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Modeling and Forecasting of Quality Indicators and Cost of Quality in Stainless Steel Cold Rolling Plants Using Grey System Theory

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Case Report

Keywords: Grey system theory, GM(1,1), cost of quality, cold rolling, forecasting

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Abstract

This study models five operational quality indicators and cost of quality per packed tonne from an anonymised stainless steel cold rolling facility in Türkiye during 2019–2025. Six independent GM(1,1) models are estimated using accumulated generation and least squares. In-sample errors, expanding-window one-step forecasts, naïve and linear trend benchmarks, and background-weight sensitivity are examined. The small sample and exponential extrapolation limit predictive reliability. Results are conditional projections rather than causal estimates or validated savings.

Keywords: Grey system theory, GM(1,1), cost of quality, cold rolling, forecasting.

1. Introduction

Quality costs encompass both the resources used to achieve conformity and the economic consequences of nonconformity (Feigenbaum, 1991; Juran and Godfrey, 1999; Crosby, 1979). In production management, declining defect rates do not necessarily imply reduced rework resource consumption. Operational and economic series must therefore be assessed separately. This study aims to develop a transparent forecasting model from short annual series and evaluate its performance against simple alternatives.

2. Literature review

Grey system theory was developed by Deng (1982) to model systems under incomplete information. The accumulated generation operation smooths fluctuations in short series; GM(1,1) employs a univariate first-order dynamic structure (Liu and Lin, 2010; Liu et al., 2017; Kayacan et al., 2010). However, its exponential structure is not suitable for every data-generating process. Discrete grey models and alternative formulations address these limitations (Xie and Liu, 2009; Liu et al., 2016).

The quality economics literature distinguishes prevention, appraisal and failure costs (Campanella, 1999; Schiffauerova and Thomson, 2006; Wood, 2013). Measuring quality costs can support managerial improvement, but measurement scope and classification are important (Plunkett and Dale, 1988; Dale and Plunkett, 1999; Sower et al., 2007; Jafari and Love, 2013). Process control requires higher-frequency observations (Montgomery, 2020; Box et al., 2015).

Forecasting performance should not be assessed solely by training errors. Simple benchmarks, chronological validation and scale-independent error measures are necessary (Hyndman and Koehler, 2006; Tashman, 2000; Hyndman and Athanasopoulos, 2021; Makridakis et al., 2018; Armstrong, 2001). With small samples, model complexity should be limited and performance differences should not be presented as definitive superiority (Chatfield, 2000).

3. Materials and methods

Source: internal quality report of the anonymised facility, dated 22 January 2026, pp. 6, 7, 9, 11, 13 and 15. The first five indicators are percentages; the final indicator is expressed in USD per packed tonne. The series may overlap and their percentages are therefore not aggregated. Absolute costs and tonnage are not used.

Table 1. Indicators

Year	NonPrime	ReWork	2.Quality	Scrap	Claim	CoQ/T
2019	4.774	10.374	1.056	3.730	2.549	186.196
2020	4.536	5.807	0.431	3.202	1.159	97.234
2021	1.947	9.976	0.424	3.910	0.609	84.279
2022	2.207	14.630	0.333	4.445	0.894	99.885
2023	3.149	16.556	0.605	4.961	1.183	105.051
2024	1.723	22.150	0.690	3.990	1.050	72.470
2025	0.856	22.656	0.525	4.028	0.406	75.993

For each series, $x^{(0)}(k)$ is a positive observation, k=1,…,7. The accumulated series and background value are:

$$x^{(1)}(k) = \Sigma_{i=1}^{k} x^{(0)}(i) \quad (1)$$

$$z^{(1)}(k) = \lambda\, x^{(1)}(k) + (1-\lambda)x^{(1)}(k-1) \quad (2)$$

$$x^{(0)}(k) + a\, z^{(1)}(k) = b \quad (3)$$

$$\theta = (a,b)^{T} = (B^{T}B)^{-1}B^{T}Y \quad (4)$$

The rows of B are $[-z^{(1)}(k), 1]$, and the elements of Y are $x^{(0)}(k)$, k=2,…,7. The standard model uses $\lambda=0.5$. The whitened equation and its solution are:

$$dx^{(1)}/dt + a\, x^{(1)} = b \quad (5)$$

$$\hat{x}^{(1)}(k) = (x^{(0)}(1)-b/a)\, \exp[-a(k-1)] + b/a \quad (6)$$

$$\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1),\ k\geq 2 \quad (7)$$

The first observation serves as the initial condition and is excluded from error calculations. Percentage projections outside the 0–100 range are considered physically invalid and are not clipped. The conventional admissibility interval for the level ratio $\lambda_k = x^{(0)}(k-1)/x^{(0)}(k)$, $(\exp(-2/8), \exp(2/8)) = (0.7788, 1.2840)$, is used diagnostically; a violation does not automatically establish model validity or invalidity.

$$\mathrm{MAPE} = (100/m)\, \Sigma\, |(x_k - \hat{x}_k)/x_k| \quad (8)$$

$$\mathrm{MAE} = (1/m)\, \Sigma\, |x_k - \hat{x}_k| \quad (9)$$

Validation: observations from 2019–2022 are used to forecast 2023, those from 2019–2023 to forecast 2024, and those from 2019–2024 to forecast 2025. The last observation and a least-squares linear trend fitted to the same training window serve as benchmarks. Three error observations are insufficient for a statistical test of superiority.

4. Results

Non-Prime

The change over 2019–2025 was -82.07%. The development coefficient is a=0.219537, and the input coefficient is b=5.365400. In-sample MAPE is 30.25%; the 2026 projection is 1.038, and the 2030 projection is 0.431. A positive a produces an exponential declining trend. This result is a mathematical extrapolation of the historical series, not the effect of a process intervention.

