left(|\overline{AI}| - |\overline{CI}|\right) \frac{\sin(\widehat{ACD})}{\sin(\widehat{ADC})} \\ &= \left(|\overline{AI}| - |\overline{CI}|\right) \frac{\sin(2\pi/3 - \psi - \phi)}{\sin(\pi/3)}. \end{aligned} \tag{A15}$$

AB_0FC is a parallelogram as well, so $|\overline{AC}| = |\overline{B_0F}| = |\overline{HF}| - |\overline{HB_0}|$ and $|\overline{CI}| = |\overline{HB_0}|$.

The $\triangle AB_0H$ has angles $\widehat{HAB_0} = \widehat{HAB} - \widehat{B_0AB} = \widehat{HAB} - (\widehat{B_0AD} + \widehat{DAB}) = \pi/2 - (\pi/3 + \psi) = \pi/6 - \psi$, $\widehat{AHB_0} = \widehat{AHG} - \widehat{FHG} = \pi/2 - \phi$ and $\widehat{HB_0A} = \pi - \widehat{HAB_0} - \widehat{AHB_0} = \pi/3 + \psi + \phi$. By the law of sines, we have $|\overline{HB_0}| = \frac{\sin(\widehat{HAB_0})|\overline{AH}|}{\sin(\widehat{HB_0A})} = \frac{\sin(\pi/6-\psi)s}{\sin(\pi/3+\psi+\phi)}$. Hence,

$$\begin{aligned}
|\overline{AD}| &= \left(|\overline{AI}| - |\overline{CI}|\right)\frac{\sin(2\pi/3-\psi-\phi)}{\sin(\pi/3)} \qquad \text{[from (A15)]}\\
&= \left(\sqrt{(2s)^2+(vT-s)^2} - \frac{s\sin(\pi/6-\psi)}{\sin(\pi/3+\psi+\phi)}\right)\frac{\sin(2\pi/3-\psi-\phi)}{\sin(\pi/3)}\\
&= \sqrt{(2s)^2+(vT-s)^2}\frac{\sin(2\pi/3-\psi-\phi)}{\sin(\pi/3)} - \frac{s\sin(\pi/6-\psi)}{\sin(\pi/3)}\\
&= \frac{2\sqrt{(2s)^2+(vT-s)^2}\sin(2\pi/3-\psi-\phi) - 2s\sin(\pi/6-\psi)}{\sqrt{3}}\\
&= \frac{2\sqrt{(2s)^2+(vT-s)^2}\left(\sin(\frac{2\pi}{3}-\psi)\cos(\phi) - \cos\left(\frac{2\pi}{3}-\psi\right)\sin(\phi)\right)}{\sqrt{3}}\\
&\quad - \frac{2s\sin(\frac{\pi}{6}-\psi)}{\sqrt{3}}\\
&= \frac{2\sqrt{(2s)^2+(vT-s)^2}\left(\frac{\sin(2\pi/3-\psi)(vT-s)}{\sqrt{(2s)^2+(vT-s)^2}} - \frac{2s\cos(2\pi/3-\psi)}{\sqrt{(2s)^2+(vT-s)^2}}\right)}{\sqrt{3}}\\
&\quad - \frac{2s\sin(\pi/6-\psi)}{\sqrt{3}}\\
&= \frac{2\left(\sin(2\pi/3-\psi)(vT-s) - 2s\cos(2\pi/3-\psi)\right) - 2s\sin(\pi/6-\psi)}{\sqrt{3}}\\
&= \frac{2\cos(\pi/6-\psi)(vT-s) + 2s\sin(\pi/6-\psi)}{\sqrt{3}} \qquad \text{(A16)}
\end{aligned}$$

Above we used $\sin(2\pi/3-\psi) = \cos(\pi/6-\psi)$, $\cos(2\pi/3-\psi) = -\sin(\pi/6-\psi)$, $\sin(2\pi/3-\psi-\phi) = \sin(\pi/3+\psi+\phi)$, $\sin(2\pi/3-\psi-\phi) = \sin(2\pi/3-\psi)\cos(\phi) - \cos(2\pi/3-\psi)\sin(\phi)$, $\sin(\arctan(y/x)) = \frac{y}{\sqrt{x^2+y^2}}$, and $\cos(\arctan(y/x)) = \frac{x}{\sqrt{x^2+y^2}}$.

Therefore, the number of lines parallel to the y_h-axis intersecting $\overline{B_0F}$ for integer x_h values is

$$\begin{aligned}
n_l^+ &= \left\lfloor \frac{|\overline{B_0F}|}{e} + 1 \right\rfloor = \left\lfloor \frac{|\overline{AC}|}{e} + 1 \right\rfloor = \left\lfloor \frac{|\overline{AD}|}{d} + 1 \right\rfloor \qquad \text{[from (A14)]}\\
&= \left\lfloor \frac{2\cos(\pi/6-\psi)(vT-s) + 2s\sin(\pi/6-\psi)}{\sqrt{3}d} + 1 \right\rfloor \qquad \text{[from (A16)]}
\end{aligned}$$

- Case $\psi > \frac{\pi}{6}$: Figure A9 shows this case. Observe that when $\psi > \frac{\pi}{6}$, $\overline{EA}$ is on the left side of the y_h-axis. Also, note that we are considering now the diagonal $\overline{EG}$, because the y_h-axis does not intersect the diagonal $\overline{HF}$

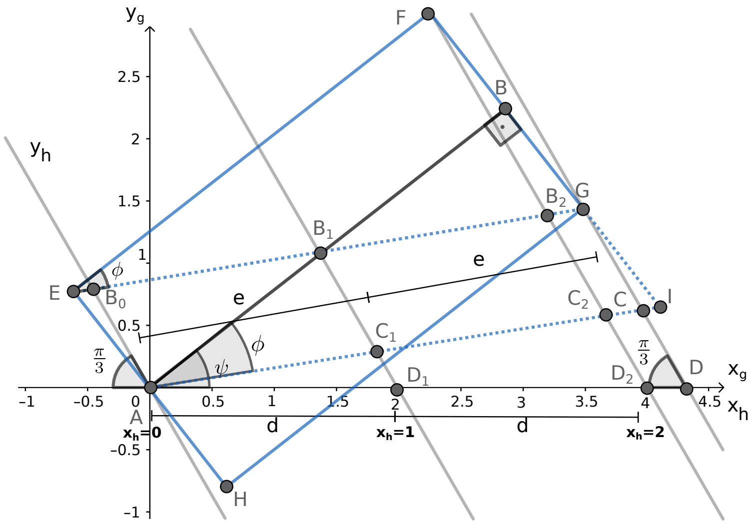


Fig. A9: The goal is to count how many points named B_i lie in the diagonal $\overline{EG}$. The triangles AD_iC_i for any $i \in \{1, 2\}$ and ADC are similar. $\overline{AD_i}$ and $\overline{AC_i}$ have distance $i \cdot d$ and $i \cdot e$, respectively. In this example, $d = 2$ and there are three points lying over $\overline{EG}$.

for these values of ψ. Then, we have to consider $\overline{B_0G}$ to count n_l^+. Additionally, $|B_0G| = |AC|$, due to the AB_0GC parallelogram properties. As in the previous case, for $i \in \{1, \dots, n_l^+ - 1\}, \triangle AD_iC_i \sim \triangle ADC$, $\widehat{CAD} = \widehat{BAD} - \widehat{BAC} = \psi - \phi$, $\widehat{ADC} = \pi/3$, $\widehat{ACD} = \pi - \widehat{CAD} - \widehat{ADC} = 2\pi/3 - \psi + \phi$, and $\frac{|\overline{B_0G}|}{e} = \frac{|\overline{AC}|}{e} = \frac{|\overline{AD}|}{d}$, by the similarity of these triangles as we showed in the previous case. Also, we have $\widehat{EAB_0} = \widehat{DAE} - \widehat{DAB_0} = \psi + \pi/2 - 2\pi/3 = \psi - \pi/6$, $\widehat{B_0EA} = \widehat{FEA} - \widehat{FEB_0} = \pi/2 - \phi$, $\widehat{EB_0A} = \pi - \widehat{B_0EA} - \widehat{EAB_0} = \pi - (\pi/2 - \phi) - (\psi - \pi/6) = 2\pi/3 + \phi - \psi$. Thus, by the law of sines, $\frac{|\overline{B_0E}|}{\sin(\widehat{EAB_0})} = \frac{|\overline{EA}|}{\sin(\widehat{EB_0A})} \Leftrightarrow |\overline{B_0E}| = \frac{s \sin(\widehat{EAB_0})}{\sin(\widehat{EB_0A})} = \frac{s \sin(\psi - \pi/6)}{\sin(2\pi/3 + \phi - \psi)}$. $EAIG$ and B_0ACG are parallelograms sharing the points G and A, so $|\overline{B_0E}| = |\overline{CI}|$. By following similar steps as before, we get

$$n_l^+ = \left\lfloor \frac{2\cos(\pi/6 - \psi)(vT - s) + 2s\sin(\psi - \pi/6)}{\sqrt{3}d} + 1 \right\rfloor.$$

We used this time $\sin(2\pi/3 - \psi) = \cos(\psi - \pi/6)$ and $\cos(2\pi/3 - \psi) = \sin(\psi - \pi/6)$.

For the final result on (A12), we simplified using the fact that when $\psi \leq \pi/6$, $\sin(|\psi - \pi/6|) = \sin(\pi/6 - \psi)$, otherwise, $\sin(|\psi - \pi/6|) = \sin(\psi - \pi/6)$.

For n_l^-, we also calculate how many lines parallel to the y_h-axis projected over the x-axis are distant from each other by d on this axis and are inside the rectangle.

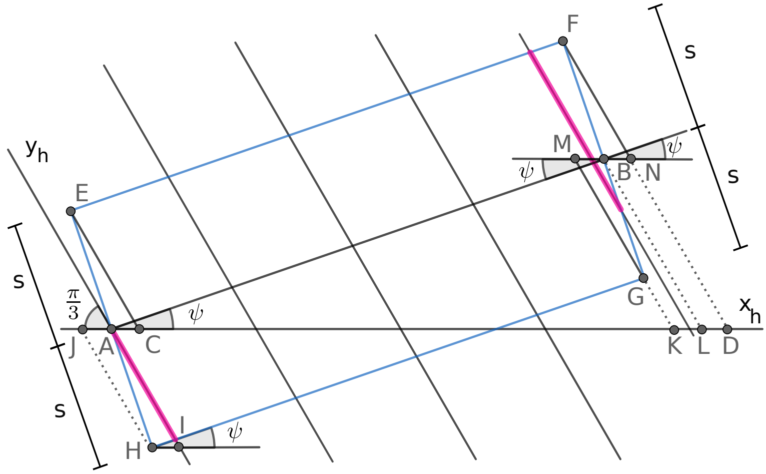


Fig. A10: In this example $\psi \leq \pi/6$. The pink line on the left side is an example of one satisfying Lemma A3, while the one on the right side, Lemma A4. The triangles ACE, HIA, BMG and BNF are congruent, because their respective angles are equal – due to parallelism – and $|\overline{EA}| = |\overline{AH}| = |\overline{GB}| = |\overline{FB}| = s$. In this example, except for $\overleftrightarrow{JH}$, $\overleftrightarrow{EC}$, $\overleftrightarrow{MG}$, $\overleftrightarrow{BL}$ and $\overleftrightarrow{FD}$, the lines parallel-to-y_h are distant by d on the projection over the x-axis and can have robots on them.

However, we consider only those on the left side of the point A, i.e., commencing from the one whose intersection with the x-axis is at $(-d, 0)$, equivalently, $(-1, 0)$ on the (x_h, y_h) coordinate system. We also have two cases here.

- Case $\psi \leq \pi/6$: Figure A10 shows the $\triangle HIA$ on the left side of the rectangle EFGH. As the robots are over the parallel-to-y_h lines distant by d on the projection over the x-axis, we want to know how many parallel lines intersect $\overline{HI}$ (equivalently, how many such lines intersect $\overline{JA}$ due to parallelism), excluding $\overleftrightarrow{AI}$ (because it was already counted on n_l^+). Thus,

$$n_l^- = \left\lfloor \frac{|HI|}{d} \right\rfloor .$$

  We have $|\overline{AH}| = s$, $\widehat{H} = \pi/2 + \psi$, $\widehat{I} = \pi/3$ and $\widehat{A} = \pi - \widehat{I} - \widehat{H} = \pi - \pi/3 - (\pi/2 + \psi) = \pi/6 - \psi$. By the law of sines on the angles opposite to the sides $\overline{AH}$ and $\overline{HI}$, results the following

$$|HI| = \frac{|AH| \sin(\widehat{A})}{\sin(\widehat{I})} = \frac{s \sin\left(\frac{\pi}{6} - \psi\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{2s \sin\left(\frac{\pi}{6} - \psi\right)}{\sqrt{3}}. \tag{A17}$$

  Thus,

$$n_l^- = \left\lfloor \frac{2s \sin\left(\frac{\pi}{6} - \psi\right)}{\sqrt{3}d} \right\rfloor .$$

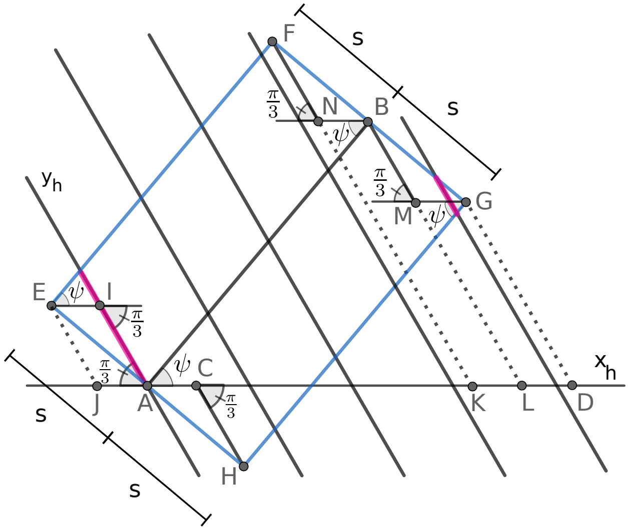


Fig. A11: In this example $\psi > \pi/6$, the side EH has an angle greater than zero with the y_h-axis. The pink line on the left side is an example of one satisfying Lemma A3, while the one on the right side, Lemma A4. The triangles AIE, HCA, FNB and BMG are congruent, because their respective angles are equal – due to parallelism – and $|\overline{EA}| = |\overline{AH}| = |\overline{GB}| = |\overline{FB}| = s$. Except for $\overleftrightarrow{EJ}, \overleftrightarrow{CH}, \overleftrightarrow{FK}, \overleftrightarrow{BL}$ and $\overleftrightarrow{GD}$, the lines parallel-to-y_h are distant by d on the projection over the x-axis and can have robots on them.

