Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Stochastic Analysis of Quality Indicators and Cost of Quality in Stainless Steel Cold Rolling Plants Using Markov Chains

Hamdi EKİCİ

ekchamdi@gmail.com

École Supérieure de Gestion et de Commerce International: ESGCI https://orcid.org/0000-0002-5096-8066

Case Report

Keywords: Markov chains, cost of quality, stainless steel, cold rolling, stochastic modeling, operations research

Posted Date: November 27th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11894431/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: The authors declare no competing interests.

Stochastic Analysis of Quality Indicators and Cost of Quality in Stainless Steel Cold Rolling Plants Using Markov Chains

Hamdi Ekici

École Supérieure de Gestion, Paris, France
h_ekici@stu-esg.fr

Abstract

This study examines seven operational and economic quality indicators from a stainless steel cold rolling facility in Türkiye over 2019–2025. First-order discrete-time Markov chains with three performance states are estimated using maximum likelihood and compared with Laplace-smoothed models. Absorbing states occur in the conventional models for rework, second-quality production and cost of quality per ton. The estimated stationary cost-of-quality-to-turnover ratios are 3.424% and 4.642%, respectively.

Keywords: Markov chains, cost of quality, stainless steel, cold rolling, stochastic modeling, operations research.

1. Introduction

In the iron and steel industry, economic losses arising from quality non-conformities are significant in terms of operational performance due to the high costs of raw materials, energy, labour and capital. Surface defects, geometric non-conformities and deviations from customer specifications can lead to consequences such as rework, second-grade products, scrap and customer claims. The systematic measurement of quality costs supports improvement initiatives (Schiffauerova and Thomson, 2006). However, annual averages do not directly explain transitions between performance states. Markov chains enable the probabilistic modelling of these transitions (Norris, 1997).

1.1. Research questions

What are the transition probabilities for the quality indicators? Which indicators remain in states of poor performance? What are the long-term cost projections? To what extent does the Laplace correction alter the results? How can these results contribute to the prioritisation of economic interventions?

2. Literature review

The Prevention–Assessment–Failure (PAF) model classifies quality costs as prevention, assessment, internal failure and external failure costs (Schiffauerova and Thomson, 2006). Tsai (1998) discussed the measurement of quality costs within the framework of activity-based costing. Quality management requires not only the reduction of non-conformities but also the economic efficiency of prevention and evaluation activities.

$$\mathrm{CoQ} = \mathrm{C_p} + \mathrm{C_a} + \mathrm{C_iF} + \mathrm{C_eF} \quad (1)$$

Anderson and Goodman (1957) examined methods for estimating transition probabilities and making statistical inferences in Markov chains. The Markov property states that, given the current state, the next state is conditionally independent of past states (Norris, 1997). The contribution of this study is not a

new Markov theory, but rather the combined application of existing methods to operational and economic quality indicators. It is not claimed to be the first study of its kind in the literature.

The measurement of quality costs is linked to the maturity of the quality system and the scope of the costs (Sower et al., 2007; Yang, 2008). The identification of hidden failure costs also broadens the scope of economic evaluation (Snieska et al., 2013). Plewa et al. (2016) have contributed to the empirical examination of quality costs in modern manufacturing. The conceptual foundations of the PAF approach have been addressed by Feigenbaum (1991), Crosby (1979) and Juran and Godfrey (1999). This tradition requires that costs not be limited solely to final non-conformities. Reis et al. (2025) compared inspection strategies using real production data in terms of quality costs; Bertolini and Castellano (2026) developed the Quality Cost Deployment approach for the systematic reduction of total quality costs. These studies support the rationale for the concurrent monitoring of cost ratios alongside operational indicators in the present research.

The primary sources on the mechanics of cold rolling are Roberts (1978) and Lenard (2014), whilst Previati and Ballo (2026) evaluate the flatness prediction in terms of physical, numerical and data-driven methods. Annual quality rates are not a direct measurement of these physical mechanisms. Fontalvo Herrera et al. (2026) examined the preventive capacity of the occupational safety system in the steel industry using Six Sigma and Markov chains. This study, however, focuses on product quality and the economics of quality rather than occupational safety. Other applications of Markov methods in quality control can be seen in CUSUM and EWMA control charts (Brook and Evans, 1972; Lucas and Crosier, 1982; Lucas and Saccucci, 1990). These methods differ from the current annual state model; the comparison is limited to the level of stochastic approaches. Duncan (1956) discusses the economic design of control charts, whilst Montgomery (2020) addresses the general methods of statistical quality control.

The structural properties of finite chains have been addressed by Kemeny and Snell (1960) and Levin and Peres (2017). Billingsley (1961) is one of the fundamental sources on statistical inference in Markov processes. Whilst Gelman et al. (2013) provide a general framework for Bayesian modelling, Hussein and Sulaiman (2026) compared classical and Dirichlet-based estimates of transition matrices in sparse data. In this study, only the finite-mean transition matrix is used; a full Bayesian stationary distribution inference is not performed. The foundations of decision optimisation were developed by Bellman (1957), Howard (1960) and Puterman (1994). Drent et al. (2024) examined deterioration-dependent production policies, whilst Zhao et al. (2025) investigated condition-based maintenance within the context of Markov decision processes.

3. Method

In this quantitative, longitudinal, single-organisation case study, the annual indicators from TRX Metal's internal quality report dated 22 January 2026 were used. There are seven observations and six transitions for each indicator.

Table 1. Quality indicators for 2019–2025

Year	NonPrime (%)	ReWork (%)	2. Quality (%)	Scrap (%)	Claim (%)	CoQ/Turnover	CoQ/Ton
2019	4,774	10,374	1,056	3,730	2,549	8,945	186,196
2020	4,536	5,807	0,431	3,202	1,159	4,687	97,234
2021	1,947	9,976	0,424	3,910	0,609	3,043	84,279
2022	2,207	14,630	0,333	4,445	0,894	2,870	99,885
2023	3,149	16,556	0,605	4,961	1,183	4,007	105,051
2024	1,723	22,150	0,690	3,990	1,050	3,236	72,470

Year	NonPrime (%)	ReWork (%)	2. Quality (%)	Scrap (%)	Claim (%)	CoQ/Turnover	CoQ/Ton
2025	0,856	22,656	0,525	4,028	0,406	3,318	75,993

Source: TRX Metal (2026).

3.1. State space

Performance is classified as 'good' (İ) if $x \leq a$, 'average' (O) if $a < x \leq b$, and 'poor' (K) if $x > b$. The thresholds have been determined for exploratory analysis; they are not the organisation's approved targets.

Table 2. Threshold values

Indicator	a	b
Non-Prime	2,00	3,00
Rework	10,00	15,00
Second quality	0,50	0,75
Scrap	4,00	4,50
Claim	0,75	1,25
CoQ/Turnover	3,50	5,00
CoQ/Ton	80,00	110,00

$$S = \{İ, O, K\} \tag{2}$$

3.2. Transition estimation and stationary distribution

N_{ij} is the number of observed transitions from state i to state j. Maximum likelihood estimate (Anderson and Goodman, 1957):

$$p_{ij} = N_{ij} / \Sigma_j N_{ij} \tag{3}$$

The distribution for the next period, $v_n = v_0 P^n$, satisfies the conditions for a steady-state distribution: $\pi P = \pi$ and $\Sigma_i \pi_i = 1$. The initial state is the 2025 observation. The existence of a steady-state distribution does not, in itself, guarantee convergence (Norris, 1997).

$$v_n = v_0 P^n \tag{4}$$

$$\pi P = \pi, \ \Sigma_i \pi_i = 1 \tag{5}$$

3.3. Laplace correction and reward model

Symmetric Dirichlet with additive correction: $\tilde{p}_{ij} = (N_{ij} + \alpha)/(N_i + 3\alpha)$. The classical model uses $\alpha = 0$, whilst the Laplace model uses $\alpha = 1$. Positive transitions are irreducible and form a non-periodic chain. The state reward r_i is the arithmetic mean of the values observed in that state. The long-run expected value is $E(X) = \Sigma_i \pi_i r_i$. The steady-state distribution of the corrected matrix is not the exact Bayesian final expectation.

$$\tilde{p}_{ij} = (N_{ij} + \alpha)/(N_i + 3\alpha) \tag{6}$$

3.4. Feasibility function and calculation algorithm

When the initial state is held constant for the observed sequence of states $s_0,\ldots,s_6$, $L(P) = \prod_i \prod_j p_{ij}^{N_{ij}}$; $\log L(P) = \Sigma_i \Sigma_j N_{ij} \ln(p_{ij})$. Under the constraint that $\Sigma_j p_{ij} = 1$ in each row, the Lagrange multiplier yields $N_{ij}/p_{ij} - \lambda_i = 0$. From the row sums, $\lambda_i = N_i$ and $p_{ij} = N_{ij}/N_i$ are obtained. Zero counts are boundary solutions; the term $0 \ln 0$ is taken to be zero (Anderson and Goodman, 1957).

When the Dirichlet(α, α, α) prior and the multinomial likelihood are combined, the infinite row becomes Dirichlet($N_{i1}+\alpha$, $N_{i2}+\alpha$, $N_{i3}+\alpha$). The average of the virtual counts is of the form $(N_{ij}+\alpha)/(N_i+3\alpha)$. For $\alpha=1$,

one virtual transition is added to each cell; nine virtual counts correspond to a total of six real transitions. Consequently, the effect of the correction is significant (Gelman et al., 2013).

For a stationary distribution, the linear system $(P^T-I)\pi^T=0$ is solved by replacing the last row with (1,1,1) and the right-hand side with (0,0,1). The reward is $r_i = \Sigma_t\ x_t \cdot 1(s_t=i)/m_i$; where m_i is the number of observations for the relevant state. v_0 is the unit vector for the 2025 state. $v_{k+1} = v_kP$ and $E(X_k) = v_kr$ are calculated. The steady-state value πr need not be the same as the finite-period forecast.

It has been verified that the sum of the transition counts for each indicator is 6, the sum of the rows in each matrix is 1, and the sum of each probability vector is 1. The calculations were performed using unrounded values; the figures in the tables have been rounded for presentation purposes.

$$L(P) = \prod_i \prod_j p_{ij}^{Nij} \quad (7)$$

$$\ell(P) = \Sigma_i \Sigma_j N_{ij} \ln(p_{ij}) \quad (8)$$

$$(P^T - I)\pi^T = 0 \quad (9)$$

$$r_i = \Sigma_t x_t\, 1(s_t = i) / m_i \quad (10)$$

$$E(X_n) = v_n r;\ \ E(X\infty) = \pi r \quad (11)$$

4. Findings

Table 3. Historical performance figures

Indicator	**2019**	**2020**	**2021**	**2022**	**2023**	**2024**	**2025**
Non-Prime	K	K	İ	O	K	İ	İ
Rework	O	İ	İ	O	K	K	K
Second quality	K	İ	İ	İ	O	O	O
Scrap	İ	İ	İ	O	K	İ	O
Claim	K	O	İ	O	O	O	İ
CoQ/Turnover	K	O	İ	İ	O	İ	İ
CoQ/Ton	K	O	O	O	O	İ	İ

4.1. Non-Prime

Table 4. Transition frequencies and probabilities (row and column order: good, average, poor).

Status	**Frequencies**	**Classic P**	**Laplace P**
İ	1 / 1 / 0	0,500 / 0,500 / 0,000	0,400 / 0,400 / 0,200
O	0 / 0 / 1	0,000 / 0,000 / 1,000	0,250 / 0,250 / 0,500
K	2 / 0 / 1	0,667 / 0,000 / 0,333	0,500 / 0,167 / 0,333

Uniform distribution (classical): 44.44% / 22.22% / 33.33%. Laplace: 39.06% / 28.12% / 32.81%. The expected values are 2.545 and 2.573, respectively.

Calculation steps and finite-period projection

The six transitions observed: 2019→2020: K→K; 2020→2021: K→I; 2021→2022: I→O; 2022→2023: O→K; 2023→2024: K→I; 2024→2025: I→I.

Row I: N = (1, 1, 0), sum = 2. Classical: (1/2, 1/2, 0/2); Laplace: (2/5, 2/5, 1/5).

Row O: N = (0, 0, 1), sum = 1. Classical: (0/1, 0/1, 1/1); Laplace: (1/4, 1/4, 2/4).

Row K: N = (2, 0, 1), sum = 3. Classical: (2/3, 0/3, 1/3); Laplace: (3/6, 1/6, 2/6).

Classical equilibrium equations: πI = (1/2)πI + (0)πO + (2/3)πK; πO = (1/2)πI + (0)πO + (0)πK; πK = (0)πI + (1)πO + (1/3)πK; πI + πO + πK = 1. Solution: (4/9, 2/9, 1/3).

Laplace's equations of equilibrium: πI = (2/5)πI + (1/4)πO + (1/2)πK; πO = (2/5)πI + (1/4)πO + (1/6)πK; πK = (1/5)πI + (1/2)πO + (1/3)πK; πI + πO + πK = 1. Solution: (25/64, 9/32, 21/64).

rI = (1,947 + 1,723 + 0,856)/3 = 1,508667.

rO = (2,207)/1 = 2,207000.

rK = (4,774 + 4,536 + 3,149)/3 = 4,153000.

Classic long-term expectation: (4/9) × 1.508667 + (2/9) × 2.207000 + (1/3) × 4.153000 = 2.545296.

Laplace long-term expectation: (25/64) × 1.508667 + (9/32) × 2.207000 + (21/64) × 4.153000 = 2.572745.

The initial state is $v_0 = (1, 0, 0)$. The first step, v_1, is the matrix row corresponding to the initial state. Each component of the second step is calculated as $\Sigma_i v_{1i} p_{ij}$. For example, in the classical model, the probability of the 2027 state is: (1/2) × (1/2) + (1/2) × (0) + (0) × (2/3) = 1/4.

Table 5. Classical and Laplace I, O, K and E values

Year	Classical I/O/K (%)	E(X)	Laplace I/O/K (%)	E(X)
2026	50,00 / 50,00 / 0,00	1,858	40,00 / 40,00 / 20,00	2,317
2027	25,00 / 25,00 / 50,00	3,005	36,00 / 29,33 / 34,67	2,630
2028	45,83 / 12,50 / 41,67	2,698	39,07 / 27,51 / 33,42	2,585
2029	50,69 / 22,92 / 26,39	2,367	39,22 / 28,07 / 32,71	2,570
2030	42,94 / 25,35 / 31,71	2,524	39,06 / 28,16 / 32,78	2,572

The 2025 value is 0.856; the change from 2019 to 2025 is –3.918 units, representing a relative change of –82.07 per cent. The Laplace correction shifts the probability of the adverse scenario from 33.33 per cent to 32.81 per cent; the difference is –0.52 percentage points. This difference reflects sensitivity to the model assumption; it is not a newly observed change.

$$\pi O = \pi \dot{I}/2; \quad \pi K = 3\pi \dot{I}/4 \tag{12}$$

$$\pi \dot{I}(1 + 1/2 + 3/4) = 1 \tag{13}$$

$$\pi = (4/9, 2/9, 1/3) \tag{14}$$

$$v_2 = (1/2, 1/2, 0)P = (1/4, 1/4, 1/2) \tag{15}$$

In the Laplace solution, the second equilibrium equation is 3πO/4 = 2πİ/5 + πK/6, and the first equation is 3πİ/5 = πO/4 + πK/2. When solved with normalisation, πI = 25/64, πO = 18/64 and πK = 21/64 are found. The determinants of both models, the component products for all years and the prize calculations are given in Appendix A.

4.2. Rework

Table 6. Transition frequencies and probabilities (row and column order: good, average, poor).

Status	Frequencies	Classic P	Laplace P
I	1 / 1 / 0	0,500 / 0,500 / 0,000	0,400 / 0,400 / 0,200
O	1 / 0 / 1	0,500 / 0,000 / 0,500	0,400 / 0,200 / 0,400
K	0 / 0 / 2	0,000 / 0,000 / 1,000	0,200 / 0,200 / 0,600

Please refer to Appendix B for a detailed numerical solution.

Uniform distribution (classical): 0.00% / 0.00% / 100.00%. Laplace: 31.58% / 26.32% / 42.11%. The expected values are 20.454 and 14.394, respectively.

4.3. Second quality

Table 7. Transition frequencies and probabilities (row and column order: good, average, poor).

Status	Frequencies	Clşassic P	Laplace P
I	2 / 1 / 0	0,667 / 0,333 / 0,000	0,500 / 0,333 / 0,167
O	0 / 2 / 0	0,000 / 1,000 / 0,000	0,200 / 0,600 / 0,200
K	1 / 0 / 0	1,000 / 0,000 / 0,000	0,500 / 0,250 / 0,250

Please refer to Appendix C for a detailed numerical solution.

Uniform distribution (classical): 0.00% / 100.00% / 0.00%. Laplace: 37.04% / 43.21% / 19.75%. The expected values are 0.607 and 0.617, respectively.

4.4. Scrap

Tablo 8. Transition frequencies and probabilities (row and column order: good, average, poor).

Status	Frequencies	Classic P	Laplace P
I	2 / 2 / 0	0,500 / 0,500 / 0,000	0,429 / 0,429 / 0,143
O	0 / 0 / 1	0,000 / 0,000 / 1,000	0,250 / 0,250 / 0,500
K	1 / 0 / 0	1,000 / 0,000 / 0,000	0,500 / 0,250 / 0,250

Please refer to Appendix D for a detailed numerical solution.

Uniform distribution (classical): 50.00% / 25.00% / 25.00%. Laplace: 39.20% / 32.00% / 28.80%. The expected values are 4.153 and 4.238, respectively.