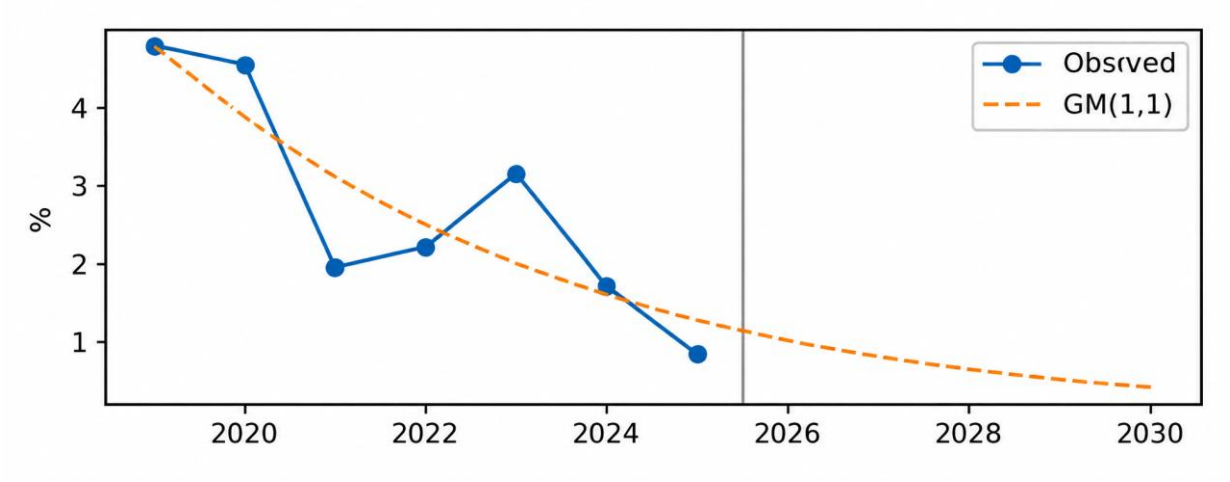

Figure 1. NonPrime-year

Rework

The change over 2019–2025 was 118.39%. The development coefficient is a=-0.214962, and the input coefficient is b=5.399599. In-sample MAPE is 14.04%; the 2026 projection is 30.914, and the 2030 projection is 73.044. A negative a produces an exponential increasing trend. This result is a mathematical extrapolation of the historical series, not the effect of a process intervention.

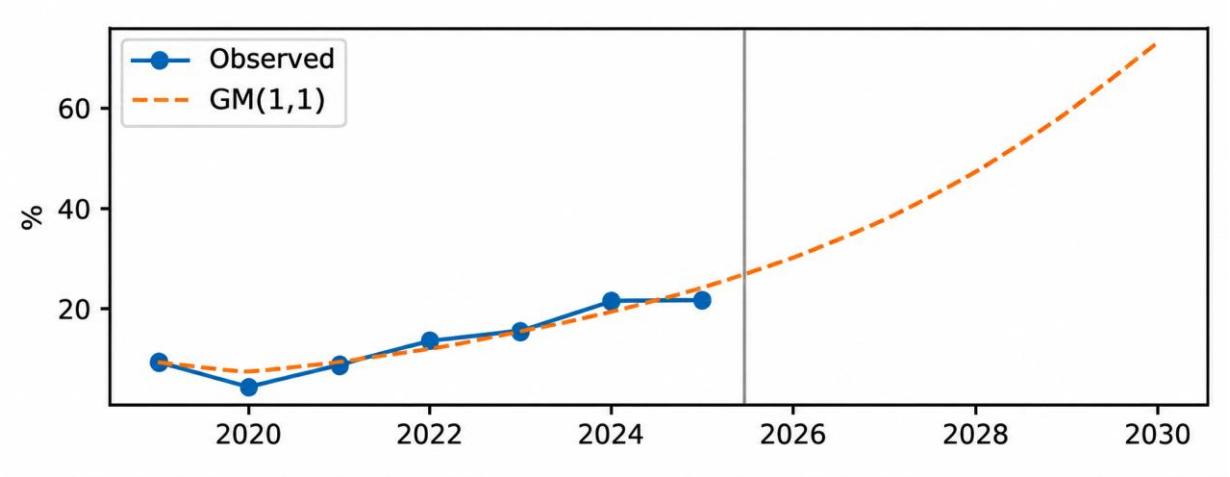

Figure 2. Rework-year

Second-quality production

The change over 2019–2025 was -50.28%. The development coefficient is a=-0.086186, and the input coefficient is b=0.291758. In-sample MAPE is 17.08%; the 2026 projection is 0.670, and the 2030 projection is 0.946. A negative a produces an exponential increasing trend. This result is a mathematical extrapolation of the historical series, not the effect of a process intervention.