- Case $\psi > \pi/6$. Figure A11 illustrates this case. The reasoning is similar to the previous case, but now we use $\triangle EIA$. Then, $|\overline{EA}| = s, \widehat{E} = \pi/2 - \psi$, $\widehat{I} = 2\pi/3$ and $\widehat{A} = \pi - \widehat{I} - \widehat{E} = \pi - 2\pi/3 - (\pi/2 - \psi) = \psi - \pi/6$. Consequently,

$$n_l^- = \left\lfloor \frac{|\overline{EI}|}{d} \right\rfloor = \left\lfloor \frac{2s \sin\left(\psi - \frac{\pi}{6}\right)}{\sqrt{3}d} \right\rfloor.$$

For the final result in (A13), we use the absolute value inside the sine function to combine both cases. □

By the previous lemma, we have calculated the interval of integer x_h values needed for counting the robots inside the rectangle. In the next lemma, we get the equation for the number of robots at the rectangular part (N_R) ranging from these integer x_h values. Although the proposition we are now proving gives the throughput in terms of θ, we are first going to calculate this number in terms of ψ.

Lemma A2. *For $\psi \in [0, \pi/3)$,*

$$N_R(T,\psi) = \sum_{x_h=-n_l^-}^{n_l^+-1} \left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right).$$

If for some x_h $\lfloor Y_2^R(x_h)\rfloor < \lceil Y_1^R(x_h)\rceil$, we assume the respective summand for this x_h being zero.

Proof By the previous lemma and knowing that the positions of the robots are integer coordinates over the hexagonal grid coordinate space,

$$N_R(T,\psi) = \sum_{x_h=-n_l^-}^{n_l^+-1} \sum_{y_h=\lceil Y_1^R(x_h)\rceil}^{\lfloor Y_2^R(x_h)\rfloor} 1 = \sum_{x_h=-n_l^-}^{n_l^+-1} \left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right).$$

since (A10) and (A11) give the minimum (Y_1^R) and maximum (Y_2^R) y_h coordinates for a given x_h value such that the robot is inside the rectangle. Note that the last summation can only be used when $\lfloor Y_2^R(x_h)\rfloor \geq \lceil Y_1^R(x_h)\rceil$, otherwise a negative number of robots would be accounted. □

In special, for $\psi = \pi/6$, by (A10) and (A11),

$$N(T,\pi/6) = \sum_{x_h=0}^{\left\lfloor \frac{2(vT-s)}{\sqrt{3}d}\right\rfloor} \left(\left\lfloor \frac{\sqrt{3}y_2 + dx_h}{2d} \right\rfloor - \left\lceil \frac{\sqrt{3}y_1 + dx_h}{2d} \right\rceil + 1 \right). \tag{A18}$$

If $\psi \neq \pi/6$, each parallel-to-y_h-axis line intersects two segments of the rectangle EFGH. The y_h-components of the two intersections of a rectangle side and such lines are the values of $Y_1^R(x_h)$ and $Y_2^R(x_h)$ for a given x_h. Hence, the set of x_h integer values $\{-n_l^-, \dots, n_l^+ - 1\}$ will be cut in disjoint subsets based on the max and min outcomes of (A10) and (A11). That is, $Y_1^R(x_h)$ and $Y_2^R(x_h)$, respectively; equivalently, which two sides of the rectangle the parallel-to-y_h-axis line corresponding to $(x_h, 0)$ intersects. The following lemmas describe each subset: $\{-n_l^-, \dots, n_l^-\}$ in Lemma A3; $\{n_l^- + 1, \dots, K' - 1\}$ in Lemma A6; $\{K', \dots, n_l^+ - 1\}$ in Lemma A4, for an integer K' defined later.

Lemma A3. *Consider parallel-to-y_h-axis lines inside the rectangle EFGH intersecting the x_h-axis at $(x_h, 0)$, for $x_h \in \mathbb{Z}$. The two following statements are equivalent:*

(I) If $\psi < \pi/6$,

$$Y_1^R(x_h) = \frac{\frac{2y_1}{d} + 2\tan(\psi)x_h}{\sqrt{3} + \tan(\psi)} \text{ and } Y_2^R(x_h) = \frac{-2x_h}{\sqrt{3}\tan(\psi) - 1}, \tag{A19}$$

and, if $\psi > \pi/6$,

$$Y_1^R(x_h) = \frac{-2x_h}{\sqrt{3}\tan(\psi) - 1} \text{ and } Y_2^R(x_h) = \frac{\frac{2y_2}{d} + 2\tan(\psi)x_h}{\sqrt{3} + \tan(\psi)}. \tag{A20}$$

(II) $x_h \in \{-n_l^-, \dots, n_l^-\}$.

Proof $(I) \Rightarrow (II)$: Let $\psi < \pi/6$. By (A10) and (A11), (A19) is equivalent to

$$\frac{\sin(\psi)x_h - \frac{s}{d}}{\cos\left(\frac{\pi}{6} - \psi\right)} \geq \frac{\frac{vT-s}{d} - \cos(\psi)x_h}{\sin\left(\psi - \frac{\pi}{6}\right)} \text{ and } \frac{\sin(\psi)x_h + \frac{s}{d}}{\cos\left(\frac{\pi}{6} - \psi\right)} \geq \frac{-\cos(\psi)x_h}{\sin\left(\psi - \frac{\pi}{6}\right)}.$$

From the second inequality, we have

$$\begin{aligned}
&\frac{\sin(\psi)x_h + \frac{s}{d}}{\cos\left(\frac{\pi}{6} - \psi\right)} \geq \frac{-\cos(\psi)x_h}{\sin\left(\psi - \frac{\pi}{6}\right)} \\
&\Leftrightarrow \left(\frac{\sin(\psi)}{\cos\left(\frac{\pi}{6} - \psi\right)} + \frac{\cos(\psi)}{\sin\left(\psi - \frac{\pi}{6}\right)}\right) x_h \geq -\frac{s}{d\cos\left(\frac{\pi}{6} - \psi\right)} \\
&\Leftrightarrow \left(\frac{\sin(\psi)\sin\left(\psi - \frac{\pi}{6}\right) + \cos(\psi)\cos\left(\frac{\pi}{6} - \psi\right)}{\cos\left(\frac{\pi}{6} - \psi\right)\sin\left(\psi - \frac{\pi}{6}\right)}\right) x_h \geq -\frac{s\sin\left(\psi - \frac{\pi}{6}\right)}{d\cos\left(\frac{\pi}{6} - \psi\right)\sin\left(\psi - \frac{\pi}{6}\right)} \\
&\qquad \Leftrightarrow \left(\frac{\cos(\frac{\pi}{6})}{\cos\left(\frac{\pi}{6} - \psi\right)\sin\left(\psi - \frac{\pi}{6}\right)}\right) x_h \geq -\frac{s\sin\left(\psi - \frac{\pi}{6}\right)}{d\cos\left(\frac{\pi}{6} - \psi\right)\sin\left(\psi - \frac{\pi}{6}\right)} \\
&\qquad \Leftrightarrow \left(\frac{\frac{\sqrt{3}}{2}}{\cos\left(\frac{\pi}{6} - \psi\right)\sin\left(\psi - \frac{\pi}{6}\right)}\right) x_h \geq -\frac{s\sin\left(\psi - \frac{\pi}{6}\right)}{d\cos\left(\frac{\pi}{6} - \psi\right)\sin\left(\psi - \frac{\pi}{6}\right)} \\
&\qquad \Leftrightarrow x_h \leq \frac{-2s\sin\left(\psi - \frac{\pi}{6}\right)}{\sqrt{3}d} = \frac{2s\sin\left(\frac{\pi}{6} - \psi\right)}{\sqrt{3}d}.
\end{aligned}$$

The change of inequality sign above is due to $\cos\left(\frac{\pi}{6} - \psi\right)\sin\left(\psi - \frac{\pi}{6}\right) < 0$ for $\psi < \pi/6$. As $x_h \in \mathbb{Z}$,

$$x_h \leq \left\lfloor \frac{2s\sin\left(\frac{\pi}{6} - \psi\right)}{\sqrt{3}d} \right\rfloor = n_l^-.$$

The lower value on x_h is obtained by Lemma A1, as to be inside the rectangle EFGH $x_h \geq -n_l^-$. For $\psi > \pi/6$, we obtain the same result by a similar reasoning, but without changing the inequality sign since in this case $\cos\left(\frac{\pi}{6} - \psi\right)\sin\left(\psi - \frac{\pi}{6}\right) > 0$.

$(II) \Rightarrow (I)$: From (A6), (A7) (i.e., the line equations for $\overleftrightarrow{HG}$, $\overleftrightarrow{EH}$ and $\overleftrightarrow{EF}$), (A10) and (A11) (i.e., the definitions of Y_1^R and Y_2^R), we have, if $\psi < \pi/6$,

$$(x_h, Y_1^R(x_h)) \in \overleftrightarrow{HG} \Leftrightarrow Y_1^R(x_h) = \frac{\frac{2y_1}{d} + 2\tan(\psi)x_h}{\sqrt{3} + \tan(\psi)},$$

$$(x_h, Y_2^R(x_h)) \in \overleftrightarrow{EH} \Leftrightarrow Y_2^R(x_h) = \frac{-2x_h}{\sqrt{3}\tan(\psi) - 1},$$

and, if $\psi > \pi/6$,

$$(x_h, Y_1^R(x_h)) \in \overleftrightarrow{EH} \Leftrightarrow Y_1^R(x_h) = \frac{-2x_h}{\sqrt{3}\tan(\psi) - 1},$$

$$(x_h, Y_2^R(x_h)) \in \overleftrightarrow{EF} \Leftrightarrow Y_2^R(x_h) = \frac{\frac{2y_2}{d} + 2\tan(\psi)x_h}{\sqrt{3} + \tan(\psi)}.$$

Then, we prove this part by showing that for all $x_h \in \{-n_l^-, \ldots, n_l^-\}$, the line parallel to the y_h-axis intercepting the point $(x_h, 0)$ intercepts both sides $\overline{EH}$ and $\overline{HG}$ (and no other), if $\psi < \pi/6$ (Figure A10), and, if $\psi > \pi/6$, both sides $\overline{EH}$ and $\overline{EF}$ (and no other) (Figure A11).

- Case $\psi < \pi/6$: Figure A10 shows the triangles HIA, ACE and BMG inside the rectangle EFGH. As the robots are over the parallel lines to the y_h-axis, which are distant by d when projected over the x-axis, we want to know how many such parallel lines intersect $\overline{HI}$ (equivalently, how many such lines intersect $\overline{JA}$ due to parallelism) or $\overline{AC}$. For such parallel lines that intersect $\overline{HI}$, Lemma A1 showed that for every $x_h \in \{-n_l^-, \ldots, -1\}$ the line parallel to y_h-axis intersecting $(x_h, 0)$ is inside the rectangle. Also, these lines intersect the sides $\overline{EH}$ and $\overline{HG}$, as any line parallel to $\overleftrightarrow{AI}$ which is on its left side intersects the sides $\overline{EH}$ and $\overline{HG}$ if it is inside the rectangle. For the case where such parallel lines intersect $\overline{AC}$, we need to know the maximum integer value, M, such that these parallel lines still intersect the sides $\overline{EH}$ and $\overline{HG}$ for any $x_h \in \{0, \ldots, M\}$. Starting from point A (that is, when $x_h = 0$), we have
$$M = \left\lfloor \frac{|\overline{AC}|}{d} \right\rfloor.$$
We have $|\overline{AH}| = |\overline{EA}| = s$, $\overleftrightarrow{AI} \parallel \overleftrightarrow{EC}$, $\overleftrightarrow{HI} \parallel \overleftrightarrow{AC}$, and $\overleftrightarrow{AH} \parallel \overleftrightarrow{AE}$ (as E, A and H are collinear), then $\widehat{IHA} = \widehat{CAE}$, $\widehat{AIH} = \widehat{ECA}$, and $\widehat{HAI} = \widehat{AEC}$. Thus, $\triangle HIA \cong \triangle ACE$, then $|\overline{AC}| = |\overline{HI}|$, whose value has been previously calculated in Lemma A1, leading to
$$M = \left\lfloor \frac{2s\sin\left(\frac{\pi}{6} - \psi\right)}{\sqrt{3}d} \right\rfloor = n_l^-.$$
Hence, for any $x_h \in \{0, \ldots, n_l^-\}$, those parallel lines intersect the sides $\overline{EH}$ and $\overline{HG}$.
- Case $\psi > \pi/6$: Figure A11 illustrates this case. The reasoning is similar to the previous case, but using that $\triangle AIE \cong \triangle HCA$. As the value for $|\overline{EI}|/d$ also has been calculated in Lemma A1 for this figure, then
$$M = \left\lfloor \frac{2s\sin\left(\psi - \frac{\pi}{6}\right)}{\sqrt{3}d} \right\rfloor = n_l^-.$$
Consequently, for any $x_h \in \{-n_l^-, \ldots, n_l^-\}$, the parallel-to-$y_h$-axis line at $(x_h, 0)$ intersects the sides $\overline{EH}$ and $\overline{EF}$ in this case. □

The next lemma will define the integer K' mentioned before. This number will be compared with the integer x_h coordinate of the point $(n_l^+ - 1, 0)$ intersected by the rightmost parallel-to-y_h-axis line inside the rectangle EFGH. Assuming $\theta \neq \pi/6$, if this rightmost line intersects a point on the x_h-axis with an integer coordinate less than K', then no parallel-to-y_h-axis line intersects the rectangle right side $\overline{FG}$. However, if the intersection point coordinate is greater than or equal to K', then at least one parallel line crosses $\overline{FG}$.

