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Harmonic Identifiability and Singular Conditioning in Quadratic Jerk Systems with PID/PIDD Control

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Abstract

We investigate the identifiability of proportional–integral–derivative controllers with an optional second-derivative term from the periodic output of a known quadratic jerk system under single-frequency excitation. In the linear model, the complex fundamental harmonic determines only two independent combinations of the four controller gains. For the nonlinear model, an explicit complex polynomial expression in the excitation frequency characterizes whether this ambiguity persists. Under nonzero excitation, if the expression is nonzero, every periodic solution with the excitation period and a periodic integral state has at least two distinct positive output harmonics, and its complete output uniquely determines the gains. For every Hurwitz controller at a zero of this expression, an exact sinusoidal solution exists at every forcing amplitude. The free mean of the integral state absorbs the constant quadratic contribution whenever it is nonzero. This solution is locally unique and exponentially stable for sufficiently small excitation amplitudes. We derive uniform expansions of the first two complex Fourier coefficients, including remainder estimates that vanish with the harmonic generation coefficient. For small nonzero amplitudes at nondegenerate frequencies on compact subsets of the Hurwitz region, two singular values of the four-dimensional observation Jacobian scale linearly with the excitation amplitude, and the remaining two scale quadratically with the amplitude and linearly with the modulus of the harmonic generation coefficient. At nondegenerate frequencies, a uniform two-sided estimate establishes injectivity from the first two complex harmonics on each compact Hurwitz set for sufficiently small nonzero amplitudes. On the small-amplitude periodic branch, the first two harmonics capture the squared separation of complete output waveforms for arbitrary pairs of controllers in a compact Hurwitz set under common excitation, up to a sharp relative error quadratic in the excitation amplitude. On gain sets containing a ball,

we establish matching deterministic minimax bounds under bounded noise for both the two-harmonic and complete-output observations. Analytical examples and Fourier–Galerkin computations illustrate the results.

Keywords: parameter identifiability, quadratic jerk systems, PID control, PIDD control, periodic solutions, harmonic generation, singular values, conditioning

1 Introduction

Different controller gains may produce nearly indistinguishable output trajectories, even when the plant is known and the closed-loop systems are stable. This ambiguity is especially relevant when the performance of several candidate controllers is evaluated using a single periodic disturbance or reference experiment. In this setting it is important to distinguish between the existence of an accurate fit to an observed response and the ability to identify the controller parameters from that response. Neither a small output error nor the convergence of a numerical optimization algorithm guarantees the latter property.

Frequency-domain identification and the role of excitation in parameter estimation are well established; see, for example, [1]. Nonlinear distortions and higher harmonics are also important in frequency-response measurements [2]. Harmonic probing and higher-order frequency-response functions have been used for nonlinear parameter estimation, including the assessment of the small amplitudes of higher harmonics [3]. Related identifiability questions arise in nonlinear feedback systems operating in a limit cycle [4]. These results provide a foundation for the analysis of harmonic identifiability.

The problem considered here is more specific. The plant is a known third-order system with a homogeneous quadratic nonlinearity, the controller belongs to a PID/PIDD family, and the unknown quantities are the controller gains. The excitation is a single sinusoid of known amplitude, frequency and phase, and the observation is the periodic scalar output with a fixed time origin. In the linear PIDD model, two gains can vary independently along an affine set while preserving the steady-state complex fundamental harmonic. The corresponding ambiguity is one-dimensional for PID feedback. We ask whether the quadratic terms remove this ambiguity, whether there are frequencies at which the ambiguity persists exactly in the nonlinear model, and how sensitive the identification problem becomes near such frequencies.

Our first result is an explicit algebraic classification in terms of a complex coefficient $\Gamma_Q(\omega)$ associated with the quadratic nonlinearity Q. A nonzero coefficient ensures that any periodic output of the stipulated period has sufficient harmonic content to identify all four gains. If the coefficient vanishes, an exact sinusoidal solution persists and the linear ambiguity is retained on the small-amplitude periodic branch. The free mean of the integral state compensates for the constant quadratic contribution whenever it is nonzero.

The second result concerns uniform expansions of the first two complex Fourier coefficients. Their nonlinear remainders contain a factor $|\Gamma_Q(\omega)|$, which is crucial near

a frequency at which harmonic generation vanishes. The third result uses these expansions to quantify the singular values of the observation Jacobian and to establish a two-sided estimate for differences of observations from arbitrary pairs of controllers in a compact subset of the Hurwitz region. In particular, the relevant small scales are the excitation amplitude and the product of its square with $|\Gamma_Q(\omega)|$. The estimates remain valid when the latter coefficient approaches zero at an arbitrarily fast rate relative to the amplitude.

A fourth result shows that the first two harmonics capture the separation of complete periodic waveforms, including the weakly observed gain directions, with a sharp relative energy error of order $O(\varepsilon^2)$. The resulting deterministic minimax bounds establish the same identification scale for the complete waveform and the first two harmonics.

The analysis uses a deterministic observation model with known plant coefficients and excitation, including its phase. The measured quantity is the scalar periodic output. The small-amplitude results concern the periodic branch emanating from the nominal equilibrium.

2 Closed-Loop Model and Linear Frequency Equivalence

Consider the third-order system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -a_0 x_1 - a_1 x_2 - a_2 x_3 + Q(x_1, x_2, x_3) + u + F(t), \\ y &= x_1, \end{aligned} \tag{1}$$

where $a_0, a_1, a_2 \in \mathbb{R}$ are known and

$$Q(x) = q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + q_{12}x_1x_2 + q_{13}x_1x_3 + q_{23}x_2x_3 \tag{2}$$

is a known real homogeneous quadratic form. The forcing is

$$F(t) = \varepsilon \cos(\omega t), \qquad \omega > 0, \qquad T = 2\pi/\omega. \tag{3}$$

The time origin is fixed by (3). Throughout, i denotes the imaginary unit, $\mathrm{i}^2 = -1$. We allow signed amplitudes ε for analytic expansions and take $\varepsilon > 0$ in the identification estimates.

For a zero reference, the proportional–integral–derivative feedback with an additional second-derivative term (PIDD) and the integral state are

$$u = -K_p x_1 - K_d x_2 - K_{dd} x_3 + K_i x_4, \qquad \dot{x}_4 = -x_1. \tag{4}$$

The gain vector is $K = (K_p, K_i, K_d, K_{dd})^\top \in \mathbb{R}^4$. Proportional–integral–derivative (PID) feedback is the restriction $K_{dd} = 0$. Introduce the real coefficients

$$A = a_2 + K_{dd}, \quad B = a_1 + K_d, \quad C = a_0 + K_p, \quad D = K_i. \tag{5}$$

Writing $X = (x_1, x_2, x_3, x_4)^\top$ and $e_3 = (0, 0, 1, 0)^\top$, the closed-loop system is

$$\dot{X} = M_K X + e_3\big(Q(x_1, x_2, x_3) + F(t)\big), \qquad M_K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -C & -B & -A & D \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

The characteristic polynomial and Hurwitz region are

$$P_K(s) = s^4 + As^3 + Bs^2 + Cs + D, \qquad \mathcal{H} = \{K \in \mathbb{R}^4 : P_K \text{ is Hurwitz}\}. \tag{7}$$

By the Routh–Hurwitz criterion, membership in $\mathcal{H}$ is equivalent to

$$A, B, C, D > 0, \qquad ABC - C^2 - A^2 D > 0. \tag{8}$$

Indeed, the last inequality also implies $AB > C$ when the coefficients are positive. In particular, the nominal unforced origin of (6) is locally exponentially stable for $K \in \mathcal{H}$. The forced periodic branch is analyzed in Section 4.

For fixed $B, C, D > 0$, the condition on A can be written as $DA^2 - BCA + C^2 < 0$. There exists an admissible positive A if and only if $B^2 > 4D$, in which case the interval is

$$\frac{C}{2D}\big(B - \sqrt{B^2 - 4D}\big) < A < \frac{C}{2D}\big(B + \sqrt{B^2 - 4D}\big). \tag{9}$$

If constraints on the gains are imposed, this interval must be intersected with the corresponding admissible interval for $a_2 + K_{dd}$. Thus, for fixed values of the other gains, nominal stability requires the second-derivative gain to lie in the stated interval.

