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Research Article

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Testing Slater determinants with limited quantum memory

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Joint measurements on two copies test whether a fermionic state is a Slater determinant at a cost independent of the number n of orbitals. We determine the quantum memory required when copies arrive one at a time. For fixed particle or hole number and infidelity threshold, the optimal copy cost with an r-level register is $\Theta(\min\{\sqrt{n}, n/r\})$ for every $1 \le r \le n$, including registers that remain coherent throughout. A cost independent of n requires $\log_2 n + O(1)$ qubits, achieved by storing one particle and exploiting Pauli blocking. For the j-particle reduction of the first copy, we identify the measurement with maximal rejection probability on every input among tests that accept every determinant with certainty. At fixed filling, perfectly complete tests with registers cleared after each pair need $\Theta(n)$ qubits for a bounded copy cost. The same optimal trade-off holds for purity and bipartite product-state testing, with corresponding memory lower bounds for pure fermionic Gaussian states. Tunneling between lattice layers and atom counting retain at least e^{-2} of the optimal rejection probability with one stored particle and certify a mean-field preparation without identifying its orbitals. On a half-filled ten-site Hubbard chain at $U/t = 0.25$, counting with at most three stored particles reduces the copy budget by a factor of 76 relative to our separate-measurement estimator calibrated with a normal approximation.

I. INTRODUCTION

A Slater determinant describes independent fermions in orthogonal orbitals. It is the reference state of mean-field theory and a standard starting point for simulating fermions on quantum processors [1]. Learning the closest determinant requires identifying its orbitals [2]. Joint measurements on two copies can instead test for a determinant at a cost independent of the number n of orbitals [3–5]. When the copies of an N-fermion state arrive one at a time, these tests store a complete copy in $\lceil \log_2 \binom{n}{N} \rceil$ qubits. Yaghubi Rad *et al.* ask how the cost of testing Gaussianity trades off against the available quantum memory [3], and Arunachalam and Schatzki, who determine this trade-off for stabilizer states, ask the same for fermionic Gaussian states [6]. We answer this question for Slater determinants with a fixed number of particles or holes, and show how the answer changes at fixed filling.

For a fixed number k of particles or holes and a fixed infidelity threshold, the optimal number T of copies with an r-level register, $1 \le r \le n$, is

$$T = \Theta(\min\{\sqrt{n}, n/r\}). \tag{1}$$

Separate measurements in random orbital bases attain the first branch and swap tests inside blocks of orbitals attain the second. The lower bound holds at every filling, even with quantum coherence across all copies and unrestricted classical feedback [Fig. 1(a)]. A cost independent of n therefore takes $\log_2 n + O(1)$ qubits, the memory of one particle in place of a complete copy.

One stored particle suffices because of Pauli blocking [Fig. 1(b)]. A particle stored from a determinant lies in an occupied orbital of the next copy and cannot enter that copy. Partial occupation permits transfer with a probability proportional to the one-body defect Δ, which bounds the infidelity. For a test that stores j particles of the first copy in each pair, discards the rest and never rejects a determinant, we find the measurement that is optimal on every input, together with its rejection probability. Tunneling between two layers followed by atom counting retains a constant fraction of this probability [Fig. 1(c)]. For one stored particle the fraction is $\alpha_k \ge e^{-2}$ on every input.

At fixed filling, a test that accepts every determinant with certainty and clears its register after each pair needs $\Theta(n)$ qubits to keep the copy cost independent of n. At half filling, storing j particles gives a cost within a logarithmic factor of the general lower bound on typical inputs, pure states drawn once from Haar measure and supplied for every copy.

Tunneling and counting certify a mean-field preparation without reconstructing its occupied orbitals. The rejection frequency gives upper and lower bounds on the infidelity, and pairing copies from two preparations measures the distance between their one-body matrices. At fixed k and j, the stored-particle test needs of order Δ^{-1} copies near a determinant, compared with Δ^{-2} for our separate-measurement estimator. On the half-filled ten-site Hubbard chain at $U/t = 0.25$, counting with at most three stored particles reduces the copy budget by a factor of 76 relative to the estimator calibrated with a normal approximation.

With the dimension D in place of n, Eq. (1) also governs purity and product testing. For purity it was established for a register cleared after each pair [7, 8], whereas the lower bound for a coherent register had exponent 1/15 [6]; the prediction and affinity method of Ref. [9], applied to a repeated Haar state, gives the matching bound. For pure Gaussian targets the lower bound shows that a cost independent of the number of modes

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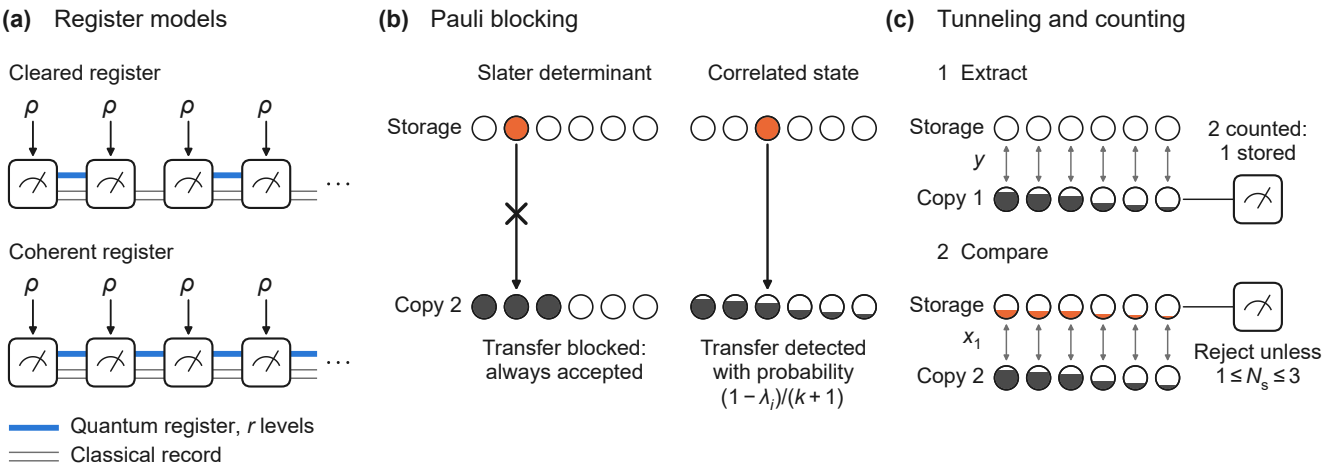

FIG. 1. (a) Register models. Copies of ρ arrive one at a time. A quantum register of r levels and a classical record pass from one copy to the next. A cleared register is empty after each pair of copies; a coherent register may stay coherent across all copies. (b) Pauli blocking with one stored particle, for $k = 3$ particles in six orbitals. In the rows of the copies the dark area of each disc is the occupation λ_i of a natural orbital; orange marks the stored particle. A particle stored from a determinant lies in an occupied orbital of the second copy and cannot be transferred. For $\sqrt{0.7}|123\rangle + \sqrt{0.2}|145\rangle + \sqrt{0.1}|246\rangle$ the particle is stored in natural orbital i with probability λ_i/k and its transfer is then detected with probability $(1-\lambda_i)/(k+1)$, which gives $p_1 = \Delta/[k(k+1)]$. (c) Tunneling and counting (Proposition 2). First, each particle of the first copy tunnels into an empty storage layer with probability y. Counting the $k - j'$ particles left behind shows that j' are stored; here $k - j' = 2$ and $j' = 1$, and the orange areas $\lambda_i/3$ show the state of the stored particle. Next, the storage layer and the second copy are coupled with tunneling probability $x_{j'}$ and the storage layer is counted. A count N_s outside $[j', k]$ rejects.

requires quantum memory, even for adaptive measurements. This answers the single-copy question of Ref. [3] and the second part of Question 10.6 of Ref. [10]. Our separate-measurement test improves the existing copy upper bounds of Refs. [11, 12] by at least one power of n. Table I summarizes the results. Section II states them, Secs. III–V prove the Slater theorems, and Secs. VI and VII treat Gaussian targets and Hubbard chains.

II. SETTING AND RESULTS

A. Copies, registers and infidelity

We consider N fermions in n orbitals. A Slater determinant is a state $a^\dagger(\phi_1)\cdots a^\dagger(\phi_N)|0\rangle$, where $a^\dagger(\phi)$ creates a fermion in orbital ϕ and $\phi_1, \ldots, \phi_N$ are orthonormal. A test receives T independent copies of an unknown state ρ, pure or mixed. The infidelity of ρ is

$$e = 1 - \max_{\phi} \langle\phi|\rho|\phi\rangle, \tag{2}$$

where the maximum ranges over determinants ϕ in all orbital bases. A determinant is fixed, up to a phase, by its occupied subspace, the span of $\phi_1, \ldots, \phi_N$. The targets are pure: a mixture of distinct determinants has $e > 0$. For pure ρ the trace distance to the nearest determinant is $\sqrt{e}$. The test must accept if $e = 0$ and reject if $e \geq \varepsilon$, and we call ε the infidelity threshold.