Lemma A4. *Consider parallel-to-y_h-axis lines inside the rectangle EFGH intersecting the x_h-axis at $(x_h, 0)$, for $x_h \in \mathbb{Z}$, and $K' = \left\lceil \frac{2(vT-s)\cos(\psi-\pi/6) - 2s\sin(|\psi-\pi/6|)}{\sqrt{3}d} \right\rceil$. Then, the two statements below are equivalent:*

(I) If $\psi < \pi/6$

$$Y_1^R(x_h) = \frac{\frac{2(vT-s)}{d\cos(\psi)} - 2x_h}{\sqrt{3}\tan(\psi) - 1} \text{ and } Y_2^R(x_h) = \frac{\frac{2y_2}{d} + 2\tan(\psi)x_h}{\sqrt{3} + \tan(\psi)}, \tag{A21}$$

and, if $\psi > \pi/6$

$$Y_1^R(x_h) = \frac{\frac{2y_1}{d} + 2\tan(\psi)x_h}{\sqrt{3} + \tan(\psi)} \text{ and } Y_2^R(x_h) = \frac{\frac{2(vT-s)}{d\cos(\psi)} - 2x_h}{\sqrt{3}\tan(\psi) - 1}. \tag{A22}$$

(II) $x_h \in \{K', \ldots, n_l^+ - 1\}$.

Proof $(I) \Rightarrow (II)$: By contrapositive, assume $x_h \notin \{K', \ldots, n_l^+ - 1\}$. By Lemma A1, there is no $x_h > n_l^+ - 1$, so $x_h < K'$. For the case of $\psi < \pi/6$, observe in Figure A10 the point K on the x_h-axis. This point corresponds to the intersection of $\overleftrightarrow{MG}$ on the x_h-axis, which is the first parallel-to-y_h-axis crossing the rectangle right side $\overline{FG}$. The point D on the x_h-axis is the projection of the point F on this axis. By (A16), $|\overline{AD}| = \frac{2\cos(\pi/6-\psi)(vT-s)+2s\sin(|\psi-\pi/6|)}{\sqrt{3}}$. Because of the parallelism, we have $|\overline{MN}| = |\overline{KD}|$. Due to the congruence of triangles ACE, HIA, BMG and BNF and (A17), $|\overline{BM}| = |\overline{BN}| = |\overline{HI}| = \frac{2s\sin(|\psi-\pi/6|)}{\sqrt{3}}$. Thus, $|\overline{KD}| = |\overline{MN}| = |\overline{BM}| + |\overline{BN}| = \frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}}$. Since $|\overline{AK}| = |\overline{AD}| - |\overline{KD}| = \frac{2\cos(\pi/6-\psi)(vT-s)-2s\sin(|\psi-\pi/6|)}{\sqrt{3}}$, the point K is located on the (x_h, y_h) coordinate space at $\left(\frac{2\cos(\pi/6-\psi)(vT-s)}{\sqrt{3}d} - \frac{2s\sin(|\psi-\pi/6|)}{\sqrt{3}d}, 0\right)$, as K is on the x-axis and to convert it to (x_h, y_h) coordinate space we only need to divide the x-coordinate by d. On the x_h-axis, the nearest point on the right of K with integer x_h is $(\lceil K \rceil, 0) = (K', 0)$. As we assumed $x_h < K'$, no parallel-to-y_h-axis crossing a integer $(x_h, 0)$ point inside the rectangle intersects $\overline{FG}$. Thus, no such parallel line has $Y_1^R(x_h) = \frac{\frac{2(vT-s)}{d\cos(\psi)} - 2x_h}{\sqrt{3}\tan(\psi)-1}$, which is the y_h-coordinate of the intersection of this line with $\overleftrightarrow{FG}$.

In the case of $\psi > \pi/6$, using a similar argument in the Figure A11 concludes the desired result, but here we use $|\overline{NB}| + |\overline{MG}| = |\overline{KD}|$ and the congruence of triangles AIE, HCA, FNB and BMG. As we assumed $x_h < K'$, no parallel-to-y_h-axis intersecting a integer point $(x_h, 0)$ inside the rectangle crosses $\overline{FG}$, so for such line $Y_2^R(x_h) \neq \frac{\frac{2(vT-s)}{d\cos(\psi)} - 2x_h}{\sqrt{3}\tan(\psi)-1}$.

$(II) \Rightarrow (I)$: If $x_h \in \{K', \ldots, n_l^+ - 1\}$ then the lines parallel-to-y_h-axis inside the rectangle intersecting the x_h-axis at $(x_h, 0)$ are on the right of point K or intersecting it. Hence, these lines intersect $\overline{EF}$ and $\overline{FG}$, if $\psi < \pi/6$. By applying (A6), (A7) (for the line equations for $\overleftrightarrow{EF}$ and $\overleftrightarrow{FG}$), (A10) and (A11) (for the definitions of Y_1^R and Y_2^R), we have (A21). A similar argument is used in the case of $\psi > \pi/6$, but for $\overline{FG}$ and $\overline{HG}$ intersections, yielding (A22). □

The lemma below characterises when a parallel-to-y_h-axis line touches only the sides EH and FG of the rectangle. Intuitively, if this happens we have a rectangle with a small width. Thus, on rectangles with a large width, no such lines are crossing the sides EH and FG, for $\psi \neq \pi/6$. We will use this lemma on the Lemma A6, for completing the disjoint subsets based on the possible max and min outcomes of Y_1^R and Y_2^R.

Lemma A5. *If $vT - s > 2s\tan(|\psi - \frac{\pi}{6}|)$, then there is not a $x_h \in \{-n_l^-, \ldots, n_l^+ - 1\}$ such that,*

$$Y_1^R(x_h) = \frac{\frac{2(vT-s)}{d\cos(\psi)} - 2x_h}{\sqrt{3}\tan(\psi) - 1} \text{ and } Y_2^R(x_h) = \frac{-2x_h}{\sqrt{3}\tan(\psi) - 1}, \text{ if } \psi < \pi/6;$$

$$Y_1^R(x_h) = \frac{-2x_h}{\sqrt{3}\tan(\psi) - 1} \text{ and } Y_2^R(x_h) = \frac{\frac{2(vT-s)}{d\cos(\psi)} - 2x_h}{\sqrt{3}\tan(\psi) - 1}, \text{ if } \psi > \pi/6.$$

Proof This proof is by contrapositive. Assume $\psi < \pi/6$. By (A10) and (A11), we have a x_h such that

$$\frac{\frac{y_1}{d} + \tan(\psi)x_h}{\frac{\sqrt{3}+\tan(\psi)}{2}} \leq \frac{\frac{vT-s}{d\cos(\psi)} - x_h}{\frac{\sqrt{3}\tan(\psi)-1}{2}} \text{ and } \frac{\frac{y_2}{d} + \tan(\psi)x_h}{\frac{\sqrt{3}+\tan(\psi)}{2}} \geq \frac{-x_h}{\frac{\sqrt{3}\tan(\psi)-1}{2}}.$$

Since $\frac{\sqrt{3}\tan(\psi)-1}{2} < 0$, the signs of inequalities change, then we have the following implication

$$\frac{\frac{y_1}{d}\frac{\sqrt{3}\tan(\psi)-1}{2}}{\frac{\sqrt{3}+\tan(\psi)}{2}} - \frac{vT-s}{d\cos(\psi)} \geq -x_h - \frac{\tan(\psi)x_h\frac{\sqrt{3}\tan(\psi)-1}{2}}{\frac{\sqrt{3}+\tan(\psi)}{2}} \text{ and}$$

$$\frac{\frac{y_2}{d}\frac{\sqrt{3}\tan(\psi)-1}{2}}{\frac{\sqrt{3}+\tan(\psi)}{2}} \leq -x_h - \frac{\tan(\psi)x_h\frac{\sqrt{3}\tan(\psi)-1}{2}}{\frac{\sqrt{3}+\tan(\psi)}{2}}$$

$$\Rightarrow \frac{\frac{y_2}{d}\frac{\sqrt{3}\tan(\psi)-1}{2}}{\frac{\sqrt{3}+\tan(\psi)}{2}} \leq \frac{\frac{y_1}{d}\frac{\sqrt{3}\tan(\psi)-1}{2}}{\frac{\sqrt{3}+\tan(\psi)}{2}} - \frac{vT-s}{d\cos(\psi)},$$

by the transitivity of $\leq$ under the real numbers. Also, we have the following equivalences

$$\begin{aligned}
&\frac{\frac{y_2}{d}\frac{\sqrt{3}\tan(\psi)-1}{2}}{\frac{\sqrt{3}+\tan(\psi)}{2}} \leq \frac{\frac{y_1}{d}\frac{\sqrt{3}\tan(\psi)-1}{2}}{\frac{\sqrt{3}+\tan(\psi)}{2}} - \frac{vT-s}{d\cos(\psi)}\\
\Leftrightarrow\ &\frac{vT-s}{d\cos(\psi)} \leq \frac{y_1-y_2}{d}\frac{\sqrt{3}\tan(\psi)-1}{\sqrt{3}+\tan(\psi)}\\
\Leftrightarrow\ &\frac{vT-s}{d\cos(\psi)} \leq -\frac{2s}{d\cos(\psi)}\frac{\sqrt{3}\tan(\psi)-1}{\sqrt{3}+\tan(\psi)} && \text{[By (A4)]}\\
\Leftrightarrow\ &\frac{vT-s}{2s} \leq \frac{1-\sqrt{3}\tan(\psi)}{\sqrt{3}+\tan(\psi)} \Leftrightarrow \frac{vT-s}{2s} \leq \frac{1}{\tan(\pi/3+\psi)}\\
\Leftrightarrow\ &\frac{vT-s}{2s} \leq \cot(\pi/3+\psi) \Leftrightarrow \frac{vT-s}{2s} \leq -\tan(\psi+5\pi/6)\\
\Leftrightarrow\ &vT-s \leq 2s\tan(\pi/6-\psi).
\end{aligned}$$

Above we used the equalities $\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$, $\cot(a) = -\tan(a+\pi/2)$ and $-\tan(\pi-a) = \tan(a)$ for any real a and b.

For the case $\psi > \pi/6$, using similar arguments we get the same result, but we do not change the signs of inequalities due to $\frac{\sqrt{3}\tan(\psi)-1}{2} > 0$ in this case. The conclusion is reached after we combine the two cases using absolute values inside the tangent. $\square$

The next lemma completes the properties of $N_R(T,\psi)$ that are useful for calculating its limit when T tends to infinity.

Lemma A6. *Let* $K' = \left\lceil \frac{2(vT-s)\cos(\psi-\pi/6)-2s\sin(|\psi-\pi/6|)}{\sqrt{3}d} \right\rceil$. *If* $vT-s > 2s\tan(|\psi-\pi/6|)$, *then* $x_h \in \{n_l^- + 1, \ldots, K'-1\}$ *if and only if*

$$Y_1^R(x_h) = \frac{\frac{y_1}{d} + \tan(\psi)x_h}{\frac{\sqrt{3}+\tan(\psi)}{2}} \text{ and } Y_2^R(x_h) = \frac{\frac{y_2}{d} + \tan(\psi)x_h}{\frac{\sqrt{3}+\tan(\psi)}{2}}.$$

Proof Excluding the case when $\psi = \pi/6$, (A10) and (A11) give four combinations of possible outcomes for the values of $Y_1^R(x_h)$ and $Y_2^R(x_h)$ based on the results of min and max. When $vT-s > 2s\tan(|\psi-\pi/6|)$, by Lemma A5, we do not have the case when they are on the sides EH and FG. For the given values of x_h on the hypothesis, neither Lemma A3 nor Lemma A4 applies, excluding other two combinations of results for $Y_1^R(x_h)$ and $Y_2^R(x_h)$. Finally, Lemma A1 shows that every parallel-to-y_h-axis line crosses the x_h-axis at $(x_h, 0)$ for $x_h \in \{-n_l^-, \ldots, n_l^+ - 1\}$, so the remaining combination yields the desired equivalence. $\square$

Now we present the calculation of $N_S(T,\theta)$. Here we are using θ instead of $\psi = \pi/3 - \theta$ for easiness of presentation. We denote (l_x, l_y) the position of the last robot inside a rectangle of width $vT-s$ and height $2s$ whose left side is at (x_0, y_0). Here *last* means the robot with highest x coordinate value.

However, if two robots have the same x coordinate value, we take the robot whose y coordinate is nearer to y_0. Let Z be the set of robot positions inside the rectangle above for $vT - s > 0$.