2.1 Frequency response and equivalence of gains

The transfer function from F to y in the linearization of (6) is

$$H_K(s) = \frac{s}{P_K(s)}, \qquad L_K(s) = \frac{P_K(s)}{s} = s^3 + As^2 + Bs + C + \frac{D}{s}, \quad s \neq 0. \tag{10}$$

The function H_K is nonzero at every nonzero imaginary argument for $K \in \mathcal{H}$. For two such gain vectors $K, \widehat{K}$, let $\Delta K = \widehat{K} - K$. Their linear steady-state outputs under the same nonzero forcing (3) coincide if and only if $L_K(\mathrm{i}\omega) = L_{\widehat{K}}(\mathrm{i}\omega)$. Direct calculation gives

$$\Delta K_p - \omega^2 \Delta K_{dd} = 0, \qquad \Delta K_i - \omega^2 \Delta K_d = 0. \tag{11}$$

The equivalent gain vectors lie in the affine plane

$$K + \operatorname{span}\{(0, \omega^2, 1, 0)^\top, (\omega^2, 0, 0, 1)^\top\}, \tag{12}$$

intersected with the Hurwitz region and any prescribed gain constraints. Since $\mathcal{H}$ is open, this equivalence set has local dimension two in the unconstrained PIDD case, and dimension one in the PID case. These are equivalence sets for the complex steady-state response, including its phase; amplitude-only measurements impose different conditions.

Conversely, if two controllers have the same complex responses at two distinct positive frequencies ω_1 and ω_2, then (11) holds at both frequencies. Subtracting the corresponding equations shows that $\Delta K_d = \Delta K_{dd} = 0$, and consequently $\Delta K_i = \Delta K_p = 0$. Thus two complex frequency-response values uniquely determine the four gains when the plant coefficients are known.

For later use define the frequency coordinates

$$z_1(K,\omega) = L_K(\mathrm{i}\omega), \qquad z_2(K,\omega) = L_K(2\mathrm{i}\omega). \tag{13}$$

We identify $\mathbb{C}^2$ with $\mathbb{R}^4$ by taking real and imaginary parts in this order. The real derivative of the affine map $K \mapsto (z_1, z_2)$ is

$$S_\omega = \begin{pmatrix} 1 & 0 & 0 & -\omega^2 \\ 0 & -1/\omega & \omega & 0 \\ 1 & 0 & 0 & -4\omega^2 \\ 0 & -1/(2\omega) & 2\omega & 0 \end{pmatrix}, \qquad |\det S_\omega| = \frac{9}{2}\omega^2. \tag{14}$$

In particular, both S_ω and its inverse are uniformly bounded when ω lies in a compact interval contained in $(0,\infty)$.

2.2 Stability within an equivalence plane

Let $\tau = \omega^2$ and fix $P_K(\mathrm{i}\omega) = R + \mathrm{i}\omega\ell$ for some real R, ℓ. On the corresponding equivalence plane,

$$D = R - \tau^2 + B\tau, \qquad C = \ell + A\tau. \tag{15}$$

Substitution gives the identity

$$ABC - C^2 - A^2 D = A\ell(B - 2\tau) - \ell^2 - A^2 R. \tag{16}$$

Together with positivity of A, B, C, D, this determines the stable portion of the plane. For $\ell > 0$, it requires $B > 2\tau + \ell/A + AR/\ell$; for $\ell < 0$ the inequality is reversed. For $\ell = 0$ it requires $R < 0$. This characterization concerns the nominal linearization. Equivalent linear frequency responses can have different characteristic polynomials and different stability margins.

3 Exact Nonlinear Identifiability and Degenerate Frequencies

Every T-periodic solution of the full system (6) satisfies

$$\int_0^T y(t)\,dt = 0 \tag{17}$$

because $\dot{x}_4 = -y$ and x_4 is periodic. The full state is assumed periodic in what follows; the unknown mean of the integral state is not prescribed. For any periodic scalar function v, write

$$[v]_n = \frac{1}{T}\int_0^T v(t)e^{-\mathrm{i}n\omega t}\,dt, \qquad n \in \mathbb{Z}, \tag{18}$$

and denote the output coefficients by $Y_n = [y]_n$. For real functions $[v]_{-n} = \overline{[v]_n}$, where the overbar denotes complex conjugation.

The scalar output equation is

$$y^{(3)} + A\ddot{y} + B\dot{y} + Cy - Dx_4 = Q(y, \dot{y}, \ddot{y}) + F(t), \qquad \dot{x}_4 = -y. \tag{19}$$

At every nonzero harmonic, periodicity gives $[x_4]_n = -Y_n/(\mathrm{i}n\omega)$. Thus the exact Fourier identity

$$L_K(\mathrm{i}n\omega)Y_n = [Q(y, \dot{y}, \ddot{y})]_n + [F]_n, \qquad n \neq 0, \tag{20}$$

holds without any truncation or approximation.

Define the second-harmonic generation coefficient

$$\begin{aligned}\Gamma_Q(\omega) = {} & q_{11} - (q_{22} + q_{13})\omega^2 + q_{33}\omega^4 \\ & + \mathrm{i}\omega(q_{12} - q_{23}\omega^2).\end{aligned} \tag{21}$$

For a sinusoidal output $v(t) = he^{\mathrm{i}\omega t} + \overline{h}e^{-\mathrm{i}\omega t}$, the quadratic term has the exact expansion

$$Q(v, \dot{v}, \ddot{v}) = Q_0 + \Gamma_Q(\omega)h^2e^{2\mathrm{i}\omega t} + \overline{\Gamma_Q(\omega)h^2}e^{-2\mathrm{i}\omega t}, \tag{22}$$

where

$$Q_0 = 2|h|^2\big(q_{11} + (q_{22} - q_{13})\omega^2 + q_{33}\omega^4\big). \tag{23}$$

These formulas follow by multiplying the Fourier representations of v, $\dot{v}$ and $\ddot{v}$. In particular, $\Gamma_Q(\omega)$ characterizes the generation of a second harmonic by the quadratic form along a sinusoidal output.

Theorem 1 (Exact identifiability and harmonic degeneracy) *Fix the known plant, $\omega > 0$ and $\varepsilon \neq 0$.*

(i) If $\Gamma_Q(\omega) \neq 0$, every T-periodic solution of (6) has at least two distinct positive output harmonics with nonzero Fourier coefficients. If two gain vectors produce the same complete T-periodic output, with a periodic integral state in both systems, then the gain

vectors are equal. These assertions do not require nominal stability or a small forcing amplitude.

(ii) If $\Gamma_Q(\omega) = 0$, then for every $K \in \mathcal{H}$ and every amplitude ε, there exists an exact T-periodic solution with output

$$y(t) = \varepsilon \operatorname{Re}\big(H_K(\mathrm{i}\omega)e^{\mathrm{i}\omega t}\big). \tag{24}$$

For sufficiently small $|\varepsilon|$, this solution is locally unique near the origin and locally exponentially stable. On this small-amplitude periodic branch, two Hurwitz gain vectors give identical outputs if and only if (11) holds, whenever the amplitude lies in the domains of both branches.

Proof The solutions considered are smooth because the vector field is polynomial in the state and the forcing is smooth. By (17), a periodic output with no positive harmonics is identically zero. Then (19) would require a constant integral state to cancel a nonconstant forcing, which is impossible.

Suppose the output has precisely one positive harmonic, of index $n \geq 1$. Then its quadratic term has Fourier support contained in $\{0, 2n, -2n\}$. If $n \geq 2$, the Fourier identity (20) at index 1 gives $0 = \varepsilon/2$, a contradiction. Hence $n = 1$. At index 2, (22) then gives $0 = \Gamma_Q(\omega)Y_1^2$, again impossible if $\Gamma_Q(\omega) \neq 0$.

Now suppose two controllers generate an identical complete output y. Their known nonlinear terms and forcing coincide. By subtracting (20), we obtain

$$\big(L_{\widehat{K}}(\mathrm{i}n\omega) - L_K(\mathrm{i}n\omega)\big)Y_n = 0 \tag{25}$$

for every $n \neq 0$. Two distinct positive indices with $Y_n \neq 0$ yield two instances of (11) at distinct frequencies. As shown in Section 2, these conditions imply $\widehat{K} = K$. The subtraction does not require dividing by L_K and is valid even for gains outside $\mathcal{H}$.