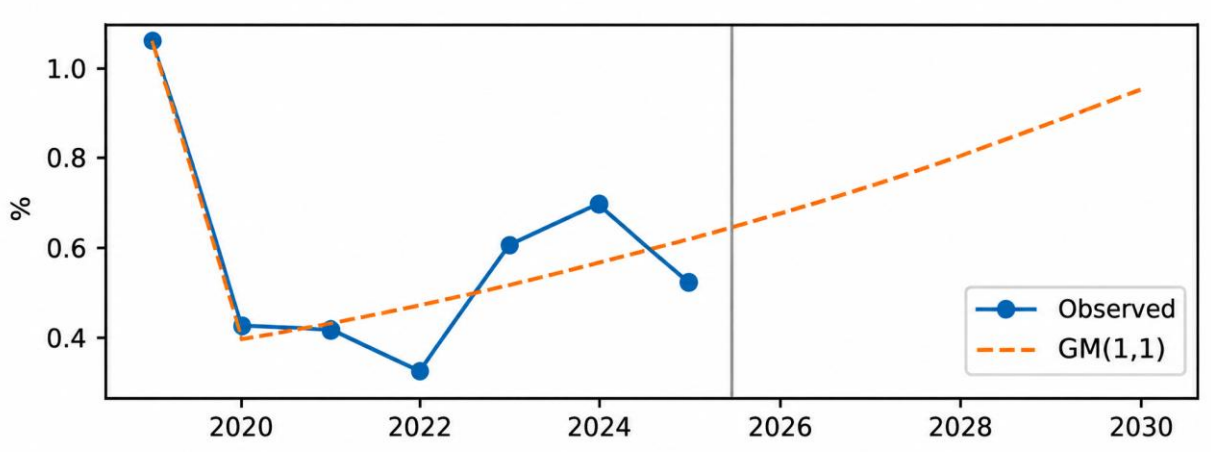

Figure 3. Second quality-year

Scrap

The change over 2019–2025 was 7.99%. The development coefficient is a=-0.031294, and the input coefficient is b=3.601433. In-sample MAPE is 10.19%; the 2026 projection is 4.557, and the 2030 projection is 5.165. A negative a produces an exponential increasing trend. This result is a mathematical extrapolation of the historical series, not the effect of a process intervention.

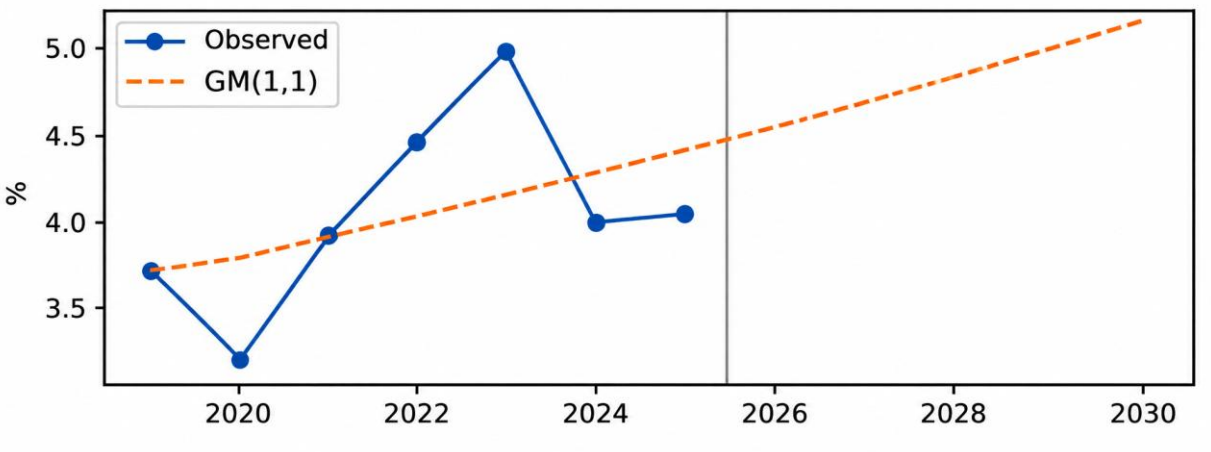

Figure 4. Scrap-year

Claim

The change over 2019–2025 was -84.07%. The development coefficient is a=0.063384, and the input coefficient is b=1.224439. In-sample MAPE is 34.67%; the 2026 projection is 0.704, and the 2030 projection is 0.546. A positive a produces an exponential declining trend. This result is a mathematical extrapolation of the historical series, not the effect of a process intervention.

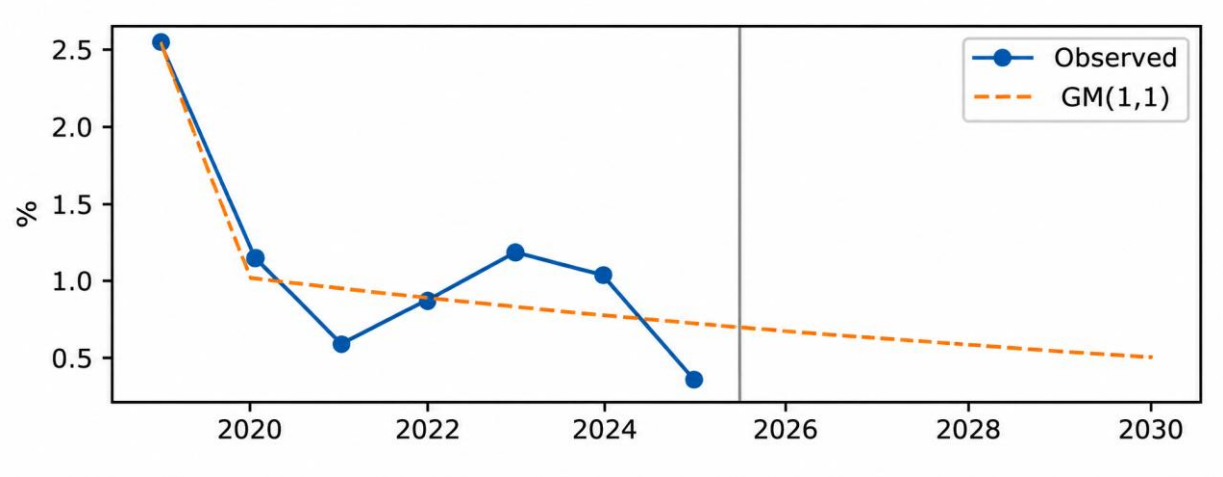

Figure 5. Claim-year

CoQ/ton

The change over 2019–2025 was -59.19%. The development coefficient is a=0.041671, and the input coefficient is b=108.529808. In-sample MAPE is 10.13%; the 2026 projection is 76.866, and the 2030 projection is 65.065. A positive a produces an exponential declining trend. This result is a mathematical extrapolation of the historical series, not the effect of a process intervention.