The test has a quantum register with r levels, that is, with a state space of dimension r, and we call $q = \log_2 r$ its size in qubits. It also keeps a classical record of all earlier outcomes and may adapt every measurement to it. For $r = 1$ it has no quantum memory and measures the copies separately. We use two models [Fig. 1(a)]. With a *cleared register* the copies are processed in R pairs, $T = 2R$. An instrument on the first copy of a pair leaves at most r levels in the register for each classical outcome, and a measurement of the register and the second copy follows. The register is empty before the next pair. With a *coherent register* the r levels may stay coherent across all T copies. A cleared register is a special case of a coherent one, so upper bounds are strongest in the first model and lower bounds in the second. A test has *perfect completeness* if it never rejects a determinant. A *typical input* is a pure state drawn once from the Haar measure on the N-particle sector and supplied for every copy. Theorems are stated for error at most $1/3$ on each side. The numbers of copies in Figs. 2 and 3 are for target error $1/6$ on each side.

The one-body matrix $\gamma_{ii'} = \mathrm{Tr}(\rho a_{i'}^\dagger a_i)$, with a_i the annihilator of orbital i, has eigenvalues $1 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0$, the natural occupations, with eigenvectors called natural orbitals. The occupation number of natural orbital i has variance $\lambda_i(1-\lambda_i)$, and the one-body defect is the sum of these variances [15],

$$\Delta = \mathrm{Tr}[\gamma(I-\gamma)] = \sum_i \lambda_i(1-\lambda_i). \tag{3}$$

For $N > n/2$ particle–hole conjugation maps determinants to determinants and γ to $I - \gamma^{\mathsf{T}}$, which leaves

TABLE I. Results and earlier work. T counts copies; protocols that use R pairs consume $2R$ copies. Costs are at a fixed infidelity threshold unless ε appears. In rows with errors on both sides, lower bounds hold for a coherent register and upper bounds use a cleared one. The memory column gives bounds on the register size for a cost independent of n (of D for purity and product testing); the second row gives the register used by the j-particle test.

Target and regime	Copies T or pairs R	Memory (qubits)	Earlier results
Slater determinants, fixed k; error 1/3 on each side	$T=\Theta(\min\{\sqrt{n}, n/r\})$ for $1\le r\le n$ (Theorem 1)	$\log_2 n + O(1)$	Two full copies, cost independent of n [3–5]; dependence on memory posed as open [3, 6]
Slater determinants, j stored particles, the others discarded; perfect completeness	Exact R for each input, Eq. (7). Worst case $\Theta(k(k-j+2)/(j\varepsilon))$ (Theorem 2)	$\lceil \log_2 \binom{n}{j} \rceil$	$j=k$: $R=\Theta(1/\varepsilon)$ for pure inputs [4, 5]
Slater determinants, fixed filling; perfect completeness, cleared register	$R=\Omega(k/(1+\ln r))$. On typical inputs, $R=\Theta(k/j)$ with j stored particles (Theorem 3)	$\Theta(n)$	A complete copy, at most n qubits, gives a cost independent of n [4, 5]
Purity testing in dimension D and product testing on $\mathbb{C}^D\otimes\mathbb{C}^D$; error 1/3 on each side	$T=\Theta(\min\{\sqrt{D}, D/r\})$ for $1\le r\le D$ (Theorem 4)	$\log_2 D + O(1)$	Cleared register [7, 8]; coherent register, $\Omega((D/r)^{1/15})$ for purity [6]; product testing at $r=1$ [13]
Pure Gaussian states; error 1/3 on each side	$T=\Omega(\min\{\sqrt{n}, n/r\})$; separate measurements, $T=O(n^2/\varepsilon^2+n^{3/2}/\varepsilon)$ (Proposition 3)	at least $\log_2 n - O(1)$	Separate measurements, $O(n^3/\varepsilon^2+n^{5/2}/\varepsilon)$ [11] and $O(n^4/\varepsilon^2)$ [12]; growth with n asked in Refs. [3, 10]
Pure Gaussian states; perfect completeness, cleared register	$R=\Omega(n/(1+\ln r))$ (Proposition 3)	$\Theta(n)$	Two full copies, $R=O(1/\varepsilon)$ for pure inputs [3, 5, 14]

e and Δ unchanged. We therefore work with $k=\min(N, n-N)$ particles, $1\le k\le n/2$, so that $\operatorname{Tr}\gamma = k$ and $\Delta = k - \operatorname{Tr}\gamma^2$. The defect vanishes exactly on determinants and obeys $e\le\Delta$. Indeed, the expected number $N_{\mathrm{out}}=\sum_{i>k}\lambda_i$ of particles outside the k leading natural orbitals bounds the probability of finding a particle there, so $e\le N_{\mathrm{out}}$, and splitting the sum in Eq. (3) at $i=k$ gives $\Delta\ge(\lambda_k+1-\lambda_{k+1})N_{\mathrm{out}}\ge N_{\mathrm{out}}$. Conversely, let P project onto the occupied subspace of a closest determinant. The number of particles outside this subspace is at most k and vanishes on that determinant, so its mean $\operatorname{Tr}[(I-P)\gamma]$ is at most ke, and $\Delta = 2\operatorname{Tr}[(I-P)\gamma] - \|\gamma - P\|_F^2 \le 2ke$, where $\|\cdot\|_F$ is the Frobenius norm. The defect therefore measures the infidelity up to a factor $2k$,

$$\Delta/(2k)\le e\le\Delta. \tag{4}$$

For $k=1$ every pure state is a determinant, so pure alternatives require $k\ge 2$.

B. Stored particles and layer pseudospin

The channel E_j moves j of the k particles of a copy into the register and traces out the other $k-j$. For $I=\{i_1<\cdots<i_j\}$ set $a_I = a_{i_j}\cdots a_{i_1}$ and let $|I\rangle$ be the corresponding occupation state of the register. Then

$$V^{(j)} = \sum_{|I|=j} |I\rangle\otimes a_I,$$
$$E_j(\rho) = \binom{k}{j}^{-1} \operatorname{Tr}_{\wedge^{k-j}}[V^{(j)}\rho V^{(j)\dagger}]. \tag{5}$$

The output $E_j(\rho)$ is the normalized j-particle reduction. Its register has $r=\binom{n}{j}$ levels, and $E_1(\rho)=\gamma/k$.

We call the register the storage layer and the incoming copy the fresh layer, with annihilators a_i and b_i and particle numbers N_{s} and N_{f}. The operators

$$J_- = \sum_i b_i^\dagger a_i, \qquad J_+ = J_-^\dagger, \qquad J_z = (N_{\mathrm{s}} - N_{\mathrm{f}})/2 \tag{6}$$

obey the angular-momentum commutation relations and commute with every rotation of the orbitals applied to both layers. The associated angular momentum J, the layer pseudospin, acts on the layer index independently of the physical spin. Two complete copies have $J_z=0$ and $J\in\{0,\dots,k\}$, and P_J denotes the probability of pseudospin J in $\rho\otimes\rho$. For every ρ it satisfies $\sum_J J(J+1)P_J = \Delta$ [15], so a determinant has $J=0$. For j stored and k fresh particles, Π_j denotes the projector onto $(\ker J_-)^\perp$, the states from which a particle can be transferred into the fresh layer.

C. Results

Theorem 1 (Register law). Fix $0 < \varepsilon < 1$ and let $1 \le r \le n$. Consider tests that decide $e = 0$ versus $e \ge \varepsilon$ with error at most $1/3$ on each side. (i) With a coherent register, $\Omega(\min\{\sqrt{n}, n/r\})$ copies are necessary for every $2 \le k \le n/2$, even if all alternatives are pure, and for every $1 \le k \le n/2$ if mixed alternatives are allowed. The implied constant and the smallest admissible n depend only on ε, uniformly in k. (ii) For fixed k a cleared register attains this bound, so the optimal number of copies obeys Eq. (1).

Corollary 1 (One-particle memory). For the alternatives of Theorem 1 at fixed k and ε, a number of copies bounded independently of n is possible if and only if $r = \Omega(n)$. In either register model $\log_2 n - O(1)$ qubits are necessary and $\lceil \log_2 n \rceil$ suffice. Indeed, $\sqrt{n}$ grows with n, so a bounded cost forces $n/r = O(1)$, and a fixed fraction of n levels attains a bounded cost by Eq. (1). Storing one particle, $r = n$, does so with perfect completeness (Theorem 2).