For (ii), take $h = \varepsilon H_K(\mathrm{i}\omega)/2$ and set

$$y = he^{\mathrm{i}\omega t} + \overline{h}e^{-\mathrm{i}\omega t}, \qquad x_4 = -\frac{h}{\mathrm{i}\omega}e^{\mathrm{i}\omega t} + \frac{\overline{h}}{\mathrm{i}\omega}e^{-\mathrm{i}\omega t} - \frac{Q_0}{D}, \tag{26}$$

where Q_0 is given by (23). Since $D > 0$ and $L_K(\mathrm{i}\omega) \neq 0$, these expressions are well defined. Direct differentiation gives $\dot{x}_4 = -y$. By (22), the quadratic term is identically Q_0. The constant part of $-Dx_4$ on the left of (19) equals Q_0; at the nonzero harmonics the equation is precisely the linear frequency-response identity. Hence (26), together with $x_2 = \dot{y}$, $x_3 = \ddot{y}$, is an exact solution.

At $\varepsilon = 0$, the matrix M_K is Hurwitz. The derivative of the time-T map there is $e^{M_K T}$, whose eigenvalues have modulus less than one. The implicit function theorem applied to the fixed-point equation yields a unique periodic solution near the origin for sufficiently small $|\varepsilon|$. Its stability follows from continuity of the eigenvalues of the derivative of the time-T map. More explicitly, a fixed point with all multipliers of modulus less than one is locally exponentially stable for the discrete time-T map; continuous dependence over each bounded time interval then gives local exponential stability of the periodically forced solution. The explicit solution (26) approaches zero as $\varepsilon \to 0$ and therefore belongs to this unique branch. The equivalence assertion follows from (24) and (11). □

Remark 1 The identification in Theorem 1(ii) uses the periodic output with a free mean of the integral state. The mean of x_4 in (26) generally depends on the controller. Local uniqueness and exponential stability are established for sufficiently small amplitudes.

3.1 Classification of degenerate frequencies

Call a positive frequency degenerate if $\Gamma_Q(\omega) = 0$. Setting $z = \omega^2$, degeneracy is equivalent to the simultaneous equations

$$R_Q(z) := q_{33}z^2 - (q_{22}+q_{13})z + q_{11} = 0, \qquad I_Q(z) := q_{12} - q_{23}z = 0, \qquad z > 0. \tag{27}$$

Proposition 1 (Algebraic classification of degeneracy) *If I_Q is not identically zero, there is at most one degenerate positive frequency. If $I_Q \equiv 0$ and R_Q is not identically zero, there are at most two. All positive frequencies are degenerate if and only if*

$$Q(x) = \gamma(x_2^2 - x_1x_3), \qquad \gamma \in \mathbb{R}. \tag{28}$$

Any isolated positive zero of Γ_Q has order one or two. Here the order is the integer m such that $|\Gamma_Q(\omega)|$ is bounded above and below by positive multiples of $|\omega - \omega_*|^m$ near the zero ω_*.

Proof The first two assertions follow from the degrees of the nonzero real polynomials in (27). If both polynomials vanish identically, then $q_{33} = q_{11} = q_{12} = q_{23} = 0$ and $q_{22} + q_{13} = 0$, which gives (28); the converse is immediate. Since $\omega \mapsto \omega^2$ has nonzero derivative at a positive frequency and the factor ω in the imaginary part of (21) is nonzero there, the order of a common zero is the smaller of the orders of the non-identically-zero polynomials that vanish there. It is therefore one or two. Equivalently, this follows by taking the first nonzero Taylor coefficient of the complex function at the zero. □

For (28), a sinusoid $y = M\cos(\omega t + \phi)$ satisfies $\dot{y}^2 - y\ddot{y} = M^2\omega^2$. Thus the nonlinearity is nontrivial when $\gamma \neq 0$, but contributes only a constant along a sinusoid at every frequency. It does not resolve the gain ambiguity on the periodic branch identified in Theorem 1(ii).

4 Uniform Expansion of the Periodic Response

Let $\mathcal{K} \subset \mathcal{H}$ be a nonempty compact set and let $W = [\omega_-, \omega_+]$ with $0 < \omega_- \leq \omega_+ < \infty$. Identify Q with its vector of six coefficients and allow Q to range in a compact set $\mathcal{Q} \subset \mathbb{R}^6$. This additional uniformity will be useful in controlling the remainder as $\Gamma_Q(\omega)$ tends to zero. All constants in this section may depend on the compact sets and known nominal plant coefficients, but not on the individual $K, \omega, Q, \varepsilon$ within the stated ranges. Derivatives with respect to K are taken in the ambient four-dimensional gain space, using the locally defined periodic branch on an open neighborhood of $\mathcal{K}$.

Theorem 2 (Uniform harmonic expansions) *There exists $\varepsilon_0 > 0$ such that for $K \in \mathcal{K}$, $\omega \in W$, $Q \in \mathcal{Q}$, and $|\varepsilon| < \varepsilon_0$, the full system has a locally unique, locally exponentially stable T-periodic solution near the origin. This branch, after the change of time variable $\theta = \omega t$, is real analytic in the parameters on a neighborhood of the indicated sets. Its output Fourier coefficients satisfy*

$$Y_1 = \frac{\varepsilon}{2z_1} + R_1, \tag{29}$$

$$Y_2 = \frac{\varepsilon^2 \Gamma_Q(\omega)}{4 z_1^2 z_2} + R_2, \tag{30}$$

where z_1, z_2 are defined in (13). There is a constant $C_* > 0$ such that

$$|R_1| + \|D_K R_1\| \le C_* |\Gamma_Q(\omega)|\, |\varepsilon|^3, \tag{31}$$

$$|R_2| + \|D_K R_2\| \le C_* |\Gamma_Q(\omega)|\, |\varepsilon|^4. \tag{32}$$

The same bounds hold on a sufficiently small fixed compact neighborhood of $\mathcal{K}$ contained in $\mathcal{H}$. In particular, the estimates do not require any positive lower bound on $|\Gamma_Q(\omega)|$.

Proof We divide the proof into three steps.

Step 1: Periodic branch and uniform regularity. In the time variable $\theta = \omega t$, (6) becomes

$$\frac{dX}{d\theta} = \frac{1}{\omega}\big(M_K X + e_3(Q(x_1, x_2, x_3) + \varepsilon \cos\theta)\big). \tag{33}$$

The vector field is analytic in the state and all real parameters for $\omega > 0$. The time-2π map of (33) is analytic on a neighborhood of $X = 0, \varepsilon = 0$ for every (K, ω, Q) under consideration. This local statement follows from analytic dependence of solutions of ordinary differential equations; solutions close to the zero solution exist throughout the fixed finite interval. The fixed-point equation has derivative $e^{2\pi M_K/\omega} - I$ with respect to the initial state at $\varepsilon = 0$. This matrix is invertible for every $K \in \mathcal{H}$. The analytic implicit function theorem therefore yields a locally unique analytic periodic branch through zero.

For clarity, the compactness argument can be applied on a slightly larger compact gain set $\mathcal{K}^+$ satisfying $\mathcal{K} \subset \operatorname{int}\mathcal{K}^+ \subset \mathcal{K}^+ \subset \mathcal{H}$, and an enlarged compact interval of positive frequencies. We may also enlarge the compact coefficient set to a closed ball. The zero solution is common to all parameters at $\varepsilon = 0$. A finite covering by implicit-function neighborhoods, followed by shrinking the state and amplitude neighborhoods, gives a common amplitude threshold and a common ball of uniqueness near zero. To see why the branches agree, choose the ball inside the uniqueness neighborhoods of the finite covering; the candidate branches lie in that ball for uniformly small amplitudes by continuity. On overlaps their fixed points must then be identical. All finitely many parameter derivatives used below are bounded on smaller compact neighborhoods, including the mixed derivatives in ε, K, Q. The stability follows as in Theorem 1; compactness also gives a uniform separation of the multipliers at zero from the unit circle. The Fourier coefficients can now be calculated over the fixed interval $[0, 2\pi]$, so their dependence on the parameters is analytic as well.