4.5. Claim

Tablo 9. Transition frequencies and probabilities (row and column order: good, average, poor).

Status	Frequencies	Classic P	Laplace P
I	0 / 1 / 0	0,000 / 1,000 / 0,000	0,250 / 0,500 / 0,250

Status	Frequencies	Classic P	Laplace P
O	2 / 2 / 0	0,500 / 0,500 / 0,000	0,429 / 0,429 / 0,143
K	0 / 1 / 0	0,000 / 1,000 / 0,000	0,250 / 0,500 / 0,250

Please refer to Appendix E for a detailed numerical solution.

Uniform distribution (classical): 33.33% / 66.67% / 0.00%. Laplace: 33.33% / 46.67% / 20.00%. The expected values are 0.883 and 1.179, respectively.

4.6. CoQ/Turnover

Tablo 10. Transition frequencies and probabilities (row and column order: good, average, poor).

Status	Frequencies	Classic P	Laplace P
I	2 / 1 / 0	0,667 / 0,333 / 0,000	0,500 / 0,333 / 0,167
O	2 / 0 / 0	1,000 / 0,000 / 0,000	0,600 / 0,200 / 0,200
K	0 / 1 / 0	0,000 / 1,000 / 0,000	0,250 / 0,500 / 0,250

Please refer to Appendix F for a detailed numerical solution.

Uniform distribution (classical): 75.00% / 25.00% / 0.00%. Laplace: 48.39% / 32.26% / 19.35%. The expected values are 3.424 and 4.642, respectively.

4.7. CoQ/Ton

Tablo 11. Transition frequencies and probabilities (row and column order: good, average, poor).

Status	Frequencies	Classic P	Laplace P
I	1 / 0 / 0	1,000 / 0,000 / 0,000	0,500 / 0,250 / 0,250
O	1 / 3 / 0	0,250 / 0,750 / 0,000	0,286 / 0,571 / 0,143
K	0 / 1 / 0	0,000 / 1,000 / 0,000	0,250 / 0,500 / 0,250

Please refer to Appendix G for a detailed numerical solution.

Uniform distribution (classical): 100.00% / 0.00% / 0.00%. Laplace: 35.44% / 44.30% / 20.25%. The expected values are 74.231 and 106.823, respectively.

5. Comparative model results

Table 12. Comparative model results for quality indicators

Indicator	2025	Classic E(X)	Laplace E(X)	Classic poor %	Laplace poor %
Non-Prime	0,856	2,545	2,573	33,33	32,81
Rework	22,656	20,454	14,394	100,00	42,11
Second quality	0,525	0,607	0,617	0,00	19,75
Scrap	4,028	4,153	4,238	25,00	28,80
Claim	0,406	0,883	1,179	0,00	20,00
CoQ/Turnover	3,318	3,424	4,642	0,00	19,35
CoQ/Ton	75,993	74,231	106,823	0,00	20,25

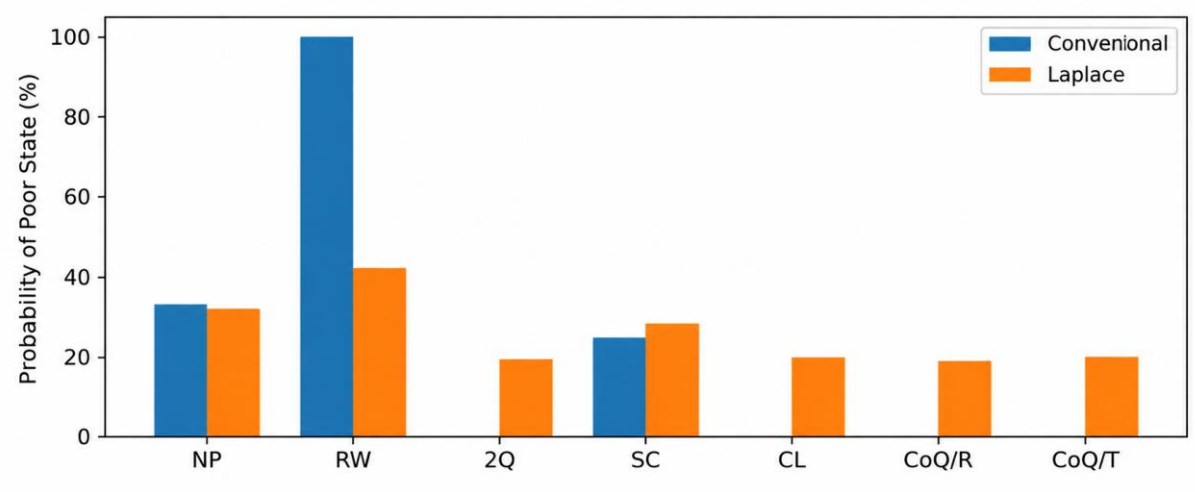

Figure 1: Comparison of the classical and Laplace models.

5.1. Sensitivity analysis

The effect of the a priori correction factor has been examined using the following alternative scenarios. The values were obtained by multiplying the mean values by a stationary distribution.

Table 13. Sensitivity analysis indicators

Indicator	**α=0**	**α=0,5**	**α=1**	**α=2**	**α=3**
Non-Prime	2,545	2,555	2,573	2,591	2,600
Rework	20,454	15,011	14,394	14,029	13,897
Second quality	0,607	0,595	0,617	0,641	0,653
Scrap	4,153	4,217	4,238	4,258	4,269
Claim	0,883	1,095	1,179	1,253	1,286
CoQ/Turnover	3,424	4,271	4,642	4,962	5,104
CoQ/Ton	74,231	101,312	106,823	111,453	113,531

5.2. Sensitivity to state thresholds

Both thresholds were reduced and increased by 10 per cent. In each scenario, the states were reclassified, and the counts and rewards were recalculated. The comparison was carried out with α = 1. If a state has never been observed, its reward is undefined; the expected value is not reported for that scenario.

Table 14. Sensitivity to state thresholds indicators

Indicator	**Threshold −%10**	**Basic**	**Threshold +%10**
Non-Prime	2,480	2,573	2,705
Rework	13,460	14,394	15,415
Second quality	0,587	0,617	0,625
Scrap	3,987	4,238	4,269
Claim	1,036	1,179	1,179
CoQ/Turnover	4,168	4,642	4,642
CoQ/Ton	Tanımsız	106,823	108,053

These scenarios do not serve as a substitute for the validation of thresholds as organisational objectives. As a change in thresholds alters both the transition matrix and the state rewards, the direction of the outcomes cannot be predetermined.

6. Economical analysis

Alternative cost estimates for annual turnover R and packaging tonnage Q are calculated as E(CoQ) = R·E(CoQ/Turnover)/100 and E(CoQ) = Q·E(CoQ/Ton). As the two estimates express the same cost using different denominators, they should not be added together. Example scenario: USD 120 million turnover and 50,000 tonnes of packaging.

Table 15. Economical analysis

.	**Classic (USD)**	**Laplace (USD)**
CoQ/Turnover	4,109,175	5,569,984
CoQ/Ton	3,711,575	5,341,169

The economic impact of a one percentage point improvement can be calculated using the formula ΔC = Q·c·Δx/100. Based on a common denominator of 50,000 tonnes and hypothetical unit costs, scrap (1,200 USD/tonne) shows a potential saving of 600,000 USD, claims (180) 90,000 USD, non-prime (120) 60,000 USD and rework (24) 12,000 USD. Overlapping losses should not be added together.

$$\mathrm{E(CoQ)} = \mathrm{R\,E(CoQ/Turnover)/100} \tag{16}$$

$$\mathrm{E(CoQ)} = \mathrm{Q\,E(CoQ/Ton)} \tag{17}$$

$$\Delta C = \mathrm{Qc\Delta x/100} \tag{18}$$

7. Discussion

During the 2019–2025 period, whilst the Non-Prime and Claim rates fell, the Rework rate increased. The hypothesis that rework may play a role in maintaining the final product quality could be investigated; however, causality cannot be established without controlling for product mix, raw material quality, tonnage and process conditions. In the classical model, the state absorbers are ‘poor’ for rework, ‘moderate’ for second-grade products and ‘good’ for CoQ/Ton. These may stem from limited observations. The Laplace correction eliminates the state absorbers but does not provide additional empirical evidence. A managerial risk assessment should consider probability, economic impact and uncertainty together.

The economic impact of the model selection should be assessed independently of the ranking of operational indicators. The fact that there is a difference of approximately 1.46 million USD in the turnover-based scenario and approximately 1.63 million USD in the tonnage-based scenario supports the use of alternative model scenarios in the budget rather than a single-point estimate. These are not confidence intervals, and the two differences cannot be added together.

When the increase in the rework rate is considered alongside the decline in non-prime and reclamation rates, the distinction between final quality and first-pass yield becomes significant. The loss of capacity caused by rework cannot be measured solely in terms of direct processing costs. To assess this, first-pass yield on a reel-by-reel basis, rework time and final grade must be taken into account.

The zero transition probabilities of the classical model should not be interpreted as physical impossibilities. The Laplace model, on the other hand, alters the structural assumption by assigning a positive probability to unobserved transitions; consequently, it has not been proven to be more accurate on the basis of the available data.

8. Further research: Markov decision processes

With three states for each of the five operational indicators, there are $3^5 = 243$ possible combined states. The available data is insufficient to estimate a matrix of this size. Dimension reduction and parametric transition models could be investigated. Using a Markov decision process (S, A, P, C, γ), decisions regarding maintenance, control, prevention and reprocessing can be linked to cost minimisation. The Bellman optimality equation:

$$V*(s) = \min_a [C(s,a) + \gamma \Sigma_{s'} P(s'|s,a)V*(s')] \quad (19)$$

As there is no transition data associated with the decision, this model has not been solved numerically.

9. Limitations

Six transitions per indicator are insufficient to reliably test for Markov property and time homogeneity. The thresholds have been selected by the researcher. Data from a single firm cannot be generalised to the sector as a whole. Annual aggregated indicators do not represent the physical transitions of individual materials. Monthly or weekly data, alternative thresholds, uncertainty intervals and out-of-sample validation are required.

10. Results

This study has enabled an exploratory analysis of seven years' worth of operational indicators using three-state Markov chains. The results should be evaluated by distinguishing between the observed performance and the model's conditional long-term behaviour.

10.1. Key points by indicator

Non-Prime — 2019: 4.774; 2025: 0.856; stationary expected value: 2.545 under the classical model and 2.573 under the Laplace-smoothed model. The substantial decline in the non-prime rate warrants further investigation into product classification practices and process improvements. Nevertheless, the probability of being in the poor state remains non-zero in both models. The sustainability of this improvement should be assessed using data on raw materials, product mix, and production line-specific defects.

Rework — 2019: 10.374; 2025: 22.656; stationary expected value: 20.454 under the classical model and 14.394 under the Laplace-smoothed model. The rework indicator represents the most pronounced operational warning signal. The absorbing nature of the poor state in the classical model is based solely on two observed transitions from the poor state to itself and does not imply that future improvement is impossible. Rework hours, production capacity losses, and first-pass yield should be monitored jointly.

Second-Quality Products — 2019: 1.056; 2025: 0.525; stationary expected value: 0.607 under the classical model and 0.617 under the Laplace-smoothed model. For second-quality products, the intermediate state is absorbing in the classical model, whereas the Laplace-smoothed model allows

transitions to alternative states. The consistency of classification criteria and product mix across the years should be verified.

Scrap — 2019: 3.730; 2025: 4.028; stationary expected value: 4.153 under the classical model and 4.238 under the Laplace-smoothed model. Both models retain a non-zero probability of the scrap rate being in the poor state. Under the assumed unit cost structure, scrap exhibits the highest economic impact per percentage-point change among the quality indicators. The underlying causes of scrap generation, metal recovery value, and net marginal economic loss should be quantified.

Claim — 2019: 2.549; 2025: 0.406; stationary expected value: 0.883 under the classical model and 1.179 under the Laplace-smoothed model. The decline in the claim rate represents a favourable historical finding; however, the zero probability of the poor state in the classical model does not guarantee sustained customer satisfaction. Delays in complaint reporting, sales tonnage, and the distribution of claims across customers should be examined.

CoQ/Turnover — 2019: 8.945; 2025: 3.318; stationary expected value: 3.424 under the classical model and 4.642 under the Laplace-smoothed model. The ratio of cost of quality to sales revenue is influenced by selling prices and product mix. The discrepancy between the classical and Laplace-smoothed stationary expected values highlights model risk in budget planning. An increase in sales revenue alone should not be interpreted as evidence of process improvement.

CoQ/Ton — 2019: 186.196; 2025: 75.993; stationary expected value: 74.231 under the classical model and 106.823 under the Laplace-smoothed model. The absorbing nature of the good state for the cost of quality per ton in the classical model results from the last two observations being classified as good and a single observed transition from the good state to itself. The Laplace-smoothed model yields a higher-cost scenario. The consistency between packaging tonnage and the scope of the costs included in the analysis should be verified.

10.2. Managerial implications and implementation plan

Operational and economic priorities should be distinguished. While rework may affect capacity utilisation and cycle time, scrap may result in direct material losses. The potential relationship between declining non-prime and claim rates and increasing rework remains a hypothesis; the seven-year time series does not establish a causal relationship.

In the first stage, monthly quality indicator and tonnage records should be consolidated using a common data dictionary, and the underlying causes of scrap and rework should be categorised by production line, product, alloy, and thickness. In the second stage, thresholds should be defined based on organisational objectives, and transition matrices should be estimated over a training period and subsequently validated using probability calibration and forecast error metrics. In the third stage, cost coefficients should be verified against financial records, and intervention costs should be compared with expected savings.

The separate models developed for the seven indicators do not constitute a joint multivariate Markov chain. Multiplying the probabilities requires an assumption of independence, which has not been tested in the present study. CoQ/Revenue and CoQ/Ton represent different normalisations of the same total cost; aggregating their economic outcomes would therefore result in double counting.

To extend the framework to a Markov decision process (MDP), the costs of actions such as maintenance, process adjustments, inspection, and training, together with their corresponding action-dependent transition probabilities, should be recorded. The Bellman equation presented in the current study constitutes a proposal for future research rather than a computationally derived optimal policy.

10.3. Integrated assessment of quantitative results

Non-Prime — Historical change, risk, and decision criterion

The absolute change between 2019 and 2025 was calculated as $0.856 - 4.774 = -3.918$, while the relative change was $(-3.918/4.774) \times 100 = -82.07\%$. The arithmetic mean over the seven-year period was 2.742, with minimum and maximum values of 0.856 and 4.774, respectively. The difference between the 2025 observation and the Laplace-smoothed stationary expected value is 1.717 units. This conditional comparison suggests that the performance observed in the most recent year should not be regarded as a sustainable long-term performance level.

Under the Laplace-smoothed model, the probability of being in the poor state was estimated at 20.00% for 2026, 32.78% for 2030, and 32.81% under stationary conditions. Given that the stationary probability of the poor state under the classical model is 33.33%, the difference between the two methods is −0.52 percentage points. The difference between their stationary expected values is 0.027 units. These values do not represent confidence intervals but rather a comparison between two distinct estimation approaches.

It should be determined whether the decline in the non-prime rate results from a genuine improvement in first-pass yield or from increased rework activity. Initial inspection results, the number of rework operations, and final product classifications should be recorded jointly at the coil level. Performance evaluation should therefore consider not only the final non-prime rate but also first-pass yield and total conversion costs.

Rework — Historical change, risk, and decision criterion

The absolute change between 2019 and 2025 was calculated as 22.656 − 10.374 = 12.282, while the relative change was (12.282/10.374) × 100 = 118.39%. The arithmetic mean over the seven-year period was 14.593, with minimum and maximum values of 5.807 and 22.656, respectively. The difference obtained by subtracting the 2025 observation from the Laplace-smoothed stationary expected value is −8.262 units. This conditional comparison suggests that the performance observed in the most recent year should not be regarded as a sustainable long-term performance level.

Under the Laplace-smoothed model, the probability of being in the poor state was estimated at 60.00% for 2026, 42.30% for 2030, and 42.11% under stationary conditions. Given that the stationary probability of the poor state under the classical model is 100.00%, the difference between the two methods is −57.89 percentage points. The difference between their stationary expected values is −6.060 units. These values do not represent confidence intervals but rather a comparison between two distinct estimation approaches.

In addition to rework tonnage, the number of repeated processing passes, production line hours, energy consumption, labour costs, and the opportunity cost of production capacity should be monitored. The absorbing structure resulting from the two observed transitions from the poor state to itself does not constitute evidence of intervention effectiveness. In a pilot improvement study, first-pass yield should be compared before and after the rework reduction intervention within the same product family.

Second quality — Historical change, risk, and decision criterion

The absolute change between 2019 and 2025 was calculated as 0.525 − 1.056 = −0.531, while the relative change was (−0.531/1.056) × 100 = −50.28%. The arithmetic mean over the seven-year period was 0.581, with minimum and maximum values of 0.333 and 1.056, respectively. The difference obtained by subtracting the 2025 observation from the Laplace-smoothed stationary expected value is 0.092 units. This conditional comparison suggests that the performance observed in the most recent year should not be regarded as a sustainable long-term performance level.

Under the Laplace-smoothed model, the probability of being in the poor state was estimated at 20.00% for 2026, 19.77% for 2030, and 19.75% under stationary conditions. Given that the stationary probability of the poor state under the classical model is 0.00%, the difference between the two methods is 19.75 percentage points. The difference between their stationary expected values is 0.011 units. These values do not represent confidence intervals but rather a comparison between two distinct estimation approaches.