Theorem 2 (Stored particles). Let $1 \le j \le k$. In each independent pair the register receives $E_j(\rho)$ from the first copy, a measurement of the register and the second copy follows, and the register is cleared. Each measurement may depend on the earlier outcomes. (i) In a single pair, any event of probability zero for every determinant has an effect bounded above by Π_j. For every input ρ and every R, the largest rejection probability of a perfectly complete test with R pairs is

$$1 - (1 - p_j)^R, \qquad p_j = \mathrm{Tr}[\Pi_j (E_j(\rho) \otimes \rho)], \tag{7}$$

attained by measuring Π_j on every pair. For $0 < p_j < 1$ the smallest number of pairs with miss probability at most δ is $\lceil \ln \delta / \ln(1-p_j) \rceil$, and for $p_j = 1$ one pair suffices. By part (iii), every input with $e > 0$ has $p_j > 0$. (ii) The rejection probability per pair is

$$p_j = \sum_{J=0}^{k} P_J [1 - a_J(k,j)],$$
$$a_J(k,j) = \prod_{v=k-j+1}^{k} \left(1 - \frac{J(J+1)}{v(v+1)}\right). \tag{8}$$

It is nondecreasing in j, with

$$p_1 = \frac{\Delta}{k(k+1)}, \qquad p_k = \sum_{J>0} P_J, \tag{9}$$

and $p_{k-1} = p_k$ for pure inputs when $k \ge 2$. (iii) Every input has $p_j \ge je/[3k(k-j+2)]$. Put $D = n - k + 1$ and $d = \lfloor (n-k+2)/2 \rfloor$. For $0 < \varepsilon \le 1 - 1/D$ there is a mixed input with $e = \varepsilon$ and $p_j \le 2j\varepsilon/[k(k-j+2)]$, and for $k \ge 2$ and $0 < \varepsilon \le 1 - 1/d$ a pure input with $e = \varepsilon$ and $p_j \le 4j\varepsilon/[k(k-j+2)]$. In these ranges, rejecting every input with $e \ge \varepsilon$ with probability at least $2/3$ takes $\Theta(k(k-j+2)/(j\varepsilon))$ pairs.

For $j \le k/2$ the number of stored particles times the worst-case number of pairs is of order k^2/ε. Storing all but a bounded number of particles gives $\Theta(1/\varepsilon)$ pairs, the cost of the joint test on two full copies [4, 5]; if $k - j$ grows with k, so does the worst-case number of pairs.

Theorem 3 (Memory at fixed filling). Let $n \ge 2k$ and $0 < \varepsilon < 1$. (i) A perfectly complete test with an r-level cleared register that rejects every pure input with $e \ge \varepsilon$ with probability at least $2/3$ uses

$$R \ge c\, k/(1 + \ln r) \tag{10}$$

pairs, for all sufficiently large k and an absolute constant $c > 0$. At $n = 2k$ the bound holds even if the rejection probability $2/3$ is required only on average over typical inputs. (ii) Let $\bar{\rho} = I/\binom{n}{k}$ be the maximally mixed state of the sector. For the test of Theorem 2 at every filling and every $1 \le j \le k$,

$$p_j(\bar{\rho}) = \frac{j(n-k)}{(n-j+1)(k+1)},$$
$$1 - \mathbb{E}p_j = [1 - p_j(\bar{\rho})] \, \frac{1 + 1/\binom{n-j}{k-j}}{1 + 1/\binom{n}{k}}, \tag{11}$$

where $\mathbb{E}$ is the mean over typical inputs. Thus $p_j(\bar{\rho}) - 1/\binom{n-j}{k-j} \le \mathbb{E}p_j \le p_j(\bar{\rho})$ and $j/[2(k+1)] \le p_j(\bar{\rho}) \le j/(k+1)$. For a fixed miss probability the test takes $\Theta(k/j)$ pairs, simultaneously for all $1 \le j \le k$, on every typical input outside a set of measure exponentially small in k.

In terms of $q = \log_2 r$, Eq. (10) reads $R(1+q) \ge ck$. At half filling a register for j particles has $q \le j \log_2(2ek/j)$, so storing j particles is within a factor $O(\log(2ek/j))$ of the bound on typical inputs. Storing a fixed fraction of the particles thus gives a bounded number of pairs on typical inputs, whereas a bound uniform over all alternatives in the E_j model requires $k - j = O(1)$ by Theorem 2(iii).

Corollary 2 (Qubits for a cost independent of n). Fix ε and require a number of copies bounded independently of n. For the alternatives of Theorem 1 at fixed k, this takes $\log_2 n + O(1)$ qubits in either register model, with or without perfect completeness. At fixed filling $k/n \in (0, 1/2]$, a perfectly complete test with a cleared register needs $\Theta(n)$ qubits. The first assertion combines Corollary 1 with Theorem 2 at $j = 1$. For the second, Eq. (10) forces $\ln r = \Omega(k) = \Omega(n)$, and storing a complete copy takes $\lceil \log_2 \binom{n}{k} \rceil \le n$ qubits and gives a constant cost by Theorem 2 at $j = k$.

At the other end of the scale, consider perfectly complete tests with a cleared register. With a classical record alone such a test accepts every state, and with two levels it accepts every pure state. A three-level test rejects every pure input with $e > 0$ with positive probability (Supplemental Material [16], Sec. S7).

III. THE REGISTER LAW AT FIXED PARTICLE NUMBER

Separate measurements attain the $\sqrt{n}$ branch of Eq. (1), swap tests inside blocks of orbitals attain the n/r branch, and a comparison of two sources bounds every test with a coherent register.

A. Separate measurements

Measure all occupations of a batch of m copies in one Haar-random orbital basis. Each copy gives an occupation vector $X \in \{0,1\}^n$ with k ones. With m_i the number of copies in which orbital i is occupied, set

$$s = \frac{\sum_i m_i(m_i-1)}{m(m-1)}, \qquad \widehat{\Delta} = k + k^2 - (n+1)s. \quad (12)$$

Using all k occupied orbitals of each copy improves on the inner-product estimator of Ref. [17] applied to one orbital per copy, which costs $O(k^2\sqrt{n}/\varepsilon + k^4/\varepsilon^2)$ copies.

Proposition 1 (Separate measurements). For $k \geq 1$ and $0 < \varepsilon < 1$, averaging $\widehat{\Delta}$ over independent batches and rejecting above $\varepsilon/2$ distinguishes a determinant from a pure or mixed input with $e \geq \varepsilon$ using $O(k\sqrt{n}/\varepsilon + k^2/\varepsilon^2)$ copies, with error at most $1/6$ on each side. The batch variance is at most $32k^2n/m^2$ for $2 \leq m \leq n$ (Appendix B).

Two different copies contribute the scalar product of their occupation vectors. Given the basis, the mean of this product is the sum of the squared orbital occupations in that basis. Averaging over bases gives $(n+1)\mathbb{E}s = k^2 + \operatorname{Tr}\gamma^2$, so $\widehat{\Delta}$ is an unbiased estimator of Δ. The variance of s has a term from the choice of basis, a term of order $1/m$ from products that share a copy, and a term of order $1/m^2$ from single products. For $m \leq n$ their sum obeys the bound of Proposition 1. Taking m of order $k\sqrt{n}/\varepsilon$, capped at n, and enough batches to bring the variance below $\varepsilon^2/24$ proves the proposition. As n grows at fixed k a single batch suffices, which gives the $\sqrt{n}$ branch.

B. Swap tests inside orbital blocks

Choose an integer $1 \leq b \leq \min(r,n)$, pad γ/k with zeros to dimension $\bar{n} = b\lceil n/b \rceil$, and partition a fresh Haar-random basis into $\bar{n}/b$ blocks of b orbitals. Extract one particle from each copy of a pair, so that each is in the state γ/k. Measure the block of the first particle and keep its state inside the block, which takes b levels, then measure the block of the second. If the blocks differ record $Z = 0$. Otherwise measure the swap S of the two particles and record its eigenvalue $Z = \pm 1$. We call this the block test. The Haar-averaged block-match projector is $c_I I + c_S \mathsf{S}$ by orbital invariance. Its trace and trace against swap are $\bar{n}b$ and $\bar{n}$. Thus

$$\begin{aligned} c_I &= (\bar{n}b-1)/(\bar{n}^2-1), \quad c_S = (\bar{n}-b)/(\bar{n}^2-1), \\ \mathbb{E}Z &= c_I \operatorname{Tr}[(\gamma/k)^2] + c_S, \\ \mathbb{E}Z^2 &= c_I + c_S \operatorname{Tr}[(\gamma/k)^2]. \end{aligned} \quad (13)$$

The mean gap from a determinant is at least $c_I\varepsilon/k^2$ and each variance is at most $(b+1)/(\bar{n}+1)$. A midpoint threshold gives per-side error at most $1/6$ with

$$2\left\lceil \frac{24(b+1)k^4}{(\bar{n}+1)c_I^2\varepsilon^2} \right\rceil = O\left(\frac{k^4 n}{b\varepsilon^2}\right) \quad (14)$$

copies. Choosing between the block test and Proposition 1 proves Theorem 1(ii). Both use a cleared register.