Step 2: Leading coefficients and parity. Expand the output as $y = \varepsilon y^{[1]} + \varepsilon^2 y^{[2]} + O(\varepsilon^3)$, where the superscripts indicate powers of ε, not time derivatives. At first order the response is linear, with

$$y^{[1]}(t) = h_1 e^{\mathrm{i}\omega t} + \overline{h_1} e^{-\mathrm{i}\omega t}, \qquad h_1 = \frac{1}{2 z_1}. \tag{34}$$

The leading second-harmonic formula is a specialization of the quadratic frequency-response (Volterra) construction; see [5]. The coefficient at 2ω of the second-order forcing is $\Gamma_Q(\omega) h_1^2$ by (22). Equation (20) thus gives the second-order output coefficient

$$h_2 = \frac{\Gamma_Q(\omega) h_1^2}{z_2} = \frac{\Gamma_Q(\omega)}{4 z_1^2 z_2}. \tag{35}$$

There is no zero output harmonic by (17). At order two the only possible nonzero output harmonics are ± 2. The constant quadratic contribution may affect the mean of the integral state.

The change $\varepsilon \mapsto -\varepsilon$ corresponds to a phase shift $\theta \mapsto \theta + \pi$. The shifted solution belongs to the same local branch, by uniqueness in the ball. Consequently $Y_n(-\varepsilon) = (-1)^n Y_n(\varepsilon)$. The first coefficient is odd and the second is even in ε. Analyticity and the computed leading coefficients give (29)–(30) initially with $R_1 = O(|\varepsilon|^3)$ and $R_2 = O(|\varepsilon|^4)$, also after taking the required derivatives in K, Q.

Step 3: Remainders near degeneracy. For each fixed Q, ω define a modified quadratic form

$$Q^{(0)}(x) = Q(x) - \operatorname{Re}\Gamma_Q(\omega)x_1^2 - \frac{\operatorname{Im}\Gamma_Q(\omega)}{\omega}x_1x_2. \tag{36}$$

By (21), $\Gamma_{Q^{(0)}}(\omega) = 0$. Moreover, in the Euclidean norm of coefficient vectors,

$$\|Q - Q^{(0)}\| \le \max\{1, \omega_-^{-1}\}|\Gamma_Q(\omega)|. \tag{37}$$

All line segments from Q to $Q^{(0)}$, with $(Q, \omega) \in \mathcal{Q} \times W$, belong to a fixed bounded coefficient ball, on which the construction in Step 1 applies with a possibly smaller ε_0.

For $Q^{(0)}$, the exact sinusoidal solution in Theorem 1 belongs to the small periodic branch. Thus $R_1 = R_2 = 0$ for $Q^{(0)}$, as functions of K, ε at the chosen frequency. Their derivatives with respect to the gains also vanish. Taylor's formula in ε, with the leading coefficients subtracted and the parity established in Step 2, shows that $D_Q R_1$ and $D_Q D_K R_1$ are uniformly $O(|\varepsilon|^3)$, while $D_Q R_2$ and $D_Q D_K R_2$ are uniformly $O(|\varepsilon|^4)$. The necessary mixed derivatives are uniformly bounded by Step 1. Integrating the coefficient derivatives along the segment between $Q^{(0)}$ and Q, and using (37), proves (31)–(32). The argument applies uniformly, including at exact degeneracy. □

Remark 2 On the compact sets in Theorem 2, $|z_1|$ and $|z_2|$ are bounded above and bounded away from zero. Hence there are positive constants c, C such that for sufficiently small $\varepsilon > 0$,

$$c\varepsilon \le |Y_1| \le C\varepsilon, \qquad c\varepsilon^2|\Gamma_Q(\omega)| \le |Y_2| \le C\varepsilon^2|\Gamma_Q(\omega)|. \tag{38}$$

The second inequality includes $\Gamma_Q(\omega) = 0$, in which case $Y_2 = 0$ exactly. In particular, when $\Gamma_Q(\omega) \neq 0$, the first two harmonics themselves are nonzero on the small-amplitude branch. The amplitude threshold can be chosen independently of the distance to degeneracy.

5 Singular Conditioning of Gain Identification

We now study the observation map

$$\mathcal{M}_{\varepsilon,\omega,Q}(K) = (Y_1, Y_2) \in \mathbb{C}^2 \simeq \mathbb{R}^4, \tag{39}$$

with the Euclidean norm $\|(v_1, v_2)\| = (|v_1|^2 + |v_2|^2)^{1/2}$. The gain coordinates and the real and imaginary components of the observations are not rescaled when defining its Jacobian. Thus singular values below are with respect to these fixed coordinates and norms. The equivalent notations $f \asymp g$ and $f = \Theta(g)$ for positive functions mean $cg \le f \le Cg$ for positive constants c, C independent of the parameters in the specified range. At $\Gamma_Q(\omega) = 0$, we state the rank separately.

Theorem 3 (Uniform singular conditioning and two-sided identification estimate) *Under the assumptions of Theorem 2, there exists a common $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ and*

$\Gamma_Q(\omega) \neq 0$, *the real Jacobian* $D_K \mathcal{M}_{\varepsilon,\omega,Q}(K)$ *is invertible. Its singular values, in decreasing order, satisfy*

$$\sigma_1, \sigma_2 = \Theta(\varepsilon), \tag{40}$$

$$\sigma_3, \sigma_4 = \Theta(\varepsilon^2 |\Gamma_Q(\omega)|). \tag{41}$$

In particular, with the Euclidean operator norm and condition number,

$$\|(D_K \mathcal{M}_{\varepsilon,\omega,Q})^{-1}\| = \Theta\left(\frac{1}{\varepsilon^2 |\Gamma_Q(\omega)|}\right), \qquad \operatorname{cond}(D_K \mathcal{M}_{\varepsilon,\omega,Q}) = \Theta\left(\frac{1}{\varepsilon |\Gamma_Q(\omega)|}\right). \tag{42}$$

Moreover, there exist positive constants c_*, C^* *such that for all* $K, \widehat{K} \in \mathcal{K}$*, including the case* $\Gamma_Q(\omega) = 0$*,*

$$c_* \mathcal{D}(K, \widehat{K}) \leq \|\mathcal{M}_{\varepsilon,\omega,Q}(K) - \mathcal{M}_{\varepsilon,\omega,Q}(\widehat{K})\| \leq C^* \mathcal{D}(K, \widehat{K}), \tag{43}$$

where

$$\begin{aligned} \mathcal{D}(K, \widehat{K}) &= \varepsilon d_1 + \varepsilon^2 |\Gamma_Q(\omega)| d_2, \\ d_j &= |z_j(K, \omega) - z_j(\widehat{K}, \omega)|, \qquad j = 1, 2. \end{aligned} \tag{44}$$

All constants are uniform over K, ω, Q *on the specified compact sets. Thus the first two complex harmonics uniquely determine the gains on* $\mathcal{K}$ *at every nondegenerate frequency for sufficiently small nonzero amplitude. At degeneracy the Jacobian in the ambient gain space has rank two and the equivalence relation is exactly* (11).

Proof Write $\Gamma = \Gamma_Q(\omega)$. For $\Gamma \neq 0$, the normalized observation map is

$$\mathcal{N}(K) = \left(\frac{Y_1}{\varepsilon}, \frac{Y_2}{\varepsilon^2 \Gamma}\right). \tag{45}$$

Theorem 2 yields, in the C^1 norm in the gain variables,

$$\mathcal{N}(K) = G(z_1, z_2) + O(\varepsilon^2), \qquad G(z_1, z_2) = \left(\frac{1}{2z_1}, \frac{1}{4z_1^2 z_2}\right). \tag{46}$$

Indeed, the remainder in the second component divided by $\varepsilon^2 \Gamma$ is uniformly $O(\varepsilon^2)$, regardless of the size or phase of Γ. The first component has remainder $O(|\Gamma|\varepsilon^2)$, which is also uniformly $O(\varepsilon^2)$ since Γ is bounded. For $z_1, z_2 \neq 0$, the map G is explicitly invertible: if $(v_1, v_2) = G(z_1, z_2)$, then

$$z_1 = \frac{1}{2v_1}, \qquad z_2 = \frac{v_1^2}{v_2}. \tag{47}$$

Its real derivative and inverse derivative are uniformly bounded on the relevant compact sets, since z_1, z_2, v_1, v_2 stay bounded and bounded away from zero. With (14), this proves that $D_K \mathcal{N}$ is invertible and that its norm and inverse norm are uniformly bounded for small ε.