The low absolute level of the second-quality product rate does not necessarily indicate a limited economic impact. The selling price differential between prime and second-quality products, reclassification costs, and inventory holding times should be quantified. Classification criteria and product mix should be reviewed to assess the persistence of the intermediate state.

Scrap — Historical change, risk, and decision criterion

The absolute change between 2019 and 2025 was calculated as 4.028 − 3.730 = 0.298, while the relative change was (0.298/3.730) × 100 = 7.99%. The arithmetic mean over the seven-year period was 4.038, with minimum and maximum values of 3.202 and 4.961, respectively. The difference obtained by subtracting the 2025 observation from the Laplace-smoothed stationary expected value is 0.210 units. This conditional comparison suggests that the performance observed in the most recent year should not be regarded as a sustainable long-term performance level.

Under the Laplace-smoothed model, the probability of being in the poor state was estimated at 50.00% for 2026, 28.84% for 2030, and 28.80% under stationary conditions. Given that the stationary probability of the poor state under the classical model is 25.00%, the difference between the two methods is 3.80 percentage points. The difference between their stationary expected values is 0.085 units. These values do not represent confidence intervals but rather a comparison between two distinct estimation approaches.

The net economic loss associated with scrap should be determined by deducting metal recovery revenue. Edge trimming losses, head and tail losses, and scrap generated by process-related defects should be distinguished, while the preventable scrap rate should be monitored separately. Owing to the high unit cost of scrap, even marginal reductions in the scrap rate may represent a significant economic priority.

Claim — Historical change, risk, and decision criterion

The absolute change between 2019 and 2025 was calculated as 0.406 − 2.549 = −2.143, while the relative change was (−2.143/2.549) × 100 = −84.07%. The arithmetic mean over the seven-year period was 1.121, with minimum and maximum values of 0.406 and 2.549, respectively. The difference obtained by subtracting the 2025 observation from the Laplace-smoothed stationary expected value is 0.773 units. This conditional comparison suggests that the performance observed in the most recent year should not be regarded as a sustainable long-term performance level.

Under the Laplace-smoothed model, the probability of being in the poor state was estimated at 25.00% for 2026, 20.00% for 2030, and 20.00% under stationary conditions. Given that the stationary probability of the poor state under the classical model is 0.00%, the difference between the two methods is 20.00 percentage points. The difference between their stationary expected values is 0.295 units. These values do not represent confidence intervals but rather a comparison between two distinct estimation approaches.

The production year and the reporting year of customer claims may differ. Reporting delay-adjusted claim rates should be established by sales cohort, customer, and defect type. A low claim rate is not the sole indicator of customer satisfaction; product returns, discounts, and warranty costs should also be monitored.

CoQ/Turnover — Historical change, risk, and decision criterion

The absolute change between 2019 and 2025 was calculated as 3.318 − 8.945 = −5.627, while the relative change was (−5.627/8.945) × 100 = −62.91%. The arithmetic mean over the seven-year period was 4.301, with minimum and maximum values of 2.870 and 8.945, respectively. The difference obtained by subtracting the 2025 observation from the Laplace-smoothed stationary expected value is 1.324 units. This conditional comparison suggests that the performance observed in the most recent year should not be regarded as a sustainable long-term performance level.

Under the Laplace-smoothed model, the probability of being in the poor state was estimated at 16.67% for 2026, 19.35% for 2030, and 19.35% under stationary conditions. Given that the stationary probability of the poor state under the classical model is 0.00%, the difference between the two methods is 19.35 percentage points. The difference between their stationary expected values is 1.217 units. These values do not represent confidence intervals but rather a comparison between two distinct estimation approaches.

The revenue-based cost of quality ratio is sensitive to fluctuations in selling prices and exchange rates. Recalculating this ratio using sales revenue at constant prices may help isolate price effects from operational effects. In budget planning scenarios, sales revenue and quality costs should be evaluated under a consistent set of assumptions.

CoQ/Ton — Historical change, risk, and decision criterion

The absolute change between 2019 and 2025 was calculated as 75.993 − 186.196 = −110.203, while the relative change was (−110.203/186.196) × 100 = −59.19%. The arithmetic mean over the seven-year period was 103.015, with minimum and maximum values of 72.470 and 186.196, respectively. The difference obtained by subtracting the 2025 observation from the Laplace-smoothed stationary expected value is 30.830 units. This conditional comparison suggests that the performance observed in the most recent year should not be regarded as a sustainable long-term performance level.

Under the Laplace-smoothed model, the probability of being in the poor state was estimated at 25.00% for 2026, 20.28% for 2030, and 20.25% under stationary conditions. Given that the stationary probability of the poor state under the classical model is 0.00%, the difference between the two methods is 20.25 percentage points. The difference between their stationary expected values is 32.592 units. These values do not represent confidence intervals but rather a comparison between two distinct estimation approaches.

The cost of quality per ton is an indicator more closely aligned with the physical scale of production; however, packaging tonnage and the scope of the costs included in the analysis must be consistent. Costs and tonnage should be matched at the product family level to isolate the effect of product mix. The resulting economic outcomes should not be aggregated with those derived from the revenue-based cost of quality indicator, as this would result in double counting.

10.4. Economic decision-making and research implementation

The difference between the models, amounting to approximately USD 1.46 million under the USD 120 million revenue scenario and USD 1.63 million under the 50,000-ton scenario, demonstrates the sensitivity of budget forecasts to model selection. These two differences cannot be aggregated because they are derived from different normalisations of the same underlying cost. To ensure that both methods account for the same actual cost, the revenue-to-tonnage ratio and the scope of the costs included in the analysis must be aligned.

Rework may be regarded as the operational priority, whereas scrap may be considered the economic priority under the assumed marginal cost coefficients. However, this prioritisation does not constitute a definitive investment decision. Intervention costs, expected avoidable losses, capacity implications, and implementation time should be evaluated jointly. For example, a reduction of 0.25 percentage points in the scrap rate would yield a potential gross saving of 50,000 × 0.0025 × 1,200 = USD 150,000, whereas an equivalent reduction in the rework rate would yield 50,000 × 0.0025 × 24 = USD 3,000. To determine the actual net economic benefit, implementation costs must be deducted.

If 84 monthly observations are obtained, a maximum of 83 consecutive transitions can be calculated. Although this represents a substantial increase compared with the six transitions available in the present study, it remains necessary to verify whether each state is observed with sufficient frequency. The training and testing periods should be separated chronologically, and predicted state probabilities should be evaluated against observed outcomes and benchmarked against a simple last-observation forecasting model. Sensitivity to prior assumptions and sensitivity to state classification thresholds should be reported separately.

The principal scientific contribution of this study lies in integrating the historical state transitions of quality indicators with their corresponding economic implications within a transparent analytical framework.

References

Anderson, T.W. and Goodman, L.A. (1957), "Statistical inference about Markov chains", *The Annals of Mathematical Statistics*, Vol. 28 No. 1, pp. 89–110. https://doi.org/10.1214/aoms/1177707039

Bellman, R. (1957), *Dynamic programming*, Princeton University Press.

Bertolini, M. and Castellano, D. (2026), "Quality cost deployment (QCD): A lean-inspired methodology for systematic reduction of total quality costs", *International Journal of Lean Six Sigma*, Vol. 17 No. 8, pp. 133–160. https://doi.org/10.1108/IJLSS-09-2025-0252

Billingsley, P. (1961), *Statistical inference for Markov processes*, University of Chicago Press.

Brook, D. and Evans, D.A. (1972), "An approach to the probability distribution of CUSUM run length", *Biometrika*, Vol. 59 No. 3, pp. 539–549.

Crosby, P.B. (1979), *Quality is free: The art of making quality certain*, McGraw-Hill.

Drent, C., Drent, M. and Arts, J. (2024), "Condition-based production for stochastically deteriorating systems: Optimal policies and learning", *Manufacturing & Service Operations Management*, Vol. 26 No. 3, pp. 1137–1156. https://doi.org/10.1287/msom.2022.0473

Duncan, A.J. (1956), "The economic design of X-bar charts used to maintain current control of a process", *Journal of the American Statistical Association*, Vol. 51 No. 274, pp. 228–242.

Feigenbaum, A.V. (1991), *Total quality control (3rd ed.)*, McGraw-Hill.

Fontalvo Herrera, T.J., Delahoz-Domínguez, E.J. and Rodelo Barrios, N. (2026), "Evaluation and prediction methods for a steel company using Six Sigma metrics, capability indicators, and Markov chains", *Eng*, Vol. 7 No. 8, p. 383. https://doi.org/10.3390/eng7080383

Gelman, A., Carlin, J.B., Stern, H.S., Dunson, D.B., Vehtari, A. and Rubin, D.B. (2013), *Bayesian data analysis (3rd ed.)*, CRC Press.

Howard, R.A. (1960), *Dynamic programming and Markov processes*, MIT Press.

Hussein, R. and Sulaiman, M. (2026), "Intelligent and Bayesian models for estimating the transition probability matrix of Markov chains: Applications to the analysis of sustainable development indicators", *Statistics, Optimization & Information Computing*, Vol. 16 No. 1, pp. 448–463. https://doi.org/10.19139/soic-2310-5070-3445

Juran, J.M. and Godfrey, A.B. (Eds.) (1999), *Juran's quality handbook (5th ed.)*, McGraw-Hill.

Kemeny, J.G. and Snell, J.L. (1960), *Finite Markov chains*, Van Nostrand.

Lenard, J.G. (2014), *Primer on flat rolling (2nd ed.)*, Elsevier.

Levin, D.A. and Peres, Y. (2017), *Markov chains and mixing times (2nd ed.)*, American Mathematical Society.

Lucas, J.M. and Crosier, R.B. (1982), "Fast initial response for CUSUM quality-control schemes: Give your CUSUM a head start", *Technometrics*, Vol. 24 No. 3, pp. 199–205.

Lucas, J.M. and Saccucci, M.S. (1990), "Exponentially weighted moving average control schemes: Properties and enhancements", *Technometrics*, Vol. 32 No. 1, pp. 1–12.

Montgomery, D.C. (2020), *Introduction to statistical quality control (8th ed.)*, Wiley.

Norris, J.R. (1997), *Markov chains*, Cambridge University Press. https://doi.org/10.1017/CBO9780511810633

Plewa, M., Kaiser, G. and Hartmann, E. (2016), "Is quality still free? Empirical evidence on quality cost in modern manufacturing", *International Journal of Quality & Reliability Management*, Vol. 33 No. 9, pp. 1270–1285.

Previati, G. and Ballo, F. (2026), "Strip flatness prediction in cold rolling: A review of models, challenges, and emerging directions", *International Journal of Advanced Manufacturing Technology*. https://doi.org/10.1007/s00170-026-19263-3

Puterman, M.L. (1994), *Markov decision processes: Discrete stochastic dynamic programming*, Wiley.

Reis, A.M., Dall-Orsoletta, A., Nunes, E., Costa, L. and Sousa, S. (2025), "Quality costs and Industry 4.0: Inspection strategy modelling and reviewing", *International Journal of Advanced Manufacturing Technology*, Vol. 136, pp. 3883–3897. https://doi.org/10.1007/s00170-024-13184-9

Roberts, W.L. (1978), *Cold rolling of steel*, Marcel Dekker.

Schiffauerova, A. and Thomson, V. (2006), "A review of research on cost of quality models and best practices", *International Journal of Quality & Reliability Management*, Vol. 23 No. 6, pp. 647–669. https://doi.org/10.1108/02656710610672470

Snieska, V., Daunoriene, A. and Zekeviciene, A. (2013), "Hidden costs in the evaluation of quality failure costs", *Engineering Economics*, Vol. 24 No. 3, pp. 176–186.

Sower, V.E., Quarles, R. and Broussard, E. (2007), "Cost of quality usage and its relationship to quality system maturity", *International Journal of Quality & Reliability Management*, Vol. 24 No. 2, pp. 121–140.

TRX Metal Sanayi ve Ticaret A.Ş. (2026), *2019–2025 quality indicators report – detailed* [Yayımlanmamış kurum içi rapor].

Tsai, W.H. (1998), "Quality cost measurement under activity-based costing", *International Journal of Quality & Reliability Management*, Vol. 15 No. 7, pp. 719–752. https://doi.org/10.1108/02656719810218202

Yang, C.C. (2008), "Improving the definition and quantification of quality costs", *Total Quality Management & Business Excellence*, Vol. 19 No. 3, pp. 175–191.

Zhao, X., Chen, P. and Tang, L.C. (2025), "Condition-based maintenance via Markov decision processes: A review", *Frontiers of Engineering Management*, Vol. 12 No. 2, pp. 330–342. https://doi.org/10.1007/s42524-024-4130-7

Appendix A. Non-Prime: Complete Numerical Solution

Detailed Numerical Calculations
Classical Model: Solution of Linear Equations

Coefficient Matrix of the Equilibrium System A: (-1/2, 0, 2/3); (1/2, -1, 0); (1, 1, 1). Right-Hand Side $b=(0, 0, 1)^T$.

det(A) = (-1/2)×[(-1)×(1)−(0)×(1)] − (0)×[(1/2)×(1)−(0)×(1)] + (2/3)×[(1/2)×(1)−(-1)×(1)] = 3/2.

det(A_İ) = (0)×[(-1)×(1)−(0)×(1)] − (0)×[(0)×(1)−(0)×(1)] + (2/3)×[(0)×(1)−(-1)×(1)] = 2/3; πİ = (2/3)/(3/2) = 4/9 = 44,4444%.

det(A_O) = (-1/2)×[(0)×(1)−(0)×(1)] − (0)×[(1/2)×(1)−(0)×(1)] + (2/3)×[(1/2)×(1)−(0)×(1)] = 1/3; πO = (1/3)/(3/2) = 2/9 = 22,2222%.

det(A_K) = (-1/2)×[(-1)×(1)−(0)×(1)] − (0)×[(1/2)×(1)−(0)×(1)] + (0)×[(1/2)×(1)−(-1)×(1)] = 1/2; πK = (1/2)/(3/2) = 1/3 = 33,3333%.

Control: 4/9 + 2/9 + 1/3 = 1. Expected Value: (4/9)×(2263/1500) + (2/9)×(2207/1000) + (1/3)×(4153/1000) = 68723/27000 = 2,545296.

2026: v = (1, 0, 0) × P.

İ: (1)×(1/2) + (0)×(0) + (0)×(2/3) = 1/2 = 50,0000%.

O: (1)×(1/2) + (0)×(0) + (0)×(0) = 1/2 = 50,0000%.

K: (1)×(0) + (0)×(1) + (0)×(1/3) = 0 = 0,0000%.

E(X) = (1/2)×(2263/1500) + (1/2)×(2207/1000) + (0)×(4153/1000) = 1,857833.

2027: v = (1/2, 1/2, 0) × P.

İ: (1/2)×(1/2) + (1/2)×(0) + (0)×(2/3) = 1/4 = 25,0000%.

O: (1/2)×(1/2) + (1/2)×(0) + (0)×(0) = 1/4 = 25,0000%.

K: (1/2)×(0) + (1/2)×(1) + (0)×(1/3) = 1/2 = 50,0000%.

E(X) = (1/4)×(2263/1500) + (1/4)×(2207/1000) + (1/2)×(4153/1000) = 3,005417.

2028: v = (1/4, 1/4, 1/2) × P.

İ: (1/4)×(1/2) + (1/4)×(0) + (1/2)×(2/3) = 11/24 = 45,8333%.

O: (1/4)×(1/2) + (1/4)×(0) + (1/2)×(0) = 1/8 = 12,5000%.

K: (1/4)×(0) + (1/4)×(1) + (1/2)×(1/3) = 5/12 = 41,6667%.

E(X) = (11/24)×(2263/1500) + (1/8)×(2207/1000) + (5/12)×(4153/1000) = 2,697764.

2029: v = (11/24, 1/8, 5/12) × P.

İ: (11/24)×(1/2) + (1/8)×(0) + (5/12)×(2/3) = 73/144 = 50,6944%.

O: (11/24)×(1/2) + (1/8)×(0) + (5/12)×(0) = 11/48 = 22,9167%.

K: (11/24)×(0) + (1/8)×(1) + (5/12)×(1/3) = 19/72 = 26,3889%.

E(X) = (73/144)×(2263/1500) + (11/48)×(2207/1000) + (19/72)×(4153/1000) = 2,366512.

2030: v = (73/144, 11/48, 19/72) × P.

İ: (73/144)×(1/2) + (11/48)×(0) + (19/72)×(2/3) = 371/864 = 42,9398%.

O: (73/144)×(1/2) + (11/48)×(0) + (19/72)×(0) = 73/288 = 25,3472%.

K: (73/144)×(0) + (11/48)×(1) + (19/72)×(1/3) = 137/432 = 31,7130%.

E(X) = (371/864)×(2263/1500) + (73/288)×(2207/1000) + (137/432)×(4153/1000) = 2,524271.