The classical record stores the block label, and the quantum register holds the particle within its block. A match has probability of order b/n and gives an outcome of order one. The gap between the two means and the variance are then both of order b/n, which gives the factor n/b in Eq. (14). Figure 2(a) illustrates Eq. (1) for a two-particle example with $n = 2^{20}$: for every register size it shows the cheapest of the three tests of this section and Sec. IV. Up to 13 qubits this is the separate-measurement estimator with 6597 copies. From 14 to 19 qubits the block test is cheaper, and its cost halves with each further qubit, from 6550 copies to 204 at 19 qubits. With 20 qubits the register holds one particle, and the optimal test of Sec. IV takes 20 copies. Figure S1 of the Supplemental Material shows each test separately, together with two further perfectly complete instruments.

C. Lower bound with a coherent register

Compare two sources in dimension D: a Haar-random pure state supplied on every copy, and the maximally mixed state I/D. Let $\mathrm{TV}_T(D,r)$ be the largest total variation distance between the distributions of the two classical records, including a final measurement of the register, over all tests with T copies and r levels. After t copies, information about the state resides in the classical record and in the register. Conditional on any record, the density of the vector relative to Haar measure has relative entropy at most $t \ln 4$. This follows by applying the symmetric-power second moment to the positive effect of the record. A transport bound on the sphere converts that entropy into a squared prediction error of order t/D for the next copy. Even for a uniform posterior an arbitrary encoding in the register incurs an additional squared error at most $(r-1)/(D+1)$.

Both comparisons preserve the memory marginal, which lets squared errors accumulate. Mix the two hypotheses with opposite weights $3/4$ and $1/4$ at the beginning of the experiment. For any record their unnormalized memory states then dominate one another within a factor three. Because each memory marginal is preserved, the linear variation of the square-root affinity

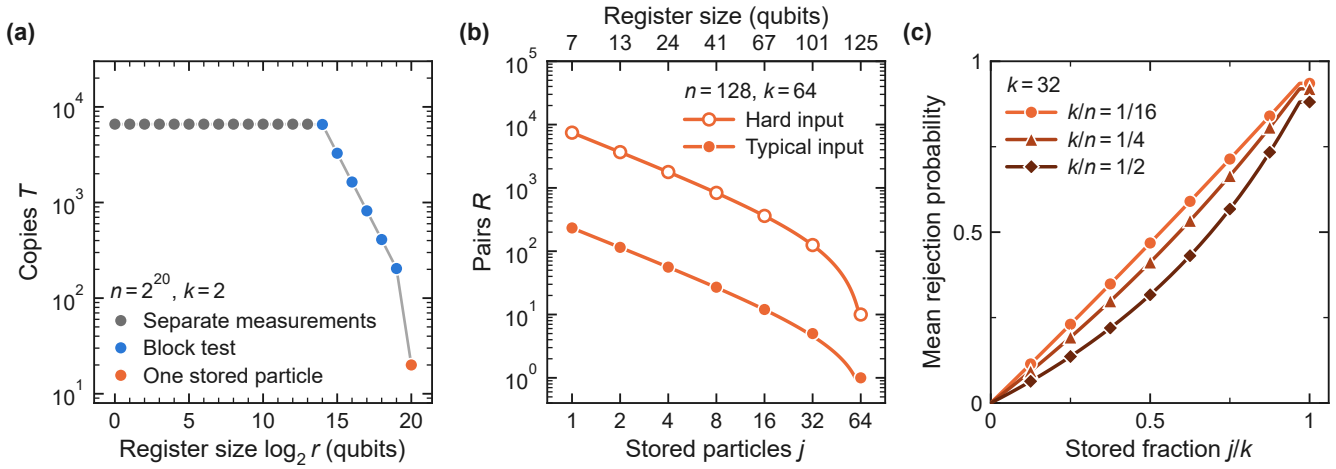

FIG. 2. Memory against cost. (a) Fixed particle number, $n = 2^{20}$ and $k = 2$: copies used by the cheapest of the three tests of Secs. III and IV to distinguish a determinant from the pair state $(|12\rangle + |34\rangle)/\sqrt{2}$ at target error 1/6 on each side, for registers of 0 to 20 qubits. Separate measurements ignore the register, the block test is cheapest at 14–19 qubits, and the optimal test with one stored particle needs 20 qubits. (b) Half filling, $n = 128$ and $k = 64$: pairs needed for miss probability 1/6 against the number j of stored particles; the upper axis gives the register size $\lceil \log_2 \binom{n}{j} \rceil$ in qubits. The hard input is the pure state with 62 occupied orbitals left untouched (spectators) and the pair state of panel (a) on four further orbitals, for which $e = 1/2$. For the typical input the miss probability is averaged over the input (Supplemental Material, Sec. S8). Lines show $\max\{1, \ln 6/[-\ln(1-p)]\}$ for every j, with $p = p_j$ for the hard input and $p = \mathbb{E}p_j$ for the typical one; symbols give the exact minimum pair counts. (c) Mean rejection probability per pair on a typical input, Eq. (11), against the stored fraction j/k for $k = 32$ at three fillings. Lines join the values at every j, and symbols mark every fourth. The means at $j = k - 1$ and $j = k$ coincide, as $p_{k-1} = p_k$ for every pure input.

cancels, leaving a loss bounded by the squared prediction errors. Summing this loss and comparing the two mixtures at the final measurement yields

$$\mathrm{TV}_T(D, r)^2 \le C[(r-1)T + T^2]/D.$$

Appendix C gives the proof, which follows the prediction and affinity method of Ref. [9] and covers continuous measurement records.

Theorem 4 (Purity and product testing with a coherent register). Fix $0 < \varepsilon < 1$. For $1 \le r \le D$, distinguishing pure states on $\mathbb{C}^D$ from states with largest eigenvalue at most $1 - \varepsilon$, with error at most 1/3 on each side, takes $\Theta(\min\{\sqrt{D}, D/r\})$ copies; the lower bound holds already for the single alternative I/D. The same law holds for distinguishing product pure states on $\mathbb{C}^D \otimes \mathbb{C}^D$ from pure states of product fidelity at most $1 - \varepsilon$. The upper bounds use a cleared register and the lower bounds allow a coherent one. For $r \ge D$ both costs are bounded as D grows.

Supplemental Material, Sec. S2, gives explicit constants for a cleared register.

For purity, use the separate-measurement estimator and the block test with one particle in D orbitals. Its defect is $1 - \mathrm{Tr}\,\rho^2 \ge 1 - \lambda_{\max}(\rho) \ge \varepsilon$ on every alternative. For a pure bipartite input, apply exactly these two measurements to subsystem A of each copy. The largest eigenvalue of its reduced state ρ_A is the largest squared Schmidt coefficient, so $\mathrm{Tr}\,\rho_A^2 \le \lambda_{\max}(\rho_A) \le 1 - \varepsilon$ on the far hypothesis. Discarding the other subsystem is a local operation on the arriving copy. This proves both upper bounds with a cleared register.

For the product lower bound compare two independent repeated Haar factors with one repeated global Haar vector. Replacing the two factors one at a time and then comparing the global vector to the maximally mixed source costs $2\,\mathrm{TV}_T(D, r) + \mathrm{TV}_T(D^2, r)$. The global vector is far from products with probability tending to one, by the mean purity of one subsystem. Finally, adding $k - 1$ occupied orbitals that are left untouched (spectators) to one particle in $D = n - k + 1$ orbitals, or $k - 2$ occupied orbitals to two particles in two groups of $d = \lfloor (n - k + 2)/2 \rfloor$ orbitals, embeds the purity and the product problem into the Slater problem with the same fidelity (Appendix C). Since $d \ge n/4$ and $D > n/2$ for every $k \le n/2$, this proves Theorem 1(i) at every filling. For purity this improves the bound $\Omega((D/r)^{1/15})$ that follows from Eq. (223) of Ref. [6]. A cleared register was treated in Ref. [7], and product testing with $r = 1$ in Ref. [13].

IV. PAULI BLOCKING WITH STORED PARTICLES

A. The optimal test

A determinant with occupied subspace L leaves in the register the uniform mixture on $\bigwedge^j L$. Every orbital in L is occupied in the fresh layer, so J_- annihilates the support of $E_j(|\phi\rangle\langle\phi|)\otimes|\phi\rangle\langle\phi|$ for every determinant ϕ. As the occupied subspace varies these supports span a subspace of $\ker J_-$ that is invariant under rotations of the orbitals. With j stored and k fresh particles the kernel is a single irreducible representation of these rotations, with Young diagram $(2^j,1^{k-j})$ (Appendix A), so the span is all of $\ker J_-$. An outcome density with zero likelihood on every determinant therefore vanishes on $\ker J_-$, and integrating all such outcomes of one measurement gives an effect at most Π_j.

