Appendices

A Derivation of the thin shell model

The following analysis of an overpressured viscous shell of thickness h was first presented by Sparks *et al.*[1]. The Navier-Stokes equations in spherical coordinates:

continuity:

$$\frac{1}{r}\frac{\delta}{\delta r}\left(r^2 u\right) = 0 \tag{1}$$

r-momentum:

$$\frac{\delta u}{\delta t} + u\frac{\delta u}{\delta r} = -\frac{1}{\sigma}\frac{\mathrm{d}p}{\mathrm{d}r} + \nu\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2, \tag{2}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{2u}{r^2} \tag{3}$$

θ-momentum:

$$0 = -\frac{1}{\sigma r}\frac{\mathrm{d}p}{\mathrm{d}\theta} \tag{4}$$

λ-momentum:

$$0 = \frac{1}{\rho r \sin\theta}\frac{\mathrm{d}p}{\mathrm{d}\theta}. \tag{5}$$

This assumes radial symmetry. The bubble is small enough that the conditions at infinity can be neglected, and so only feels the local pressure field.

From the continuity equation

$$\frac{1}{r}\frac{\delta}{\delta r}\left(r^2 u\right) = 0 \tag{6}$$

it follows that $r^2 u$ must be a function of t (since $\delta/\delta r = 0$ and must therefore be a constant)

$$r^2 u = f(t) \Rightarrow u = f(t)/r^2 \tag{7}$$

The particle velocity at the bubble interface $r = R(t)$ is $R'(t)$. Therefore

$$u = R' = \frac{f(t)}{R^2} \Rightarrow u = \frac{R^2 R'}{r^2} \tag{8}$$

This form can be substituted into the r-momentum equation.

Similarly, the Navier-Stokes equations can be assumed to produce an equation for p:

$$p = \rho\left(\frac{2RR' + R^2 R''}{r^2} - \frac{R^4 R'^2}{r^5}\right) + p(\alpha, t) \tag{9}$$

Assuming that $p = p_i$ for $r < R$ and $p = p_e$ for $r > R + h$ in this case two equations for continuity of stress are valid applied across each of the two interfaces:

$$-p_i = p|_{r=R} + 2\mu\left.\frac{\delta u}{\delta r}\right|_{r=R} \tag{10}$$

$$-p_e = p|_{r=R+h} + 2\mu \frac{\delta u}{\delta r}\bigg|_{r=R+h} \tag{11}$$

The conservation of mass allows that the mass of the shell may be expressed as the difference between the mass of the whole and the mass of the bubble:

$$(R+h)^3 - R^3 = \frac{3M}{4\pi\rho} \tag{12}$$

and the Ideal Gas Law also applies.

Substituting 9 into the equations for internal and external pressure the following expression can be obtained:

$$p_i - p_e = r\left(\frac{3R'^2}{2} - R'^2\left(\frac{2R}{R+h} - \frac{R^4}{2(R+h)^4}\right) + R''\left(R - \frac{R^2}{R+h}\right)\right) + 4\mu R'\left(\frac{1}{R} - \frac{R^2}{(R+h)^3}\right). \tag{13}$$

Expanding using the binomial theorem yields the approximation

$$\frac{p_i - p_e}{\rho} \approx \frac{3h^2R'^2}{R^2} + hR'' + 12\frac{\mu hR}{R^2} \tag{14}$$

Binomial expansion of 12 will result in an approximation

$$h \sim \frac{M}{4\pi\rho R^2}. \tag{15}$$

Using this expression follows

$$\frac{p_i - p_e}{\rho} \approx \frac{3M^2}{(4\pi\rho)^2}\frac{R'^2}{R^6} + \left(\frac{M}{4\pi\rho}\right)\frac{R''}{R^2} + \frac{3M\mu}{\pi\rho}\frac{R'}{R^?} \tag{16}$$

Neglecting those terms related to inertial stress in the viscous analysis and using the binomial expansion 12, Eq. 16 can be re-expressed as

$$P_i - P_e = \frac{12\mu hR'}{R^2}. \tag{17}$$

B Analytical Solution Constants

From Recktenwald *et al.*[2], The coefficient ζ_n is obtained by calculating the roots of the equation

$$1 - \zeta_n \cot(\zeta_n) = Bi \tag{18}$$

where Bi, is the biot number, taken to be

$$Bi = \frac{hR_p}{k_b} \tag{19}$$

where all variables have been previously defined. These constants were subsequently used to calculate C with

$$C_n = \frac{4\left[\sin(\zeta_n) - \zeta_n \cos(\zeta_n)\right]}{2\zeta_n - \sin(2\zeta_n)} \tag{20}$$

References

1. Barclay, J., Riley, D. S. & Sparks, R. S. J., Analytical models for pore growth during decompression of high viscosity magmas, *Bulletin of Volcanology* **57**, 422–431 (1995).

2. Recktenwald, G., Transient, One-Dimensional Heat Conduction in a Convectively Cooled Sphere. (Portland State University, Oregon, 2006).