Lemma A7. *Let $c_x = x_0 + vT - s$, and*

$$(l_x, l_y) = \begin{cases} \operatorname{argmin}_{(x,y)\in Z} |vT - s + x_0 - x| + |y_0 - y| & \text{if } T > \frac{s}{v}, \\ (x_0, y_0) & \text{otherwise.} \end{cases}$$

Then,

$$N_S(T, \theta) = \sum_{x_h=B}^{U} \left(\lfloor Y_2^S(x_h) \rfloor - \lceil Y_1^S(x_h) \rceil + 1 \right),$$

for $\lfloor Y_2^S(x_h) \rfloor \geq \lceil Y_1^S(x_h) \rceil$ (if for some x_h $\lfloor Y_2^S(x_h) \rfloor < \lceil Y_1^S(x_h) \rceil$, we assume the respective summand for this x_h being zero),

$$B = \begin{cases} \left\lceil \frac{2(\sin(\pi/3 - \theta)(c_x - l_x) + \cos(\pi/3 - \theta)(y_0 - l_y - s))}{\sqrt{3}d} \right\rceil, & \text{if } T > \frac{s}{v}, \\ \left\lceil -\frac{2\sqrt{2svT - (vT)^2}}{\sqrt{3}d} \sin\left(\theta + \frac{\pi}{6}\right) \right\rceil, & \text{otherwise,} \end{cases} \tag{A23}$$

if $T > \frac{s}{v}$ or $\arctan\left(\frac{\frac{s}{2} - \sin(\theta)(vT - s)}{\frac{\sqrt{3}s}{2} + \cos(\theta)(vT - s)} \right) \leq \frac{\pi}{2} - \theta$,

$$U = \left\lfloor \frac{2(\sin(\pi/3 - \theta)(c_x - l_x) + \cos(\pi/3 - \theta)(y_0 - l_y) + s)}{\sqrt{3}d} \right\rfloor, \tag{A24}$$

otherwise,

$$U = \left\lfloor \frac{2\sqrt{2svT - (vT)^2}}{\sqrt{3}d} \cos\left(\theta - \frac{\pi}{3}\right) \right\rfloor. \tag{A25}$$

Also,

$$Y_1^S(x_h) = \frac{dx_h - C_{-\theta,x} + \sqrt{3}C_{-\theta,y} - \sqrt{\Delta(x_h)}}{2d},$$

$$Y_2^S(x_h) = \begin{cases} \min(L(x_h), C_2(x_h)) - 1, & \text{if } \min(L(x_h), C_2(x_h)) = \lfloor L(x_h) \rfloor \\ & \text{and } T > \frac{s}{v}, \\ \min(L(x_h), C_2(x_h)), & \text{otherwise,} \end{cases}$$

for

$$C_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} c_x - l_x \\ y_0 - l_y \end{bmatrix},$$

$$\Delta(x_h) = 4s^2 - \left(\sqrt{3}(dx_h - C_{-\theta,x}) - C_{-\theta,y}\right)^2,$$

$$C_2(x_h) = \frac{dx_h - C_{-\theta,x} + \sqrt{3}C_{-\theta,y} + \sqrt{\Delta(x_h)}}{2d},$$

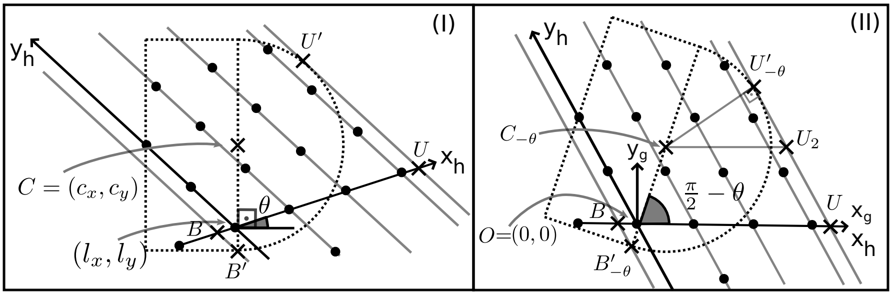


Fig. A12: In the space (I) the robots are in the standard coordinate system and the semicircle with centre at $C = (c_x, c_y)$ has the lowest point at $B^{'}$. $\overleftrightarrow{CB^{'}}$ has angle $\frac{\pi}{2}$ with the usual x-axis, however the x_h-axis here has angle θ with it. In (II) we rotate by $-\theta$ with (l_x, l_y) as centre of rotation. After this rotation, $B^{'}$, $U^{'}$ and C become $B^{'}_{-\theta}$, $U^{'}_{-\theta}$ and $C_{-\theta}$, respectively, and $\overleftrightarrow{C_{-\theta}B^{'}_{-\theta}}$ has angle $\frac{\pi}{2} - \theta$ in relation to the x_g-axis and x_h-axis, which are now coincident lines despite their scale being different. B and U are the minimum and maximum values of the x_h-axis coordinate for a line parallel to the y_h-axis on the hexagonal grid coordinate system.

and

$$L(x_h) = \begin{cases} \dfrac{\sin\left(\frac{\pi}{2} - \theta\right)(dx_h - C_{-\theta,x}) + \cos\left(\frac{\pi}{2} - \theta\right) C_{-\theta,y}}{d \sin\left(\frac{5\pi}{6} - \theta\right)}, & \text{if } T > \dfrac{s}{v}, \\ \dfrac{\sin\left(\frac{\pi}{2} - \theta\right) x_h}{\sin\left(\frac{5\pi}{6} - \theta\right)}, & \text{otherwise.} \end{cases}$$

Proof Assume $T > \frac{s}{v}$, as shown in Figure A4 (III). The robots are located in the usual Euclidean space. Instead of it, we use in this proof a similar coordinate system transformation for positioning the robots in a hexagonal grid as integer coordinates as we did in the rectangular part. As in the previous lemmas, we call this coordinate system space coordinates (x_h, y_h). However, here we are using a (x_h, y_h) coordinate system with a different origin and inclination.

In order to do so, we first redefine a (x_g, y_g) coordinate space, that is, we perform rotation by $-\theta$ on the usual Euclidean space about (l_x, l_y). The origin of the (x_g, y_g) coordinate system is at (l_x, l_y). The transformation for (x_g, y_g) coordinate system used here is similar to the depicted in the Figure A5, but here we are using $-\theta$ and (l_x, l_y) instead of $-\psi$ and (x_0, y_0), i.e.,

$$\begin{bmatrix} x_g \\ y_g \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x - l_x \\ y - l_y \end{bmatrix}.$$

As the coordinate space (x_g, y_g) is already translated to the point (l_x, l_y), the transformation from the new (x_h, y_h) to the new (x_g, y_g) is the same as in (A3). We repeat it below for convenience:

$$\begin{bmatrix} x_g \\ y_g \end{bmatrix} = \begin{bmatrix} d & -\frac{d}{2} \\ 0 & \frac{\sqrt{3}d}{2} \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix}. \tag{A26}$$

Despite these differences, we will keep using the notation (x_g, y_g) and (x_h, y_h) as we did before for a clean presentation.

Figure A12 shows how the semicircle with centre at $C = (c_x, c_y) = (x_0 + vT - s, y_0)$ will be after the rotation by $-\theta$ about (l_x, l_y), that is,

$$C_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} c_x - l_x \\ c_y - l_y \end{bmatrix}. \tag{A27}$$

Hereafter we will use the subscript $-\theta$ on every point presented on the usual Euclidean space to denote the corresponding point on the (x_g, y_g) coordinate space.

We compute first the upper and lower values, U and B, of x_h lying on the semicircle. For getting the U value on the x_h-axis, we draw a line parallel to the y_h-axis on the rightmost semicircle boundary at the point $U^{'}$ in order to reach the x_h-axis (Figure A12 (I)). The corresponding point on the (x_g, y_g) space is denoted by $U^{'}_{-\theta}$ (Figure A12 (II)). We compute $U^{'}_{-\theta}$, then we take its x_h-value on the hexagonal grid coordinate system. $\triangle U^{'}_{-\theta} C_{-\theta} U_2$ in Figure A12 (II) has $|U^{'}_{-\theta} C_{-\theta}| = s$ and $\widehat{U^{'}_{-\theta} C_{-\theta} U_2} = \pi - \widehat{C_{-\theta} U^{'}_{-\theta} U_2} - \widehat{U^{'}_{-\theta} U_2 C_{-\theta}} = \pi - \pi/2 - \pi/3 = \pi/6$. Hence,

$$\begin{aligned} U^{'}_{-\theta} &= C_{-\theta} + s(\cos(\pi/6), \sin(\pi/6)) \\ &= \Bigg(\cos(\theta)(c_x - l_x) + \sin(\theta)(c_y - l_y) + \frac{\sqrt{3}s}{2}, \\ &\qquad \cos(\theta)(c_y - l_y) - \sin(\theta)(c_x - l_x) + \frac{s}{2}\Bigg). \end{aligned}$$

The inverse transformation from (A26) is

$$\begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} \frac{1}{d} & \frac{1}{\sqrt{3}d} \\ 0 & \frac{2}{\sqrt{3}d} \end{bmatrix} \begin{bmatrix} x_g \\ y_g \end{bmatrix}. \tag{A28}$$

Applying the transformation of (A28) to the point $U^{'}_{-\theta}$ we get its x_h-axis coordinate

$$\begin{aligned} U = &\frac{1}{d}\left(\cos(\theta)(c_x - l_x) + \sin(\theta)(c_y - l_y) + \frac{\sqrt{3}s}{2}\right) + \\ &\frac{1}{\sqrt{3}d}\left(\cos(\theta)(c_y - l_y) - \sin(\theta)(c_x - l_x) + \frac{s}{2}\right) \\ = &\frac{2(\sin(\pi/3 - \theta)(c_x - l_x) + \cos(\pi/3 - \theta)(c_y - l_y) + s)}{\sqrt{3}d} \end{aligned} \tag{A29}$$

As we need the integer coordinate less or equal to this value, we apply the floor function to yield the desired result in (A24).

For getting the B value on the x_h-axis, we draw a line parallel to the y_h-axis on the lower semicircle corner at the point $B^{'}$ in order to reach the x_h-axis (Figure A12 (I)). We perform a calculation similar to the previous paragraph but using $B^{'}_{-\theta}$ (Figure A12 (II)). We have $\widehat{C_{-\theta} O U} = \pi/2 - \theta$ (as this is the same angle of $\overleftrightarrow{CB'}$ with x_h-axis in Figure A12 (I) which coincides with x_g-axis in the Figure A12 (II)).

Then, as the vector $C_{-\theta}B'_{-\theta}$ is pointed downwards, it has negative angle with the x_g-axis, that is, $-\widehat{B_{-\theta}OU} = -(\pi - \widehat{C_{-\theta}OU}) = -(\pi - (\pi/2 - \theta)) = -\pi/2 - \theta$ with x_g-axis. Also, $\left|\overline{C_{-\theta}B'_{-\theta}}\right| = s$. Consequently,

$$\begin{aligned}\overrightarrow{C_{-\theta}B'_{-\theta}} = B'_{-\theta} - C_{-\theta} = s(\cos(-\pi/2 - \theta), \sin(-\pi/2 - \theta)) \Leftrightarrow \\ B'_{-\theta} = C_{-\theta} + s(\cos(-\pi/2 - \theta), \sin(-\pi/2 - \theta)) \\ = (\cos(\theta)(c_x - l_x) + \sin(\theta)(c_y - l_y - s), \\ \cos(\theta)(c_y - l_y - s) - \sin(\theta)(c_x - l_x)).\end{aligned}$$

Using (A28) on $B'_{-\theta}$ we get,

$$\begin{aligned}B &= \frac{1}{d}\left(\cos(\theta)(c_x - l_x) + \sin(\theta)(c_y - l_y - s)\right) + \\ &\quad \frac{1}{\sqrt{3}d}\left(\cos(\theta)(c_y - l_y - s) - \sin(\theta)(c_x - l_x)\right) \\ &= \frac{2(\sin(\pi/3 - \theta)(c_x - l_x) + \cos(\pi/3 - \theta)(c_y - l_y - s))}{\sqrt{3}d}.\end{aligned}$$

Then, we apply the ceiling function on this value to get an integer coordinate greater or equal to it in order to obtain (A23) for $T > \frac{s}{v}$.

On the hexagonal grid coordinate system, for each x_h from B to U, we need to find the minimum and maximum y_h – namely $Y_1^S(x_h)$ and $Y_2^S(x_h)$, respectively – of a line parallel to y_h-axis intercepting the x_h-axis and lying on the semicircle. Depending on the angle of $\overleftrightarrow{C_{-\theta}B'_{-\theta}}$ with the x_h-axis, the minimum and maximum y_h can be either on the semicircle arc or $\overleftrightarrow{C_{-\theta}B'_{-\theta}}$. Due to $\theta \in [0, \pi/3)$, the angle of $\overleftrightarrow{C_{-\theta}B'_{-\theta}}$ is in $(\frac{\pi}{6}, \frac{\pi}{2}]$. Thus, the minimum y_h value is at the semicircle arc, otherwise the minimum angle of $\overleftrightarrow{C_{-\theta}B'_{-\theta}}$ would be $2\pi/3$, which is the y_h-axis angle with the x_h-axis. However, the maximum y_h value could be either on $\overleftrightarrow{C_{-\theta}B'_{-\theta}}$ or on the circle, thus we take the lowest, since we want the y_h value on the boundary of the semicircle.

Let $C_1(x_h)$ and $C_2(x_h)$ be functions that respectively return the lowest and the highest y_h value at the circle centred at $C_{-\theta}$ and radius s for a x_h coordinate value of a parallel-to-y_h-axis line assuming it intersects the circle. Then, a point (x_g, y_g) on the Euclidean space is on that circle if

$$\begin{aligned}(x_g - C_{-\theta,x})^2 + (y_g - C_{-\theta,y})^2 = s^2 \Leftrightarrow \\ \left(dx_h - \frac{dy_h}{2} - C_{-\theta,x}\right)^2 + \left(\frac{\sqrt{3}dy_h}{2} - C_{-\theta,y}\right)^2 = s^2,\end{aligned}$$

by (A26).

Isolating y_h and solving the two degree polynomial we get

$$y_{h_1} = C_1(x_h) = \frac{dx_h - C_{-\theta,x} + \sqrt{3}C_{-\theta,y} - \sqrt{\Delta(x_h)}}{2d} \text{ and} \tag{A30}$$

$$y_{h_2} = C_2(x_h) = \frac{dx_h - C_{-\theta,x} + \sqrt{3}C_{-\theta,y} + \sqrt{\Delta(x_h)}}{2d}, \tag{A31}$$

for $0 \leq \Delta(x_h) = 4s^2 - \left(\sqrt{3}(dx_h - C_{-\theta,x}) - C_{-\theta,y}\right)^2$. $\Delta(x_h)$ cannot be negative, otherwise the lines would not intersect this circle, contradicting our assumption.