Fix an input ρ. In a perfectly complete test almost every record that ends in rejection contains an outcome whose likelihood vanishes on every determinant (Appendix E). The pairs are independent, each in the state $E_j(\rho)\otimes\rho$, so whatever the earlier outcomes such an outcome occurs in the next pair with probability at most p_j. The probability that none occurs in R pairs is at least $(1-p_j)^R$, and the test rejects with probability at most $1-(1-p_j)^R$. Measuring Π_j on every pair and rejecting at its first occurrence attains this value for every input, so the same projective measurement is optimal also among tests that adapt to earlier outcomes. This proves Theorem 2(i).

For perfectly complete Gaussianity tests on two complete copies of a pure input of definite parity, the corresponding optimality of the Bell-sampling projector was shown in Ref. [18]; part (i) holds for every number of stored particles and for mixed inputs. The discarded particles can also supply a classical record. With one stored particle, measuring the other $k-1$ in a random orbital basis and recording the outcome raises p_1 to p_{rec} with $p_1\le p_{\mathrm{rec}}\le[(n+2)/(n-k+1)]\,p_1\le 2p_1$ (Supplemental Material, Sec. S6).

B. Rejection probability

Write $h=k-j$. After extraction $J_z=-h/2$, and the layer pseudospin takes the values $h/2+i$ with $0\le i\le j$. On pseudospin $h/2+i$,

$$J_+^sJ_-^s=i^{\underline{s}}(h+i+1)^{\overline{s}}I,$$

where underlining and overlining denote falling and rising factorials. The kernel is the component $i=0$. A specified set of s particles survives extraction with probability $j^{\underline{s}}/k^{\underline{s}}$. Tracing out the other $k-j$ particles multiplies every s-body matrix element by this factor (Appendix A). The resulting moment is

$$\langle J_+^sJ_-^s\rangle_j=\frac{j^{\underline{s}}}{k^{\underline{s}}}\sum_{J=s}^{k}\frac{(J+s)!}{(J-s)!}\,P_J.$$

The projector onto the kernel is

$$K_0=\sum_{s=0}^{j}\frac{(-1)^sJ_+^sJ_-^s}{s!(h+2)^{\overline{s}}}.$$

On pseudospin $h/2+i$ its eigenvalue is one for $i=0$ and zero for all $i>0$ by Chu–Vandermonde. Substituting these moments gives a terminating Pfaff–Saalschütz sum with upper parameters $-J,J+1,-j$ and lower parameters $h+2,-k$, which equals $a_J(k,j)$ in Eq. (8). Appendix A gives the adjoint-channel and summation details. Each further stored particle contributes one more factor, until a factor vanishes at $v=J$.

C. Worst-case inputs

Every input has $P_{>0}\ge e/6$, where $P_{>0}=\sum_{J>0}P_J$. For a pure input this is shown in the proof of Theorem 5.1 of Ref. [4]. For a mixture, let λ be its largest eigenvalue and e_ψ the infidelity of a corresponding eigenvector. Odd J has weight $(1-\operatorname{Tr}\rho^2)/2\ge(1-\lambda)/2$, since the swap of the two copies acts as $(-1)^J$. The component $\psi\otimes\psi$ contributes at least $\lambda^2e_\psi/6$ at even $J>0$. Using $e\le 1-\lambda+\lambda e_\psi$ gives $6P_{>0}-e\ge(1-\lambda)(2-\lambda e_\psi)\ge 0$. Every positive pseudospin has $1-a_J\ge 1-a_1=2j/[k(h+2)]$, which proves $p_j\ge je/[3k(h+2)]$.

The factor $k(h+2)$ is attained by the following inputs, which we call hard inputs. Leave $k-1$ spectator orbitals filled and place one particle in a mixture on $D=n-k+1$ active orbitals. Give its spectrum the weights $w=(1-\varepsilon,\varepsilon/(D-1),\dots,\varepsilon/(D-1))$. The first weight is the largest if and only if $\varepsilon\le 1-1/D$, and then $e=\varepsilon$ (Appendix C). The spectators contribute pseudospin zero and the active particle contributes only pseudospins zero and one. Thus $p_j=\Delta j/[k(h+2)]$, with $\Delta=1-\sum w_i^2\le 2\varepsilon$.

For a pure hard input leave $k-2$ spectators filled and use a pair $\sum_i\sqrt{w_i}\,a_i\wedge b_i$ on two disjoint groups of $d=\lfloor(n-k+2)/2\rfloor$ orbitals. Use the same weights with D replaced by d. The largest squared overlap with a determinant equals the largest Schmidt weight (Appendix C), so $e=\varepsilon$ for $\varepsilon\le 1-1/d$. Each active occupation occurs twice, giving $\Delta=2(1-\sum w_i^2)\le 4\varepsilon$. The only pseudospins are zero and two, hence $p_j=(\Delta/6)(1-a_2)$. The smallest positive transfer eigenvalue after extraction is $h+2$, so the first moment $j\Delta/k$ bounds p_j above by $j\Delta/[k(h+2)]$. Fewer than d active pairs may be used. With two pairs and $\varepsilon=1/2$ the active part is the pair state $(|12\rangle+|34\rangle)/\sqrt{2}$ of Fig. 2, for which $\Delta=1$.

By Theorem 2(i), miss probability δ on these inputs takes $\lceil\ln\delta/\ln(1-p_j)\rceil\ge(1/p_j-1)\ln(1/\delta)$ pairs, while $\lceil 3k(h+2)\ln(1/\delta)/(je)\rceil$ pairs suffice for every input. With $\delta=1/3$ this is part (iii).

D. Tunneling and counting

Proposition 2 (Tunneling and counting). Suppose that k is known and that the number of particles in each layer can be counted. Couple each orbital of the first copy to an empty orbital of the storage layer with tunneling probability $y = j/k$. Counting the $k - j'$ particles left behind shows that $j' \sim \mathrm{Bin}(k, y)$ particles are stored, in the state $E_{j'}(\rho)$. Keep the pairs with $1 \le j' \le j$ and discard the others. Couple the storage layer to the second copy with tunneling probability $x_{j'} = 1/(k-j'+2)$ and reject if the count N_s of the storage layer lies outside $[j', k]$. (i) This test is perfectly complete, and its rejection probability per pair obeys $c_0\, p_j \le p_j^{\mathrm{count}} \le p_j$ with an absolute constant $c_0 > 0$. (ii) For one stored particle the following identity holds for every input,

$$p_1^{\mathrm{count}} = \alpha_k p_1 = \frac{\alpha_k \Delta}{k(k+1)},$$
$$\alpha_k = \Big(1 - \frac{1}{k}\Big)^{k-1} \left[\Big(\frac{k}{k+1}\Big)^k + \frac{k}{(k+1)^k}\right] \ge \mathrm{e}^{-2}, \tag{15}$$

with $\alpha_1 = 1$. (iii) For $j = k$, $p_k^{\mathrm{count}} \ge \frac{3}{4}\, p_k$ for every input. For $N > n/2$ the same sequence is applied to holes, starting from a filled storage layer.

On a determinant the two layers have layer pseudospin $(k - j')/2$. The coupling rotates the pseudospin and preserves its magnitude, so every count lies in $[j', k]$. A larger pseudospin reaches counts outside this interval, and at the tunneling probability $x_{j'}$ a fixed fraction of the weight of $\Pi_{j'}$ falls outside it (Appendix A). At most j particles are retained until the next copy, and discarded pairs count toward the cost. Exact evaluation for all $2 \le k \le 40$ gives $p_j^{\mathrm{count}} \ge \alpha_k p_j$ for every j and every input, so one stored particle is the least favorable case there (Supplemental Material, Sec. S3).

For $j = 1$, $\Pr(j' = 1) = ky(1 - y)^{k-1}$. Conditional on this outcome a rejection means that all $k+1$ particles are found in the fresh layer or all in the storage layer. The two events have the effects $\eta_\mathrm{f}\Pi_1$ and $\eta_\mathrm{s}\Pi_1$, with $\eta_\mathrm{f} = (k + 1)x_1(1 - x_1)^k$ and $\eta_\mathrm{s} = (k + 1)(1 - x_1)x_1^k$, because their amplitudes are multiples of the map W that sends $v \otimes \psi$ to $v \wedge \psi$, and $W^\dagger W = (k + 1)\Pi_1$. With $y = 1/k$ and $x_1 = 1/(k + 1)$ the product $\Pr(j' = 1)(\eta_\mathrm{f} + \eta_\mathrm{s})$ is α_k, and each factor exceeds e^{-1}. Thus $\alpha_{10} \simeq 0.149$, and α_k decreases to $\mathrm{e}^{-2} \simeq 0.135$. For $k = 2$, balanced tunneling $x_1 = y = 1/2$ gives the larger fraction $3/8$. For $j = k$ the storage layer holds a complete copy and $x_k = 1/2$. Layer pseudospin $J \ge 1$ is then rejected with probability $1 - \mathcal{L}_J(0)^2 \ge 3/4$, where $\mathcal{L}_J$ is the Legendre polynomial, which gives part (iii) (Appendix A). Applied to two complete copies, the same interference measures Rényi entropies [19], and with bosonic atoms in an optical lattice it has been realized [20].