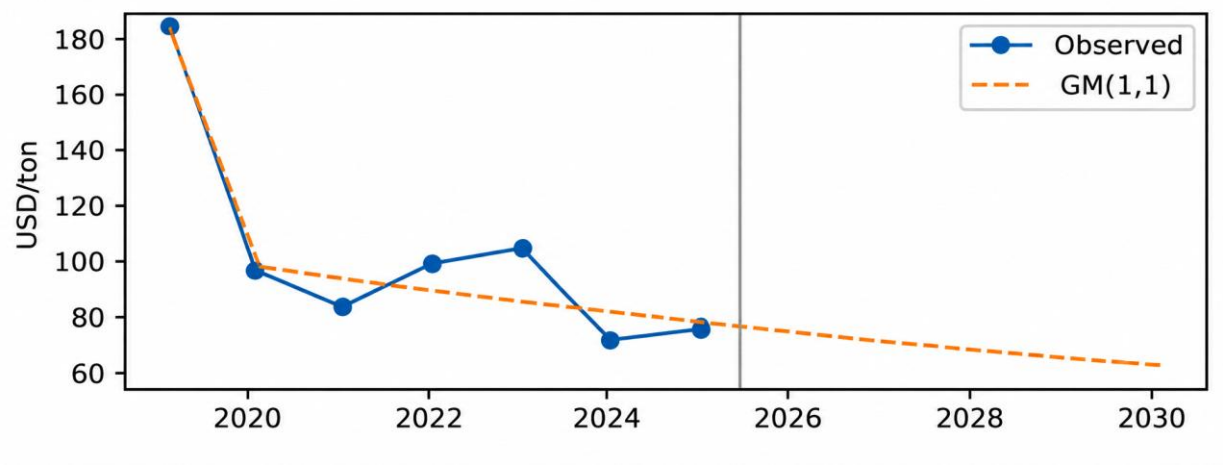

Figure 6. CoQ/ton-year

Table 2. Indicators and MAPE

Indicator	a	b	MAPE %	Ratio violations /6
Non-Prime	0.219537	5.365400	30.25	4
Rework	-0.214962	5.399599	14.04	4
Second quality	-0.086186	0.291758	17.08	3
Scrap	-0.031294	3.601433	10.19	0
Claim	0.063384	1.224439	34.67	5
CoQ/ton	0.041671	108.529808	10.13	2

5. Comparative forecasting and sensitivity analysis

Table 3. Indicators and GM MAPE

Indicator	GM MAPE %	Naive %	Trend %
Non-Prime	51.19	71.32	37.44
Rework	23.30	13.04	11.04
Second quality	36.76	29.57	57.46
Scrap	23.79	11.89	18.63
Claim	73.65	65.24	66.68
CoQ/ton	22.09	18.17	29.89

The lowest error across the three validation years may vary by indicator. Model selection should not rely solely on training fit. This comparison is exploratory.

Table 4. Forecasting future Indicator values

Indicator	2026	2027	2028	2029	2030
Non-Prime	1.038	0.834	0.669	0.537	0.431
Rework	30.914	38.328	47.520	58.915	73.044
Second quality	0.670	0.731	0.797	0.868	0.946
Scrap	4.557	4.702	4.851	5.006	5.165
Claim	0.704	0.661	0.620	0.582	0.546
CoQ/ton	76.866	73.729	70.720	67.833	65.065

Background weights of 0.4 and 0.6 are predetermined sensitivity scenarios; no optimal weight search was performed.

Table 5. Indicators and future value

Indicator	2026 λ=0.4	2026 λ=0.5	2026 λ=0.6	2030 λ=0.4	2030 λ=0.6
Non-Prime	1.033	1.038	1.045	0.432	0.432
Rework	32.547	30.914	29.404	78.217	68.345
Second quality	0.677	0.670	0.664	0.954	0.938
Scrap	4.567	4.557	4.546	5.169	5.160
Claim	0.694	0.704	0.715	0.533	0.560
CoQ per tonne	76.504	76.866	77.228	64.688	65.445

6. Economic analysis

The observed 2025 unit cost of USD 75.993 per tonne is used as the reference. The differences below represent projected unit costs relative to that reference, not realised savings.

Table 6. Forecasting future USD/ton values

Year	GM USD/ton	Difference from reference (USD/ton)
2026	76.866	0.873
2027	73.729	-2.264
2028	70.720	-5.273
2029	67.833	-8.160
2030	65.065	-10.928

7. Discussion

The opposing trends in Non-Prime and Claim relative to Rework warrant investigation of quality classification and rework policies. However, these independent models do not establish causal relationships among indicators. Fluctuations in scrap and second-quality production may not be captured by the unidirectional exponential model. The high initial cost of quality may influence the initial condition and coefficients. Grey forecasts should be treated as alternative scenarios rather than control targets.

8. Limitations and future research

Seven annual observations and three out-of-sample errors are insufficient to establish reliable prediction intervals. Product mix, prices, maintenance, raw materials and process conditions are not included. Monthly data would enable comparisons of rolling-window, discrete GM, seasonal and multivariate grey models. Markov analysis examines state probabilities, whereas this study examines continuous indicator values; direct comparison using the same error metric would require recalculating Markov reward forecasts over identical validation windows.