Laplace (α=1) model: Solution of Linear Equations

Coefficient Matrix of the Equilibrium System A: (-3/5, 1/4, 1/2); (2/5, -3/4, 1/6); (1, 1, 1). Right-Hand Side b=(0, 0, 1)^{T}.

det(A) = (-3/5)×[(-3/4)×(1)−(1/6)×(1)] − (1/4)×[(2/5)×(1)−(1/6)×(1)] + (1/2)×[(2/5)×(1)−(-3/4)×(1)] = 16/15.

det(A_İ) = (0)×[(-3/4)×(1)−(1/6)×(1)] − (1/4)×[(0)×(1)−(1/6)×(1)] + (1/2)×[(0)×(1)−(-3/4)×(1)] = 5/12; πİ = (5/12)/(16/15) = 25/64 = 39,0625%.

det(A_O) = (-3/5)×[(0)×(1)−(1/6)×(1)] − (0)×[(2/5)×(1)−(1/6)×(1)] + (1/2)×[(2/5)×(1)−(0)×(1)] = 3/10; πO = (3/10)/(16/15) = 9/32 = 28,1250%.

det(A_K) = (-3/5)×[(-3/4)×(1)−(0)×(1)] − (1/4)×[(2/5)×(1)−(0)×(1)] + (0)×[(2/5)×(1)−(-3/4)×(1)] = 7/20; πK = (7/20)/(16/15) = 21/64 = 32,8125%.

Control: 25/64 + 9/32 + 21/64 = 1. Expected Value: (25/64)×(2263/1500) + (9/32)×(2207/1000) + (21/64)×(4153/1000) = 493967/192000 = 2,572745.

2026: v = (1, 0, 0) × P.

İ: (1)×(2/5) + (0)×(1/4) + (0)×(1/2) = 2/5 = 40,0000%.

O: (1)×(2/5) + (0)×(1/4) + (0)×(1/6) = 2/5 = 40,0000%.

K: (1)×(1/5) + (0)×(1/2) + (0)×(1/3) = 1/5 = 20,0000%.

E(X) = (2/5)×(2263/1500) + (2/5)×(2207/1000) + (1/5)×(4153/1000) = 2,316867.

2027: v = (2/5, 2/5, 1/5) × P.

İ: (2/5)×(2/5) + (2/5)×(1/4) + (1/5)×(1/2) = 9/25 = 36,0000%.

O: (2/5)×(2/5) + (2/5)×(1/4) + (1/5)×(1/6) = 22/75 = 29,3333%.

K: (2/5)×(1/5) + (2/5)×(1/2) + (1/5)×(1/3) = 26/75 = 34,6667%.

E(X) = (9/25)×(2263/1500) + (22/75)×(2207/1000) + (26/75)×(4153/1000) = 2,630213.

2028: v = (9/25, 22/75, 26/75) × P.

İ: (9/25)×(2/5) + (22/75)×(1/4) + (26/75)×(1/2) = 293/750 = 39,0667%.

O: (9/25)×(2/5) + (22/75)×(1/4) + (26/75)×(1/6) = 619/2250 = 27,5111%.

K: (9/25)×(1/5) + (22/75)×(1/2) + (26/75)×(1/3) = 376/1125 = 33,4222%.

E(X) = (293/750)×(2263/1500) + (619/2250)×(2207/1000) + (376/1125)×(4153/1000) = 2,584581.

2029: v = (293/750, 619/2250, 376/1125) × P.

İ: (293/750)×(2/5) + (619/2250)×(1/4) + (376/1125)×(1/2) = 17647/45000 = 39,2156%.

O: (293/750)×(2/5) + (619/2250)×(1/4) + (376/1125)×(1/6) = 37901/135000 = 28,0748%.

K: (293/750)×(1/5) + (619/2250)×(1/2) + (376/1125)×(1/3) = 22079/67500 = 32,7096%.

E(X) = (17647/45000)×(2263/1500) + (37901/135000)×(2207/1000) + (22079/67500)×(4153/1000) = 2,569674.

2030: v = (17647/45000, 37901/135000, 22079/67500) × P.

İ: (17647/45000)×(2/5) + (37901/135000)×(1/4) + (22079/67500)×(1/2) = 1054613/2700000 = 39,0597%.

O: (17647/45000)×(2/5) + (37901/135000)×(1/4) + (22079/67500)×(1/6) = 2280679/8100000 = 28,1565%.

K: (17647/45000)×(1/5) + (37901/135000)×(1/2) + (22079/67500)×(1/3) = 1327741/4050000 = 32,7837%.

E(X) = (1054613/2700000)×(2263/1500) + (2280679/8100000)×(2207/1000) + (1327741/4050000)×(4153/1000) = 2,572204.

Appendix B. Rework: Complete Numerical Solution

Calculation Steps and Finite-Horizon Projection

Six Observed Transitions: 2019→2020: O→İ; 2020→2021: İ→İ; 2021→2022: İ→O; 2022→2023: O→K; 2023→2024: K→K; 2024→2025: K→K.

İ satırı: N = (1, 1, 0), toplam = 2. Klasik: (1/2, 1/2, 0/2); Laplace: (2/5, 2/5, 1/5).

O satırı: N = (1, 0, 1), toplam = 2. Klasik: (1/2, 0/2, 1/2); Laplace: (2/5, 1/5, 2/5).

K satırı: N = (0, 0, 2), toplam = 2. Klasik: (0/2, 0/2, 2/2); Laplace: (1/5, 1/5, 3/5).

Classical Equilibrium Equations: $\pi İ = (1/2)\pi İ + (1/2)\pi O + (0)\pi K$; $\pi O = (1/2)\pi İ + (0)\pi O + (0)\pi K$; $\pi K = (0)\pi İ + (1/2)\pi O + (1)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (0, 0, 1).

Laplace Equilibrium Equations: $\pi İ = (2/5)\pi İ + (2/5)\pi O + (1/5)\pi K$; $\pi O = (2/5)\pi İ + (1/5)\pi O + (1/5)\pi K$; $\pi K = (1/5)\pi İ + (2/5)\pi O + (3/5)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (6/19, 5/19, 8/19).

$rİ = (5{,}807 + 9{,}976)/2 = 7{,}891500$.

$rO = (10{,}374 + 14{,}630)/2 = 12{,}502000$.

$rK = (16{,}556 + 22{,}150 + 22{,}656)/3 = 20{,}454000$.

Classical Long-Run Expected Value: $(0)\times 7{,}891500 + (0)\times 12{,}502000 + (1)\times 20{,}454000 = 20{,}454000$.

Laplace Long-Run Expected Value: $(6/19)\times 7{,}891500 + (5/19)\times 12{,}502000 + (8/19)\times 20{,}454000 = 14{,}394263$.

The initial state vector is $v_0 = (0, 0, 1)$. The first-step vector, v_1, corresponds to the row of the transition matrix associated with the initial state. Each component of the second-step vector is calculated using $\Sigma_i v_{1i}p_{ij}$. For example, the probability of being in the good state in 2027 under the classical model is calculated as follows: $(0) \times (1/2) + (0) \times (1/2) + (1) \times (0) = 0$.

Year	Classic I/O/K (%)	E(X)	Laplace I/O/K (%)	E(X)
2026	0,00 / 0,00 / 100,00	20,454	20,00 / 20,00 / 60,00	16,351
2027	0,00 / 0,00 / 100,00	20,454	28,00 / 24,00 / 48,00	15,028
2028	0,00 / 0,00 / 100,00	20,454	30,40 / 25,60 / 44,00	14,599
2029	0,00 / 0,00 / 100,00	20,454	31,20 / 26,08 / 42,72	14,461
2030	0,00 / 0,00 / 100,00	20,454	31,46 / 26,24 / 42,30	14,416

The observed value in 2025 was 22.656, with an absolute change of 12.282 units and a relative change of 118.39% over the 2019–2025 period. Laplace smoothing reduces the stationary probability of the poor state from 100.00% to 42.11%, corresponding to a difference of −57.89 percentage points. This difference reflects sensitivity to model assumptions rather than a newly observed change in actual performance.

Detailed Numerical Calculations

Classical Model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-1/2, 1/2, 0); (1/2, -1, 0); (1, 1, 1). Right-Hand Side Vector $b = (0, 0, 1)^T$.

$\det(A) = (-1/2)\times[(-1)\times(1)-(0)\times(1)] - (1/2)\times[(1/2)\times(1)-(0)\times(1)] + (0)\times[(1/2)\times(1)-(-1)\times(1)] = 1/4$.

$\det(A_İ) = (0)\times[(-1)\times(1)-(0)\times(1)] - (1/2)\times[(0)\times(1)-(0)\times(1)] + (0)\times[(0)\times(1)-(-1)\times(1)] = 0$; $\pi İ = (0)/(1/4) = 0 = 0{,}0000\%$.

det(A_O) = (-1/2)×[(0)×(1)−(0)×(1)] − (0)×[(1/2)×(1)−(0)×(1)] + (0)×[(1/2)×(1)−(0)×(1)] = 0; πO = (0)/(1/4) = 0 = 0,0000%.

det(A_K) = (-1/2)×[(-1)×(1)−(0)×(1)] − (1/2)×[(1/2)×(1)−(0)×(1)] + (0)×[(1/2)×(1)−(-1)×(1)] = 1/4; πK = (1/4)/(1/4) = 1 = 100,0000%.

Control: 0 + 0 + 1 = 1. Expected Value: (0)×(15783/2000) + (0)×(6251/500) + (1)×(10227/500) = 10227/500 = 20,454000.

2026: v = (0, 0, 1) × P.

İ: (0)×(1/2) + (0)×(1/2) + (1)×(0) = 0 = 0,0000%.

O: (0)×(1/2) + (0)×(0) + (1)×(0) = 0 = 0,0000%.

K: (0)×(0) + (0)×(1/2) + (1)×(1) = 1 = 100,0000%.

E(X) = (0)×(15783/2000) + (0)×(6251/500) + (1)×(10227/500) = 20,454000.

2027: v = (0, 0, 1) × P.

İ: (0)×(1/2) + (0)×(1/2) + (1)×(0) = 0 = 0,0000%.

O: (0)×(1/2) + (0)×(0) + (1)×(0) = 0 = 0,0000%.

K: (0)×(0) + (0)×(1/2) + (1)×(1) = 1 = 100,0000%.

E(X) = (0)×(15783/2000) + (0)×(6251/500) + (1)×(10227/500) = 20,454000.

2028: v = (0, 0, 1) × P.

İ: (0)×(1/2) + (0)×(1/2) + (1)×(0) = 0 = 0,0000%.

O: (0)×(1/2) + (0)×(0) + (1)×(0) = 0 = 0,0000%.

K: (0)×(0) + (0)×(1/2) + (1)×(1) = 1 = 100,0000%.

E(X) = (0)×(15783/2000) + (0)×(6251/500) + (1)×(10227/500) = 20,454000.

2029: v = (0, 0, 1) × P.

İ: (0)×(1/2) + (0)×(1/2) + (1)×(0) = 0 = 0,0000%.

O: (0)×(1/2) + (0)×(0) + (1)×(0) = 0 = 0,0000%.

K: (0)×(0) + (0)×(1/2) + (1)×(1) = 1 = 100,0000%.

E(X) = (0)×(15783/2000) + (0)×(6251/500) + (1)×(10227/500) = 20,454000.

2030: v = (0, 0, 1) × P.

İ: (0)×(1/2) + (0)×(1/2) + (1)×(0) = 0 = 0,0000%.

O: (0)×(1/2) + (0)×(0) + (1)×(0) = 0 = 0,0000%.

K: (0)×(0) + (0)×(1/2) + (1)×(1) = 1 = 100,0000%.

E(X) = (0)×(15783/2000) + (0)×(6251/500) + (1)×(10227/500) = 20,454000.

Laplace (α=1) model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-3/5, 2/5, 1/5); (2/5, -4/5, 1/5); (1, 1, 1). Right-Hand Side Vector b =$(0, 0, 1)^T$.

det(A) = (-3/5)×[(-4/5)×(1)−(1/5)×(1)] − (2/5)×[(2/5)×(1)−(1/5)×(1)] + (1/5)×[(2/5)×(1)−(-4/5)×(1)] = 19/25.

det(A_İ) = (0)×[(-4/5)×(1)−(1/5)×(1)] − (2/5)×[(0)×(1)−(1/5)×(1)] + (1/5)×[(0)×(1)−(-4/5)×(1)] = 6/25; πİ = (6/25)/(19/25) = 6/19 = 31,5789%.

det(A_O) = (-3/5)×[(0)×(1)−(1/5)×(1)] − (0)×[(2/5)×(1)−(1/5)×(1)] + (1/5)×[(2/5)×(1)−(0)×(1)] = 1/5; πO = (1/5)/(19/25) = 5/19 = 26,3158%.

det(A_K) = (-3/5)×[(-4/5)×(1)−(0)×(1)] − (2/5)×[(2/5)×(1)−(0)×(1)] + (0)×[(2/5)×(1)−(-4/5)×(1)] = 8/25; πK = (8/25)/(19/25) = 8/19 = 42,1053%.

Control: 6/19 + 5/19 + 8/19 = 1. Expected Value: (6/19)×(15783/2000) + (5/19)×(6251/500) + (8/19)×(10227/500) = 273491/19000 = 14,394263.

2026: v = (0, 0, 1) × P.

İ: (0)×(2/5) + (0)×(2/5) + (1)×(1/5) = 1/5 = 20,0000%.

O: (0)×(2/5) + (0)×(1/5) + (1)×(1/5) = 1/5 = 20,0000%.

K: (0)×(1/5) + (0)×(2/5) + (1)×(3/5) = 3/5 = 60,0000%.

E(X) = (1/5)×(15783/2000) + (1/5)×(6251/500) + (3/5)×(10227/500) = 16,351100.

2027: v = (1/5, 1/5, 3/5) × P.

İ: (1/5)×(2/5) + (1/5)×(2/5) + (3/5)×(1/5) = 7/25 = 28,0000%.

O: (1/5)×(2/5) + (1/5)×(1/5) + (3/5)×(1/5) = 6/25 = 24,0000%.

K: (1/5)×(1/5) + (1/5)×(2/5) + (3/5)×(3/5) = 12/25 = 48,0000%.

E(X) = (7/25)×(15783/2000) + (6/25)×(6251/500) + (12/25)×(10227/500) = 15,028020.

2028: v = (7/25, 6/25, 12/25) × P.

İ: (7/25)×(2/5) + (6/25)×(2/5) + (12/25)×(1/5) = 38/125 = 30,4000%.

O: (7/25)×(2/5) + (6/25)×(1/5) + (12/25)×(1/5) = 32/125 = 25,6000%.

K: (7/25)×(1/5) + (6/25)×(2/5) + (12/25)×(3/5) = 11/25 = 44,0000%.

E(X) = (38/125)×(15783/2000) + (32/125)×(6251/500) + (11/25)×(10227/500) = 14,599288.

2029: v = (38/125, 32/125, 11/25) × P.

İ: (38/125)×(2/5) + (32/125)×(2/5) + (11/25)×(1/5) = 39/125 = 31,2000%.

O: (38/125)×(2/5) + (32/125)×(1/5) + (11/25)×(1/5) = 163/625 = 26,0800%.

K: (38/125)×(1/5) + (32/125)×(2/5) + (11/25)×(3/5) = 267/625 = 42,7200%.

E(X) = (39/125)×(15783/2000) + (163/625)×(6251/500) + (267/625)×(10227/500) = 14,460618.

2030: v = (39/125, 163/625, 267/625) × P.

İ: (39/125)×(2/5) + (163/625)×(2/5) + (267/625)×(1/5) = 983/3125 = 31,4560%.

O: (39/125)×(2/5) + (163/625)×(1/5) + (267/625)×(1/5) = 164/625 = 26,2400%.

K: (39/125)×(1/5) + (163/625)×(2/5) + (267/625)×(3/5) = 1322/3125 = 42,3040%.

E(X) = (983/3125)×(15783/2000) + (164/625)×(6251/500) + (1322/3125)×(10227/500) = 14,415735.

Appendix C. Second-Quality Products: Complete Numerical Solution

Calculation Steps and Finite-Horizon Projection

Six Observed Transitions: 2019→2020: K→İ; 2020→2021: İ→İ; 2021→2022: İ→İ; 2022→2023: İ→O; 2023→2024: O→O; 2024→2025: O→O.

Row İ: N = (2, 1, 0), total = 3. Classic: (2/3, 1/3, 0/3); Laplace: (3/6, 2/6, 1/6).

Row O: N = (0, 2, 0), total = 2. Classic: (0/2, 2/2, 0/2); Laplace: (1/5, 3/5, 1/5).

Row K: N = (1, 0, 0), total = 1. Classic: (1/1, 0/1, 0/1); Laplace: (2/4, 1/4, 1/4).

Classical Equilibrium Equations: $\pi İ = (2/3)\pi İ + (0)\pi O + (1)\pi K$; $\pi O = (1/3)\pi İ + (1)\pi O + (0)\pi K$; $\pi K = (0)\pi İ + (0)\pi O + (0)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (0, 1, 0).

Laplace Equilibrium Equations: $\pi İ = (1/2)\pi İ + (1/5)\pi O + (1/2)\pi K$; $\pi O = (1/3)\pi İ + (3/5)\pi O + (1/4)\pi K$; $\pi K = (1/6)\pi İ + (1/5)\pi O + (1/4)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (10/27, 35/81, 16/81).

$rİ = (0{,}431 + 0{,}424 + 0{,}333)/3 = 0{,}396000$.

$rO = (0{,}605 + 0{,}690 + 0{,}525)/3 = 0{,}606667$.

$rK = (1{,}056)/1 = 1{,}056000$.

Classical Long-Run Expected Value: $(0)\times 0{,}396000 + (1)\times 0{,}606667 + (0)\times 1{,}056000 = 0{,}606667$.

Laplace Long-Run Expected Value: $(10/27)\times 0{,}396000 + (35/81)\times 0{,}606667 + (16/81)\times 1{,}056000 = 0{,}617399$.