Since $\Pi_1 = W^\dagger W/(k+1)$, a pair whose two copies come from different states, with one-body matrices γ_s and γ_f and defects Δ_s and Δ_f, is rejected by the optimal test with probability

$$\frac{k - \mathrm{Tr}(\gamma_\mathrm{s}\gamma_\mathrm{f})}{k(k+1)} = \frac{\Delta_\mathrm{s} + \Delta_\mathrm{f} + \|\gamma_\mathrm{s} - \gamma_\mathrm{f}\|_F^2}{2k(k+1)}, \tag{16}$$

and by tunneling and counting with α_k times this probability.

V. MEMORY AT FIXED FILLING

At fixed filling the optimal test with one stored particle needs a growing number of pairs: since $\mathrm{Tr}\,\gamma^2 \ge k^2/n$, every input has $p_1 = \Delta/[k(k+1)] \le (1-k/n)/(k+1) < 1/k$. Theorem 3 extends this to every perfectly complete test with a cleared register. At half filling the mean probability that any perfectly complete test with an r-level first-copy instrument rejects a pair in a newly drawn typical state is $O((1+\ln r)/k)$, and storing j particles comes close to this on typical inputs.

A. Typical inputs

Let $\mathcal{N} = \binom{n}{k}$ and let S be the swap of the two copies. For a typical input both copies of a pair are in the same pure state ψ, and $\mathbb{E}\,\psi\psi^\dagger \otimes \psi\psi^\dagger = (I + \mathsf{S})/[\mathcal{N}(\mathcal{N} + 1)]$. The identity term is proportional to two copies of the maximally mixed state $\bar\rho$. The mean acceptance is the trace of the image of this moment under $E_j \otimes \mathrm{id}$ against $K_0 = I - \Pi_j$.

With j stored and k fresh particles the kernel carries one irreducible representation of the orbital rotations, with highest-weight vector $|1 \cdots j\rangle \otimes |1 \cdots k\rangle$. The operator J_- maps its orthogonal complement one-to-one onto the space with $j-1$ stored and $k+1$ fresh particles, so

$$\mathrm{Tr}\, K_0 = \binom{n}{k}\binom{n}{j} - \binom{n}{k+1}\binom{n}{j-1}.$$

For the identity term, covariance and trace preservation give $E_j(I) = \mathcal{N} I/\binom{n}{j}$. Its trace against the kernel is therefore $\mathcal{N}\,\mathrm{Tr}\, K_0/\binom{n}{j}$. For the swap term, orbital invariance makes its compression scalar on the kernel. To find the scalar, evaluate it on the highest weight vector. The matrix element of S between the determinant in the fresh layer and itself is the projector onto that determinant in the storage layer. The extracted state of that determinant is uniform on the j-wedges of its occupied subspace. The highest weight matrix element is consequently $1/\binom{k}{j}$, making the second trace $\mathrm{Tr}\, K_0/\binom{k}{j}$.

Adding the two traces and using $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$ proves Eq. (11). The identity term gives $p_j(\bar\rho)$, and since $\binom{n-j}{k-j} \le \mathcal{N}$ the swap term lowers the mean by at most $1/\binom{n-j}{k-j}$. At fixed j/k and k/n the mean tends to $j(n-k)/[k(n-j)]$. For $j \le k/2$ the rejection probability of a typical input lies between half and three halves of

its mean outside a set of exponentially small measure, because it is the expectation of a bounded effect on two copies (Appendix D). Theorem 2(i) then gives $\Theta(k/j)$ pairs. For larger j the probability is at least its value at $\lfloor k/2 \rfloor$, which is bounded below by a constant, so a bounded number of pairs suffices. Figure 2(b) compares a typical input with a hard input, and panel (c) shows the means at three fillings.

B. Lower bound for every first-copy instrument

Start at $n = 2k$. By Appendix E, almost every record that ends in rejection contains an outcome whose likelihood vanishes on every determinant. After refining the first outcome to one Kraus operator, its effect is Γ of rank at most r. The effect M of the pair outcome on the two copies obeys $M \le \Gamma \otimes I$ and vanishes on the span of the states $\phi \otimes \phi$ with ϕ a determinant, which is the sector of layer pseudospin zero, with projector Q_0.

Set $H = I - (1 + k/2)Q_0$. Since $\operatorname{Tr} HM = \operatorname{Tr} M$, a weighted Cauchy–Schwarz inequality gives $\operatorname{Tr} M \le \|(\Gamma^{1/4} \otimes I)H(\Gamma^{1/4} \otimes I)\|_F^2$. This reduces the bound to a quadratic form in $\sqrt{\Gamma}$, regarded as a vector in two copies of the sector. Covariance under orbital rotations makes the form a scalar Λ_{k-J} on each pseudospin J, and the scalars follow from the spectrum of the Johnson association algebra. For $J \le k/8$ the multiplier satisfies $|\Lambda_{k-J}| = O(1/k + J^2/k^2)$ (Appendix D). The rank of Γ limits the weight at large J.

The vector has Schmidt rank at most r, and its pseudospin distribution obeys $\Pr(J \ge s) \le 6r \exp[-s^2/(4k)]$ for every real $s \ge 0$. Integrating in s^2, with the probability capped at one, gives $\mathbb{E}J^2 \le 4k[1 + \ln(6r)]$. Averaging the bound on Λ_{k-J} over this distribution gives a probability $O((1 + \ln r)/k)$ for the pair. If $\ln r$ is below a small multiple of k, the contribution above $k/8$ is exponentially small. The trace against the swap part of the Haar moment contributes at most $r/\mathcal{N}$ to the probability for a pair in a newly drawn typical state. For larger r, Eq. (10) follows from $R \ge 1$. Appendix D supplies the finite constants.

The same pure state is supplied for all pairs. After a record of a copies the conditional two-copy moment contains the moment of a pair in a newly drawn typical state of weight $\mathcal{N}(\mathcal{N}+1)/[(\mathcal{N}+a)(\mathcal{N}+a+1)]$. The rest has mass at most $2a/\mathcal{N}$. Summing over pairs bounds mean rejection by $Rp_* + 2R(R-1)/\mathcal{N}$, where p_* bounds the probability that a pair in a newly drawn typical state produces an outcome of probability zero for every determinant. For large k a typical input has $e \ge \varepsilon$ with probability tending to one, by a net over the determinants (Appendix D). The test must then reject with probability at least $2/3 - o(1)$ on average, which forces Eq. (10).

For $n > 2k$ restrict the inputs to $2k$ fixed orbitals. The projection of a determinant onto the k-particle states of these orbitals is a determinant of norm at most one, so such an input has the same infidelity as in the smaller system. On these inputs each first-copy instrument still leaves at most r levels, and the law of the classical record is unchanged. The bound therefore holds for all $n \ge 2k$. The hard inputs of this bound are typical states of the $2k$-orbital sector. Equation (11) refers to typical states of the full sector.

VI. GAUSSIAN TARGETS

Pure fermionic Gaussian states generalize determinants to indefinite particle number. Inputs are density operators commuting with fermion parity, targets are pure Gaussian states of either parity, $F_{\mathrm{G}} = \max_\phi \langle\phi|\rho|\phi\rangle$ over these targets, and $e = 1 - F_{\mathrm{G}}$. For Hermitian Majorana operators c_a with $\{c_a, c_b\} = 2\delta_{ab} I$, define $G_{ab} = i \operatorname{Tr}(\rho[c_a, c_b])/2$ and $\Delta_G = 2n - \|G\|_F^2$.

Proposition 3 (Pure Gaussian targets). Fix $0 < \varepsilon < 1$. (i) In either register model, deciding $e = 0$ versus $e \ge \varepsilon$ with error at most $1/3$ on each side takes $\Omega(\min\{\sqrt{n}, n/r\})$ copies for mixed and for pure alternatives. (ii) A perfectly complete test with a cleared register needs $\Omega(n/(1 + \ln r))$ pairs, even if all alternatives are pure. A cost independent of n therefore takes $\Theta(n)$ qubits, attained for pure and mixed alternatives by storing a complete copy for the joint test of Ref. [14], whose rejection probability is at least $e/8$. (iii) Separate measurements suffice with $O(n^2/\varepsilon^2 + n^{3/2}/\varepsilon)$ copies and error at most $1/6$ on each side. (iv) A $2n$-level register gives a perfectly complete test using at most $2\lceil n^2 \ln(1/\delta)/\varepsilon \rceil$ copies at miss probability δ.

For the lower bound with pure alternatives, place one fermion in each of three disjoint groups of $\lfloor n/3 \rfloor$ orbitals. A product state of the three particles is a determinant, hence Gaussian. A Haar-random state of the three particles lives in dimension $\lfloor n/3 \rfloor^3$, whereas an η-net of all pure Gaussian states has log-cardinality $O(n^2 \ln(n/\eta))$, so with probability tending to one it has $e \ge \varepsilon$ for every fixed $\varepsilon < 1$. In the half-filled sector the dimension is exponential in n and a typical state is again far from every Gaussian state. Every determinant is Gaussian, so a perfectly complete Gaussian test is a perfectly complete test for determinants, and the bound of Theorem 3 applies.