Multiplication of a complex number by Γ is the product of a rotation and multiplication by $|\Gamma|$. Hence the original Jacobian can be factored in real coordinates as

$$D_K \mathcal{M}_{\varepsilon,\omega,Q} = \begin{pmatrix} \varepsilon I_2 & 0 \\ 0 & \varepsilon^2 |\Gamma| I_2 \end{pmatrix} B_{\varepsilon,\omega,Q,K}, \tag{48}$$

where $B_{\varepsilon,\omega,Q,K}$ and its inverse have uniformly bounded norms. For small ε we have $\varepsilon|\Gamma| \leq 1$, so the singular values of the diagonal factor are $\varepsilon, \varepsilon, \varepsilon^2|\Gamma|, \varepsilon^2|\Gamma|$, in decreasing order. For invertible B and any square matrix A_0, the singular values of $A_0 B$ satisfy

$$\frac{\sigma_j(A_0)}{\|B^{-1}\|} \leq \sigma_j(A_0 B) \leq \sigma_j(A_0)\|B\|.$$

Applying these inequalities to (48) proves (40)–(42).

To prove the estimate for arbitrary pairs, let $G = (G_1, G_2)$ as in (46), and write $g_j = |G_j(z) - G_j(\widehat{z})|$, where $z = (z_1, z_2)$ and $\widehat{z} = (\widehat{z}_1, \widehat{z}_2)$ are the frequency coordinates of the two gain vectors. The identity for differences of reciprocals gives $cd_1 \le g_1 \le Cd_1$. The explicit rational formulas for G_2 and (47), with the uniform lower and upper bounds on the denominators, give

$$g_2 \le C(d_1 + d_2), \qquad d_2 \le C(g_1 + g_2). \tag{49}$$

For instance, the first bound follows by adding and subtracting $1/(4\widehat{z}_1^2 z_2)$, and the second by adding and subtracting $\widehat{v}_1^2/v_2$ in (47). These difference estimates do not require convexity of the domain of G.

Set $a_* = \varepsilon$ and $b_* = \varepsilon^2|\Gamma|$, with $0 \le b_* \le a_*$. From the bounds on g_1 and (49),

$$c(a_* d_1 + b_* d_2) \le a_* g_1 + b_* g_2 \le C(a_* d_1 + b_* d_2). \tag{50}$$

The equivalence of the Euclidean norm and the sum of moduli in $\mathbb{C}^2$ then gives a two-sided estimate with $\mathcal{D}$ for the differences of the leading observation map $(\varepsilon G_1, \varepsilon^2 \Gamma G_2)$.

It remains to absorb the remainders. We first record the pairwise estimates

$$|R_1(K) - R_1(\widehat{K})| \le C|\Gamma|\varepsilon^3 \|K - \widehat{K}\|, \tag{51}$$

$$|R_2(K) - R_2(\widehat{K})| \le C|\Gamma|\varepsilon^4 \|K - \widehat{K}\|. \tag{52}$$

Even if $\mathcal{K}$ is not convex, these follow from Theorem 2. Choose $\delta_0 > 0$ so that the closed δ_0-neighborhood of $\mathcal{K}$ lies in $\mathcal{K}^+$ from that theorem. For $\|K - \widehat{K}\| \le \delta_0$, integrate the derivative along the line segment, which lies in $\mathcal{K}^+$. For $\|K - \widehat{K}\| > \delta_0$, use the two separate uniform bounds on the remainders, and multiply the resulting constant by $\|K - \widehat{K}\|/\delta_0 \ge 1$. This proves (51)–(52) on the entire compact set.

By the inverse bound for S_ω, $\|K - \widehat{K}\| \le C(d_1 + d_2)$. The sum of the two remainder differences is therefore at most

$$C|\Gamma|\varepsilon^3(d_1 + d_2) \le C'\varepsilon(a_* d_1 + b_* d_2), \tag{53}$$

where we have restricted $\varepsilon \le 1$ and used the uniform boundedness of Γ. In detail, the coefficient of d_2 is εb_*, and that of d_1 is bounded by a constant times $\varepsilon^2 a_*$. For sufficiently small ε, (53) is less than half the lower bound in (50) (after conversion to the Euclidean norm). The triangle inequality now proves (43).

If $\Gamma \neq 0$, equality of the two observations implies $d_1 = d_2 = 0$, and (14) implies equality of the gains. If $\Gamma = 0$, the remainders vanish, the map is exactly $(\varepsilon/(2z_1), 0)$, and its derivative has rank two by (14). The same argument for (43) applies with $b_* = 0$. This completes the proof. □

5.1 Approach to a degenerate frequency

Let Q be fixed, and let $\omega_* > 0$ be an isolated zero of Γ_Q. By Proposition 1, there is an integer $m \in \{1, 2\}$ such that, near ω_*,

$$|\Gamma_Q(\omega)| = \Theta(|\omega - \omega_*|^m). \tag{54}$$

Corollary 1 *For $K \in \mathcal{K}$, $0 < \varepsilon < \varepsilon_1$ and ω sufficiently close to but different from ω_*,*

$$\sigma_{\min}(D_K \mathcal{M}_{\varepsilon,\omega,Q}) = \Theta(\varepsilon^2|\omega - \omega_*|^m), \qquad \operatorname{cond}(D_K \mathcal{M}_{\varepsilon,\omega,Q}) = \Theta\left(\frac{1}{\varepsilon|\omega - \omega_*|^m}\right). \tag{55}$$

The estimates hold in the joint limit $\varepsilon \to 0^+$ and $\omega \to \omega_*$ with no restriction on their relative rates.

Proof Choose a compact positive frequency interval containing a neighborhood of ω_*. Apply Theorem 3 and the two-sided bound on $|\Gamma_Q|$. □

5.2 Observation errors and PID restriction

For $\Gamma \neq 0$, (43), $\varepsilon|\Gamma| \leq 1$ and the invertibility of S_ω imply the uniform inverse estimate

$$\|K - \widehat{K}\| \leq \frac{C}{\varepsilon^2|\Gamma|}\|\mathcal{M}(K) - \mathcal{M}(\widehat{K})\|. \tag{56}$$

Here and below the suppressed observation indices are fixed and common to both controllers. For example, let $\mathcal{M}(K)+e$ be observed with $\|e\| \leq \eta$. Any gain vector in $\mathcal{K}$ whose observation fits these data within η then differs from K by at most $2C\eta/(\varepsilon^2|\Gamma|)$. This estimate holds for every observation error in the prescribed norm ball.

The matching deterministic lower bound is established below for the more informative observation of the complete periodic waveform; it therefore also provides the same order of lower bound for two-harmonic observations, up to the fixed normalization of their noise norms.

For PID, restrict the ambient derivative in (48) to the three-dimensional subspace $\Delta K_{dd} = 0$. The first frequency-coordinate map has rank two on this subspace and a one-dimensional kernel. At $\Gamma \neq 0$, the restricted 4×3 observation Jacobian has two singular values of order ε and one of order $\varepsilon^2|\Gamma|$. To verify this, the upper bound on its norm and the lower bound on its smallest singular value follow from the corresponding full-space bounds. The second singular value is bounded below by a constant times ε by restricting to the two-dimensional orthogonal complement of the kernel of $D_K z_1$ within the PID subspace: the derivative of $1/(2z_1)$ on this complement is uniformly bounded below, and the first-component remainder is $O(|\Gamma|\varepsilon^3)$. Finally, testing a unit vector in the one-dimensional kernel gives an upper bound $C\varepsilon^2|\Gamma|$ for the smallest singular value, by (29)–(32). These bounds are uniform since ω is confined to W. At degeneracy the restricted Jacobian has rank two. The pairwise estimate (43) and exact identification statements also apply on the PID slice.