9. Conclusion

This study examined six operational and economic quality indicators from a stainless steel cold rolling facility in Türkiye over 2019–2025 using the GM(1,1) model within Grey System Theory. Independent mathematical models were developed for each indicator, and historical trends, development coefficients, forecasting performance and 2026–2030 projections were evaluated. The aim was to investigate the mathematical representation of industrial quality performance under limited historical data and establish an analytical framework for production management.

9.1. Operational quality indicators

The findings indicate that quality performance does not evolve uniformly. Improvements in some indicators and adverse trends in others require a multidimensional approach to quality management.

9.1.1. Non-Prime

The rate declined from 4.774% in 2019 to 0.856% in 2025, a reduction of 82.07%. The positive development coefficient (a = 0.219537) indicates a long-term declining trend. The 2026 and 2030 projections are 1.038% and 0.431%, respectively. The 2026 projection exceeds the latest observation, demonstrating that a long-term decline does not imply uninterrupted improvement.

In-sample MAPE is 30.25%, while out-of-sample MAPE is 51.19%. Because annual fluctuations are inadequately represented, the results should be interpreted as trend scenarios rather than definitive performance targets.

9.1.2. Rework

The rework rate increased from 10.374% to 22.656%, an increase of 118.39%. The negative development coefficient (a = −0.214962) produces exponential growth. Projections for 2026, 2028 and 2030 are 30.914%, 47.520% and 73.044%, respectively. Because the model excludes capacity, process and managerial intervention constraints, long-term exponential growth should not be interpreted as a definitive forecast.

Declining Non-Prime and Claim rates alongside increasing Rework raise the hypothesis that final quality improvements may be associated with rework activities. The independent univariate models do not establish causality. First-pass yield, rework causes, additional processing times, energy consumption and capacity losses warrant joint investigation.

9.1.3. Second-quality production

The rate declined from 1.056% to 0.525%, a reduction of 50.28%. Nevertheless, the negative development coefficient (a = −0.086186) produces an increasing trend over the forecast horizon. Projections for 2026 and 2030 are 0.670% and 0.946%.

The opposing directions of the endpoint change and model trend demonstrate that period-to-period changes alone are insufficient. The use of the first observation as the initial condition and subsequent observations for parameter estimation is important in interpreting this discrepancy.

9.1.4. Scrap

The scrap rate increased from 3.730% to 4.028%, an increase of 7.99%. The development coefficient is −0.031294, and projections for 2026 and 2030 are 4.557% and 5.165%. With an in-sample MAPE of 10.19%, it is among the best-fitting indicators. It is the only series with no violations of the conventional level-ratio admissibility interval.

However, the last-observation benchmark produced lower out-of-sample error. Historical fit alone does not establish forecasting superiority. Given its implications for material yield and resource consumption, scrap warrants managerial priority.

9.1.5. Customer claims

The rate declined from 2.549% to 0.406%, a reduction of 84.07%. The positive development coefficient (a = 0.063384) indicates a long-term declining trend. The 2026 and 2030 projections, 0.704% and 0.546%, both exceed the latest observation.

In-sample MAPE is 34.67% and out-of-sample MAPE is 73.65%. Future multivariate models could examine product mix, customer expectations, delivery conditions, defect types and reporting delays.

9.2. Economic quality performance

Cost of quality per packed tonne declined from USD 186.196 to USD 75.993, a reduction of USD 110.203 per tonne, or approximately 59.19%. The positive development coefficient (a = 0.041671) represents a long-term declining trend.

Table 7. Forecasting future USD/ton values

Year	Cost of quality (USD/ton)
2025 (actual)	75,993
2026	76,866
2027	73,729
2028	70,720
2029	67,833
2030	65,065

The difference between the 2030 projection and the 2025 observation is USD 10.928 per tonne, equivalent to a reduction of approximately 14.38%. This is not a validated saving or the effect of a specific investment, but a conditional mathematical extrapolation of the historical trend. In-sample MAPE of 10.13% is the lowest among the six indicators; however, out-of-sample MAPE of 22.09% warrants caution when interpreting economic projections.

Declining unit quality costs alongside increasing rework and scrap suggest that the relationship between operational and economic performance may be multidimensional. Cost components, unit cost coefficients, product mix and resource utilisation efficiency warrant joint investigation.

9.3. Grey forecasting model performance

Out-of-sample errors obtained through expanding-window validation were compared with in-sample fit errors.

Table 8. Indicators and MAPE Value

Indicator	In-sample MAPE (%)	Out-of-sample MAPE (%)
Non-Prime	30,25	51,19
Rework	14,04	23,30
Second quality	17,08	36,76
Scrap	10,19	23,79
Claim	34,67	73,65
CoQ per tonne	10,13	22,09

Out-of-sample errors were higher for all indicators. Linear trend produced the lowest out-of-sample MAPE for Non-Prime and Rework, while the last-observation method performed best for second-quality production, scrap, Claim and CoQ per tonne. Classical GM(1,1) did not achieve the lowest error for any indicator during validation. The ability to model small samples does not guarantee forecasting superiority. With only three out-of-sample forecasts, definitive conclusions about statistical superiority cannot be drawn. The grey model contributes a transparent, reproducible two-parameter representation of short series.

9.4. Sensitivity analysis

Changing the background weight to 0.4, 0.5 and 0.6 revealed differing sensitivities. The 2030 Rework projection ranges from 68.345% to 78.217%, while projected unit quality costs range from USD 64.688 to USD 65.445 per tonne. The unit cost model is less sensitive to these changes. These ranges represent parameter scenarios rather than statistical confidence intervals. Forecast horizons and parameter choices require careful consideration for exponentially increasing series.