For separate measurements draw $O \in SO(2n)$ from the Haar measure and, on each copy of a batch, measure the n commuting parities $i\tilde{c}_{2v-1}\tilde{c}_{2v}$ of the rotated Majorana operators $\tilde{c} = Oc$. Each copy gives a sign vector $X \in \{\pm 1\}^n$. Let s_G average the dot products of the resulting sign vectors over distinct pairs in the batch. The unbiased estimate is $\widehat{\Delta}_G = 2n - 2(2n-1)s_G$. With a register use $W_G = \sum_a |a\rangle \otimes c_a$ to encode the first copy by the isometry $W_G/\sqrt{2n}$. Discard its Fock system and measure $W_G W_G^\dagger/(2n)$ on the register and next copy. The rejection probability per pair is $\Delta_G/(2n)^2$, and a union bound in canonical modes gives $e \le \Delta_G/4$. Appendix F proves all four bounds. Part (iii) improves the bounds

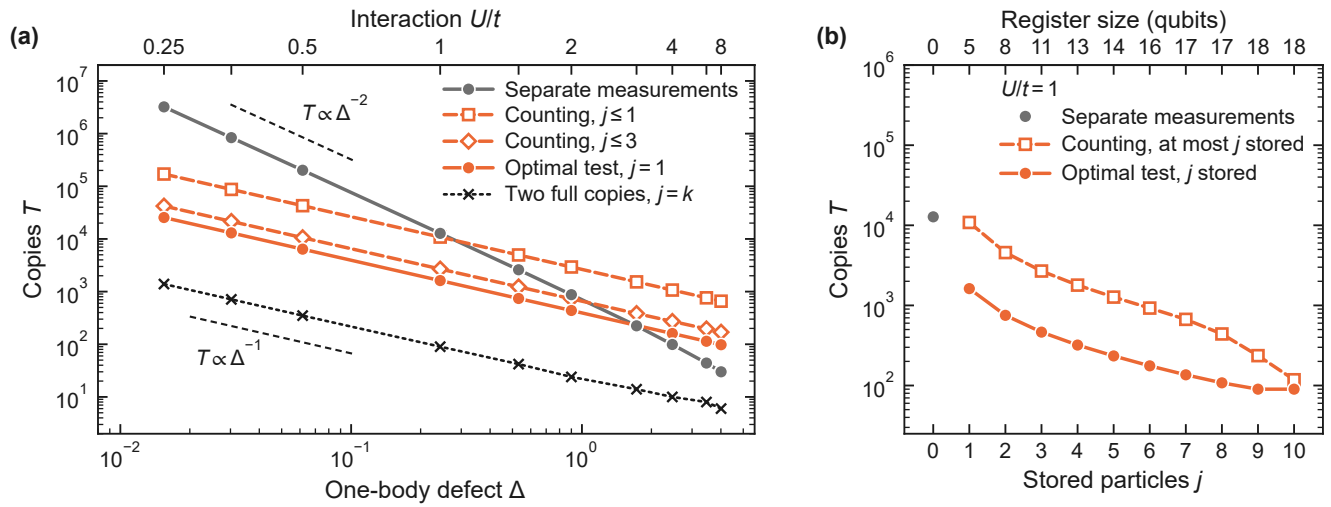

FIG. 3. Copies needed to distinguish the ground state of a half-filled Hubbard chain with ten sites and open ends from a determinant, at target error 1/6 on each side. (a) Against the one-body defect Δ; the upper axis gives U/t. Separate measurements: the estimator of Sec. III, calibrated as described in Sec. VII. Counting: tunneling and counting with at most one and at most three stored particles. Optimal test: the test of Theorem 2 with one stored particle; with $j = k$ it is a joint measurement on two full copies. The dashed segments show the power laws $T \propto \Delta^{-2}$ and $T \propto \Delta^{-1}$ that the estimator and the tests with stored particles follow near a determinant. (b) Dependence on the number j of stored particles at $U/t = 1$; the upper axis gives the register size $\lceil \log_2 \binom{20}{j} \rceil$ in qubits. Separate measurements store no particle and appear at $j = 0$. The optimal test costs the same at $j = 9$ and $j = 10$ because the state is pure.

$O(n^3/\varepsilon^2 + n^{5/2}/\varepsilon)$ of Ref. [11] and $O(n^4/\varepsilon^2)$ of Ref. [12], both restated in terms of our infidelity, by one and two powers of n in the leading term. Applying the single-copy hypothesis selection of Ref. [21] to a net of pure Gaussian states and verifying the selected candidate on further copies takes $\widetilde{O}(n^2/\varepsilon)$ copies with unrestricted classical computation (Appendix F). Part (iii) reaches the same power of n with nonadaptive measurements and a closed-form statistic. Bittel *et al.* treat the tolerant problem in trace distance with single-copy measurements [22].

VII. HUBBARD CHAINS AND A DOUBLE WELL

We now compare the costs on ground states of Hubbard chains with nearest-neighbor hopping t and on-site repulsion U. We distinguish the ground state from a determinant at target error 1/6 on each side. In tunneling and counting a single count outside the allowed interval rejects, and a determinant never produces one. The separate-measurement estimator fluctuates even on a determinant and has to resolve Δ against this noise.

Figure 3 shows the number of copies for a half-filled chain with ten sites, that is, $N = 10$ particles in $n = 20$ spin orbitals, so that $k = 10$. For a test with rejection probability p per pair it is $2\lceil \ln 6/[-\ln(1-p)] \rceil$, discarded pairs included. For the estimator we average B batches of m copies, place the threshold between the two means in proportion to their standard deviations, and choose m and B to minimize mB with the normal quantile at 5/6. Direct sampling of the measurement outcomes in nine settings with $U/t \geq 2$ agrees with this calibration: the false-rejection probabilities agree with 1/6 within their standard errors, and the miss probabilities lie between 0.14 and 0.17 (Supplemental Material, Sec. S5).

Near a determinant p_j is proportional to Δ at fixed k and j, so the tests with stored particles cost of order Δ^{-1} copies, whereas the estimator, whose variance stays positive on a determinant, costs of order Δ^{-2}. At $U/t = 0.25$ counting with at most three stored particles, which takes 11 qubits in place of the 18 of a complete copy, needs 42 168 copies, against 3 219 425 for the estimator, a factor of 76. The estimator becomes the cheaper of the two between $U/t = 2$ and 3, where both costs are below 10^3 copies. Figure 3(b) shows the dependence on the number of stored particles at $U/t = 1$. All thirteen couplings are listed in Supplemental Material, Sec. S5.

For two particles on 24 sites at $U/t = 4$, counting takes 98 copies with at most one stored particle and 44 with both, against 310 for the estimator. A double well with two particles has a closed-form solution and serves to calibrate the measurement. In the singlet sector hopping mixes the covalent and the even ionic configuration, and the one-body defect is $\Delta = U^2/(U^2 + 16t^2)$. At $U/t = 4$ the optimal test with one stored particle takes 21 pairs, or 42 copies. With $y = x_1 = 1/2$, counting rejects with probability 1/32 per pair, so 57 pairs give miss probability below 1/6. Independent false counts on a determinant keep the false-rejection probability at most 1/6 if they oc-

cur with probability at most $1-(5/6)^{1/57} \simeq 3.2\times10^{-3}$ per pair; Supplemental Material, Sec. S5, gives the number of pairs prepared in a determinant that bounds this probability. With exact counts and these tunneling probabilities, 1000 independent pairs without a count outside the interval certify $e \le \Delta < 0.048$ at 95% confidence. Coherent tunneling in such double wells has been realized in optical superlattices [23].

VIII. DISCUSSION

The stored-particle test turns Pauli blocking into a certificate of mean-field form. With exact layer-number counts, a single count outside the allowed interval certifies a departure from a determinant, and the fraction of pairs with such a count estimates its probability, which is $\alpha_k\Delta/[k(k+1)]$ with one stored particle. By Eq. (4) a confidence interval $[p_-,p_+]$ for this probability bounds the infidelity from both sides,

$$\frac{(k+1)\,p_-}{2\alpha_k} \le e \le \frac{k(k+1)\,p_+}{\alpha_k}, \tag{17}$$

and R pairs without such a count give $p_+ = \ln(1/\xi)/R$ at confidence $1-\xi$. For fixed k, precision and confidence, the certificate uses a number of pairs independent of n and requires total atom counts in the two layers. Pairing copies from two runs A and B of an experiment compares their one-body matrices. By Eq. (16) the rejection probabilities with one stored particle obey

$$k(k+1)\,(2p_{\mathrm{AB}} - p_{\mathrm{AA}} - p_{\mathrm{BB}}) = \|\gamma_{\mathrm{A}} - \gamma_{\mathrm{B}}\|_F^2, \tag{18}$$

and for two determinants $p_{\mathrm{AB}} = \sum_i \sin^2\theta_i/[k(k+1)]$, with θ_i the principal angles between their occupied subspaces. The three frequencies measured by tunneling and counting, divided by α_k, estimate these probabilities.

