Supplementary Information

for manuscript *Dynamics of microswimmers in viscoelastic liquid crystals*, by H. Chi, A. Gavrikov, L. Berlyand and I. S. Aranson

Note that all equation numbers below refer to equations in the main text, unless the number has prefix "SI-"

Supplementary Note 1: Finding numerically the translational and angular velocity of the swimmer

Recall the momentum equation

$$-\Delta\mathbf{v}+\nabla p+\alpha\mathbf{v}=\nabla\cdot\sigma \text{ in } \Omega, \tag{SI-1}$$

$$\nabla\cdot\mathbf{v}=0 \text{ in } \Omega, \tag{SI-2}$$

$$\mathbf{v}=\mathbf{v}^{*}+\omega\times r+\mathbf{V} \text{ on } \gamma. \tag{SI-3}$$

Here the velocity field $\mathbf{v}$, translational and angular velocity $\mathbf{V},\omega$ of the swimmer are all unknown. Additional equations (22) are imposed. In practice, (SI-1)-(SI-3) are decomposed into four parts:

- First part:

$$\begin{aligned}&-\Delta\mathbf{v}_1+\nabla p+\alpha\mathbf{v}_1=\nabla\cdot\sigma \text{ in } \Omega,\\ &\nabla\cdot\mathbf{v}_1=0 \text{ in } \Omega,\\ &\mathbf{v}_1=\mathbf{v}^{*} \text{ on } \gamma.\end{aligned}$$

- Second part:

$$\begin{aligned}&-\Delta\mathbf{v}_2+\nabla p+\alpha\mathbf{v}_2=0 \text{ in } \Omega,\\ &\nabla\cdot\mathbf{v}_2=0 \text{ in } \Omega,\\ &\mathbf{v}_2=1\times r \text{ on } \gamma.\end{aligned}$$

- Third part:

$$\begin{aligned}&-\Delta\mathbf{v}_3+\nabla p+\alpha\mathbf{v}_3=0 \text{ in } \Omega,\\ &\nabla\cdot\mathbf{v}_3=0 \text{ in } \Omega,\\ &\mathbf{v}_3=\mathbf{e}_1 \text{ on } \gamma.\end{aligned}$$

- Last part:

$$\begin{aligned}&-\Delta\mathbf{v}_4+\nabla p+\alpha\mathbf{v}_4=0 \text{ in } \Omega,\\ &\nabla\cdot\mathbf{v}_4=0 \text{ in } \Omega,\\ &\mathbf{v}_4=\mathbf{e}_2 \text{ on } \gamma.\end{aligned}$$

where each part is solved explicitly using Boundary Integral Method introduced in the next section. Here $\mathbf{v}=\mathbf{v}_1+\omega\mathbf{v}_2+\mathbf{V}_1\mathbf{v}_3+\mathbf{V}_2\mathbf{v}_4$. Denote $\mathbf{F}_i$ and $\mathbf{T}_i$ is the force exerted on the swimmer by $\mathbf{v}_i$. Then (22) can be rewritten into three linear equations for $\omega,\mathbf{V}_1$ and $\mathbf{V}_2$. One can get $\omega,\mathbf{V}_1$ and $\mathbf{V}_2$ at the current time step by solving this linear system.

Supplementary Note 2: Details of Boundary Integral Method

After the tensor order parameter $\mathbf{Q}^{n+1}$ and comformation tensor $\mathbf{C}^{n+1}$ have been updated through (6)-(7), we numerically solve momentum equation (9) for $\mathbf{v}^{n+1}$ using Boundary Integral Method [1]. So one can consider the boundary value problem

$$\begin{aligned}
&-\Delta \mathbf{v} + \nabla p + \alpha \mathbf{v} = \nabla \cdot \sigma \text{ in } \Omega,\\
&\nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\\
&\mathbf{v} = \mathbf{v}^* \text{ on } \gamma.
\end{aligned}$$

Here γ is the surface of the swimmer and $\mathbf{v}^*$ represents the slip velocity introduced in (1). Stress σ depends on given data $\mathbf{Q}^{n+1}$ and $\mathbf{C}^{n+1}$. Denote P as the domain inside of the swimmer. Introduce Green's function $G^i(x;y)$ of the Stokes problem in periodic box $\Omega \cup P$. Here $G^i(x;y)$ corresponds to the Dirac force along direction of the unit vector $\mathbf{e}_i$. Applying Green's Second Identity to $\mathbf{v}$ and $G^i(x;y)$, one can obtain

$$-\int_\Omega \nabla G^i(x;y) : \sigma dx + \int_\gamma G^i(x;y) \cdot (\nabla \mathbf{v} + \nabla \mathbf{v}^T - pI + \sigma) \cdot \vec{\boldsymbol{\nu}} dS(x) \tag{SI-4}$$

$$-\int_\gamma \mathbf{v} \cdot (\nabla G^i(x;y) + \nabla G^i(x;y)^T - qI) \cdot \vec{\boldsymbol{\nu}} dS(x) \tag{SI-5}$$

$$= \begin{cases} u^i(y) - \int_\Omega \mathbf{v}^i dx/|B|, & y \in \Omega,\\ \frac{1}{2}\mathbf{v}^i(y) - \int_\Omega \mathbf{v}^i dx/|B|, & y \in \gamma,\\ 0 - \int_\Omega \mathbf{v}^i dx/|B|, & y \in P. \end{cases} \tag{SI-6}$$