The initial state vector is $v_0 = (0, 1, 0)$. The first-step vector, v_1, corresponds to the row of the transition matrix associated with the initial state. Each component of the second-step vector is calculated using $\Sigma_i v_{1i}p_{ij}$. For example, the probability of being in the good state in 2027 under the classical model is calculated as follows: $(0) \times (2/3) + (1) \times (0) + (0) \times (1) = 0$.

Year	Classic İ/O/K (%)	E(X)	Laplace İ/O/K (%)	E(X)
2026	0,00 / 100,00 / 0,00	0,607	20,00 / 60,00 / 20,00	0,654
2027	0,00 / 100,00 / 0,00	0,607	32,00 / 47,67 / 20,33	0,631
2028	0,00 / 100,00 / 0,00	0,607	35,70 / 44,35 / 19,95	0,621
2029	0,00 / 100,00 / 0,00	0,607	36,69 / 43,50 / 19,81	0,618
2030	0,00 / 100,00 / 0,00	0,607	36,95 / 43,28 / 19,77	0,618

The observed value in 2025 was 0.525, with an absolute change of −0.531 units and a relative change of −50.28% over the 2019–2025 period. Laplace smoothing increases the stationary probability of the poor state from 0.00% to 19.75%, corresponding to a difference of 19.75 percentage points. This difference reflects sensitivity to model assumptions rather than a newly observed change in actual performance.

Detailed Numerical Calculations

Classical Model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-1/3, 0, 1); (1/3, 0, 0); (1, 1, 1). Right-Hand Side Vector $b = (0, 0, 1)^T$.

$\det(A) = (-1/3)\times[(0)\times(1)-(0)\times(1)] - (0)\times[(1/3)\times(1)-(0)\times(1)] + (1)\times[(1/3)\times(1)-(0)\times(1)] = 1/3$.

$\det(A_İ) = (0)\times[(0)\times(1)-(0)\times(1)] - (0)\times[(0)\times(1)-(0)\times(1)] + (1)\times[(0)\times(1)-(0)\times(1)] = 0$; $\pi İ = (0)/(1/3) = 0 = 0{,}0000\%$.

$\det(A_O) = (-1/3)\times[(0)\times(1)-(0)\times(1)] - (0)\times[(1/3)\times(1)-(0)\times(1)] + (1)\times[(1/3)\times(1)-(0)\times(1)] = 1/3$; $\pi O = (1/3)/(1/3) = 1 = 100{,}0000\%$.

det(A_K) = (-1/3)×[(0)×(1)−(0)×(1)] − (0)×[(1/3)×(1)−(0)×(1)] + (0)×[(1/3)×(1)−(0)×(1)] = 0; πK = (0)/(1/3) = 0 = 0,0000%.

Control: 0 + 1 + 0 = 1. Expected Value: (0)×(99/250) + (1)×(91/150) + (0)×(132/125) = 91/150 = 0,606667.

2026: v = (0, 1, 0) × P.

İ: (0)×(2/3) + (1)×(0) + (0)×(1) = 0 = 0,0000%.

O: (0)×(1/3) + (1)×(1) + (0)×(0) = 1 = 100,0000%.

K: (0)×(0) + (1)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (0)×(99/250) + (1)×(91/150) + (0)×(132/125) = 0,606667.

2027: v = (0, 1, 0) × P.

İ: (0)×(2/3) + (1)×(0) + (0)×(1) = 0 = 0,0000%.

O: (0)×(1/3) + (1)×(1) + (0)×(0) = 1 = 100,0000%.

K: (0)×(0) + (1)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (0)×(99/250) + (1)×(91/150) + (0)×(132/125) = 0,606667.

2028: v = (0, 1, 0) × P.

İ: (0)×(2/3) + (1)×(0) + (0)×(1) = 0 = 0,0000%.

O: (0)×(1/3) + (1)×(1) + (0)×(0) = 1 = 100,0000%.

K: (0)×(0) + (1)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (0)×(99/250) + (1)×(91/150) + (0)×(132/125) = 0,606667.

2029: v = (0, 1, 0) × P.

İ: (0)×(2/3) + (1)×(0) + (0)×(1) = 0 = 0,0000%.

O: (0)×(1/3) + (1)×(1) + (0)×(0) = 1 = 100,0000%.

K: (0)×(0) + (1)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (0)×(99/250) + (1)×(91/150) + (0)×(132/125) = 0,606667.

2030: v = (0, 1, 0) × P.

İ: (0)×(2/3) + (1)×(0) + (0)×(1) = 0 = 0,0000%.

O: (0)×(1/3) + (1)×(1) + (0)×(0) = 1 = 100,0000%.

K: (0)×(0) + (1)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (0)×(99/250) + (1)×(91/150) + (0)×(132/125) = 0,606667.

Laplace (α=1) model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-1/2, 1/5, 1/2); (1/3, -2/5, 1/4); (1, 1, 1). Right-Hand Side Vector b =$(0, 0, 1)^T$.

det(A) = (-1/2)×[(-2/5)×(1)−(1/4)×(1)] − (1/5)×[(1/3)×(1)−(1/4)×(1)] + (1/2)×[(1/3)×(1)−(-2/5)×(1)] = 27/40.

det(A_İ) = (0)×[(-2/5)×(1)−(1/4)×(1)] − (1/5)×[(0)×(1)−(1/4)×(1)] + (1/2)×[(0)×(1)−(-2/5)×(1)] = 1/4; πİ = (1/4)/(27/40) = 10/27 = 37,0370%.

det(A_O) = (-1/2)×[(0)×(1)−(1/4)×(1)] − (0)×[(1/3)×(1)−(1/4)×(1)] + (1/2)×[(1/3)×(1)−(0)×(1)] = 7/24; πO = (7/24)/(27/40) = 35/81 = 43,2099%.

det(A_K) = (-1/2)×[(-2/5)×(1)−(0)×(1)] − (1/5)×[(1/3)×(1)−(0)×(1)] + (0)×[(1/3)×(1)−(-2/5)×(1)] = 2/15; πK = (2/15)/(27/40) = 16/81 = 19,7531%.

Control: 10/27 + 35/81 + 16/81 = 1. Expected Value: (10/27)×(99/250) + (35/81)×(91/150) + (16/81)×(132/125) = 37507/60750 = 0,617399.

2026: v = (0, 1, 0) × P.

İ: (0)×(1/2) + (1)×(1/5) + (0)×(1/2) = 1/5 = 20,0000%.

O: (0)×(1/3) + (1)×(3/5) + (0)×(1/4) = 3/5 = 60,0000%.

K: (0)×(1/6) + (1)×(1/5) + (0)×(1/4) = 1/5 = 20,0000%.

E(X) = (1/5)×(99/250) + (3/5)×(91/150) + (1/5)×(132/125) = 0,654400.

2027: v = (1/5, 3/5, 1/5) × P.

İ: (1/5)×(1/2) + (3/5)×(1/5) + (1/5)×(1/2) = 8/25 = 32,0000%.

O: (1/5)×(1/3) + (3/5)×(3/5) + (1/5)×(1/4) = 143/300 = 47,6667%.

K: (1/5)×(1/6) + (3/5)×(1/5) + (1/5)×(1/4) = 61/300 = 20,3333%.

E(X) = (8/25)×(99/250) + (143/300)×(91/150) + (61/300)×(132/125) = 0,630618.

2028: v = (8/25, 143/300, 61/300) × P.

İ: (8/25)×(1/2) + (143/300)×(1/5) + (61/300)×(1/2) = 357/1000 = 35,7000%.

O: (8/25)×(1/3) + (143/300)×(3/5) + (61/300)×(1/4) = 887/2000 = 44,3500%.

K: (8/25)×(1/6) + (143/300)×(1/5) + (61/300)×(1/4) = 399/2000 = 19,9500%.

E(X) = (357/1000)×(99/250) + (887/2000)×(91/150) + (399/2000)×(132/125) = 0,621101.

2029: v = (357/1000, 887/2000, 399/2000) × P.

İ: (357/1000)×(1/2) + (887/2000)×(1/5) + (399/2000)×(1/2) = 7339/20000 = 36,6950%.

O: (357/1000)×(1/3) + (887/2000)×(3/5) + (399/2000)×(1/4) = 17399/40000 = 43,4975%.

K: (357/1000)×(1/6) + (887/2000)×(1/5) + (399/2000)×(1/4) = 7923/40000 = 19,8075%.

E(X) = (7339/20000)×(99/250) + (17399/40000)×(91/150) + (7923/40000)×(132/125) = 0,618364.

2030: v = (7339/20000, 17399/40000, 7923/40000) × P.

İ: (7339/20000)×(1/2) + (17399/40000)×(1/5) + (7923/40000)×(1/2) = 147803/400000 = 36,9507%.

O: (7339/20000)×(1/3) + (17399/40000)×(3/5) + (7923/40000)×(1/4) = 1038769/2400000 = 43,2820%.

K: (7339/20000)×(1/6) + (17399/40000)×(1/5) + (7923/40000)×(1/4) = 474413/2400000 = 19,7672%.

E(X) = (147803/400000)×(99/250) + (1038769/2400000)×(91/150) + (474413/2400000)×(132/125) = 0,617644.

Appendix D. Scrap: Complete Numerical Solution

Calculation Steps and Finite-Horizon Projection

Six Observed Transitions: 2019→2020: İ→İ; 2020→2021: İ→İ; 2021→2022: İ→O; 2022→2023: O→K; 2023→2024: K→İ; 2024→2025: İ→O.

Row İ: N = (2, 2, 0), toplam = 4. Classic: (2/4, 2/4, 0/4); Laplace: (3/7, 3/7, 1/7).

Row O: N = (0, 0, 1), toplam = 1. Classic: (0/1, 0/1, 1/1); Laplace: (1/4, 1/4, 2/4).

Row K: N = (1, 0, 0), toplam = 1. Classic: (1/1, 0/1, 0/1); Laplace: (2/4, 1/4, 1/4).

Classical Equilibrium Equations: $\pi İ = (1/2)\pi İ + (0)\pi O + (1)\pi K$; $\pi O = (1/2)\pi İ + (0)\pi O + (0)\pi K$; $\pi K = (0)\pi İ + (1)\pi O + (0)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (1/2, 1/4, 1/4).

Laplace Equilibrium Equations: $\pi İ = (3/7)\pi İ + (1/4)\pi O + (1/2)\pi K$; $\pi O = (3/7)\pi İ + (1/4)\pi O + (1/4)\pi K$; $\pi K = (1/7)\pi İ + (1/2)\pi O + (1/4)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (49/125, 8/25, 36/125).

rİ = (3,730 + 3,202 + 3,910 + 3,990)/4 = 3,708000.

rO = (4,445 + 4,028)/2 = 4,236500.

rK = (4,961)/1 = 4,961000.

Classical Long-Run Expected Value: (1/2)×3,708000 + (1/4)×4,236500 + (1/4)×4,961000 = 4,153375.

Laplace Long-Run Expected Value: (49/125)×3,708000 + (8/25)×4,236500 + (36/125)×4,961000 = 4,237984.

The initial state vector is $v_0 = (0, 1, 0)$. The first-step vector, v_1, corresponds to the row of the transition matrix associated with the initial state. Each component of the second-step vector is calculated using $\Sigma_i v_{1i}p_{ij}$. For example, the probability of being in the good state in 2027 under the classical model is calculated as follows: (0) × (1/2) + (0) × (0) + (1) × (1) = 1.

Year	Classic İ/O/K (%)	E(X)	Laplace İ/O/K (%)	E(X)
2026	0,00 / 0,00 / 100,00	4,961	25,00 / 25,00 / 50,00	4,467
2027	100,00 / 0,00 / 0,00	3,708	41,96 / 29,46 / 28,57	4,222
2028	50,00 / 50,00 / 0,00	3,972	39,64 / 32,49 / 27,87	4,229
2029	25,00 / 25,00 / 50,00	4,467	39,05 / 32,08 / 28,88	4,239
2030	62,50 / 12,50 / 25,00	4,087	39,19 / 31,97 / 28,84	4,238

The observed value in 2025 was 4.028, with an absolute change of 0.298 units and a relative change of 7.99% over the 2019–2025 period. Laplace smoothing increases the stationary probability of the poor state from 25.00% to 28.80%, corresponding to a difference of 3.80 percentage points. This difference reflects sensitivity to model assumptions rather than a newly observed change in actual performance.

Detailed Numerical Calculations

Classical Model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-1/2, 0, 1); (1/2, -1, 0); (1, 1, 1). Right-Hand Side Vector $b=(0, 0, 1)^T$.

det(A) = (-1/2)×[(-1)×(1)−(0)×(1)] − (0)×[(1/2)×(1)−(0)×(1)] + (1)×[(1/2)×(1)−(-1)×(1)] = 2.

det(A_İ) = (0)×[(-1)×(1)−(0)×(1)] − (0)×[(0)×(1)−(0)×(1)] + (1)×[(0)×(1)−(-1)×(1)] = 1; $\pi İ$ = (1)/(2) = 1/2 = 50,0000%.

det(A_O) = (-1/2)×[(0)×(1)−(0)×(1)] − (0)×[(1/2)×(1)−(0)×(1)] + (1)×[(1/2)×(1)−(0)×(1)] = 1/2; πO = (1/2)/(2) = 1/4 = 25,0000%.

det(A_K) = (-1/2)×[(-1)×(1)−(0)×(1)] − (0)×[(1/2)×(1)−(0)×(1)] + (0)×[(1/2)×(1)−(-1)×(1)] = 1/2; πK = (1/2)/(2) = 1/4 = 25,0000%.

Control: 1/2 + 1/4 + 1/4 = 1. Expected Value: (1/2)×(927/250) + (1/4)×(8473/2000) + (1/4)×(4961/1000) = 33227/8000 = 4,153375.

2026: v = (0, 1, 0) × P.

İ: (0)×(1/2) + (1)×(0) + (0)×(1) = 0 = 0,0000%.

O: (0)×(1/2) + (1)×(0) + (0)×(0) = 0 = 0,0000%.

K: (0)×(0) + (1)×(1) + (0)×(0) = 1 = 100,0000%.

E(X) = (0)×(927/250) + (0)×(8473/2000) + (1)×(4961/1000) = 4,961000.

2027: v = (0, 0, 1) × P.

İ: (0)×(1/2) + (0)×(0) + (1)×(1) = 1 = 100,0000%.

O: (0)×(1/2) + (0)×(0) + (1)×(0) = 0 = 0,0000%.

K: (0)×(0) + (0)×(1) + (1)×(0) = 0 = 0,0000%.

E(X) = (1)×(927/250) + (0)×(8473/2000) + (0)×(4961/1000) = 3,708000.

2028: v = (1, 0, 0) × P.

İ: (1)×(1/2) + (0)×(0) + (0)×(1) = 1/2 = 50,0000%.

O: (1)×(1/2) + (0)×(0) + (0)×(0) = 1/2 = 50,0000%.

K: (1)×(0) + (0)×(1) + (0)×(0) = 0 = 0,0000%.

E(X) = (1/2)×(927/250) + (1/2)×(8473/2000) + (0)×(4961/1000) = 3,972250.

2029: v = (1/2, 1/2, 0) × P.

İ: (1/2)×(1/2) + (1/2)×(0) + (0)×(1) = 1/4 = 25,0000%.

O: (1/2)×(1/2) + (1/2)×(0) + (0)×(0) = 1/4 = 25,0000%.

K: (1/2)×(0) + (1/2)×(1) + (0)×(0) = 1/2 = 50,0000%.

E(X) = (1/4)×(927/250) + (1/4)×(8473/2000) + (1/2)×(4961/1000) = 4,466625.

2030: v = (1/4, 1/4, 1/2) × P.

İ: (1/4)×(1/2) + (1/4)×(0) + (1/2)×(1) = 5/8 = 62,5000%.

O: (1/4)×(1/2) + (1/4)×(0) + (1/2)×(0) = 1/8 = 12,5000%.

K: (1/4)×(0) + (1/4)×(1) + (1/2)×(0) = 1/4 = 25,0000%.

E(X) = (5/8)×(927/250) + (1/8)×(8473/2000) + (1/4)×(4961/1000) = 4,087313.

Laplace (α=1) model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-4/7, 1/4, 1/2); (3/7, -3/4, 1/4); (1, 1, 1). Right-Hand Side Vector b=$(0, 0, 1)^T$.

det(A) = (-4/7)×[(-3/4)×(1)−(1/4)×(1)] − (1/4)×[(3/7)×(1)−(1/4)×(1)] + (1/2)×[(3/7)×(1)−(-3/4)×(1)] = 125/112.

det(A_İ) = (0)×[(-3/4)×(1)−(1/4)×(1)] − (1/4)×[(0)×(1)−(1/4)×(1)] + (1/2)×[(0)×(1)−(-3/4)×(1)] = 7/16; πİ = (7/16)/(125/112) = 49/125 = 39,2000%.

det(A_O) = (-4/7)×[(0)×(1)−(1/4)×(1)] − (0)×[(3/7)×(1)−(1/4)×(1)] + (1/2)×[(3/7)×(1)−(0)×(1)] = 5/14; πO = (5/14)/(125/112) = 8/25 = 32,0000%.

det(A_K) = (-4/7)×[(-3/4)×(1)−(0)×(1)] − (1/4)×[(3/7)×(1)−(0)×(1)] + (0)×[(3/7)×(1)−(-3/4)×(1)] = 9/28; πK = (9/28)/(125/112) = 36/125 = 28,8000%.

Control: 49/125 + 8/25 + 36/125 = 1. Expected Value: (49/125)×(927/250) + (8/25)×(8473/2000) + (36/125)×(4961/1000) = 132437/31250 = 4,237984.

2026: v = (0, 1, 0) × P.