We denote $L(x_h)$ a function that returns the y_h component of the line $\overleftrightarrow{C_{-\theta}B_{-\theta}}$ for a given x_h. The $\overleftrightarrow{C_{-\theta}B_{-\theta}}$ equation for a point in the space (x_g, y_g) is

$$\tan\left(\frac{\pi}{2}-\theta\right) = \frac{y_g - C_{-\theta,y}}{x_g - C_{-\theta,x}} \Leftrightarrow \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{y_g - C_{-\theta,y}}{x_g - C_{-\theta,x}} \Leftrightarrow$$

$$\sin\left(\frac{\pi}{2}-\theta\right)\left(dx_h - \frac{dy_h}{2} - C_{-\theta,x}\right) = \cos\left(\frac{\pi}{2}-\theta\right)\left(\frac{\sqrt{3}dy_h}{2} - C_{-\theta,y}\right) \Leftrightarrow$$

$$y_h = \frac{\sin\left(\frac{\pi}{2}-\theta\right)(dx_h - C_{-\theta,x}) + \cos\left(\frac{\pi}{2}-\theta\right)C_{-\theta,y}}{d\left(\frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{2}-\theta\right) + \sin\left(\frac{\pi}{2}-\theta\right)\frac{1}{2}\right)} \Leftrightarrow$$

$$L(x_h) = y_h = \frac{\sin\left(\frac{\pi}{2}-\theta\right)(dx_h - C_{-\theta,x}) + \cos\left(\frac{\pi}{2}-\theta\right)C_{-\theta,y}}{d\sin\left(\frac{5\pi}{6}-\theta\right)}.$$

We have that $Y_1^S(x_h) = C_1(x_h)$ and $Y_2^S(x_h)$ can be either $\min(L(x_h), C_2(x_h))$ or $\min(L(x_h), C_2(x_h)) - 1$. As $T > \frac{s}{v}$, we can have a number of robots inside the rectangle $N_R(T, \theta) \geq 1$. If, for some x_h, $Y'(x_h) = \min(L(x_h), C_2(x_h)) = \lfloor L(x_h) \rfloor$, then the robot on $\left(x_h, Y'(x_h)\right)$ is on the line $\overleftrightarrow{C_{-\theta}B'_{-\theta}}$. As this line belongs to the rectangle, the robot was already counted by $N_R(T, \theta)$. Hence,

$$Y_2^S(x_h) = \begin{cases} \min(L(x_h), C_2(x_h)) - 1, & \text{if } \min(L(x_h), C_2(x_h)) = \lfloor L(x_h) \rfloor \\ & \text{and } T > \dfrac{s}{v}, \\ \min(L(x_h), C_2(x_h)), & \text{otherwise.} \end{cases}$$

The number of robots inside the semicircle is the number of integer coordinates (x_h, y_h) for x_h ranging from B to U and $y_h \in \left[\left\lceil Y_1^S(x_h) \right\rceil, \left\lfloor Y_2^S(x_h) \right\rfloor\right]$ for each x_h. Thus,

$$N_S(T, \theta) = \sum_{x_h=B}^{U} \sum_{y_h=\lceil Y_1^S(x_h) \rceil}^{\lfloor Y_2^S(x_h) \rfloor} 1 = \sum_{x_h=B}^{U} \left(\lfloor Y_2^S(x_h) \rfloor - \lceil Y_1^S(x_h) \rceil + 1\right).$$

Heed that the last summation can only be used when $\left\lfloor Y_2^S(x_h) \right\rfloor \geq \left\lceil Y_1^S(x_h) \right\rceil$, otherwise a negative number of robots would be summed.

Now, assume $T \leq \frac{s}{v}$. Then, the semicircle has centre at $C = (c_x, c_y) = (x_0 - (s - vT), y_0)$ (Figure A13). Now, as we do not have the rectangle part, we consider the *last* robot of the rectangular part being the first robot to arrive at the target region, so $(l_x, l_y) = (x_0, y_0)$, and, by (A27),

$$C_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} c_x - x_0 \\ c_y - y_0 \end{bmatrix}. \tag{A32}$$

In the usual Euclidean coordinate space before the rotation about (x_0, y_0), we consider the line $\overleftrightarrow{OA}$ perpendicular to the x-axis at $O = (x_0, y_0)$. This line represents the perpendicular axis such that we wish to count all the robots from it to the arc of the semicircle on its right. From Figure A13 (I),

$$r = |\overline{AO}| = \sqrt{|\overline{CA}|^2 - |\overline{CO}|^2} = \sqrt{s^2 - (s - vT)^2} = \sqrt{2svT - (vT)^2}.$$

After the rotation by $-\theta$ about the point O, the maximum value for x_h is defined by the point U. The point U is chosen depending on the angles $\widehat{U'_{-\theta}OU}$ and $\widehat{A_{-\theta}OU}$. When the angle $\widehat{U'_{-\theta}OU}$ is greater than $\widehat{A_{-\theta}OU}$, the value of U is calculated in

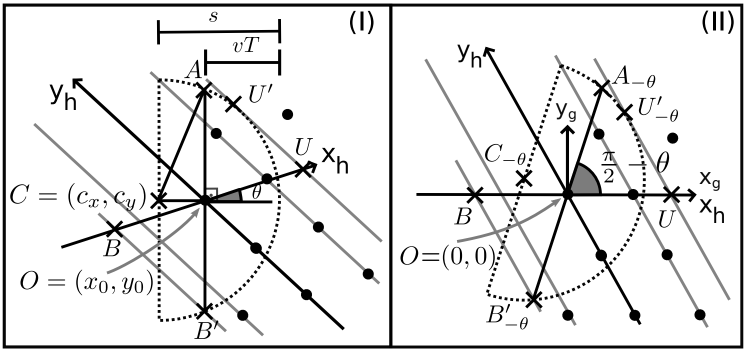


Fig. A13: Similar to the coordinate spaces of Figure A12, but for $T \leq \frac{s}{v}$. The rotation and hexagonal grid system centres are now (x_0, y_0). Notice also in (I) that $\triangle CAO$ is right with hypotenuse $\overline{CA}$ measuring s, and the horizontal cathetus $\overline{CO}$ measures $s - vT$.

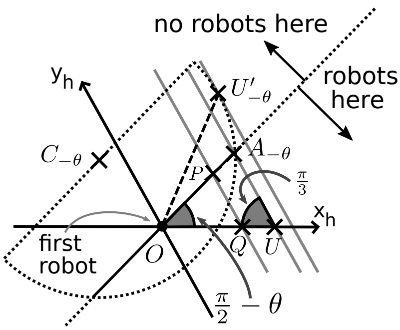


Fig. A14: An example of when the angle $\widehat{U'_{-\theta}OU}$ is greater than $\widehat{A_{-\theta}OU}$. We only consider robots inside the semicircle below the line $\overleftrightarrow{OA_{-\theta}}$, otherwise the robot on O would not be the first robot by assumption. In this case, any line parallel to y_h-axis crossing the semicircle below $\overline{OA_{-\theta}}$ must have its x_h-axis coordinate less than or equal to U, for example Q projected from P.

relation to $A_{-\theta}$, because the line parallel to y_h-axis intercepting $U'_{-\theta}$ is not inside the semicircle below $\overleftrightarrow{OA_{-\theta}}$ (Figure A14). For comparison, Figure A13 (II) shows an example where we choose U as the x_h-axis intersection with the line parallel to y_h-axis at $U'_{-\theta}$. As we saw before for the case $T > \frac{s}{v}$ (Figure A12 (II)), the angle of $\overline{C_{-\theta}U'_{\theta}}$ in relation to x_h-axis is $\pi/6$, consequently,

$$\begin{aligned} U'_{-\theta} &= C_{-\theta} + s\left(\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right)\right) \\ &= \left(\frac{\sqrt{3}s}{2} + \cos(\theta)(vT - s), \frac{s}{2} - \sin(\theta)(vT - s)\right), \end{aligned}$$

from (A32), and $\widehat{U'_{-\theta}OU}$ measures

$$\arctan\left(\frac{U'_{-\theta,y}}{U'_{-\theta,x}}\right) = \arctan\left(\frac{\frac{s}{2} - \sin(\theta)(vT-s)}{\frac{\sqrt{3}s}{2} + \cos(\theta)(vT-s)}\right).$$

$\widehat{A_{-\theta}OU}$ measures $\frac{\pi}{2} - \theta$, as show in Figure A13 (II). Thence,

$$A_{-\theta} = \left(r\cos\left(\frac{\pi}{2} - \theta\right), r\sin\left(\frac{\pi}{2} - \theta\right)\right).$$

If $\arctan\left(\frac{U'_{-\theta,y}}{U'_{-\theta,x}}\right) \leq \widehat{A_{-\theta}OU} = \frac{\pi}{2} - \theta$, we apply (A28) on $U'_{-\theta}$ to get its x_h-axis coordinate

$$\begin{aligned} U &= \frac{1}{d}\left(\frac{\sqrt{3}s}{2} + \cos(\theta)(vT-s)\right) + \frac{1}{\sqrt{3}d}\left(\frac{s}{2} - \sin(\theta)(vT-s)\right) \\ &= \cos(\theta)\frac{vT-s}{d} - \sin(\theta)\frac{vT-s}{\sqrt{3}d} + \frac{2s}{\sqrt{3}d} \\ &= \sqrt{3}\cos(\theta)\frac{vT-s}{\sqrt{3}d} - \sin(\theta)\frac{vT-s}{\sqrt{3}d} + \frac{2s}{\sqrt{3}d} \\ &= \frac{2\sin(\pi/3-\theta)(vT-s)}{\sqrt{3}d} + \frac{2s}{\sqrt{3}d}, \end{aligned}$$

followed by applying floor function to it, as we need the integer coordinate less or equal to this value. This is the same as (A29) by using $(l_x, l_y) = (x_0, y_0)$, then we also have (A24) when $\arctan\left(\frac{\frac{s}{2}-\sin(\theta)(vT-s)}{\frac{\sqrt{3}s}{2}+\cos(\theta)(vT-s)}\right) \leq \frac{\pi}{2} - \theta$.

If $\arctan\left(\frac{U'_{-\theta,y}}{U'_{-\theta,x}}\right) > \frac{\pi}{2} - \theta$, then there are no robots to consider on the parallel lines to y_h-axis between $U'_{-\theta}$ and $A_{-\theta}$, otherwise the robot at (x_0, y_0) would not be the first to arrive at the target region. Thus, if $\arctan\left(\frac{U'_{-\theta,y}}{U'_{-\theta,x}}\right) > \frac{\pi}{2} - \theta$, we use the x_h-coordinate for the point $A_{-\theta}$ on the hexagonal grid space, that is,

$$\begin{aligned} U &= \frac{1}{d}\left(r\cos\left(\frac{\pi}{2} - \theta\right)\right) + \frac{1}{\sqrt{3}d}\left(r\sin\left(\frac{\pi}{2} - \theta\right)\right) \\ &= \frac{2r}{\sqrt{3}d}\left(\frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{2} - \theta\right) + \frac{1}{2}\sin\left(\frac{\pi}{2} - \theta\right)\right) \\ &= \frac{2r}{\sqrt{3}d}\cos\left(\theta - \frac{\pi}{3}\right). \end{aligned}$$

then we apply the floor function to yield the desired result in (A25).

Now we will find the minimum value for an integer x_h such that a parallel-to-y_h-axis line is inside the semicircle and starting from the right of $\overleftrightarrow{OA}$ or on it. For the calculation of B, from Figure A13 (II), similarly to how we previously did,

$$\begin{aligned} B'_{-\theta} &= O + r(\cos(-(\pi/2+\theta)), \sin(-(\pi/2+\theta)) \\ &= (r\cos(\pi/2+\theta), -r\sin(\pi/2+\theta)) \\ &= (-r\sin(\theta), -r\cos(\theta)), \end{aligned}$$

and, by (A28), as B is the x_h-coordinate of the $B_{-\theta}$,

$$\begin{aligned} B &= \frac{1}{d}(-r\sin(\theta)) + \frac{1}{\sqrt{3}d}(-r\cos(\theta)) = -\frac{2r}{\sqrt{3}d}\left(\frac{1}{2}\cos(\theta) + \frac{\sqrt{3}}{2}\sin(\theta)\right) \\ &= -\frac{2r}{\sqrt{3}d}\sin\left(\theta + \frac{\pi}{6}\right). \end{aligned}$$

Also, we apply the ceiling function to yield the desired result in (A23).

In this case, $C_1(x_h)$ and $C_2(x_h)$ are equal to (A30) and (A31), but $L(x_h)$ is different from the previous case. The line $\overleftrightarrow{OA_{-\theta}}$ for a point (x_g, y_g) in the Euclidean space is

$$\begin{aligned} y_g = \tan\left(\frac{\pi}{2} - \theta\right) x_g \Leftrightarrow y_g &= \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} x_g \Leftrightarrow \\ \frac{\sqrt{3} d y_h}{2} &= \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} \left(d x_h - \frac{d y_h}{2}\right) \Leftrightarrow \\ y_h &= \frac{2\sin\left(\frac{\pi}{2} - \theta\right) x_h}{\sqrt{3}\cos\left(\frac{\pi}{2} - \theta\right) + \sin\left(\frac{\pi}{2} - \theta\right)} \Leftrightarrow \\ L(x_h) = y_h &= \frac{\sin\left(\frac{\pi}{2} - \theta\right) x_h}{\sin\left(\frac{5\pi}{6} - \theta\right)}. \end{aligned}$$

□

We have $\lim_{T\to\infty} f_h(T,\theta) = \lim_{T\to\infty} \frac{N_R(T,\theta)}{T} + \lim_{T\to\infty} \frac{N_S(T,\theta) - 1}{T}$, by Definition 2. As shown below, this limit needs only the rectangle part, because N_S is limited by a semicircle with finite radius.