5.3 Information in the complete periodic waveform

Use the fixed phase variable $\theta = \omega t$ and let $\mathcal{Y}(K)(\theta) = y(\theta/\omega; K)$ denote the small periodic branch for a common experiment (ε, ω, Q). On real 2π-periodic records use the root-mean-square (RMS) norm

$$\|v\|_{\mathrm{rms}}^2 = \frac{1}{2\pi}\int_0^{2\pi} |v(\theta)|^2\, d\theta.$$

Let P_{12} be the orthogonal projection onto the real span of $\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta$.

Theorem 4 (Two-sided separation estimates for complete periodic outputs) *Under the hypotheses of Theorem 3, set* $\Delta y = \mathcal{Y}(K) - \mathcal{Y}(\widehat{K})$ *and* $\Delta m = \mathcal{M}(K) - \mathcal{M}(\widehat{K})$. *There are*

$C > 0$ and $\varepsilon_* > 0$, uniform on the same compact sets, such that for $0 < \varepsilon < \varepsilon_*$ and every pair $K, \widehat{K} \in \mathcal{K}$,

$$\|(I - P_{12})\Delta y\|_{\mathrm{rms}} \le C\varepsilon^3 |\Gamma_Q(\omega)| \, \|K - \widehat{K}\|, \tag{57}$$

$$2\|\Delta m\|^2 \le \|\Delta y\|_{\mathrm{rms}}^2 \le 2(1 + C\varepsilon^2)\|\Delta m\|^2. \tag{58}$$

Both bounds include $\Gamma_Q(\omega) = 0$. For the ambient derivatives $J_y = D_K\mathcal{Y}$ and $J_m = D_K\mathcal{M}$ one also has

$$2J_m^* J_m \preceq J_y^* J_y \preceq 2(1 + C\varepsilon^2) J_m^* J_m,$$

where the adjoints use the indicated real Hilbert space norms and $\preceq$ denotes the positive-semidefinite order on symmetric matrices.

Proof Write $U = (I - P_{12})\mathcal{Y}$. On the fixed phase interval the implicit-function construction in Theorem 2 yields bounded continuous mixed parameter derivatives of $\mathcal{Y}$ of every fixed finite order on slightly enlarged compact sets. Indeed the initial point of the branch and the finite-time flow have these derivatives, uniformly for $0 \le \theta \le 2\pi$. Their composition therefore admits the same Taylor estimates in the rms norm. The projection is bounded in that norm.

For every K, Q, ω, the coefficients of orders zero, one and two in the Taylor expansion of U vanish. The zero-order state is zero. The order-one output has support at ± 1, and the order-two output has support at ± 2 since its mean vanishes by (17). The same three coefficients vanish after differentiation in Q and after mixed differentiation in K, Q. Taylor's formula consequently gives

$$\|D_Q U\| + \|D_K D_Q U\| \le C\varepsilon^3.$$

At the coefficient vector $Q^{(0)}$ of (36) the small branch is exactly sinusoidal for every nearby gain, so $U = D_K U = 0$ there. Integration along $[Q^{(0)}, Q]$ and (37) give

$$\|U\|_{\mathrm{rms}} + \|D_K U\| \le C|\Gamma_Q(\omega)|\varepsilon^3.$$

Choose $d > 0$ so that the closed d-neighborhood of $\mathcal{K}$ is inside the enlarged compact Hurwitz set. For pairs at distance at most d, integrate $D_K U$ along the segment. For pairs farther apart, bound the two values of U separately and use $1 \le \|K - \widehat{K}\|/d$. This proves (57) without a connectedness assumption.

Uniform invertibility of S_ω gives $\|K - \widehat{K}\| \le C(d_1 + d_2)$ in the notation of (44). Since $\varepsilon|\Gamma_Q(\omega)| \le 1$ for small ε,

$$\varepsilon^3|\Gamma_Q(\omega)|(d_1 + d_2) \le \varepsilon\big(\varepsilon d_1 + \varepsilon^2|\Gamma_Q(\omega)|d_2\big) \le C\varepsilon\|\Delta m\|$$

by (43). Parseval gives $\|\Delta y\|_{\mathrm{rms}}^2 = 2\|\Delta m\|^2 + \|(I - P_{12})\Delta y\|_{\mathrm{rms}}^2$, proving (58). Apply this inequality on an enlarged compact Hurwitz neighborhood to K and $K + hv$, divide by h^2 and pass to the limit for every $v \in \mathbb{R}^4$. This proves the quadratic-form assertion even when K is isolated in the original compact set. □

Sharpness of the relative energy bound.

Take $Q = x_1^2$, $\omega = 1$ and the two controllers in Example 2. Their common first-order coefficient is $h_1 = 1/(2z_1)$. Put $z_3 = L_K(3\mathrm{i})$. At order two $h_2 = h_1^2/z_2$. At order three the coefficient of the forcing at index three is $2h_1h_2$, obtained by multiplying the order-one and order-two outputs. Thus the third output harmonic is

$$Y_3(K) = \frac{2h_1^3}{z_2 z_3}\varepsilon^3 + O(\varepsilon^5),$$

where the fourth-order coefficient vanishes by amplitude parity. For the stated controllers $z_3 = -32 - 28\mathrm{i}/3$ and $\widehat{z}_3 = -36 - 8\mathrm{i}$. Direct multiplication with the two frequency coordinates at 2ω gives $\mathrm{Re}(z_2 z_3) = 1250/3$ and $\mathrm{Re}(\widehat{z}_2\widehat{z}_3) = 520$. Hence $|\Delta Y_3|$ has a nonzero cubic leading term. Also $|\Delta Y_2|$ has a nonzero quadratic leading term, and $\Delta Y_1 = O(\varepsilon^3)$. Parseval and (58) give

$$\frac{\|\Delta y\|_{\mathrm{rms}}^2}{2\|\Delta m\|^2} - 1 = \Theta(\varepsilon^2).$$

Thus the relative order in (58) is attained.

Bounded errors in a complete waveform.

Suppose $\mathcal{K}$ contains a fixed closed ball in gain space of positive radius. At $\Gamma_Q(\omega) \neq 0$ let $b_* = \varepsilon^2|\Gamma_Q(\omega)|$ and observe $\mathcal{Y}(K) + e$ with $\|e\|_{\mathrm{rms}} \le \eta$. For estimators $\mathcal{E}$ mapping the observed record to a gain vector, the minimax deterministic error satisfies

$$\inf_{\mathcal{E}} \sup_{\substack{K\in\mathcal{K} \\ \|e\|_{\mathrm{rms}}\le\eta}} \|\mathcal{E}(\mathcal{Y}(K) + e) - K\| \asymp \min\{1, \eta/b_*\}.$$

The constants may depend on the specified compact sets and the interior ball. For the upper bound, a minimum-residual fit exists by compactness. The two predicted records are then within 2η, so (58) and (56) give $C\eta/b_*$. The diameter gives the other bound. For the lower bound, let K_0 be the center of the ball, choose a unit $v \in \ker D_K z_1$, and take $h = c\min\{1, \eta/b_*\}$. Choose $c > 0$ small enough to keep $K_0 + hv$ in the ball. The two controllers have $d_1 = 0$ and $d_2 \le Ch$ by (14). Equations (43) and (58) bound their distance between complete records by $Cb_*h \le 2\eta$ after a further fixed decrease of c. Their midpoint is an admissible observation for both, so one error is at least $h/2$. At exact degeneracy the same interior equivalence set contains distinct controllers with identical noiseless records. For PID the proof uses a ball relative to the PID slice and its one-dimensional kernel. The observation model here specifies a complete period, a known phase, and a deterministic bound in the rms norm.

6 Analytical Examples

The following examples illustrate both the persistence and the removal of ambiguity. They are established directly from the formulas above.

Example 1 Table 1 lists quadratic forms illustrating the different degeneracy scenarios. For the first form in Table 1, Theorem 1(i) gives exact identifiability from the complete periodic output at any positive frequency, conditional only on existence of the stipulated periodic solution. The second and fourth forms illustrate the distinct orders in (55). In the last case, the nonlinearity is not zero but all frequencies are degenerate; a single sinusoidal experiment cannot identify all four gains on the small-amplitude branch.

Table 1 Harmonic generation coefficients and their positive zeros.