İ: (0)×(3/7) + (1)×(1/4) + (0)×(1/2) = 1/4 = 25,0000%.

O: (0)×(3/7) + (1)×(1/4) + (0)×(1/4) = 1/4 = 25,0000%.

K: (0)×(1/7) + (1)×(1/2) + (0)×(1/4) = 1/2 = 50,0000%.

E(X) = (1/4)×(927/250) + (1/4)×(8473/2000) + (1/2)×(4961/1000) = 4,466625.

2027: v = (1/4, 1/4, 1/2) × P.

İ: (1/4)×(3/7) + (1/4)×(1/4) + (1/2)×(1/2) = 47/112 = 41,9643%.

O: (1/4)×(3/7) + (1/4)×(1/4) + (1/2)×(1/4) = 33/112 = 29,4643%.

K: (1/4)×(1/7) + (1/4)×(1/2) + (1/2)×(1/4) = 2/7 = 28,5714%.

E(X) = (47/112)×(927/250) + (33/112)×(8473/2000) + (2/7)×(4961/1000) = 4,221719.

2028: v = (47/112, 33/112, 2/7) × P.

İ: (47/112)×(3/7) + (33/112)×(1/4) + (2/7)×(1/2) = 1243/3136 = 39,6365%.

O: (47/112)×(3/7) + (33/112)×(1/4) + (2/7)×(1/4) = 1019/3136 = 32,4936%.

K: (47/112)×(1/7) + (33/112)×(1/2) + (2/7)×(1/4) = 437/1568 = 27,8699%.

E(X) = (1243/3136)×(927/250) + (1019/3136)×(8473/2000) + (437/1568)×(4961/1000) = 4,228939.

2029: v = (1243/3136, 1019/3136, 437/1568) × P.

İ: (1243/3136)×(3/7) + (1019/3136)×(1/4) + (437/1568)×(1/2) = 34285/87808 = 39,0454%.

O: (1243/3136)×(3/7) + (1019/3136)×(1/4) + (437/1568)×(1/4) = 28167/87808 = 32,0779%.

K: (1243/3136)×(1/7) + (1019/3136)×(1/2) + (437/1568)×(1/4) = 6339/21952 = 28,8766%.

E(X) = (34285/87808)×(927/250) + (28167/87808)×(8473/2000) + (6339/21952)×(4961/1000) = 4,239356.

2030: v = (34285/87808, 28167/87808, 6339/21952) × P.

İ: (34285/87808)×(3/7) + (28167/87808)×(1/4) + (6339/21952)×(1/2) = 963573/2458624 = 39,1916%.

O: (34285/87808)×(3/7) + (28167/87808)×(1/4) + (6339/21952)×(1/4) = 786081/2458624 = 31,9724%.

K: (34285/87808)×(1/7) + (28167/87808)×(1/2) + (6339/21952)×(1/4) = 354485/1229312 = 28,8360%.

E(X) = (963573/2458624)×(927/250) + (786081/2458624)×(8473/2000) + (354485/1229312)×(4961/1000) = 4,238290.

Appendix E. Claim: Complete Numerical Solution

Calculation Steps and Finite-Horizon Projection

Six Observed Transitions: 2019→2020: K→O; 2020→2021: O→İ; 2021→2022: İ→O; 2022→2023: O→O; 2023→2024: O→O; 2024→2025: O→İ.

Row İ: N = (0, 1, 0), toplam = 1. Klasik: (0/1, 1/1, 0/1); Laplace: (1/4, 2/4, 1/4).

Row O: N = (2, 2, 0), toplam = 4. Klasik: (2/4, 2/4, 0/4); Laplace: (3/7, 3/7, 1/7).

Row K: N = (0, 1, 0), toplam = 1. Klasik: (0/1, 1/1, 0/1); Laplace: (1/4, 2/4, 1/4).

Classical Equilibrium Equations: $\pi İ = (0)\pi İ + (1/2)\pi O + (0)\pi K$; $\pi O = (1)\pi İ + (1/2)\pi O + (1)\pi K$; $\pi K = (0)\pi İ + (0)\pi O + (0)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (1/3, 2/3, 0).

Laplace Equilibrium Equation: $\pi İ = (1/4)\pi İ + (3/7)\pi O + (1/4)\pi K$; $\pi O = (1/2)\pi İ + (3/7)\pi O + (1/2)\pi K$; $\pi K = (1/4)\pi İ + (1/7)\pi O + (1/4)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (1/3, 7/15, 1/5).

rİ = (0,609 + 0,406)/2 = 0,507500.

rO = (1,159 + 0,894 + 1,183 + 1,050)/4 = 1,071500.

rK = (2,549)/1 = 2,549000.

Classical Long-Run Expected Value: (1/3)×0,507500 + (2/3)×1,071500 + (0)×2,549000 = 0,883500.

Laplace Long-Run Expected Value: (1/3)×0,507500 + (7/15)×1,071500 + (1/5)×2,549000 = 1,179000.

The initial state vector is $v_0 = (1, 0, 0)$. The first-step vector, v_1, corresponds to the row of the transition matrix associated with the initial state. Each component of the second-step vector is calculated using $\Sigma_i v_{1i}p_{ij}$. For example, the probability of being in the good state in 2027 under the classical model is calculated as follows: (0) × (0) + (1) × (1/2) + (0) × (0) = 1/2.

Year	Classic İ/O/K (%)	E(X)	Laplace İ/O/K (%)	E(X)
2026	0,00 / 100,00 / 0,00	1,071	25,00 / 50,00 / 25,00	1,300
2027	50,00 / 50,00 / 0,00	0,789	33,93 / 46,43 / 19,64	1,170
2028	25,00 / 75,00 / 0,00	0,930	33,29 / 46,68 / 20,03	1,180
2029	37,50 / 62,50 / 0,00	0,860	33,34 / 46,67 / 20,00	1,179
2030	31,25 / 68,75 / 0,00	0,895	33,33 / 46,67 / 20,00	1,179

The observed value in 2025 was 0.406, with an absolute change of −2.143 units and a relative change of −84.07% over the 2019–2025 period. Laplace smoothing increases the stationary probability of the poor state from 0.00% to 20.00%, corresponding to a difference of 20.00 percentage points. This difference reflects sensitivity to model assumptions rather than a newly observed change in actual performance.

Detailed Numerical Calculations

Classical Model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-1, 1/2, 0); (1, -1/2, 1); (1, 1, 1). Right-Hand Side Vector $b=(0, 0, 1)^T$.

det(A) = (-1)×[(-1/2)×(1)−(1)×(1)] − (1/2)×[(1)×(1)−(1)×(1)] + (0)×[(1)×(1)−(-1/2)×(1)] = 3/2.

det(A_İ) = (0)×[(-1/2)×(1)−(1)×(1)] − (1/2)×[(0)×(1)−(1)×(1)] + (0)×[(0)×(1)−(-1/2)×(1)] = 1/2; $\pi İ$ = (1/2)/(3/2) = 1/3 = 33,3333%.

det(A_O) = (-1)×[(0)×(1)−(1)×(1)] − (0)×[(1)×(1)−(1)×(1)] + (0)×[(1)×(1)−(0)×(1)] = 1; πO = (1)/(3/2) = 2/3 = 66,6667%.

det(A_K) = (-1)×[(-1/2)×(1)−(0)×(1)] − (1/2)×[(1)×(1)−(0)×(1)] + (0)×[(1)×(1)−(-1/2)×(1)] = 0; πK = (0)/(3/2) = 0 = 0,0000%.

Control: 1/3 + 2/3 + 0 = 1. Expected Value: (1/3)×(203/400) + (2/3)×(2143/2000) + (0)×(2549/1000) = 1767/2000 = 0,883500.

2026: v = (1, 0, 0) × P.

İ: (1)×(0) + (0)×(1/2) + (0)×(0) = 0 = 0,0000%.

O: (1)×(1) + (0)×(1/2) + (0)×(1) = 1 = 100,0000%.

K: (1)×(0) + (0)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (0)×(203/400) + (1)×(2143/2000) + (0)×(2549/1000) = 1,071500.

2027: v = (0, 1, 0) × P.

İ: (0)×(0) + (1)×(1/2) + (0)×(0) = 1/2 = 50,0000%.

O: (0)×(1) + (1)×(1/2) + (0)×(1) = 1/2 = 50,0000%.

K: (0)×(0) + (1)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (1/2)×(203/400) + (1/2)×(2143/2000) + (0)×(2549/1000) = 0,789500.

2028: v = (1/2, 1/2, 0) × P.

İ: (1/2)×(0) + (1/2)×(1/2) + (0)×(0) = 1/4 = 25,0000%.

O: (1/2)×(1) + (1/2)×(1/2) + (0)×(1) = 3/4 = 75,0000%.

K: (1/2)×(0) + (1/2)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (1/4)×(203/400) + (3/4)×(2143/2000) + (0)×(2549/1000) = 0,930500.

2029: v = (1/4, 3/4, 0) × P.

İ: (1/4)×(0) + (3/4)×(1/2) + (0)×(0) = 3/8 = 37,5000%.

O: (1/4)×(1) + (3/4)×(1/2) + (0)×(1) = 5/8 = 62,5000%.

K: (1/4)×(0) + (3/4)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (3/8)×(203/400) + (5/8)×(2143/2000) + (0)×(2549/1000) = 0,860000.

2030: v = (3/8, 5/8, 0) × P.

İ: (3/8)×(0) + (5/8)×(1/2) + (0)×(0) = 5/16 = 31,2500%.

O: (3/8)×(1) + (5/8)×(1/2) + (0)×(1) = 11/16 = 68,7500%.

K: (3/8)×(0) + (5/8)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (5/16)×(203/400) + (11/16)×(2143/2000) + (0)×(2549/1000) = 0,895250.

Laplace (α=1) model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-3/4, 3/7, 1/4); (1/2, -4/7, 1/2); (1, 1, 1). Right-Hand Side Vector b=$(0, 0, 1)^T$.

det(A) = (-3/4)×[(-4/7)×(1)−(1/2)×(1)] − (3/7)×[(1/2)×(1)−(1/2)×(1)] + (1/4)×[(1/2)×(1)−(-4/7)×(1)] = 15/14.

det(A_İ) = (0)×[(-4/7)×(1)−(1/2)×(1)] − (3/7)×[(0)×(1)−(1/2)×(1)] + (1/4)×[(0)×(1)−(-4/7)×(1)] = 5/14; πİ = (5/14)/(15/14) = 1/3 = 33,3333%.

det(A_O) = (-3/4)×[(0)×(1)−(1/2)×(1)] − (0)×[(1/2)×(1)−(1/2)×(1)] + (1/4)×[(1/2)×(1)−(0)×(1)] = 1/2; πO = (1/2)/(15/14) = 7/15 = 46,6667%.

det(A_K) = (-3/4)×[(-4/7)×(1)−(0)×(1)] − (3/7)×[(1/2)×(1)−(0)×(1)] + (0)×[(1/2)×(1)−(-4/7)×(1)] = 3/14; πK = (3/14)/(15/14) = 1/5 = 20,0000%.

Control: 1/3 + 7/15 + 1/5 = 1. Expected Value: (1/3)×(203/400) + (7/15)×(2143/2000) + (1/5)×(2549/1000) = 1179/1000 = 1,179000.

2026: v = (1, 0, 0) × P.

İ: (1)×(1/4) + (0)×(3/7) + (0)×(1/4) = 1/4 = 25,0000%.

O: (1)×(1/2) + (0)×(3/7) + (0)×(1/2) = 1/2 = 50,0000%.

K: (1)×(1/4) + (0)×(1/7) + (0)×(1/4) = 1/4 = 25,0000%.

E(X) = (1/4)×(203/400) + (1/2)×(2143/2000) + (1/4)×(2549/1000) = 1,299875.

2027: v = (1/4, 1/2, 1/4) × P.

İ: (1/4)×(1/4) + (1/2)×(3/7) + (1/4)×(1/4) = 19/56 = 33,9286%.

O: (1/4)×(1/2) + (1/2)×(3/7) + (1/4)×(1/2) = 13/28 = 46,4286%.

K: (1/4)×(1/4) + (1/2)×(1/7) + (1/4)×(1/4) = 11/56 = 19,6429%.

E(X) = (19/56)×(203/400) + (13/28)×(2143/2000) + (11/56)×(2549/1000) = 1,170366.

2028: v = (19/56, 13/28, 11/56) × P.

İ: (19/56)×(1/4) + (13/28)×(3/7) + (11/56)×(1/4) = 261/784 = 33,2908%.

O: (19/56)×(1/2) + (13/28)×(3/7) + (11/56)×(1/2) = 183/392 = 46,6837%.

K: (19/56)×(1/4) + (13/28)×(1/7) + (11/56)×(1/4) = 157/784 = 20,0255%.

E(X) = (261/784)×(203/400) + (183/392)×(2143/2000) + (157/784)×(2549/1000) = 1,179617.

2029: v = (261/784, 183/392, 157/784) × P.

İ: (261/784)×(1/4) + (183/392)×(3/7) + (157/784)×(1/4) = 3659/10976 = 33,3364%.

O: (261/784)×(1/2) + (183/392)×(3/7) + (157/784)×(1/2) = 2561/5488 = 46,6655%.

K: (261/784)×(1/4) + (183/392)×(1/7) + (157/784)×(1/4) = 2195/10976 = 19,9982%.

E(X) = (3659/10976)×(203/400) + (2561/5488)×(2143/2000) + (2195/10976)×(2549/1000) = 1,178956.

2030: v = (3659/10976, 2561/5488, 2195/10976) × P.

İ: (3659/10976)×(1/4) + (2561/5488)×(3/7) + (2195/10976)×(1/4) = 51221/153664 = 33,3331%.

O: (3659/10976)×(1/2) + (2561/5488)×(3/7) + (2195/10976)×(1/2) = 35855/76832 = 46,6668%.

K: (3659/10976)×(1/4) + (2561/5488)×(1/7) + (2195/10976)×(1/4) = 30733/153664 = 20,0001%.

E(X) = (51221/153664)×(203/400) + (35855/76832)×(2143/2000) + (30733/153664)×(2549/1000) = 1,179003.

Appendix F. CoQ/Turnover: Complete Numerical Solution

Calculation Steps and Finite-Horizon Projection

Six Observed Transitions: 2019→2020: K→O; 2020→2021: O→İ; 2021→2022: İ→İ; 2022→2023: İ→O; 2023→2024: O→İ; 2024→2025: İ→İ.

Row İ: N = (2, 1, 0), toplam = 3. Klasik: (2/3, 1/3, 0/3); Laplace: (3/6, 2/6, 1/6).

Row O: N = (2, 0, 0), toplam = 2. Klasik: (2/2, 0/2, 0/2); Laplace: (3/5, 1/5, 1/5).

Row K: N = (0, 1, 0), toplam = 1. Klasik: (0/1, 1/1, 0/1); Laplace: (1/4, 2/4, 1/4).

Classical Equilibrium Equations: $\pi İ = (2/3)\pi İ + (1)\pi O + (0)\pi K$; $\pi O = (1/3)\pi İ + (0)\pi O + (1)\pi K$; $\pi K = (0)\pi İ + (0)\pi O + (0)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (3/4, 1/4, 0).

Laplace Equilibrium Equations: $\pi İ = (1/2)\pi İ + (3/5)\pi O + (1/4)\pi K$; $\pi O = (1/3)\pi İ + (1/5)\pi O + (1/2)\pi K$; $\pi K = (1/6)\pi İ + (1/5)\pi O + (1/4)\pi K$; $\pi İ + \pi O + \pi K = 1$. Çözüm: (15/31, 10/31, 6/31).

$rİ = (3{,}043 + 2{,}870 + 3{,}236 + 3{,}318)/4 = 3{,}116750$.

$rO = (4{,}687 + 4{,}007)/2 = 4{,}347000$.

$rK = (8{,}945)/1 = 8{,}945000$.

Classical Long-Run Expected Value: $(3/4)\times3{,}116750 + (1/4)\times4{,}347000 + (0)\times8{,}945000 = 3{,}424313$.

Laplace Long-Run Expected Val: $(15/31)\times3{,}116750 + (10/31)\times4{,}347000 + (6/31)\times8{,}945000 = 4{,}641653$.

The initial state vector is $v_0 = (1, 0, 0)$. The first-step vector, v_1, corresponds to the row of the transition matrix associated with the initial state. Each component of the second-step vector is calculated using $\Sigma_i v_{1i}p_{ij}$. For example, the probability of being in the good state in 2027 under the classical model is calculated as follows: $(2/3) \times (2/3) + (1/3) \times (1) + (0) \times (0) = 7/9$.

Year	Classic İ/O/K (%)	E(X)	Laplace İ/O/K (%)	E(X)
2026	66,67 / 33,33 / 0,00	3,527	50,00 / 33,33 / 16,67	4,498
2027	77,78 / 22,22 / 0,00	3,390	49,17 / 31,67 / 19,17	4,623
2028	74,07 / 25,93 / 0,00	3,436	48,37 / 32,31 / 19,32	4,640
2029	75,31 / 24,69 / 0,00	3,421	48,40 / 32,25 / 19,35	4,641
2030	74,90 / 25,10 / 0,00	3,426	48,39 / 32,26 / 19,35	4,642

The observed value in 2025 was 3.318, with an absolute change of −5.627 units and a relative change of −62.91% over the 2019–2025 period. Laplace smoothing increases the stationary probability of the poor state from 0.00% to 19.35%, corresponding to a difference of 19.35 percentage points. This difference reflects sensitivity to model assumptions rather than a newly observed change in actual performance.