9.5. Scientific contributions

An industrial case study demonstrates the application of Grey System Theory to quality economics in stainless steel cold rolling.

Six independent GM(1,1) models examine the multidimensional nature of quality performance and its development coefficients.

Economic analysis is conducted using unit costs without disclosing absolute financial figures.

Forecasting performance is compared with simple benchmarks using chronological out-of-sample validation.

Matrix operations, parameter calculations, differential equation solutions and projections are explicitly presented in the mathematical appendices.

The advantages of small-sample modelling and the limitations of exponential forecasting are evaluated using industrial data.

The original contribution lies not in developing a new grey forecasting algorithm but in applying an existing method to industrial quality economics, evaluating it comparatively and examining its managerial implications.

9.6. Managerial and strategic implications

Rework reduction: causes should be investigated by process, product group, alloy, thickness and defect type.

First-pass yield improvement: products meeting conformity criteria without additional processing should constitute a key performance objective.

Scrap control: defect formation points should be identified and root-cause analyses conducted.

Integrated quality management: improvements in one indicator should be examined for potential cost increases in another.

Forecast updating: models should be recalibrated as new observations become available and compared with alternative methods.

Economic decision support: unit cost projections can inform budgeting scenarios; investment decisions should not rely solely on these projections.

9.7. Future research directions

Seven annual observations constitute the principal limitation. Monthly, weekly and production-batch data could improve validation capacity. Multivariate GM(1,N), discrete DGM(1,1), nonlinear grey Bernoulli models, rolling-window models and fractional-order accumulated models could be compared.

In hybrid Grey–Markov models, grey models can represent underlying trends while Markov chains can represent probabilistic transitions of forecast errors or quality states. Whether such an approach improves forecasting performance must be tested using independent observations. System dynamics modelling of feedback mechanisms and the use of grey forecasts in stochastic or multi-objective optimisation could be investigated for integrated decision support.

9.8. Overall assessment

In conclusion, Grey System Theory is one mathematical tool for examining stainless steel cold rolling quality performance under limited historical data. Substantial improvements were observed in Non-Prime, Claim and unit quality costs, while increases in Rework and scrap warrant managerial attention. Quality management should focus not only on final product conformity but also on the economic resources used to achieve it.

GM(1,1) can contribute to understanding historical trends, identifying potential risks and developing alternative economic scenarios. However, the comparative findings do not support its adoption as a primary forecasting tool without independent validation. Its principal value lies in providing mathematically structured assessments under incomplete information rather than definitive forecasts. Integrating grey forecasting, system dynamics, stochastic modelling and mathematical optimisation could provide a foundation for future research.

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Appendix A. Non-Prime: Detailed mathematical solution

Accumulated generation, background and observation values:

k	$x^{(0)}$	$x^{(1)}$	$z^{(1)}$	$-z^{(1)}$
1	4.774000	4.774000	—	—
2	4.536000	9.310000	7.042000	-7.042000
3	1.947000	11.257000	10.283500	-10.283500
4	2.207000	13.464000	12.360500	-12.360500
5	3.149000	16.613000	15.038500	-15.038500
6	1.723000	18.336000	17.474500	-17.474500
7	0.856000	19.192000	18.764000	-18.764000

$$B^TB = [[1191.724425, -80.963000], [-80.963000, 6]]$$

$$B^TY = [-172.770894, 14.418000]^T$$

$$D = (1191.724425) \times 6 - (-80.963000)^2 = 595.339181$$

$$a = [6 \times (-172.770894) - (-80.963000) \times (14.418000)] / D$$

$$b = [(1191.724425) \times (14.418000) - (-80.963000) \times (-172.770894)] / D$$

$$a = 0.219537;\ b = 5.365400;\ b/a = 24.439580$$

$$\hat{x}^{(1)}(k) = (-19.665580)\ \exp[(-0.219537)(k-1)] + (24.439580)$$

$$\hat{x}^{(0)}(8) = (-19.665580) \times [\exp(-1.536761) - \exp(-1.317224)]$$

$$\hat{x}^{(0)}(8) = 1.038369$$

Year	Observation	GM forecast	Absolute percentage error
2019	4.774000	4.774000	—
2020	4.536000	3.876279	14.544118
2021	1.947000	3.112226	59.847263
2022	2.207000	2.498776	13.220476
2023	3.149000	2.006243	36.289531
2024	1.723000	1.610793	6.512330
2025	0.856000	1.293290	51.085227
2026	—	1.038369	—
2027	—	0.833697	—
2028	—	0.669367	—
2029	—	0.537428	—
2030	—	0.431496	—

Intermediate calculations are displayed to six decimal places; results are calculated using unrounded coefficients.

Expanding-window calculations

Year	a	b	GM	Naïve	Trend	Actual
2023	0.469499	7.542514	1.034420	2.207000	0.793500	3.149000
2024	0.168760	4.846674	1.893190	3.149000	1.648900	1.723000
2025	0.182107	4.977769	1.511084	1.723000	1.140400	0.856000