Lemma A8.

$$\lim_{T\to\infty} \frac{N_S(T,\theta) - 1}{T} = 0.$$

Proof As $T \to \infty$, we have $T > \frac{s}{v}$. By Lemma A7, $c_x = x_0 + vT - s$, which is the x-axis coordinate of the right side of the rectangle. We have that the robots are distant by d, so the last robot must be at most distant by d from the point (c_x, y_0). Hence, $x_0 + vT - s - d \le l_x \le x_0 + vT - s$, and $y_0 - d \le l_y \le y_0 + d$, so $0 = c_x - (x_0 + vT - s) \le c_x - l_x \le c_x - (x_0 + vT - s - d) = d$ and $-d \le y_0 - l_y \le d$. Then, $-d \le C_{-\theta,x}, C_{-\theta,y} \le d$. Thus,

$$\begin{aligned} B &= \left\lceil \frac{2(\sin(\pi/3 - \theta)(c_x - l_x) + \cos(\pi/3 - \theta)(y_0 - l_y - s))}{\sqrt{3}d} \right\rceil \\ &\ge \left\lceil \frac{2(\cos(\pi/3 - \theta)(-d - s))}{\sqrt{3}d} \right\rceil = \left\lceil \frac{-2\cos(\pi/3 - \theta)(1 + \frac{s}{d})}{\sqrt{3}} \right\rceil \ge \left\lceil \frac{-2(1 + \frac{s}{d})}{\sqrt{3}} \right\rceil \\ &= \left\lceil -\frac{2}{\sqrt{3}} - \frac{s}{\sqrt{3}d} \right\rceil \ge -\frac{2}{\sqrt{3}} - \frac{s}{\sqrt{3}d}, \\ U &= \left\lfloor \frac{2(\sin(\pi/3 - \theta)(c_x - l_x) + \cos(\pi/3 - \theta)(y_0 - l_y) + s)}{\sqrt{3}d} \right\rfloor \\ &\le \left\lfloor \frac{2(\sin(\pi/3 - \theta)d + \cos(\pi/3 - \theta)d + s)}{\sqrt{3}d} \right\rfloor \le \left\lfloor \frac{2(2d + s)}{\sqrt{3}d} \right\rfloor \le \frac{4}{\sqrt{3}} + \frac{2s}{\sqrt{3}d}, \end{aligned}$$

and for any integer $x_h \in [B, U]$, as $\Delta(x_h)$ cannot be negative,

$$0 \le \Delta(x_h) = 4s^2 - \left(\sqrt{3}(d x_h - C_{-\theta,x}) - C_{-\theta,y}\right)^2 \le 4s^2,$$

$$\begin{aligned}\lceil Y_1^S(x_h)\rceil \geq Y_1^S(x_h) &= \frac{dx_h - C_{-\theta,x} + \sqrt{3}C_{-\theta,y} - \sqrt{\Delta(x_h)}}{2d}\\ &\geq \frac{dx_h - d - \sqrt{3}d - 2s}{2d} = \frac{x_h - 1 - \sqrt{3}}{2} - \frac{s}{d},\end{aligned}$$

and

$$\begin{aligned}\lfloor Y_2^S(x_h)\rfloor &\leq Y_2^S(x_h) \leq \min(L(x_h), C_2(x_h)) \leq C_2(x_h)\\ &= \frac{dx_h - C_{-\theta,x} + \sqrt{3}C_{-\theta,y} + \sqrt{\Delta(x_h)}}{2d} \leq \frac{dx_h + d + \sqrt{3}d + 2s}{2d}\\ &= \frac{x_h + 1 + \sqrt{3}}{2} + \frac{s}{d}.\end{aligned}$$

Thus,

$$\begin{aligned}0 \leq N_S(T,\theta) &= \sum_{x_h=B}^{U}\left(\lfloor Y_2^S(x_h)\rfloor - \lceil Y_1^S(x_h)\rceil + 1\right)\\ &\leq \sum_{x_h=B}^{U}\left(\frac{x_h + 1 + \sqrt{3}}{2} + \frac{s}{d} - \left(\frac{x_h - 1 - \sqrt{3}}{2} - \frac{s}{d}\right) + 1\right)\\ &= \sum_{x_h=B}^{U}\left(\frac{2s}{d} + \sqrt{3} + 2\right)\\ &\leq \left(\frac{4}{\sqrt{3}} + \frac{2s}{\sqrt{3}d} - \left(-\frac{2}{\sqrt{3}} - \frac{s}{\sqrt{3}d}\right) + 1\right)\left(\frac{2s}{d} + \sqrt{3} + 2\right)\\ &= \left(2\sqrt{3} + \frac{\sqrt{3}s}{d} + 1\right)\left(\frac{2s}{d} + \sqrt{3} + 2\right)\\ \Rightarrow 0 = \lim_{T\to\infty}\frac{-1}{T} &\leq \lim_{T\to\infty}\frac{N_S(T,\theta) - 1}{T}\\ &\leq \lim_{T\to\infty}\frac{1}{T}\left(\left(2\sqrt{3} + \frac{\sqrt{3}s}{d} + 1\right)\left(\frac{2s}{d} + \sqrt{3} + 2\right) - 1\right) = 0.\end{aligned}$$

Hence, the result follows from the sandwich theorem. □

As we obtained that $\lim_{T\to\infty}\frac{N_S(T,\theta) - 1}{T} = 0$, hereafter we only calculate the limit for the number of robots inside the rectangle. By Lemmas A2 to A6, if $n_l^+ - 1 < K'$ we have

$$\begin{aligned}\lim_{T\to\infty} f_h(T,\psi) &= \lim_{T\to\infty}\frac{1}{T}\sum_{x_h=-n_l^-}^{n_l^-}\left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right)\\ &+ \lim_{T\to\infty}\frac{1}{T}\sum_{x_h=n_l^-+1}^{n_l^+-1}\left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right),\end{aligned}$$

otherwise,

$$\begin{aligned}\lim_{T\to\infty} f_h(T,\psi) = &\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=-n_l^-}^{n_l^-}\left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right)\\ &+\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=n_l^-+1}^{K'-1}\left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right)\\ &+\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=K'}^{n_l^+-1}\left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right).\end{aligned}$$

To clarify, the third summation is zero in the case of $n_l^+ - 1 < K'$, while the second summation goes until $\min(n_l^+ - 1, K' - 1)$ in both cases. Each one will be individually solved assuming $\psi \neq \pi/6$. Later, we will see that the final result holds for $\psi = \pi/6$ as well. The following lemmas will be useful soon.

Lemma A9. *Assume $\psi \neq \pi/6$.*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=-n_l^-}^{n_l^-}\left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right) = 0.$$

Proof As for any x, $x - 1 < \lfloor x\rfloor \leq x \leq \lceil x\rceil < x + 1$,

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=-n_l^-}^{n_l^-}\left(Y_2^R(x_h) - Y_1^R(x_h) - 1\right)\\ &<\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=-n_l^-}^{n_l^-}\left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right)\\ &\leq\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=-n_l^-}^{n_l^-}\left(Y_2^R(x_h) - Y_1^R(x_h) + 1\right).\end{aligned}$$

By Lemma A3, the first and last summations do not depend on T, so both sides have limit equal to 0. By the sandwich theorem, we have the result. □

Lemma A10. *Assume $\psi \neq \pi/6$. For $K' = \left\lceil \frac{2(vT-s)\cos(\psi-\pi/6) - 2s\sin(|\psi-\pi/6|)}{\sqrt{3}d}\right\rceil$,*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=K'}^{n_l^+-1}\left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right) = 0.$$

Proof If $K' > n_l^+ - 1$, this limit is already zero, so we focus this proof on the other case. We have, analogously to the previous lemma,

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \sum_{x_h=K'}^{n_l^+-1} \left(Y_2^R(x_h) - Y_1^R(x_h) - 1\right) \\
&< \lim_{T\to\infty} \frac{1}{T} \sum_{x_h=K'}^{n_l^+-1} \left(\lfloor Y_2^R(x_h)\rfloor - \lceil Y_1^R(x_h)\rceil + 1\right) \\
&\le \lim_{T\to\infty} \frac{1}{T} \sum_{x_h=K'}^{n_l^+-1} \left(Y_2^R(x_h) - Y_1^R(x_h) + 1\right).
\end{aligned} \tag{A33}$$

For any constant c, we have

$$\lim_{T\to\infty} \frac{1}{T} \sum_{x_h=K'}^{n_l^+-1} c = 0, \tag{A34}$$

because the number of x_h indexes in the summation is limited by a finite number of integer outcomes that depends on T. In other words, the number of indexes in the above summation is $n_l^+ - K'$ such that $\frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}d} - 1 < n_l^+ - K' \le \frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}d} + 1$. The last inequality is obtained by counting how many x_h are used in the summation and knowing that $2y - 1 < \lfloor x+y\rfloor - \lceil x-y\rceil + 1 \le 2y+1$ for any $x, y \in \mathbb{R}$. Thus, for any T, $n_l^+ - K'$ can only range from $\left\lceil \frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}d}\right\rceil - 1$ to $\left\lfloor \frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}d}\right\rfloor + 1$. This yields to three possible integer numbers, if $\frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}d} \in \mathbb{Z}$, or four, otherwise. Thus, a finite range of outcomes, none of them having T. Hence, for all outcomes, the limit on the left side of (A34) is zero.

Assume $\psi > \pi/6$ (for $\psi < \pi/6$ the result is the same). From Lemma A4,

$$\begin{aligned}
Y_2^R(x_h) - Y_1^R(x_h) &= \frac{\frac{2(vT-s)}{d\cos(\psi)} - 2x_h}{\sqrt{3}\tan(\psi) - 1} - \frac{\frac{2y_1}{d} + 2\tan(\psi)x_h}{\sqrt{3} + \tan(\psi)} \\
&= \frac{\frac{2(vT-s)}{d\cos(\psi)}}{\sqrt{3}\tan(\psi) - 1} - \frac{\frac{2y_1}{d}}{\sqrt{3} + \tan(\psi)} - \left(\frac{2}{\sqrt{3}\tan(\psi) - 1}\right. \\
&\left. + \frac{2\tan(\psi)}{\sqrt{3} + \tan(\psi)}\right) x_h.
\end{aligned} \tag{A35}$$

For the second term above, by (A34), $\lim_{T\to\infty} \sum_{x_h=K'}^{n_l^+-1} \frac{\frac{2y_1}{d}}{\sqrt{3} + \tan(\psi)} = 0.$

For the first term,

$$\begin{aligned}
&\sum_{x_h=K'}^{n_l^+-1} \frac{1}{T} \frac{\frac{2(vT-s)}{d\cos(\psi)}}{\sqrt{3}\tan(\psi)-1} = \sum_{x_h=K'}^{n_l^+-1} \frac{2\left(v-\frac{s}{T}\right)}{d\cos(\psi)(\sqrt{3}\tan(\psi)-1)} \\
&= \sum_{x_h=K'}^{n_l^+-1} \frac{2\left(v-\frac{s}{T}\right)}{d(\sqrt{3}\sin(\psi)-\cos(\psi))} = \sum_{x_h=K'}^{n_l^+-1} \frac{v-\frac{s}{T}}{d\sin(\psi-\pi/6)} \\
&= \sum_{x_h=K'}^{n_l^+-1} \frac{v}{d\sin(\psi-\pi/6)} - \frac{1}{T}\sum_{x_h=K'}^{n_l^+-1} \frac{s}{d\sin(\psi-\pi/6)},
\end{aligned} \tag{A36}$$

due to $\frac{\sqrt{3}}{2}\sin(\psi) - \frac{1}{2}\cos(\psi) = \sin(\psi-\pi/6)$. Let L be the number of terms on the summation of (A36). As discussed above, L is an integer in $\left\{\left\lceil \frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}d}\right\rceil - 1, \ldots, \left\lfloor \frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}d}\right\rfloor + 1\right\}$, so

$$\sum_{x_h=K'}^{n_l^+-1} \frac{1}{T} \frac{\frac{2(vT-s)}{d\cos(\psi)}}{\sqrt{3}\tan(\psi)-1} = \frac{Lv}{d\sin(\psi-\pi/6)} - \frac{1}{T}\sum_{x_h=K'}^{n_l^+-1} \frac{s}{d\sin(\psi-\pi/6)}. \tag{A37}$$

Also, we have

$$\begin{aligned}
&\frac{2}{\sqrt{3}\tan(\psi)-1} + \frac{2\tan(\psi)}{\sqrt{3}+\tan(\psi)} = \frac{2(\sqrt{3}+\tan(\psi)) + 2\tan(\psi)(\sqrt{3}\tan(\psi)-1)}{(\sqrt{3}\tan(\psi)-1)(\sqrt{3}+\tan(\psi))} \\
&= \frac{2\sqrt{3}(1+\tan^2(\psi))}{(\sqrt{3}\tan(\psi)-1)(\sqrt{3}+\tan(\psi))} = \frac{2\sqrt{3}}{(\sqrt{3}\sin(\psi)-\cos(\psi))(\sqrt{3}\cos(\psi)+\sin(\psi))} \\
&= \frac{\sqrt{3}}{2\sin(\psi-\pi/6)\cos(\psi-\pi/6)}
\end{aligned}$$

as $1+\tan^2(\psi) = \sec^2(\psi)$ and $\frac{\sqrt{3}}{2}\cos(\psi) + \frac{1}{2}\sin(\psi) = \cos(\psi-\pi/6)$. Hence, for the last term in (A35),

$$\begin{aligned}
&\frac{1}{T}\sum_{x_h=K'}^{n_l^+-1} \frac{\sqrt{3}}{2\sin(\psi-\pi/6)\cos(\psi-\pi/6)} x_h \\
&= \frac{1}{T}\frac{\sqrt{3}}{2\sin(\psi-\pi/6)\cos(\psi-\pi/6)} \frac{(n_l^+-1+K')(n_l^+-K')}{2} \\
&= \frac{\sqrt{3}LG}{4T\sin(\psi-\pi/6)\cos(\psi-\pi/6)},
\end{aligned} \tag{A38}$$

for an integer $G = n_l^+ - 1 + K'$. As $2x-1 < \lfloor x+y \rfloor + \lceil x-y \rceil < 2x+1$ for any $x, y \in \mathbb{R}$, $G \in \left(\frac{4(vT-s)\cos(\psi-\pi/6)}{\sqrt{3}d} - 1, \frac{4(vT-s)\cos(\psi-\pi/6)}{\sqrt{3}d} + 1\right)$.

For the lowest bound on G, using (A37) and (A38)

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=K'}^{n_l^+-1}\left(\mathrm{Y}_2^R(x_h)-\mathrm{Y}_1^R(x_h)\right)\\
&=\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=K'}^{n_l^+-1}\left(\frac{\frac{2(vT-s)}{d\cos(\psi)}}{\sqrt{3}\tan(\psi)-1}-\left(\frac{2}{\sqrt{3}\tan(\psi)-1}+\frac{2\tan(\psi)}{\sqrt{3}+\tan(\psi)}\right)x_h\right)\\
&=\lim_{T\to\infty}\left(\frac{Lv}{d\sin(\psi-\pi/6)}-\frac{1}{T}\sum_{x_h=K'}^{n_l^+-1}\frac{s}{d\sin(\psi-\pi/6)}\right.\\
&\quad\left.-\frac{\sqrt{3}L\left(\frac{4(vT-s)\cos(\psi-\pi/6)}{\sqrt{3}d}-1\right)}{4T\sin(\psi-\pi/6)\cos(\psi-\pi/6)}\right)\\
&=\lim_{T\to\infty}\left(\frac{\sqrt{3}L}{4T\sin(\psi-\pi/6)\cos(\psi-\pi/6)}-\frac{1}{T}\sum_{x_h=K'}^{n_l^+-1}\frac{s}{d\sin(\psi-\pi/6)}\right)=0,
\end{aligned}$$

due to (A34) on the second term and, as

$$L\in\left\{\left\lceil\frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}d}\right\rceil-1,\ldots,\left\lfloor\frac{4s\sin(|\psi-\pi/6|)}{\sqrt{3}d}\right\rfloor+1\right\},$$

no element in this finite set has the term T.