Detailed Numerical Calculations

Classical Model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-1/3, 1, 0); (1/3, -1, 1); (1, 1, 1). Right-Hand Side Vector $b=(0, 0, 1)^T$.

$\det(A) = (-1/3)\times[(-1)\times(1)-(1)\times(1)] - (1)\times[(1/3)\times(1)-(1)\times(1)] + (0)\times[(1/3)\times(1)-(-1)\times(1)] = 4/3$.

$\det(A_İ) = (0)\times[(-1)\times(1)-(1)\times(1)] - (1)\times[(0)\times(1)-(1)\times(1)] + (0)\times[(0)\times(1)-(-1)\times(1)] = 1$; $\pi İ = (1)/(4/3) = 3/4 = 75{,}0000\%$.

$\det(A_O) = (-1/3)\times[(0)\times(1)-(1)\times(1)] - (0)\times[(1/3)\times(1)-(1)\times(1)] + (0)\times[(1/3)\times(1)-(0)\times(1)] = 1/3$; $\pi O = (1/3)/(4/3) = 1/4 = 25{,}0000\%$.

det(A_K) = (-1/3)×[(-1)×(1)−(0)×(1)] − (1)×[(1/3)×(1)−(0)×(1)] + (0)×[(1/3)×(1)−(-1)×(1)] = 0; πK = (0)/(4/3) = 0 = 0,0000%.

Control: 3/4 + 1/4 + 0 = 1. Expected Value: (3/4)×(12467/4000) + (1/4)×(4347/1000) + (0)×(1789/200) = 54789/16000 = 3,424313.

2026: v = (1, 0, 0) × P.

İ: (1)×(2/3) + (0)×(1) + (0)×(0) = 2/3 = 66,6667%.

O: (1)×(1/3) + (0)×(0) + (0)×(1) = 1/3 = 33,3333%.

K: (1)×(0) + (0)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (2/3)×(12467/4000) + (1/3)×(4347/1000) + (0)×(1789/200) = 3,526833.

2027: v = (2/3, 1/3, 0) × P.

İ: (2/3)×(2/3) + (1/3)×(1) + (0)×(0) = 7/9 = 77,7778%.

O: (2/3)×(1/3) + (1/3)×(0) + (0)×(1) = 2/9 = 22,2222%.

K: (2/3)×(0) + (1/3)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (7/9)×(12467/4000) + (2/9)×(4347/1000) + (0)×(1789/200) = 3,390139.

2028: v = (7/9, 2/9, 0) × P.

İ: (7/9)×(2/3) + (2/9)×(1) + (0)×(0) = 20/27 = 74,0741%.

O: (7/9)×(1/3) + (2/9)×(0) + (0)×(1) = 7/27 = 25,9259%.

K: (7/9)×(0) + (2/9)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (20/27)×(12467/4000) + (7/27)×(4347/1000) + (0)×(1789/200) = 3,435704.

2029: v = (20/27, 7/27, 0) × P.

İ: (20/27)×(2/3) + (7/27)×(1) + (0)×(0) = 61/81 = 75,3086%.

O: (20/27)×(1/3) + (7/27)×(0) + (0)×(1) = 20/81 = 24,6914%.

K: (20/27)×(0) + (7/27)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (61/81)×(12467/4000) + (20/81)×(4347/1000) + (0)×(1789/200) = 3,420515.

2030: v = (61/81, 20/81, 0) × P.

İ: (61/81)×(2/3) + (20/81)×(1) + (0)×(0) = 182/243 = 74,8971%.

O: (61/81)×(1/3) + (20/81)×(0) + (0)×(1) = 61/243 = 25,1029%.

K: (61/81)×(0) + (20/81)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (182/243)×(12467/4000) + (61/243)×(4347/1000) + (0)×(1789/200) = 3,425578.

Laplace (α=1) model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-1/2, 3/5, 1/4); (1/3, -4/5, 1/2); (1, 1, 1). Right-Hand Side Vector b =$(0, 0, 1)^T$.

det(A) = (-1/2)×[(-4/5)×(1)−(1/2)×(1)] − (3/5)×[(1/3)×(1)−(1/2)×(1)] + (1/4)×[(1/3)×(1)−(-4/5)×(1)] = 31/30.

det(A_İ) = (0)×[(-4/5)×(1)−(1/2)×(1)] − (3/5)×[(0)×(1)−(1/2)×(1)] + (1/4)×[(0)×(1)−(-4/5)×(1)] = 1/2; πİ = (1/2)/(31/30) = 15/31 = 48,3871%.

det(A_O) = (-1/2)×[(0)×(1)−(1/2)×(1)] − (0)×[(1/3)×(1)−(1/2)×(1)] + (1/4)×[(1/3)×(1)−(0)×(1)] = 1/3; πO = (1/3)/(31/30) = 10/31 = 32,2581%.

det(A_K) = (-1/2)×[(-4/5)×(1)−(0)×(1)] − (3/5)×[(1/3)×(1)−(0)×(1)] + (0)×[(1/3)×(1)−(-4/5)×(1)] = 1/5; πK = (1/5)/(31/30) = 6/31 = 19,3548%.

Control: 15/31 + 10/31 + 6/31 = 1. Expected Value: (15/31)×(12467/4000) + (10/31)×(4347/1000) + (6/31)×(1789/200) = 115113/24800 = 4,641653.

2026: v = (1, 0, 0) × P.

İ: (1)×(1/2) + (0)×(3/5) + (0)×(1/4) = 1/2 = 50,0000%.

O: (1)×(1/3) + (0)×(1/5) + (0)×(1/2) = 1/3 = 33,3333%.

K: (1)×(1/6) + (0)×(1/5) + (0)×(1/4) = 1/6 = 16,6667%.

E(X) = (1/2)×(12467/4000) + (1/3)×(4347/1000) + (1/6)×(1789/200) = 4,498208.

2027: v = (1/2, 1/3, 1/6) × P.

İ: (1/2)×(1/2) + (1/3)×(3/5) + (1/6)×(1/4) = 59/120 = 49,1667%.

O: (1/2)×(1/3) + (1/3)×(1/5) + (1/6)×(1/2) = 19/60 = 31,6667%.

K: (1/2)×(1/6) + (1/3)×(1/5) + (1/6)×(1/4) = 23/120 = 19,1667%.

E(X) = (59/120)×(12467/4000) + (19/60)×(4347/1000) + (23/120)×(1789/200) = 4,623410.

2028: v = (59/120, 19/60, 23/120) × P.

İ: (59/120)×(1/2) + (19/60)×(3/5) + (23/120)×(1/4) = 387/800 = 48,3750%.

O: (59/120)×(1/3) + (19/60)×(1/5) + (23/120)×(1/2) = 1163/3600 = 32,3056%.

K: (59/120)×(1/6) + (19/60)×(1/5) + (23/120)×(1/4) = 1391/7200 = 19,3194%.

E(X) = (387/800)×(12467/4000) + (1163/3600)×(4347/1000) + (1391/7200)×(1789/200) = 4,640175.

2029: v = (387/800, 1163/3600, 1391/7200) × P.

İ: (387/800)×(1/2) + (1163/3600)×(3/5) + (1391/7200)×(1/4) = 69697/144000 = 48,4007%.

O: (387/800)×(1/3) + (1163/3600)×(1/5) + (1391/7200)×(1/2) = 7739/24000 = 32,2458%.

K: (387/800)×(1/6) + (1163/3600)×(1/5) + (1391/7200)×(1/4) = 27869/144000 = 19,3535%.

E(X) = (69697/144000)×(12467/4000) + (7739/24000)×(4347/1000) + (27869/144000)×(1789/200) = 4,641423.

2030: v = (69697/144000, 7739/24000, 27869/144000) × P.

İ: (69697/144000)×(1/2) + (7739/24000)×(3/5) + (27869/144000)×(1/4) = 1393523/2880000 = 48,3862%.

O: (69697/144000)×(1/3) + (7739/24000)×(1/5) + (27869/144000)×(1/2) = 1393609/4320000 = 32,2595%.

K: (69697/144000)×(1/6) + (7739/24000)×(1/5) + (27869/144000)×(1/4) = 1672213/8640000 = 19,3543%.

E(X) = (1393523/2880000)×(12467/4000) + (1393609/4320000)×(4347/1000) + (1672213/8640000)×(1789/200) = 4,641640.

Appendix G. CoQ/Ton: Complete Numerical Solution

Calculation Steps and Finite-Horizon Projection

Six Observed Transitions: 2019→2020: K→O; 2020→2021: O→O; 2021→2022: O→O; 2022→2023: O→O; 2023→2024: O→İ; 2024→2025: İ→İ.

Row İ: N = (1, 0, 0), toplam = 1. Klasik: (1/1, 0/1, 0/1); Laplace: (2/4, 1/4, 1/4).

Row O: N = (1, 3, 0), toplam = 4. Klasik: (1/4, 3/4, 0/4); Laplace: (2/7, 4/7, 1/7).

Row K: N = (0, 1, 0), toplam = 1. Klasik: (0/1, 1/1, 0/1); Laplace: (1/4, 2/4, 1/4).

Classical Equilibrium Equations: $\pi İ = (1)\pi İ + (1/4)\pi O + (0)\pi K$; $\pi O = (0)\pi İ + (3/4)\pi O + (1)\pi K$; $\pi K = (0)\pi İ + (0)\pi O + (0)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (1, 0, 0).

Laplace Equilibrium Equations: $\pi İ = (1/2)\pi İ + (2/7)\pi O + (1/4)\pi K$; $\pi O = (1/4)\pi İ + (4/7)\pi O + (1/2)\pi K$; $\pi K = (1/4)\pi İ + (1/7)\pi O + (1/4)\pi K$; $\pi İ + \pi O + \pi K = 1$. Solution: (28/79, 35/79, 16/79).

$rİ = (72{,}470 + 75{,}993)/2 = 74{,}231500$.

$rO = (97{,}234 + 84{,}279 + 99{,}885 + 105{,}051)/4 = 96{,}612250$.

$rK = (186{,}196)/1 = 186{,}196000$.

Classical Long-Run Expected Value: $(1)\times 74{,}231500 + (0)\times 96{,}612250 + (0)\times 186{,}196000 = 74{,}231500$.

Laplace Long-Run Expected Value: $(28/79)\times 74{,}231500 + (35/79)\times 96{,}612250 + (16/79)\times 186{,}196000 = 106{,}823377$.

The initial state vector is $v_0 = (1, 0, 0)$. The first-step vector, v_1, corresponds to the row of the transition matrix associated with the initial state. Each component of the second-step vector is calculated using $\Sigma_i v_{1i}p_{ij}$. For example, the probability of being in the good state in 2027 under the classical model is calculated as follows: (1) × (1) + (0) × (1/4) + (0) × (0) = 1.

Year	Classic İ/O/K (%)	E(X)	Laplace İ/O/K (%)	E(X)
2026	100,00 / 0,00 / 0,00	74,231	50,00 / 25,00 / 25,00	107,818
2027	100,00 / 0,00 / 0,00	74,231	38,39 / 39,29 / 22,32	108,016
2028	100,00 / 0,00 / 0,00	74,231	36,00 / 43,21 / 20,79	107,180
2029	100,00 / 0,00 / 0,00	74,231	35,54 / 44,09 / 20,37	106,906
2030	100,00 / 0,00 / 0,00	74,231	35,46 / 44,26 / 20,28	106,840

The observed value in 2025 was 75.993, with an absolute change of −110.203 units and a relative change of −59.19% over the 2019–2025 period. Laplace smoothing increases the stationary probability of the poor state from 0.00% to 20.25%, corresponding to a difference of 20.25 percentage points. This difference reflects sensitivity to model assumptions rather than a newly observed change in actual performance.

Detailed Numerical Calculations

Classical Model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (0, 1/4, 0); (0, -1/4, 1); (1, 1, 1). Right-Hand Side Vector $b=(0, 0, 1)^T$.

$\det(A) = (0)\times[(-1/4)\times(1)-(1)\times(1)] - (1/4)\times[(0)\times(1)-(1)\times(1)] + (0)\times[(0)\times(1)-(-1/4)\times(1)] = 1/4$.

$\det(A_İ) = (0)\times[(-1/4)\times(1)-(1)\times(1)] - (1/4)\times[(0)\times(1)-(1)\times(1)] + (0)\times[(0)\times(1)-(-1/4)\times(1)] = 1/4$; $\pi İ = (1/4)/(1/4) = 1 = 100{,}0000\%$.

$\det(A_O) = (0)\times[(0)\times(1)-(1)\times(1)] - (0)\times[(0)\times(1)-(1)\times(1)] + (0)\times[(0)\times(1)-(0)\times(1)] = 0$; $\pi O = (0)/(1/4) = 0 = 0{,}0000\%$.

det(A_K) = (0)×[(-1/4)×(1)−(0)×(1)] − (1/4)×[(0)×(1)−(0)×(1)] + (0)×[(0)×(1)−(-1/4)×(1)] = 0; πK = (0)/(1/4) = 0 = 0,0000%.

Control: 1 + 0 + 0 = 1. Expected Value: (1)×(148463/2000) + (0)×(386449/4000) + (0)×(46549/250) = 148463/2000 = 74,231500.

2026: v = (1, 0, 0) × P.

İ: (1)×(1) + (0)×(1/4) + (0)×(0) = 1 = 100,0000%.

O: (1)×(0) + (0)×(3/4) + (0)×(1) = 0 = 0,0000%.

K: (1)×(0) + (0)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (1)×(148463/2000) + (0)×(386449/4000) + (0)×(46549/250) = 74,231500.

2027: v = (1, 0, 0) × P.

İ: (1)×(1) + (0)×(1/4) + (0)×(0) = 1 = 100,0000%.

O: (1)×(0) + (0)×(3/4) + (0)×(1) = 0 = 0,0000%.

K: (1)×(0) + (0)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (1)×(148463/2000) + (0)×(386449/4000) + (0)×(46549/250) = 74,231500.

2028: v = (1, 0, 0) × P.

İ: (1)×(1) + (0)×(1/4) + (0)×(0) = 1 = 100,0000%.

O: (1)×(0) + (0)×(3/4) + (0)×(1) = 0 = 0,0000%.

K: (1)×(0) + (0)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (1)×(148463/2000) + (0)×(386449/4000) + (0)×(46549/250) = 74,231500.

2029: v = (1, 0, 0) × P.

İ: (1)×(1) + (0)×(1/4) + (0)×(0) = 1 = 100,0000%.

O: (1)×(0) + (0)×(3/4) + (0)×(1) = 0 = 0,0000%.

K: (1)×(0) + (0)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (1)×(148463/2000) + (0)×(386449/4000) + (0)×(46549/250) = 74,231500.

2030: v = (1, 0, 0) × P.

İ: (1)×(1) + (0)×(1/4) + (0)×(0) = 1 = 100,0000%.

O: (1)×(0) + (0)×(3/4) + (0)×(1) = 0 = 0,0000%.

K: (1)×(0) + (0)×(0) + (0)×(0) = 0 = 0,0000%.

E(X) = (1)×(148463/2000) + (0)×(386449/4000) + (0)×(46549/250) = 74,231500.

Laplace (α=1) model: Solution of Linear Equations

Coefficient Matrix A of the Equilibrium System: (-1/2, 2/7, 1/4); (1/4, -3/7, 1/2); (1, 1, 1). Right-Hand Side Vector b=(0, 0, 1)^{T}.
det(A) = (-1/2)×[(-3/7)×(1)−(1/2)×(1)] − (2/7)×[(1/4)×(1)−(1/2)×(1)] + (1/4)×[(1/4)×(1)−(-3/7)×(1)] = 79/112.

det(A_İ) = (0)×[(-3/7)×(1)−(1/2)×(1)] − (2/7)×[(0)×(1)−(1/2)×(1)] + (1/4)×[(0)×(1)−(-3/7)×(1)] = 1/4; πİ = (1/4)/(79/112) = 28/79 = 35,4430%.

det(A_O) = (-1/2)×[(0)×(1)−(1/2)×(1)] − (0)×[(1/4)×(1)−(1/2)×(1)] + (1/4)×[(1/4)×(1)−(0)×(1)] = 5/16; πO = (5/16)/(79/112) = 35/79 = 44,3038%.

det(A_K) = (-1/2)×[(-3/7)×(1)−(0)×(1)] − (2/7)×[(1/4)×(1)−(0)×(1)] + (0)×[(1/4)×(1)−(-3/7)×(1)] = 1/7; πK = (1/7)/(79/112) = 16/79 = 20,2532%.

Control: 28/79 + 35/79 + 16/79 = 1. Expected Value: (28/79)×(148463/2000) + (35/79)×(386449/4000) + (16/79)×(46549/250) = 33756187/316000 = 106,823377.

2026: v = (1, 0, 0) × P.

İ: (1)×(1/2) + (0)×(2/7) + (0)×(1/4) = 1/2 = 50,0000%.

O: (1)×(1/4) + (0)×(4/7) + (0)×(1/2) = 1/4 = 25,0000%.

K: (1)×(1/4) + (0)×(1/7) + (0)×(1/4) = 1/4 = 25,0000%.

E(X) = (1/2)×(148463/2000) + (1/4)×(386449/4000) + (1/4)×(46549/250) = 107,817813.

2027: v = (1/2, 1/4, 1/4) × P.

İ: (1/2)×(1/2) + (1/4)×(2/7) + (1/4)×(1/4) = 43/112 = 38,3929%.

O: (1/2)×(1/4) + (1/4)×(4/7) + (1/4)×(1/2) = 11/28 = 39,2857%.