Appendix B. Rework: Detailed mathematical solution

Accumulated generation, background and observation values:

k	$x^{(0)}$	$x^{(1)}$	$z^{(1)}$	$-z^{(1)}$
1	10.374000	10.374000	—	—
2	5.807000	16.181000	13.277500	-13.277500
3	9.976000	26.157000	21.169000	-21.169000
4	14.630000	40.787000	33.472000	-33.472000
5	16.556000	57.343000	49.065000	-49.065000
6	22.150000	79.493000	68.418000	-68.418000
7	22.656000	102.149000	90.821000	-90.821000

$$B^{T}B = [[17081.644341, -276.222500], [-276.222500, 6]]$$

$$B^{T}Y = [-5163.399163, 91.775000]^{T}$$

$$D = (17081.644341) \times 6 - (-276.222500)^{2} = 26190.996541$$

$$a = [6 \times (-5163.399163) - (-276.222500) \times (91.775000)] / D$$

$$b = [(17081.644341) \times (91.775000) - (-276.222500) \times (-5163.399163)] / D$$

$$a = -0.214962;\ b = 5.399599;\ b/a = -25.118828$$

$$\hat{x}^{(1)}(k) = (35.492828)\ \exp[(0.214962)(k-1)] + (-25.118828)$$

$$\hat{x}^{(0)}(8) = (35.492828) \times [\exp(1.504736) - \exp(1.289773)]$$

$$\hat{x}^{(0)}(8) = 30.914237$$

Year	Observation	GM forecast	Absolute percentage error
2019	10.374000	10.374000	—
2020	5.807000	8.511715	46.576802
2021	9.976000	10.552952	5.783404
2022	14.630000	13.083709	10.569314
2023	16.556000	16.221380	2.021140
2024	22.150000	20.111511	9.203109
2025	22.656000	24.934555	10.057181
2026	—	30.914237	—
2027	—	38.327937	—
2028	—	47.519554	—
2029	—	58.915459	—
2030	—	73.044274	—

Intermediate calculations are displayed to six decimal places; results are calculated using unrounded coefficients.

Expanding-window calculations

Year	a	b	GM	Naïve	Trend	Actual
2023	-0.431782	0.362346	22.114803	14.630000	14.431000	16.556000
2024	-0.296423	3.073103	23.423033	16.556000	17.824700	22.150000
2025	-0.277161	3.546583	29.585707	22.150000	22.826933	22.656000

Appendix C. Second-quality production: Detailed mathematical solution

Accumulated generation, background and observation values:

k	$x^{(0)}$	$x^{(1)}$	$z^{(1)}$	$-z^{(1)}$
1	1.056000	1.056000	—	—
2	0.431000	1.487000	1.271500	-1.271500
3	0.424000	1.911000	1.699000	-1.699000
4	0.333000	2.244000	2.077500	-2.077500
5	0.605000	2.849000	2.546500	-2.546500
6	0.690000	3.539000	3.194000	-3.194000
7	0.525000	4.064000	3.801500	-3.801500

$$B^{T}B = [[39.957020, -14.590000], [-14.590000, 6]]$$

$$B^{T}Y = [-7.700480, 3.008000]^{T}$$

$$D = (39.957020) \times 6 - (-14.590000)^{2} = 26.874020$$

$$a = [6 \times (-7.700480) - (-14.590000) \times (3.008000)] / D$$

$$b = [(39.957020) \times (3.008000) - (-14.590000) \times (-7.700480)] / D$$

$$a = -0.086186;\ b = 0.291758;\ b/a = -3.385221$$

$$\hat{x}^{(1)}(k) = (4.441221)\exp[(0.086186)(k-1)] + (-3.385221)$$

$$\hat{x}^{(0)}(8) = (4.441221) \times [\exp(0.603301) - \exp(0.517115)]$$

$$\hat{x}^{(0)}(8) = 0.670452$$

Year	Observation	GM forecast	Absolute percentage error
2019	1.056000	1.056000	—
2020	0.431000	0.399749	7.250743
2021	0.424000	0.435730	2.766576
2022	0.333000	0.474950	42.627590
2023	0.605000	0.517700	14.429822
2024	0.690000	0.564297	18.217806
2025	0.525000	0.615089	17.159788
2026	—	0.670452	—
2027	—	0.730799	—
2028	—	0.796577	—
2029	—	0.868276	—
2030	—	0.946429	—

Intermediate calculations are displayed to six decimal places; results are calculated using unrounded coefficients.

Expanding-window calculations

Year	a	b	GM	Naïve	Trend	Actual
2023	0.119329	0.596791	0.310235	0.333000	0.017000	0.605000
2024	-0.109166	0.240985	0.582550	0.605000	0.269800	0.690000
2025	-0.155851	0.160319	0.766402	0.690000	0.449933	0.525000

Appendix D. Scrap: Detailed mathematical solution

Accumulated generation, background and observation values:

k	$x^{(0)}$	$x^{(1)}$	$z^{(1)}$	$-z^{(1)}$
1	3.730000	3.730000	—	—
2	3.202000	6.932000	5.331000	-5.331000
3	3.910000	10.842000	8.887000	-8.887000
4	4.445000	15.287000	13.064500	-13.064500
5	4.961000	20.248000	17.767500	-17.767500
6	3.990000	24.238000	22.243000	-22.243000
7	4.028000	28.266000	26.252000	-26.252000

$$B^{T}B = [[1777.682100, -93.545000], [-93.545000, 6]]$$

$$B^{T}Y = [-392.526928, 24.536000]^{T}$$

$$D = (1777.682100) \times 6 - (-93.545000)^{2} = 1915.425572$$

$$a = [6 \times (-392.526928) - (-93.545000) \times (24.536000)] / D$$

$$b = [(1777.682100) \times (24.536000) - (-93.545000) \times (-392.526928)] / D$$

$$a = -0.031294;\ b = 3.601433;\ b/a = -115.083581$$

$$\hat{x}^{(1)}(k) = (118.813581)\ \exp[(0.031294)(k-1)] + (-115.083581)$$

$$\hat{x}^{(0)}(8) = (118.813581) \times [\exp(0.219058) - \exp(0.187764)]$$

$$\hat{x}^{(0)}(8) = 4.557075$$

Year	Observation	GM forecast	Absolute percentage error
2019	3.730000	3.730000	—
2020	3.202000	3.776949	17.955949
2021	3.910000	3.897014	0.332111
2022	4.445000	4.020896	9.541144
2023	4.961000	4.148716	16.373395
2024	3.990000	4.280599	7.283179
2025	4.028000	4.416674	9.649311
2026	—	4.557075	—
2027	—	4.701940	—
2028	—	4.851409	—
2029	—	5.005630	—
2030	—	5.164753	—

Intermediate calculations are displayed to six decimal places; results are calculated using unrounded coefficients.