For the highest bound on G, we have the same limit. Hence, by the sandwich theorem applied on the results for both bounds of G, we get

$$\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=K'}^{n_l^+-1}\left(\mathrm{Y}_2^R(x_h)-\mathrm{Y}_1^R(x_h)\right)=0. \tag{A39}$$

Using (A39) and (A34) on the bounds of (A33) and the sandwich theorem again concludes with the desired value. □

Lemma A11. *Assume* $\psi\neq\pi/6$.

$$\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=n_l^-+1}^{\min(n_l^+-1,K'-1)}\left(\lfloor \mathrm{Y}_2^R(x_h)\rfloor-\lceil \mathrm{Y}_1^R(x_h)\rceil+1\right)$$

exists and is bounded by

$$\left(\frac{4vs}{\sqrt{3}d^2}-\frac{2v\cos(\psi-\pi/6)}{\sqrt{3}d},\frac{4vs}{\sqrt{3}d^2}+\frac{2v\cos(\psi-\pi/6)}{\sqrt{3}d}\right].$$

Proof The next lemmas will be useful for proving this lemma.

Lemma A12. *For any* $a,b>0, a\lfloor x\rfloor-b\lfloor y\rfloor<ax-by+a+b$.

Proof As mentioned before, by the definition of floor function $\lfloor x \rfloor = x - frac(x)$, where $frac$ is the function that returns the fractional part of the number x, such that $0 \leq frac(x) < 1$ (Graham et al, 1994),

$$\begin{aligned} a\lfloor x \rfloor - b\lfloor y \rfloor &= ax - afrac(x) - by + bfrac(y) \\ &< ax - by + b - afrac(x) && [\text{because } frac(y) < 1] \\ &< ax - by + b + a && [\text{as } \; -afrac(x) \leq 0 < a]. \end{aligned}$$

□

Lemma A13. *Let $c, d, A_1, B_1, A_2, B_2 \in \mathbb{R}$, $c > 0$ and $I_1 \in \mathbb{Z}$. Then, the limit below exists:*

$$\lim_{n\to\infty} \sum_{i=I_1+1}^{\lfloor cn+d \rfloor} \frac{frac(-(A_1 i + B_1)) + frac(A_2 i + B_2)}{n}.$$

Proof For convergence, we show that for $R(i) = frac(-(A_1 i+B_1))+frac(A_2 i+B_2)$, $(a_n)_{n\in\mathbb{N}^*} = \left(\sum_{i=I_1+1}^{\lfloor cn+d \rfloor} \frac{R(i)}{n}\right)_{n\in\mathbb{N}^*}$ is a Cauchy sequence. Take $\epsilon > 0$ and choose $N > \frac{4|I_1-d+1|}{\epsilon}$. Let $n, m \in \mathbb{N}^*$ and $n > m > N$. We have

$$|a_n - a_m| = \left| \sum_{i=I_1+1}^{\lfloor cn+d \rfloor} \frac{R(i)}{n} - \sum_{i=I_1+1}^{\lfloor cm+d \rfloor} \frac{R(i)}{m} \right| = \left| \frac{1}{nm} \left(m \sum_{i=I_1+1}^{\lfloor cn+d \rfloor} R(i) - n \sum_{i=I_1+1}^{\lfloor cm+d \rfloor} R(i) \right) \right|$$

$$= \left| \frac{1}{nm} \left(m \sum_{i=\lfloor cm+d \rfloor+1}^{\lfloor cn+d \rfloor} R(i) + (m-n) \sum_{i=I_1+1}^{\lfloor cm+d \rfloor} R(i) \right) \right| \qquad [\text{as } \; \lfloor cn+d \rfloor > \lfloor cm+d \rfloor]$$

$$< 2 \left| \frac{m(\lfloor cn+d \rfloor - (\lfloor cm+d \rfloor + 1) + 1) + (m-n)(\lfloor cm+d \rfloor - (I_1+1) + 1)}{nm} \right|$$

$$[\text{as } R(i) < 2 \text{ for any } i]$$

$$= 2 \left| \frac{m\lfloor cn+d \rfloor - n\lfloor cm+d \rfloor - (m-n)I_1}{nm} \right|$$

$$< 2 \left| \frac{m(cn+d) - n(cm+d) + m + n - (m-n)I_1}{nm} \right| \qquad [\text{Lemma } A12]$$

$$= 2 \left| \frac{(n-m)(I_1-d) + m + n}{nm} \right| < 2 \left| \frac{(n+m)(I_1-d) + m + n}{nm} \right|$$

$$= 2 \left| \frac{(m+n)(I_1-d+1)}{nm} \right| = 2|I_1-d+1| \frac{m+n}{nm} = 2|I_1-d+1| \left(\frac{1}{n} + \frac{1}{m} \right)$$

$$< 2|I_1-d+1| \frac{2}{N} = \frac{4|I_1-d+1|}{N} < \epsilon.$$

□

To prove the existence, we have $\lceil x \rceil = x + frac(-x)$, for any real number x[2], because

$$\begin{aligned} frac(-x) &= -x - \lfloor -x \rfloor && [\text{def. of } \lfloor -x \rfloor] \\ &= -x - (-\lceil x \rceil) && [\lfloor -x \rfloor = -\lceil x \rceil] \\ &= -x + \lceil x \rceil \\ \Leftrightarrow \lceil x \rceil &= x + frac(-x). \end{aligned}$$

Thus,

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \sum_{x_h=n_l^-+1}^{\min(n_l^+-1,K'-1)} \left(\lfloor Y_2^R(x_h) \rfloor - \lceil Y_1^R(x_h) \rceil + 1 \right) \\ &= \lim_{T\to\infty} \frac{1}{T} \sum_{x_h=n_l^-+1}^{\min(n_l^+-1,K'-1)} \left(Y_2^R(x_h) - Y_1^R(x_h) - frac\left(-Y_1^R(x_h)\right) \right. \\ &\quad \left. -frac\left(Y_2^R(x_h)\right) + 1 \right) \\ &= \lim_{T\to\infty} \frac{1}{T} \sum_{x_h=n_l^-+1}^{\min(n_l^+-1,K'-1)} \left(Y_2^R(x_h) - Y_1^R(x_h) + 1 \right) - \\ &\quad \lim_{T\to\infty} \frac{1}{T} \sum_{x_h=n_l^-+1}^{\min(n_l^+-1,K'-1)} \left(frac\left(-Y_1^R(x_h)\right) + frac\left(Y_2^R(x_h)\right) \right). \end{aligned}$$

The limit of the first term above exists and its value is presented below on (A42). The existence of the limit for the second term was shown by Lemma A13 for any outcome of $\min(n_l^+ - 1, K' - 1)$, because, if $\left\lfloor \frac{2(vT-s)\cos(\pi/6-\theta)+2s\sin(|\pi/6-\theta|)}{\sqrt{3}d} \right\rfloor = n_l^+ - 1 \leq K' - 1$, $c = \frac{2v\cos(\psi-\pi/6)}{\sqrt{3}d}$ and $d = \frac{2s(\sin(|\pi/6-\theta|)-\cos(\pi/6-\theta))}{\sqrt{3}d}$ on the Lemma A13. If $n_l^+ - 1 > K' - 1 = \left\lceil \frac{2(vT-s)\cos(\psi-\pi/6)-2s\sin(|\psi-\pi/6|)}{\sqrt{3}d} - 1 \right\rceil$, as for any x, $\lceil x \rceil = \lfloor x \rfloor$ or $\lceil x \rceil = \lfloor x \rfloor + 1$ depending on whether x is an integer or not, then $K' - 1 = \left\lfloor \frac{2(vT-s)\cos(\psi-\pi/6)}{\sqrt{3}d} - \frac{2s\sin(|\psi-\pi/6|)}{\sqrt{3}d} - 1 \right\rfloor$ or $K' - 1 = \left\lfloor \frac{2(vT-s)\cos(\psi-\pi/6)}{\sqrt{3}d} - \frac{2s\sin(|\psi-\pi/6|)}{\sqrt{3}d} \right\rfloor$. For both cases, on the Lemma A13 $c = \frac{2v\cos(\psi-\pi/6)}{\sqrt{3}d}$ as well, but for the former case, $d = -\frac{2s(\sin(|\psi-\pi/6|)+\cos(\pi/6-\theta))}{\sqrt{3}d} - 1$, and for the latter, $d = -\frac{2s(\sin(|\psi-\pi/6|)+\cos(\pi/6-\theta))}{\sqrt{3}d}$.

[2] Heed that using this definition of $frac$, $frac(1.7) = 0.7$ and $frac(-1.7) = 0.3$.

To get the bounds, we have

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=n_l^-+1}^{\min(n_l^+-1,K'-1)}\left(Y_2^R(x_h)-Y_1^R(x_h)-1\right)\\
&<\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=n_l^-+1}^{\min(n_l^+-1,K'-1)}\left(\lfloor Y_2^R(x_h)\rfloor-\lceil Y_1^R(x_h)\rceil+1\right)\\
&\le\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=n_l^-+1}^{\min(n_l^+-1,K'-1)}\left(Y_2^R(x_h)-Y_1^R(x_h)+1\right),
\end{aligned}\tag{A40}$$

and by Lemma A6, as $T\to\infty$,

$$Y_2^R(x_h)-Y_1^R(x_h)=\frac{\frac{y_2}{d}+\tan(\psi)x_h}{\frac{\sqrt{3}+\tan(\psi)}{2}}-\frac{\frac{y_1}{d}+\tan(\psi)x_h}{\frac{\sqrt{3}+\tan(\psi)}{2}}=\frac{2s}{d\cos(\psi-\pi/6)},$$

by (A4).

For the first limit at (A40) in the case of $\min(n_l^+-1,K'-1)=n_l^+-1$,

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=n_l^-+1}^{n_l^+-1}\left(Y_2^R(x_h)-Y_1^R(x_h)-1\right)\\
&=\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=n_l^-+1}^{n_l^+-1}\left(\frac{2s}{d\cos(\psi-\pi/6)}-1\right)\\
&=\lim_{T\to\infty}\frac{1}{T}\left(n_l^+-n_l^--1\right)\left(\frac{2s}{d\cos(\psi-\pi/6)}-1\right)\\
&=\left(\frac{2s}{d\cos(\psi-\pi/6)}-1\right)\left(\lim_{T\to\infty}\frac{1}{T}n_l^+-\lim_{T\to\infty}\frac{1}{T}(n_l^--1)\right)\\
&=\left(\frac{2s}{d\cos(\psi-\pi/6)}-1\right)\lim_{T\to\infty}\frac{1}{T}n_l^+\\
&=\left(\frac{2s}{d\cos(\psi-\pi/6)}-1\right)\frac{2v\cos(\psi-\pi/6)}{\sqrt{3}d}.\\
&=\frac{4vs}{\sqrt{3}d^2}-\frac{2v\cos(\psi-\pi/6)}{\sqrt{3}d}.
\end{aligned}\tag{A41}$$

Above we get $\lim_{T\to\infty}\frac{1}{T}n_l^+=\frac{2v\cos(\psi-\pi/6)}{\sqrt{3}d}$ by using the sandwich theorem and the inequality $x-1<\lfloor x\rfloor\le x$ to get the bounds on n_l^+.

Similarly, for the last limit at (A40) in the case of $\min(n_l^+-1,K'-1)=n_l^+-1$,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{x_h=n_l^-+1}^{n_l^+-1}\left(Y_2^R(x_h)-Y_1^R(x_h)+1\right)=\frac{4vs}{\sqrt{3}d^2}+\frac{2v\cos(\psi-\pi/6)}{\sqrt{3}d}.$$

The limits above in the case of $\min(n_l^+ - 1, K' - 1) = K' - 1$ yields the same result because of the sandwich theorem, the inequality $x \le \lceil x \rceil < x + 1$, and

$$\begin{aligned}
&\frac{2v\cos(\psi - \pi/6)}{\sqrt{3}d} = \lim_{T\to\infty} \frac{1}{T}\left(\frac{2(vT - s)\cos(\psi - \pi/6) - 2s\sin(|\psi - \pi/6|)}{\sqrt{3}d}\right) \\
&\le \lim_{T\to\infty} \frac{1}{T}(K' - 1) = \lim_{T\to\infty} \frac{1}{T}K' \\
&= \lim_{T\to\infty} \frac{1}{T}\left\lceil \frac{2(vT - s)\cos(\psi - \pi/6) - 2s\sin(|\psi - \pi/6|)}{\sqrt{3}d} \right\rceil \\
&< \lim_{T\to\infty} \frac{1}{T}\left(\frac{2(vT - s)\cos(\psi - \pi/6) - 2s\sin(|\psi - \pi/6|)}{\sqrt{3}d} + 1\right) \\
&= \frac{2v\cos(\psi - \pi/6)}{\sqrt{3}d},
\end{aligned}$$

so, $\lim_{T\to\infty} \frac{1}{T}K' = \lim_{T\to\infty} \frac{1}{T}n_l^+$. Consequently, the limit below exists and

$$\lim_{T\to\infty} \frac{1}{T} \sum_{x_h = n_l^- + 1}^{\min(n_l^+ - 1, K' - 1)} \left(Y_2^R(x_h) - Y_1^R(x_h) + 1\right) = \frac{4vs}{\sqrt{3}d^2} + \frac{2v\cos(\psi - \frac{\pi}{6})}{\sqrt{3}d}. \quad \text{(A42)}$$

Finally, using the bounds provided by (A41) and (A42) we have the expected result. □

By Lemmas A9, A10 and A11 we have for $\psi \ne \pi/6$

$$\lim_{T\to\infty} f_h(T, \psi) \in \left(\frac{4vs}{\sqrt{3}d^2} - \frac{2v\cos(\psi - \pi/6)}{\sqrt{3}d}, \frac{4vs}{\sqrt{3}d^2} + \frac{2v\cos(\psi - \pi/6)}{\sqrt{3}d}\right]. \quad \text{(A43)}$$