K: (1/2)×(1/4) + (1/4)×(1/7) + (1/4)×(1/4) = 25/112 = 22,3214%.

E(X) = (43/112)×(148463/2000) + (11/28)×(386449/4000) + (25/112)×(46549/250) = 108,016013.

2028: v = (43/112, 11/28, 25/112) × P.

İ: (43/112)×(1/2) + (11/28)×(2/7) + (25/112)×(1/4) = 1129/3136 = 36,0013%.

O: (43/112)×(1/4) + (11/28)×(4/7) + (25/112)×(1/2) = 1355/3136 = 43,2079%.

K: (43/112)×(1/4) + (11/28)×(1/7) + (25/112)×(1/4) = 163/784 = 20,7908%.

E(X) = (1129/3136)×(148463/2000) + (1355/3136)×(386449/4000) + (163/784)×(46549/250) = 107,180087.

2029: v = (1129/3136, 1355/3136, 163/784) × P.

İ: (1129/3136)×(1/2) + (1355/3136)×(2/7) + (163/784)×(1/4) = 15605/43904 = 35,5435%.

O: (1129/3136)×(1/4) + (1355/3136)×(4/7) + (163/784)×(1/2) = 38711/87808 = 44,0860%.

K: (1129/3136)×(1/4) + (1355/3136)×(1/7) + (163/784)×(1/4) = 17887/87808 = 20,3706%.

E(X) = (15605/43904)×(148463/2000) + (38711/87808)×(386449/4000) + (17887/87808)×(46549/250) = 106,906088.

2030: v = (15605/43904, 38711/87808, 17887/87808) × P.

İ: (15605/43904)×(1/2) + (38711/87808)×(2/7) + (17887/87808)×(1/4) = 871837/2458624 = 35,4604%.

O: (15605/43904)×(1/4) + (38711/87808)×(4/7) + (17887/87808)×(1/2) = 136033/307328 = 44,2631%.

K: (15605/43904)×(1/4) + (38711/87808)×(1/7) + (17887/87808)×(1/4) = 498523/2458624 = 20,2765%.

E(X) = (871837/2458624)×(148463/2000) + (136033/307328)×(386449/4000) + (498523/2458624)×(46549/250) = 106,840408.

Appendix H. Sensitivity Calculations

Detailed Numerical Calculations for Sensitivity Scenarios

Non-Prime

α=1/2: rows of P (3/7, 3/7, 1/7); (1/5, 1/5, 3/5); (5/9, 1/9, 1/3); det(A)=359/315; π=(147/359, 95/359, 117/359); E(X)=(147/359)×(2263/1500) + (95/359)×(2207/1000) + (117/359)×(4153/1000) = 2,555265.

α=1: rows of P (2/5, 2/5, 1/5); (1/4, 1/4, 1/2); (1/2, 1/6, 1/3); det(A)=16/15; π=(25/64, 9/32, 21/64); E(X)=(25/64)×(2263/1500) + (9/32)×(2207/1000) + (21/64)×(4153/1000) = 2,572745.

α=2: rows of P (3/8, 3/8, 1/4); (2/7, 2/7, 3/7); (4/9, 2/9, 1/3); det(A)=517/504; π=(192/517, 14/47, 171/517); E(X)=(192/517)×(2263/1500) + (14/47)×(2207/1000) + (171/517)×(4153/1000) = 2,591306.

α=3: rows of P (4/11, 4/11, 3/11); (3/10, 3/10, 2/5); (5/12, 1/4, 1/3); det(A)=223/220; π=(242/669, 205/669, 74/223); E(X)=(242/669)×(2263/1500) + (205/669)×(2207/1000) + (74/223)×(4153/1000) = 2,600147.

Threshold Multiplier 9/10: Thresholds 9/5, 27/10; series KKOOKİİ; N=([[1, 0, 0], [0, 1, 1], [1, 1, 1]]); π=(8/23, 15/46, 15/46); E(X)=(8/23)×(2579/2000) + (15/46)×(2077/1000) + (15/46)×(4153/1000) = 2,480043.

Threshold Multiplier 11/10: Thresholds 11/5, 33/10; series KKİOOİİ; N=([[1, 1, 0], [1, 1, 0], [1, 0, 1]]); π=(2/5, 7/20, 1/4); E(X)=(2/5)×(2263/1500) + (7/20)×(1339/500) + (1/4)×(931/200) = 2,704517.

Rework

α=1/2: rows of P (3/7, 3/7, 1/7); (3/7, 1/7, 3/7); (1/7, 1/7, 5/7); det(A)=31/49; π=(9/31, 7/31, 15/31); E(X)=(9/31)×(15783/2000) + (7/31)×(6251/500) + (15/31)×(10227/500) = 15,011210.

α=1: rows of P (2/5, 2/5, 1/5); (2/5, 1/5, 2/5); (1/5, 1/5, 3/5); det(A)=19/25; π=(6/19, 5/19, 8/19); E(X)=(6/19)×(15783/2000) + (5/19)×(6251/500) + (8/19)×(10227/500) = 14,394263.

α=2: rows of P (3/8, 3/8, 1/4); (3/8, 1/4, 3/8); (1/4, 1/4, 1/2); det(A)=55/64; π=(18/55, 16/55, 21/55); E(X)=(18/55)×(15783/2000) + (16/55)×(6251/500) + (21/55)×(10227/500) = 14,029327.

α=3: rows of P (4/11, 4/11, 3/11); (4/11, 3/11, 4/11); (3/11, 3/11, 5/11); det(A)=109/121; π=(36/109, 33/109, 40/109); E(X)=(36/109)×(15783/2000) + (33/109)×(6251/500) + (40/109)×(10227/500) = 13,897431.

Threshold Multiplier 9/10: Thresholds 9, 27/2; dizi OİOKKKK; N=([[0, 1, 0], [1, 0, 1], [0, 0, 3]]); π=(24/97, 25/97, 48/97); E(X)=(24/97)×(5807/1000) + (25/97)×(407/40) + (48/97)×(9499/500) = 13,460278.

Threshold Multiplier 11/10: Thresholds 11, 33/2; dizi İİİOKKK; N=([[2, 1, 0], [0, 0, 1], [0, 0, 2]]); π=(24/79, 20/79, 35/79); E(X)=(24/79)×(8719/1000) + (20/79)×(1463/100) + (35/79)×(10227/500) = 15,414506.

Second quality

α=1/2: rows of P (5/9, 1/3, 1/9); (1/7, 5/7, 1/7); (3/5, 1/5, 1/5); det(A)=179/315; π=(63/179, 91/179, 25/179); E(X)=(63/179)×(99/250) + (91/179)×(91/150) + (25/179)×(132/125) = 0,595277.

α=1: rows of P (1/2, 1/3, 1/6); (1/5, 3/5, 1/5); (1/2, 1/4, 1/4); det(A)=27/40; π=(10/27, 35/81, 16/81); E(X)=(10/27)×(99/250) + (35/81)×(91/150) + (16/81)×(132/125) = 0,617399.

α=2: rows of P (4/9, 1/3, 2/9); (1/4, 1/2, 1/4); (3/7, 2/7, 2/7); det(A)=197/252; π=(72/197, 76/197, 49/197); E(X)=(72/197)×(99/250) + (76/197)×(91/150) + (49/197)×(132/125) = 0,641435.

α=3: rows of P (5/12, 1/3, 1/4); (3/11, 5/11, 3/11); (2/5, 3/10, 3/10); det(A)=1103/1320; π=(396/1103, 407/1103, 300/1103); E(X)=(396/1103)×(99/250) + (407/1103)×(91/150) + (300/1103)×(132/125) = 0,653245.

Threshold Multiplier 9/10: Thresholds 9/20, 27/40; dizi KİİİOKO; N=([[2, 1, 0], [0, 0, 1], [1, 1, 0]]); π=(16/41, 40/123, 35/123); E(X)=(16/41)×(99/250) + (40/123)×(113/200) + (35/123)×(873/1000) = 0,586691.

Threshold Multiplier 11/10: Thresholds 11/20, 33/40; dizi KİİİOOİ; N=([[2, 1, 0], [1, 1, 0], [1, 0, 0]]); π=(48/103, 35/103, 20/103); E(X)=(48/103)×(1713/4000) + (35/103)×(259/400) + (20/103)×(132/125) = 0,624646.

Scrap

α=1/2: rows of P (5/11, 5/11, 1/11); (1/5, 1/5, 3/5); (3/5, 1/5, 1/5); det(A)=343/275; π=(143/343, 15/49, 95/343); E(X)=(143/343)×(927/250) + (15/49)×(8473/2000) + (95/343)×(4961/1000) = 4,216827.

α=1: rows of P (3/7, 3/7, 1/7); (1/4, 1/4, 1/2); (1/2, 1/4, 1/4); det(A)=125/112; π=(49/125, 8/25, 36/125); E(X)=(49/125)×(927/250) + (8/25)×(8473/2000) + (36/125)×(4961/1000) = 4,237984.

α=2: rows of P (2/5, 2/5, 1/5); (2/7, 2/7, 3/7); (3/7, 2/7, 2/7); det(A)=256/245; π=(95/256, 21/64, 77/256); E(X)=(95/256)×(927/250) + (21/64)×(8473/2000) + (77/256)×(4961/1000) = 4,258293.

α=3: rows of P (5/13, 5/13, 3/13); (3/10, 3/10, 2/5); (2/5, 3/10, 3/10); det(A)=1331/1300; π=(481/1331, 40/121, 410/1331); E(X)=(481/1331)×(927/250) + (40/121)×(8473/2000) + (410/1331)×(4961/1000) = 4,268684.

Threshold Multiplier 9/10: Thresholds 11/20, 33/40; dizi KİİİOOİ; N=([[2, 1, 0], [1, 1, 0], [1, 0, 0]]); π=(48/103, 35/103, 20/103); E(X)=(48/ 18/5, 81/20; dizi OİOKKOO; N=([[0, 1, 0], [1, 1, 1], [0, 1, 1]]); π=(4/15, 2/5, 1/3); E(X)=(4/15)×(1601/500) + (2/5)×(7829/2000) + (1/3)×(4703/1000) = 3,987333.

Threshold Multiplier 11/10: Thresholds 11/20, 33/40; dizi KİİİOOİ; N=([[2, 1, 0], [1, 1, 0], [1, 0, 0]]); π=(48/103, 35/103, 20/103); E(X)=(48/ 22/5, 99/20; dizi İİİOKİİ; N=([[3, 1, 0], [0, 0, 1], [1, 0, 0]]); π=(7/15, 4/15, 4/15); E(X)=(7/15)×(943/250) + (4/15)×(889/200) + (4/15)×(4961/1000) = 4,268533.

Claim

α=1/2: rows of P (1/5, 3/5, 1/5); (5/11, 5/11, 1/11); (1/5, 3/5, 1/5); det(A)=63/55; π=(1/3, 11/21, 1/7); E(X)=(1/3)×(203/400) + (11/21)×(2143/2000) + (1/7)×(2549/1000) = 1,094571.

α=1: rows of P (1/4, 1/2, 1/4); (3/7, 3/7, 1/7); (1/4, 1/2, 1/4); det(A)=15/14; π=(1/3, 7/15, 1/5); E(X)=(1/3)×(203/400) + (7/15)×(2143/2000) + (1/5)×(2549/1000) = 1,179000.

α=2: rows of P (2/7, 3/7, 2/7); (2/5, 2/5, 1/5); (2/7, 3/7, 2/7); det(A)=36/35; π=(1/3, 5/12, 1/4); E(X)=(1/3)×(203/400) + (5/12)×(2143/2000) + (1/4)×(2549/1000) = 1,252875.

α=3: rows of P (3/10, 2/5, 3/10); (5/13, 5/13, 3/13); (3/10, 2/5, 3/10); det(A)=66/65; π=(1/3, 13/33, 3/11); E(X)=(1/3)×(203/400) + (13/33)×(2143/2000) + (3/11)×(2549/1000) = 1,286455.

Threshold Multiplier 9/10: Thresholds 27/40, 9/8; dizi KKİOKOİ; N=([[0, 1, 0], [1, 0, 1], [1, 1, 1]]); π=(24/73, 25/73, 24/73); E(X)=(24/73)×(203/400) + (25/73)×(243/250) + (24/73)×(4891/3000) = 1,035726.

Threshold Multiplier 11/10: Thresholds 33/40, 11/8; dizi KOİOOOİ; N=([[0, 1, 0], [2, 2, 0], [0, 1, 0]]); π=(1/3, 7/15, 1/5); E(X)=(1/3)×(203/400) + (7/15)×(2143/2000) + (1/5)×(2549/1000) = 1,179000.

CoQ/Turnover
α=1/2: rows of P (5/9, 1/3, 1/9); (5/7, 1/7, 1/7); (1/5, 3/5, 1/5); det(A)=113/105; π=(63/113, 35/113, 15/113); E(X)=(63/113)×(12467/4000) + (35/113)×(4347/1000) + (15/113)×(1789/200) = 4,271462.

α=1: rows of P (1/2, 1/3, 1/6); (3/5, 1/5, 1/5); (1/4, 1/2, 1/4); det(A)=31/30; π=(15/31, 10/31, 6/31); E(X)=(15/31)×(12467/4000) + (10/31)×(4347/1000) + (6/31)×(1789/200) = 4,641653.

α=2: rows of P (4/9, 1/3, 2/9); (1/2, 1/4, 1/4); (2/7, 3/7, 2/7); det(A)=85/84; π=(36/85, 28/85, 21/85); E(X)=(36/85)×(12467/4000) + (28/85)×(4347/1000) + (21/85)×(1789/200) = 4,961929.

α=3: rows of P (5/12, 1/3, 1/4); (5/11, 3/11, 3/11); (3/10, 2/5, 3/10); det(A)=166/165; π=(33/83, 55/166, 45/166); E(X)=(33/83)×(12467/4000) + (55/166)×(4347/1000) + (45/166)×(1789/200) = 5,104310.

Threshold Multiplier 9/10: Thresholds 63/20, 9/2; dizi KKİİOOO; N=([[1, 1, 0], [0, 2, 0], [1, 0, 1]]); π=(5/16, 7/16, 1/4); E(X)=(5/16)×(5913/2000) + (7/16)×(10561/3000) + (1/4)×(852/125) = 4,168052.

Threshold Multiplier 11/10: Thresholds 77/20, 11/2; dizi KOİİOİİ; N=([[2, 1, 0], [2, 0, 0], [0, 1, 0]]); π=(15/31, 10/31, 6/31); E(X)=(15/31)×(12467/4000) + (10/31)×(4347/1000) + (6/31)×(1789/200) = 4,641653.

CoQ/Ton
α=1/2: rows of P (3/5, 1/5, 1/5); (3/11, 7/11, 1/11); (1/5, 3/5, 1/5); det(A)=167/275; π=(65/167, 77/167, 25/167); E(X)=(65/167)×(148463/2000) + (77/167)×(386449/4000) + (25/167)×(46549/250) = 101,311921.

α=1: rows of P (1/2, 1/4, 1/4); (2/7, 4/7, 1/7); (1/4, 1/2, 1/4); det(A)=79/112; π=(28/79, 35/79, 16/79); E(X)=(28/79)×(148463/2000) + (35/79)×(386449/4000) + (16/79)×(46549/250) = 106,823377.

α=2: rows of P (3/7, 2/7, 2/7); (3/10, 1/2, 1/5); (2/7, 3/7, 2/7); det(A)=391/490; π=(133/391, 160/391, 98/391); E(X)=(133/391)×(148463/2000) + (160/391)×(386449/4000) + (98/391)×(46549/250) = 111,452577.

α=3: rows of P (2/5, 3/10, 3/10); (4/13, 6/13, 3/13); (3/10, 2/5, 3/10); det(A)=1099/1300; π=(370/1099, 429/1099, 300/1099); E(X)=(370/1099)×(148463/2000) + (429/1099)×(386449/4000) + (300/1099)×(46549/250) = 113,531492.

Threshold Multiplier 9/10: Thresholds 72, 99; dizi KOOKKOO; N=([[0, 0, 0], [0, 2, 1], [0, 2, 1]]); π=(1/5, 7/15, 1/3); gözlenmeyen durum nedeniyle beklenti tanımsız.

Threshold Multiplier 11/10: Thresholds 88, 121; dizi KOİOOİİ; N=([[1, 1, 0], [2, 1, 0], [0, 1, 0]]); π=(25/61, 24/61, 12/61); E(X)=(25/61)×(116371/1500) + (24/61)×(30217/300) + (12/61)×(46549/250) = 108,052929.

Appendix I. Economic Calculations

Detailed Breakdown of Economic Calculations

CoQ/Turnover: Classical model: 3.424313 × 1,200,000 = USD 4,109,175.00; Laplace-smoothed model: 4.641653 × 1,200,000 = USD 5,569,983.87. Difference = USD 1,460,808.87. The multipliers presented are based on scenario assumptions.

CoQ/Ton: Classical model: 74.231500 × 50,000 = USD 3,711,575.00; Laplace-smoothed model: 106.823377 × 50,000 = USD 5,341,168.83. Difference = USD 1,629,593.83. The multipliers presented are based on scenario assumptions.

A one-percentage-point improvement corresponds to an equivalent tonnage of 50,000 × 0.01 = 500 tons. The corresponding potential savings are as follows: scrap, 500 × 1,200 = USD 600,000; claim, 500 × 180 = USD 90,000; non-prime, 500 × 120 = USD 60,000; and rework, 500 × 24 = USD 12,000. To determine actual savings, indicator-specific denominators and marginal costs must be verified, and potential overlaps must be assessed to avoid double counting.