Define inner solution $\mathbf{w}$ in P such that

$$\begin{aligned}
&-\Delta \mathbf{w} + \alpha \mathbf{w} + \nabla r = 0 \text{ in } P,\\
&\nabla \cdot \mathbf{w} = 0 \text{ in } P,\\
&\big(\nabla \mathbf{w} + \nabla \mathbf{w}^T - rI\big) \cdot \vec{\boldsymbol{\nu}} = \big(\nabla \mathbf{v} + \nabla \mathbf{v}^T - pI + \sigma\big) \cdot \vec{\boldsymbol{\nu}} \text{ on } \gamma,
\end{aligned} \tag{SI-7}$$

where $\vec{\boldsymbol{\nu}}$ is the inward normal vector, pointing towards the center of the swimmer. Using the boundary condition (SI-7), one can get

$$\int_P \mathbf{w} dx = -\int_\Omega \mathbf{v} dx. \tag{SI-8}$$

Similiarly, appling Green's Second Identity to $\mathbf{v}$ and the Green's function $G(x;y)$ of the Stokes problem in periodic box $\Omega \cup P$, one can obtain

$$\int_\gamma G^i(x;y) \cdot (\nabla \mathbf{w} + \nabla \mathbf{w}^T - rI) \cdot (-\vec{\boldsymbol{\nu}}) dS(x) \tag{SI-9}$$

$$- \int_\gamma \mathbf{w} \cdot (\nabla G^i(x;y) + \nabla G^i(x;y)^T - qI) \cdot (-\vec{\boldsymbol{\nu}}) dS(x) \tag{SI-10}$$

$$= \begin{cases} 0 - \int_P \mathbf{w}^i dx/|B|, & y \in \Omega, \\ \frac{1}{2}\mathbf{w}^i(y) - \int_P \mathbf{w}^i dx/|B|, & y \in \gamma, \\ \mathbf{w}^i(y) - \int_P \mathbf{w}^i dx/|B|, & y \in P. \end{cases} \tag{SI-11}$$

Summing up (SI-4)-(SI-6) and (SI-9)-(SI-11), we derive

$$- \int_\Omega \nabla G^i(x;y) : \sigma(x) dx - \int_\gamma (\mathbf{v} - \mathbf{w}) \cdot (\nabla G^i(x;y) + \nabla G^i(x;y)^T - qI) \cdot \vec{\boldsymbol{\nu}} dS(x) \tag{SI-12}$$

$$= \begin{cases} \mathbf{v}^i(y) - \int_\Omega \mathbf{v}^i dx/|B| - \int_P \mathbf{w}^i dx/|B|, & y \in \Omega, \\ \frac{1}{2}(\mathbf{v}^i(y) + \mathbf{w}^i(y)) - \int_\Omega \mathbf{v}^i dx/|B| - \int_P \mathbf{w}^i dx/|B|, & y \in \gamma \\ \mathbf{w}^i(y) - \int_\Omega \mathbf{v}^i dx/|B| - \int_P \mathbf{w}^i dx/|B|, & y \in P, \end{cases} \tag{SI-13}$$

$$= \begin{cases} \mathbf{v}^i(y), & y \in \Omega, \\ \frac{1}{2}(\mathbf{v}^i(y) + \mathbf{w}^i(y)), & y \in \gamma \\ \mathbf{w}^i(y), & y \in P, \end{cases} \tag{SI-14}$$

$$= \begin{cases} \mathbf{v}^i(y), & y \in \Omega, \\ \mathbf{v}^i(y) - \frac{1}{2}(\mathbf{v}^i(y) - \mathbf{w}^i(y)), & y \in \gamma, \\ \mathbf{w}^i(y), & y \in P. \end{cases} \tag{SI-15}$$

Here (SI-14) follows (SI-13) due to (SI-8). Define auxiliary function $\boldsymbol{\phi} = \mathbf{v} - \mathbf{w}$ on γ. Using the second case in (SI-15), one can solve $\boldsymbol{\phi}$ on a discrete mesh defined on γ. Then based on the first case in (SI-14), we recover $\mathbf{v}$ in Ω.

Supplementary references:

[1] D. H. Yu. *Natural Boundary Integral Method and Its Applications.* Kluwer Academic Publishers, 2002.