For $\psi = \pi/6$, by (A18),

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T} \sum_{x_h=0}^{\left\lfloor \frac{2(vT-s)}{\sqrt{3}d} \right\rfloor} \left(\frac{\sqrt{3}y_2 + dx_h}{2d} - \frac{\sqrt{3}y_1 + dx_h}{2d} - 1\right) &< \lim_{T\to\infty} f_h(T, \pi/6) \\
\le \lim_{T\to\infty} \frac{1}{T} \sum_{x_h=0}^{\left\lfloor \frac{2(vT-s)}{\sqrt{3}d} \right\rfloor} &\left(\frac{\sqrt{3}y_2 + dx_h}{2d} - \frac{\sqrt{3}y_1 + dx_h}{2d} + 1\right),
\end{aligned}$$

with

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \sum_{x_h=0}^{\left\lfloor \frac{2(vT-s)}{\sqrt{3}d} \right\rfloor} \left(\frac{\sqrt{3}y_2 + dx_h}{2d} - \frac{\sqrt{3}y_1 + dx_h}{2d} + 1\right) \\
&= \lim_{T\to\infty} \frac{1}{T} \sum_{x_h=0}^{\left\lfloor \frac{2(vT-s)}{\sqrt{3}d} \right\rfloor} \left(\frac{\sqrt{3}s}{d\cos(\pi/6)} + 1\right) = \lim_{T\to\infty} \frac{1}{T}\left\lfloor \frac{2(vT - s)}{\sqrt{3}d} + 1 \right\rfloor \left(\frac{2s}{d} + 1\right) \\
&= \frac{2v}{\sqrt{3}d}\left(\frac{2s}{d} + 1\right),
\end{aligned}$$

from (A4) and, as similarly done before, $\lim_{T\to\infty} \frac{1}{T}\left\lfloor \frac{2(vT-s)}{\sqrt{3}d}+1\right\rfloor = \frac{2v}{\sqrt{3}d}$ by using the sandwich theorem and the inequality $x-1 < \lfloor x \rfloor \leq x$ to get the bounds on the floor function; and

$$\lim_{T\to\infty} \frac{1}{T} \sum_{x_h=0}^{\left\lfloor \frac{2(vT-s)}{\sqrt{3}d} \right\rfloor} \left(\frac{\sqrt{3}y_2 + dx_h}{2d} - \frac{\sqrt{3}y_1 + dx_h}{2d} - 1 \right) = \frac{2v}{\sqrt{3}d}\left(\frac{2s}{d}-1\right).$$

Accordingly,

$$\lim_{T\to\infty} f_h(T,\pi/6) \in \left(\frac{2v}{\sqrt{3}d}\left(\frac{2s}{d}-1\right), \frac{2v}{\sqrt{3}d}\left(\frac{2s}{d}+1\right) \right],$$

which are the same values in (A43) if used $\psi = \pi/6$. Lemmas A1–A6, A9, A10 and A11 used ψ, so, after replacing ψ by $\pi/3-\theta$, we conclude the Proposition 6.

8 Proof of Lemma 2

In Figure 11 (b) we show the distance d_r from the target centre where the robots begin turning. By symmetry, this is the same distance from the target centre where the robots stop turning. From the right triangle ABC on that figure, we have $|\overline{AC}| = \sqrt{(r+s)^2-(r+d/2)^2}$ and from $\triangle ACD$, $d_r = \sqrt{(d/2)^2+|\overline{AC}|^2}$. Thus,

$$d_r = \sqrt{(d/2)^2+(r+s)^2-(r+d/2)^2} = \sqrt{s(2r+s)-rd}.$$

9 Proof of Lemma 3

From Figure 11 (b), we can see that the right triangle ABE has angle $\widehat{EAB} = \alpha/2$, hypotenuse $r+s$ and cathetus $r+d/2$. Hence, it directly follows that

$$sin(\alpha/2) = \frac{r+d/2}{r+s} \Leftrightarrow r = \frac{s\sin(\alpha/2)-d/2}{1-\sin(\alpha/2)}.$$

10 Proof of Proposition 7

The number of trajectories K must be greater or equal to 3. The reason is that for the minimum possible value for s, $s=d/2$, $K=2$ is enough to have parallel lanes. However, starting with $K=3$, curved trajectories are needed to guarantee that robots of one lane do not interfere with robots from another lane.

Also, we have K identical trajectories around the target, each taking a central angle of α. As a result, the value of α given K is $\alpha = \frac{2\pi}{K}$, implying that $0 < \alpha \leq \frac{2\pi}{3}$.

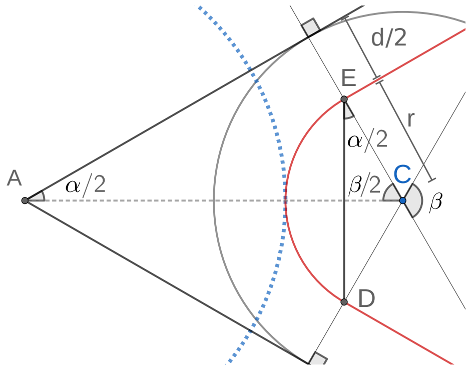


Fig. A15: The red line represents the trajectory of robots in one lane. α is the central angle for the lane. The dashed blue circle of centre A is the target. C is the centre of the circle of radius r from the circular trajectory. The grey circle of centre C has a radius of $r + d/2$. Points D and E represent the connection between the curved path and the straight path. We have $\beta = \pi - \alpha$ due to the symmetry and the fact that the sum of the angles of $\triangle ECD$ is equal to π.

Additionally, in the worst case, one robot of each lane arrives in the target region at the same time. When robots of all lanes simultaneously occupy the target region, their positions can be seen as the vertices of a regular polygon which must be inscribed in the circular target region of radius s (e.g., in Figure 12 we have a square whose sides are greater than d). The number of robots on the target region at the same time must be limited by the maximum number of sides of an inscribed regular polygon of a side with minimum side greater or equal to d. The side of a K regular polygon inscribed in a circle of radius s measures $2s\sin\left(\frac{\pi}{K}\right)$. Thence, $2s\sin\left(\frac{\pi}{K}\right) \geq d \Rightarrow \frac{\pi}{\arcsin\left(\frac{d}{2s}\right)} \geq K$.

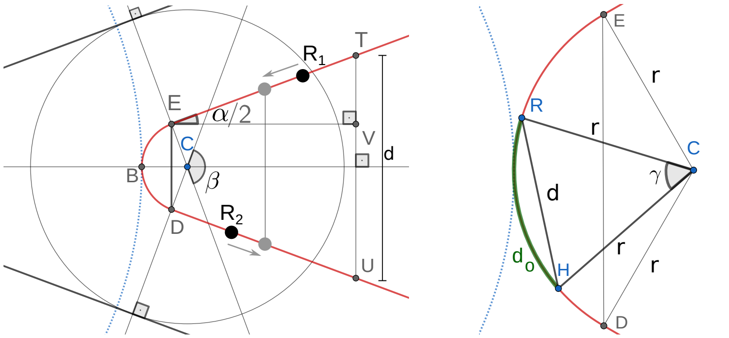
11 Proof of Proposition 8

Using the touch and run strategy, each lane is distant by at least d from each other. However, the minimum distance between robots on the same lane d_o must be checked at the beginning of the curved path, as their distance decreases if assuming linear constant velocity. We distinguish two cases based on Figure A15:

1. $|\overline{ED}| < d$: Two robots cannot be on the lane curved path;
2. $|\overline{ED}| \geq d$: More than one robot can occupy the lane curved path.

These cases affect the minimum distance between robots d_o such that they can follow the trajectory without decreasing their linear speed. In both cases, they need to satisfy the minimum distance d if they are turning on the curved path. From Figure A15,

$$|\overline{ED}| = 2r\sin\left(\frac{\beta}{2}\right) = 2r\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = 2r\cos\left(\frac{\alpha}{2}\right). \tag{A44}$$



(a) $|\overline{ED}| < d$ (b) $|\overline{ED}| \geq d$

Fig. A16: Enlargements of Figure A15. (a) The robots R_1 and R_2 are the black dots on the red line representing the trajectory. If the delay between R_1 and R_2 is less than the time for a robot to run from T to U following the red trajectory, there will be some instant which R_1 and R_2 will be vertically aligned. Their positions at that instant are represented by grey dots in front of them. Hence, their distance would be less than d. The right triangle TVE has side $\overline{TV}$, which can be measured using $\overline{ED}$. (b) d_o denotes the minimum arc length for two robots located at any two points R and H on $\widehat{ED}$ such that they are distant by at least d. γ is the angle defining the arc d_o for the circle of centre C.

In the case 1, in Figure A16 (a), we define two points T and U on the lane such that the distance between them is $|\overline{TU}| = d$ and their distances to the target are equal. The delay between one robot at T and another at U is equal to

$$\Delta t_1 = \frac{|\overline{TE}| + |\widehat{ED}| + |\overline{DU}|}{v},$$

that is, the time for running through the straight line TE, the curved path ED and the straight line DU.

For any delay less than Δt_1 between two robots, say R_1 and R_2, there is an instant of time when R_1 is on the path between B and T and R_2 is on the path between B and U and they are vertically aligned (Figure A16 (a)). In this case, the distance between R_1 and R_2 is below $|\overline{TU}|$, so they do not respect the minimum distance d between them. Hence, the minimum delay between two robots in case 1 is Δt_1.

From Figure A15, we have $|\widehat{ED}| = r\beta = r(\pi - \alpha)$. For calculating the value of $|\overline{TE}|$ and $|\overline{DU}|$ from Figure A16 (a), we observe that $|\overline{TE}| = |\overline{DU}|$ by symmetry. Thus,

$$
\begin{aligned}
|\overline{VT}| &= \frac{d}{2} - \frac{|\overline{ED}|}{2} && \text{[From Figure A16 (a)]} \\
&= \frac{d}{2} - r\cos\left(\frac{\alpha}{2}\right) && \text{[From (A44)].}
\end{aligned}
$$

As $\triangle TVE$ is right, $|\overline{TE}| = \frac{|\overline{VT}|}{\sin(\alpha/2)}$. Thence,

$$
\Delta t_1 = \frac{r(\pi - \alpha) + 2\frac{d/2 - r\cos(\alpha/2)}{\sin(\alpha/2)}}{v} = \frac{r(\pi-\alpha)}{v} + \frac{d - 2r\cos(\alpha/2)}{v\sin(\alpha/2)}
$$

and

$$
d_o = \max\left(d, v\Delta t_1\right) = \max\left(d, r(\pi-\alpha) + \frac{d - 2r\cos\left(\alpha/2\right)}{\sin(\alpha/2)}\right).
$$

Above we used max function because the result of $v\Delta t_1$ can still be less than d, depending on α, r and d.

In the case 2, we need to check the minimum distance d when two robots are on the circular part $\overset{\frown}{ED}$ in Figure A16 (b). From this figure, $\triangle CRH$ is isosceles, so $\gamma = 2\arcsin\left(\frac{d}{2r}\right)$. Thus, to keep constant velocity, the delay between two robots in this case is

$$
\Delta t_2 = \frac{d_o}{v} = \frac{r\gamma}{v} = \frac{2r}{v}\arcsin\left(\frac{d}{2r}\right).
$$

Then,

$$
d_o = \max\left(d, v\Delta t_2\right) = \max\left(d, 2r\arcsin\left(\frac{d}{2r}\right)\right).
$$

We used max function for a similar reason as exposed before. After rearranging, we have (17) and (18).

For calculating the throughput $f_t(K,T)$ for K lanes and a given time T after the arrival of the first robot, we get the number of robots reaching the target region by the time T, then we use the Definition 2. As we assume that the first robot of every lane begins at the same distance from the target, at time $T = 0$ we have K robots simultaneously arriving. Then, after d_o/v units of time, we have K more robots arriving and this keeps happening every d_o/v units of time. Denote $N(K,T)$ the total number of robots that have arrived at the target region from K lanes by time T. Thus, we have:

$$
N(K,T) = K\left\lfloor \frac{T}{\frac{d_0}{v}} + 1 \right\rfloor = K\left\lfloor \frac{vT}{d_o} + 1 \right\rfloor,
$$

so, by Definition 2,

$$
f_t(K,T) = \frac{1}{T}\left(K\left\lfloor \frac{vT}{d_o} + 1\right\rfloor - 1\right).
$$

As for every number x, $\lfloor x \rfloor = x - frac(x)$ and $0 \leq frac(x) < 1$, then distributing $\frac{1}{T}$ for each term, we get

$$\begin{aligned} f_t(K) &= \lim_{T\to\infty} f_t(K,T) = \lim_{T\to\infty} \frac{1}{T}\left(K\left(\frac{vT}{d_o}+1\right) - frac\left(\frac{vT}{d_o}+1\right) - 1\right) \\ &= \lim_{T\to\infty} \frac{K}{T}\left(\frac{vT}{d_o}+1\right) = \frac{Kv}{d_o}. \end{aligned}$$

12 Proof of Proposition 9

For any $u < \frac{\sqrt{3}+2}{4-2\sqrt{3}}$, $(2u+1)\frac{v}{d} > f_h^{min}(u)$, due to

$$\begin{aligned} &(2u+1)\frac{v}{d} > \frac{2}{\sqrt{3}}(2u-1)\frac{v}{d} \Leftrightarrow 2u+1 > \frac{2}{\sqrt{3}}(2u-1) \\ &\Leftrightarrow 2u - \frac{4}{\sqrt{3}}u > -1 - \frac{2}{\sqrt{3}} \Leftrightarrow u < \frac{-1-\frac{2}{\sqrt{3}}}{2-\frac{4}{\sqrt{3}}} = \frac{\sqrt{3}+2}{4-2\sqrt{3}}. \end{aligned} \tag{A45}$$

We have that $f_p(u) = (2u+1)\frac{v}{d}$ when $2u+1 \in \mathbb{Z}$. Also, as $u < \frac{\sqrt{3}+2}{4-2\sqrt{3}} < 7$, u can be a number satisfying $(2u+1) = \lfloor 2u+1 \rfloor$. Thus, there are some values of u such that $f_p(u) = \lfloor 2u+1 \rfloor \frac{v}{d} > f_h^{min}(u)$.

From the equivalence in (A45) and because for any x, $\lfloor x \rfloor \leq x$, it follows that for any $u \geq \frac{\sqrt{3}+2}{4-2\sqrt{3}}$, $f_p(u) \leq (2u+1)\frac{v}{d} \leq f_h^{min}(u)$.

References

Chang HC, Wang LC (2010) A simple proof of thue's theorem on circle packing. URL https://arxiv.org/abs/1009.4322, preprint arXiv:1009.4322v1, 1009.4322

Graham RL, Knuth DE, Patashnik O (1994) Concrete Mathematics: A Foundation for Computer Science, 2nd edn. Addison-Wesley Professional, Reading, MA

Red Blob Games (2021) Hexagonal grids. https://www.redblobgames.com/grids/hexagons/, accessed: 2021-11-16