Expanding-window calculations

Year	a	b	GM	Naïve	Trend	Actual
2023	-0.159786	2.399212	5.245290	4.445000	4.535000	4.961000
2024	-0.138912	2.565003	5.765307	4.961000	5.161100	3.990000
2025	-0.058929	3.308496	4.879641	3.990000	4.750867	4.028000

Appendix E. Claim: Detailed mathematical solution

Accumulated generation, background and observation values:

k	$x^{(0)}$	$x^{(1)}$	$z^{(1)}$	$-z^{(1)}$
1	2.549000	2.549000	—	—
2	1.159000	3.708000	3.128500	-3.128500
3	0.609000	4.317000	4.012500	-4.012500
4	0.894000	5.211000	4.764000	-4.764000
5	1.183000	6.394000	5.802500	-5.802500
6	1.050000	7.444000	6.919000	-6.919000
7	0.406000	7.850000	7.647000	-7.647000

$$B^{T}B = [[188.601541, -32.273500], [-32.273500, 6]]$$

$$B^{T}Y = [-27.562550, 5.301000]^{T}$$

$$D = (188.601541) \times 6 - (-32.273500)^{2} = 90.030442$$

$$a = [6 \times (-27.562550) - (-32.273500) \times (5.301000)] / D$$

$$b = [(188.601541) \times (5.301000) - (-32.273500) \times (-27.562550)] / D$$

$$a = 0.063384;\ b = 1.224439;\ b/a = 19.317675$$

$$\hat{x}^{(1)}(k) = (-16.768675)\ \exp[(-0.063384)(k-1)] + (19.317675)$$

$$\hat{x}^{(0)}(8) = (-16.768675) \times [\exp(-0.443691) - \exp(-0.380306)]$$

$$\hat{x}^{(0)}(8) = 0.704085$$

Year	Observation	GM forecast	Absolute percentage error
2019	2.549000	2.549000	—
2020	1.159000	1.029888	11.139915
2021	0.609000	0.966635	58.725011
2022	0.894000	0.907267	1.484015
2023	1.183000	0.851545	28.018165
2024	1.050000	0.799245	23.881388
2025	0.406000	0.750158	84.767945
2026	—	0.704085	—
2027	—	0.660842	—
2028	—	0.620255	—
2029	—	0.582160	—
2030	—	0.546406	—

Intermediate calculations are displayed to six decimal places; results are calculated using unrounded coefficients.

Expanding-window calculations

Year	a	b	GM	Naïve	Trend	Actual
2023	0.175433	1.583512	0.615741	0.894000	-0.076000	1.183000
2024	-0.044412	0.764646	1.072139	1.183000	0.379700	1.050000
2025	-0.039461	0.784641	1.099870	1.050000	0.526867	0.406000

Appendix F. CoQ per tonne: Detailed mathematical solution

Accumulated generation, background and observation values:

k	$x^{(0)}$	$x^{(1)}$	$z^{(1)}$	$-z^{(1)}$
1	186.196000	186.196000	—	—
2	97.234000	283.430000	234.813000	-234.813000
3	84.279000	367.709000	325.569500	-325.569500
4	99.885000	467.594000	417.651500	-417.651500
5	105.051000	572.645000	520.119500	-520.119500
6	72.470000	645.115000	608.880000	-608.880000
7	75.993000	721.108000	683.111500	-683.111500

$$B^{T}B = [[1443465.889864, -2790.145000], [-2790.145000, 6]]$$

$$B^{T}Y = [-242663.898624, 534.912000]^{T}$$

$$D = (1443465.889864) \times 6 - (-2790.145000)^{2} = 875886.218159$$

$$a = [6 \times (-242663.898624) - (-2790.145000) \times (534.912000)] / D$$

$$b = [(1443465.889864) \times (534.912000) - (-2790.145000) \times (-242663.898624)] / D$$

$$a = 0.041671;\ b = 108.529808;\ b/a = 2604.473353$$

$$\hat{x}^{(1)}(k) = (-2418.277353)\ \exp[(-0.041671)(k-1)] + (2604.473353)$$

$$\hat{x}^{(0)}(8) = (-2418.277353) \times [\exp(-0.291694) - \exp(-0.250023)]$$

$$\hat{x}^{(0)}(8) = 76.866002$$

Year	Observation	GM forecast	Absolute percentage error
2019	186.196000	186.196000	—
2020	97.234000	98.700193	1.507902
2021	84.279000	94.671818	12.331444
2022	99.885000	90.807858	9.087593
2023	105.051000	87.101603	17.086365
2024	72.470000	83.546616	15.284415
2025	75.993000	80.136723	5.452769
2026	—	76.866002	—
2027	—	73.728773	—
2028	—	70.719588	—
2029	—	67.833221	—
2030	—	65.064658	—

Intermediate calculations are displayed to six decimal places; results are calculated using unrounded coefficients.

Expanding-window calculations

Year	a	b	GM	Naïve	Trend	Actual
2023	-0.014876	88.949473	96.622369	99.885000	48.926500	105.051000
2024	-0.041951	80.899998	107.150865	105.051000	66.637300	72.470000
2025	0.029687	104.293943	83.890961	72.470000	54.561867	75.993000