$Q(x)$	$\Gamma_Q(\omega)$	Degenerate frequencies
$x_3^2 - 0.4x_1x_2$	$\omega^4 - 0.4\mathrm{i}\omega$	None
$x_1^2 + x_2^2$	$1 - \omega^2$	1 (simple)
$4x_1^2 + 5x_2^2 + x_3^2$	$(\omega^2 - 1)(\omega^2 - 4)$	1, 2 (simple)
$x_1^2 + 2x_2^2 + x_3^2$	$(1 - \omega^2)^2$	1 (double)
$x_2^2 - x_1x_3$	0	All positive frequencies

Example 2 Let $a_0 = 1$, $a_1 = 0.6$, $a_2 = 2$, and consider the two controllers

$$K = (3, 1, 5.4, 2)^\top, \qquad \widehat{K} = (3.5, 1.5, 5.9, 2.5)^\top. \tag{59}$$

The nominal polynomial for K is $(s+1)^4$, and for $\widehat{K}$ it is $s^4 + 4.5s^3 + 6.5s^2 + 4.5s + 1.5$. The latter satisfies (8), since its coefficients are positive and $4.5 \cdot 6.5 \cdot 4.5 - 4.5^2 - 4.5^2 \cdot 1.5 = 81 > 0$. At $\omega = 1$ their gain difference satisfies (11). Their frequency coordinates are

$$z_1 = \widehat{z}_1 = 4\mathrm{i}, \qquad z_2 = -12 + 3.5\mathrm{i}, \qquad \widehat{z}_2 = -13.5 + 4.25\mathrm{i}. \tag{60}$$

The linear periodic outputs coincide exactly. For $Q = x_3^2 - 0.4x_1x_2$, however, $\Gamma_Q(1) = 1 - 0.4\mathrm{i}$, and the first Fourier coefficients differ only at order $O(\varepsilon^3)$, whereas

$$Y_2(K) - Y_2(\widehat{K}) = -\frac{\varepsilon^2(1 - 0.4\mathrm{i})}{64}\left(\frac{1}{-12 + 3.5\mathrm{i}} - \frac{1}{-13.5 + 4.25\mathrm{i}}\right) + O(\varepsilon^4). \tag{61}$$

The coefficient of ε^2 is nonzero. Thus the two controllers are distinguishable by the second harmonic despite having identical linear frequency responses at the excitation frequency.

More generally, fix distinct $K, \widehat{K} \in \mathcal{H}$ satisfying (11) at a fixed nondegenerate frequency. Their z_2 coordinates are different, so the second-order output difference has a nonzero complex coefficient

$$\Delta h_2 = \frac{\Gamma_Q(\omega)}{4z_1^2}\left(\frac{1}{z_2} - \frac{1}{\widehat{z}_2}\right). \tag{62}$$

The first-order waveforms coincide, and the second-order difference consists only of the two harmonics at $\pm 2\omega$ by the proof of Theorem 2. Analytic dependence of the periodic state on ε on the fixed time interval and Parseval's identity therefore give

$$\|y(\cdot; K) - y(\cdot; \widehat{K})\|_{L^2(0,T)} = \sqrt{2T}\,|\Delta h_2|\varepsilon^2 + O(\varepsilon^3). \tag{63}$$

Since each output has norm of order ε, their relative difference is of order ε for this fixed pair.

Example 3 Keep the controllers in Example 2 and let $Q = x_1^2 + x_2^2$. At $\omega = 1$, $\Gamma_Q = 0$. The common exact periodic output is $y(t) = (\varepsilon/4)\sin t$, because $H_K(\mathrm{i}) = 1/(4\mathrm{i})$. The constant quadratic term is $Q_0 = \varepsilon^2/16$. By (26), the means of the two integral states are respectively $-\varepsilon^2/16$ and $-\varepsilon^2/24$. Hence the outputs are identical even in the nonlinear system, but the integral states differ. For sufficiently small amplitudes both periodic solutions are locally exponentially stable. This demonstrates why the available observations and the treatment of the integral state are essential to the identification problem.

7 Numerical Verification

The following computations evaluate the small periodic branch and its derivatives with respect to the controller gains. The plant coefficients and the two gain vectors are those of Example 2. We compute the periodic output directly by a Fourier–Galerkin method, with indices $-N, \dots, N$ and zero mean. All examples in this section use diagonal quadratic forms, so the associated symmetric bilinear form $\mathcal{B}_Q(v, w)$ is obtained by replacing each square of a derivative by the product of the corresponding derivatives of v and w. Products are computed by linear convolution followed by truncation, without circular aliasing. The mean of the integral state follows from the zero-frequency equation of (19).

To avoid cancellation close to degeneracy, let l denote the output of the linear model at unit amplitude, and write $y = \varepsilon l + \varepsilon^2 \Gamma_Q v$, where Γ_Q is real for the forms used here. Let $\mathcal{H}_N$ act by multiplication with $H_K(\mathrm{i}n\omega)$ on nonzero retained Fourier modes and by zero on the mean. Define b to have only the coefficients $b_2 = h_1^2$ and $b_{-2} = \overline{h_1}^2$, with $h_1 = (2z_1)^{-1}$. The truncated Fourier equations for $\Gamma_Q \neq 0$ are exactly

$$v = \mathcal{H}_N\big(b + 2\varepsilon \mathcal{B}_Q(l, v) + \varepsilon^2 \Gamma_Q \mathcal{B}_Q(v, v)\big). \tag{64}$$

The same scaled equation can be evaluated at $\Gamma_Q = 0$, where the recovered output is exactly εl; the auxiliary variable is then immaterial. Iteration starts from zero. The four derivatives with respect to the controller gains of v are computed by differentiating (64) and iterating the differentiated equations together with the state equation. In particular, $\partial_{K_j} H_K = -H_K^2 \partial_{K_j} L_K$, and $\partial_K L_K(s) = (1, 1/s, s, s^2)$ in the prescribed gain order. The real Jacobian is assembled from the first two complex output coefficients. The computations use extended precision for the Fourier iteration, where supported by the platform, and double precision for the final singular value decomposition. Results are checked against $N = 64$ and against a double-precision iteration at $N = 32$.

7.1 Simple and double degeneracy

For $K = (3, 1, 5.4, 2)^\top$, use the forms $Q_s = x_1^2 + x_2^2$ and $Q_d = x_1^2 + 2x_2^2 + x_3^2$, which have a simple and a double zero at $\omega = 1$, respectively. Take $\varepsilon = 2^{-j}$ for $j = 3, \dots, 6$ and $\omega = 1 \pm 2^{-k}$ for $k = 3, \dots, 8$. Figure 1 displays the normalized singular values. The dotted lines give the reference limits as $(\varepsilon, \omega) \to (0, 1)$ with $\Gamma_Q \neq 0$:

$$\frac{\sigma_1}{\varepsilon}, \frac{\sigma_2}{\varepsilon} \longrightarrow \frac{\sqrt{2}}{32}, \qquad \frac{\sigma_3}{\varepsilon^2 |\Gamma_Q|} \longrightarrow \frac{3}{10000\sqrt{2}}, \qquad \frac{\sigma_4}{\varepsilon^2 |\Gamma_Q|} \longrightarrow \frac{3}{20000\sqrt{2}}. \tag{65}$$

For these constants, decompose the gain space orthogonally into the kernel of $D_K z_1$ and its orthogonal complement. The strong block of the Jacobian, divided by ε, tends to the derivative of G_1 on the complement. After rotating the second complex output by the inverse phase of Γ_Q, the weak block divided by $\varepsilon^2 |\Gamma_Q|$ tends to the derivative of G_2 on the kernel. Indeed, the Jacobian block of the first Fourier coefficient on the kernel is $O(\varepsilon^3 |\Gamma_Q|)$ by (31), and block elimination uses left and right factors tending

to the identity since $\varepsilon|\Gamma_Q| \to 0$. Thus the limiting singular values are those of the two restricted blocks. At $\omega = 1$, the nonzero singular values of $D_K z_1$ are both $\sqrt{2}$, and those of $D_K z_2$ restricted to the kernel are $3/\sqrt{2}$ and $3/(2\sqrt{2})$. The corresponding complex multipliers have moduli $1/(2|z_1|^2) = 1/32$ and $1/(4|z_1|^2|z_2|^2) = 1/10000$, which yield (65).

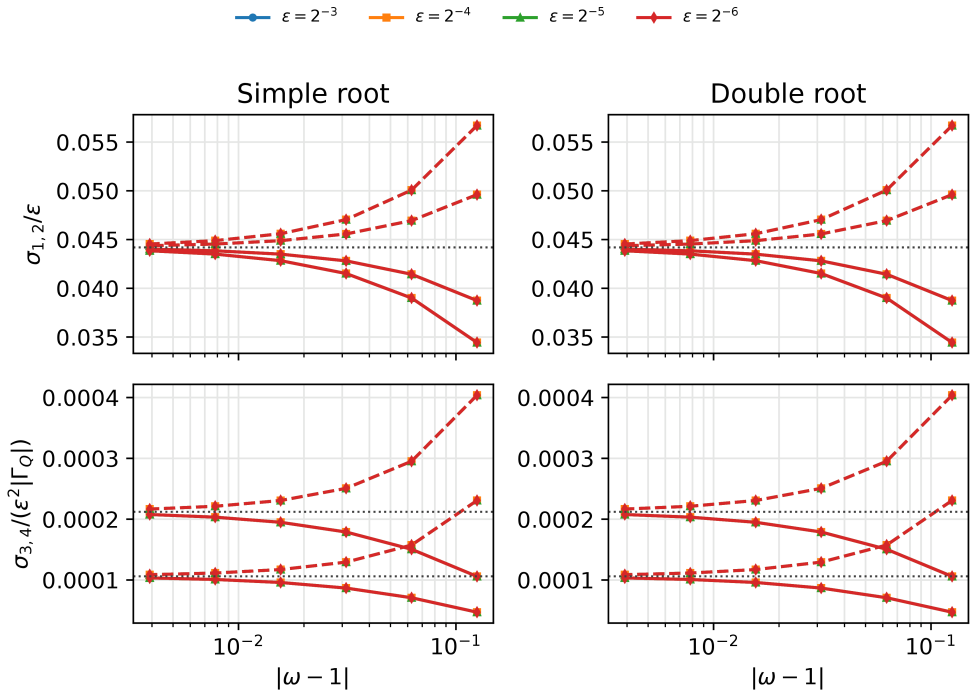


Fig. 1 Normalized singular values for simple (left) and double (right) degeneracy at $\omega = 1$. Solid and dashed curves correspond to frequencies above and below one, respectively. Dotted horizontal lines are the analytical limits in (65). Each color and marker identifies one forcing amplitude; curves at different amplitudes nearly coincide at this scale.

At the smallest amplitude and frequency offset on either side of the zero, the smallest singular value is approximately 2.0×10^{-10} for Q_s and 1.6×10^{-12} for Q_d. The normalized values remain separated from zero, consistent with Theorem 3. At $\omega = 1$ the construction gives $Y_2 = 0$ and exactly two nonzero singular values, as predicted by the analytic sinusoidal branch. Across the frequency sweep the scaled fixed-point residual is below 3×10^{-19} in extended precision; the singular values agree for $N = 32$ and $N = 64$ to the reported precision. The maximum relative difference from double-precision Fourier iteration is below 3×10^{-8}. These comparisons quantify numerical consistency across the tested truncations and arithmetic precisions.

7.2 Energy of the omitted harmonics

Take $Q = x_1^2$, $\omega = 1$ and the two gains from Example 2. Define

$$E(\varepsilon) = \frac{\sum_{n=3}^{N} |Y_n(K) - Y_n(\widehat{K})|^2}{\sum_{n=1}^{2} |Y_n(K) - Y_n(\widehat{K})|^2}. \tag{66}$$

This expression computes the omitted energy directly, avoiding subtraction of nearly equal squared norms. The amplitudes are $\varepsilon = 2^{-j}$, $j = 3, \ldots, 10$. Figure 2 gives the results. A least-squares fit of $\log E$ against $\log \varepsilon$ has slope 1.99999, and the ratio E/ε^2 at the smallest amplitude is approximately 1.8681×10^{-4}. The results for $N = 32$ and $N = 64$ agree to the reported precision. This is consistent with the sharp relative energy estimate of Theorem 4.

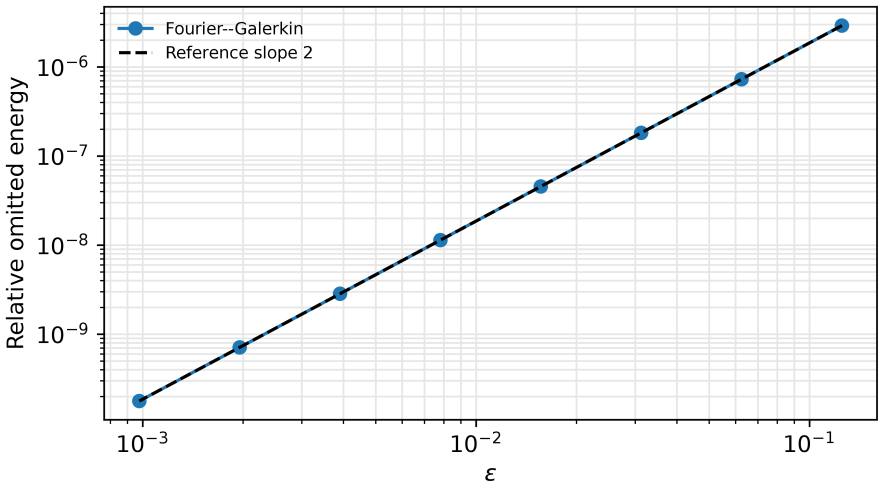


Fig. 2 Relative omitted energy for two linearly frequency-equivalent controllers with $Q = x_1^2$. The dashed reference line has slope two and is normalized at the smallest excitation amplitude.

The Supplementary Material contains the complete Python implementation, computational settings and reference numerical results. The script reproduces the results presented in Figures 1 and 2 and exports the numerical data in CSV format.

8 Discussion and Conclusions

The analysis separates three aspects of gain identification from periodic outputs. First, the linear model has an explicit affine equivalence set at a single frequency. Nominal Hurwitz stability generally does not remove this ambiguity, because the Hurwitz region is open. Second, the quadratic nonlinearity can remove the ambiguity by producing additional harmonics. For the class studied here, the coefficient $\Gamma_Q(\omega)$ provides an exact criterion: under nonzero excitation, if it is nonzero, identical complete periodic

outputs imply identical gains; if it vanishes, the integral state can compensate for the constant quadratic term and the small-amplitude periodic branch preserves the linear ambiguity. The exceptional class (28) and the algebraic classification of isolated degenerate frequencies make this distinction explicit.

Third, exact identification does not ensure well-conditioned identification. The first two complex harmonics determine the four gains for sufficiently small nonzero amplitudes at nondegenerate frequencies, but the two smallest singular values of the four-dimensional PIDD observation Jacobian are of order $\varepsilon^2|\Gamma_Q(\omega)|$. The uniform remainder estimates are important when frequency and amplitude approach degeneracy simultaneously; bounds involving only powers of the amplitude would not resolve arbitrarily small values of $|\Gamma_Q(\omega)|$. The two-sided estimate for differences of observations further quantifies the gain ambiguity on compact subsets of the Hurwitz region, including nonconvex subsets.

On the small-amplitude periodic branch, the complete-output estimate shows that the two-harmonic observation captures the squared separation of periodic output records up to a sharp relative correction of order $O(\varepsilon^2)$. For gain sets containing a ball, the deterministic minimax identification error under bounded root-mean-square (RMS) noise therefore has the same order as for two harmonics. The numerical experiments support the predicted scaling at simple and double zeros and the sharp energy estimate.

The results characterize gain identification from the complex Fourier coefficients of the periodic output under known excitation and plant coefficients, with a free mean of the periodic integral state. The uniform small-amplitude estimates hold on compact Hurwitz sets and compact intervals of strictly positive frequencies. Together, the algebraic criterion, uniform asymptotic bounds and minimax estimates quantify the persistence and removal of harmonic gain ambiguity in quadratic jerk systems.

Supplementary information. The Supplementary Material provides the complete Python implementation, computational parameters and reference numerical results.

Declarations

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Data availability. All numerical data can be reproduced using the Python code provided in the Supplementary Material.

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