At fixed particle number, a cost independent of n takes the memory of one particle, $\log_2 n + O(1)$ qubits. At fixed filling, the same cost takes $\Theta(n)$ qubits for tests that never reject a determinant and clear the register after each pair. Theorem 3 is tight within a logarithmic factor on typical inputs at half filling, and in order when the register holds a complete copy. Two questions remain at fixed filling: the optimal worst-case cost for registers of intermediate size, and the cost when errors on both sides are allowed. For the latter, Theorem 1 requires $r = \Omega(n)$ levels. Do n levels suffice for a cost independent of n?

DATA AVAILABILITY

The data that support the findings of this article are openly available on Zenodo [24].

Appendix A: Stored particles

Lemma A1 (Channel and kernel). Put $h = k - j$. Counting occupied j-subsets gives $V^{(j)\dagger}V^{(j)} = \binom{k}{j} I$. A determinant with occupied subspace L gives the uniform state on $\wedge^j L$ and every support vector tensored with $|L\rangle$ is annihilated by J_-. At fixed total $t = k + j \le n$, the exterior Pieri rule with ℓ_{s} particles in the storage layer lists exactly once each diagram $(2^l, 1^{t-2l})$ with $0 \le l \le \min(\ell_{\mathrm{s}}, t-\ell_{\mathrm{s}})$. Its multiplicity space has each value $J_z = \ell_{\mathrm{s}} - t/2$ once for $l \le \ell_{\mathrm{s}} \le t-l$, hence is pseudospin $t/2 - l$. At $\ell_{\mathrm{s}} = j$ write $l = j - i$. Only $i = 0$ is lowest weight, giving the irreducible kernel $(2^j, 1^h)$. Every higher pseudospin persists at $\ell_{\mathrm{s}} = j-1$, proving the dimension subtraction in Sec. V. The determinant supports span a nonzero invariant subspace of the kernel and thus all of it. Positivity gives $F \le \Pi_j$ for every effect F that vanishes on all determinants.

Lemma A2 (Transfer spectrum). At layer pseudospin S and $J_z = z$, J_+J_- has eigenvalue $(S+z)(S-z+1)$. At $S = h/2 + i$, $z = -h/2$, the eigenvalue is $i(h+i+1)$. Successive lowerings therefore give

$$J_+^s J_-^s\big|_i = i^{\underline{s}}(h+i+1)^{\overline{s}} I. \tag{A1}$$

Here underlining denotes a falling factorial and overlining a rising factorial, both one at $s = 0$.

Lemma A3 (Moments after extraction). For ordered s-subsets I, L' and $0 \le s \le j$, the channel adjoint satisfies

$$E_j^*(a_I^\dagger a_{L'}) = \frac{j^{\underline{s}}}{k^{\underline{s}}} a_I^\dagger a_{L'}. \tag{A2}$$

Indeed, resolving the traced-out h particles gives Kraus operators $a_C/\sqrt{\binom{k}{j}}$ for $|C| = h$, up to a common exterior-ordering sign. In each adjoint term, moving $a_C^\dagger$ past $a_I^\dagger$ and a_C past $a_{L'}$ contributes the same sign twice. The middle sum $\sum_C a_C^\dagger a_C$ acts on $k-s$ remaining particles as $\binom{k-s}{h} I$, giving Eq. (A2). Every term of $J_+^s J_-^s$ has precisely s creations followed by s annihilations in the storage layer. Applying Eq. (A2) to that layer and then evaluating on the balanced full copies gives

$$\big\langle J_+^s J_-^s \big\rangle_j = \frac{j^{\underline{s}}}{k^{\underline{s}}} \sum_{J=s}^{k} P_J \,\frac{(J+s)!}{(J-s)!}. \tag{A3}$$

The expectation on the left is in $E_j(\rho)\otimes\rho$. On balanced copies the first moment is $k - \mathrm{Tr}\,\gamma^2$ by anticommuting the fresh-layer operators and factoring the two copies, as in Ref. [15]. Thus the first moment after extraction is $j\Delta/k$. The smallest positive eigenvalue in Lemma A2 is $h+2$ and the largest is $j(k+1)$. Dividing the first moment by the two extremal positive eigenvalues gives

$$\Delta/[k(k+1)] \le p_j \le j\Delta/[k(h+2)]. \tag{A4}$$

Lemma A4 (Projector onto the kernel). On the two layers after extraction set

$$K_0 = \sum_{s=0}^{j} \frac{(-1)^s J_+^s J_-^s}{s!\,(h+2)^{\bar{s}}}. \tag{A5}$$

By Eq. (A1) its eigenvalue on i is ${}_2F_1(-i, h+i+1; h+2; 1)$. Chu–Vandermonde gives $(1-i)^{\bar{i}}/(h+2)^{\bar{i}}$, which is one at $i=0$ and zero otherwise. Hence $K_0 = I - \Pi_j$. Inserting Eq. (A3) into its expectation, the coefficient of P_J is the terminating balanced sum

$${}_3F_2\left(\begin{matrix}-J,\ J+1,\ -j\\ h+2,\ -k\end{matrix}; 1\right).$$

For Pfaff–Saalschütz take $a=-J$, $b=J+1$, $c=h+2$ and termination index j. The second denominator is $1+a+b-c-j=-k$ and every term through $\min(j,J)$ is defined. The sum is

$$\frac{(c-a)^{\bar{j}}(c-b)^{\bar{j}}}{c^{\bar{j}}(c-a-b)^{\bar{j}}} = \prod_{m=h+1}^{k} \frac{(m-J)(m+J+1)}{m(m+1)}. \tag{A6}$$

Each factor lies in $[0,1]$ until a zero occurs at $m=J$. This proves monotonicity and $p_k = P_{>0}$. For $\psi\otimes\psi$ only even J occur because the swap has eigenvalue $(-1)^J$ (the π rotation and wedge ordering phases cancel [15]). Thus $p_{k-1} = p_k$ for pure inputs.

Lemma A5 (Efficiency of counting). Each stored count j' has binomial probability and leaves the state $E_{j'}(\rho)$. Put $h = k - j'$, $m = -h/2$ and $S = h/2 + i$. The rotation changes the count by $Y = m' - m$. The component $i=0$ always stays in $[j',k]$. Fix j' and set $x = 1/(h+2)$. For $1 \le i \le 4$ the extremal output $m' = -S$ alone has probability $\binom{h+2i}{i} x^i (1-x)^{h+i}$. The three factors $\binom{h+2i}{i} x^i$, $(1-x)^{h+1}$, and $(1-x)^{i-1}$ are at least $1/i!$, $\exp(-1)$, and 2^{1-i} respectively. Their product exceeds $1/1000$.

For $i \ge 5$, write $Y = m' - m$, $\bar{Y} = \mathbb{E}Y = hx \le 1$, and $A = S(S+1) - m^2$. From the two ladder amplitudes, $\langle J_x^2\rangle = A/2$ and $v = \operatorname{Var} Y = x(1-x)[(2i+1)h + 2i(i+1)] \ge 2i+1 \ge 11$. Indeed $v - (2i+1)$ has numerator $(2i^2-4i-3)h + 2i^2 - 6i - 4$ over $(h+2)^2$. The centered variable $Z = Y - \bar{Y}$ is represented on $|S,m\rangle$ by $\cos\theta(J_z - m) + \sin\theta J_x$. In its square the single-step and even-step terms are orthogonal. The single-step term has squared norm $\cos^2\theta\, v$. Expanding the even-step term with $J_\pm$ gives $\langle J_x^4\rangle = (3A^2 - 2A + 3m^2)/8 \le A^2/2$ since $A = i(h+i+1) + h/2 \ge 3|m|$. Consequently $\mathbb{E}Z^4 \le v + 2v^2$. Interpolation gives $\mathbb{E}|Z| \ge v^{3/2}/\sqrt{\mathbb{E}Z^4}$ and $\mathbb{E}(-Z)_+ = \mathbb{E}|Z|/2$. Hence

$$\frac{\mathbb{E}(-Y)_+}{\sqrt{v}} \ge \frac{\sqrt{v}}{2\sqrt{1+2v}} - \frac{1}{\sqrt{v}} > 0.044.$$

Cauchy–Schwarz yields $\Pr(Y<0) \ge [\mathbb{E}(-Y)_+]^2/(v+\bar{Y}^2) > 1/1000$. A negative change is already outside the interval $[j',k]$. Thus the conditional rejection probability is at least $p_{j'}/1000$ for each accepted number of stored particles.

It remains to average over the number stored. Put $\tau_{j'} = j'/[(k-j'+1)(k+1)]$ and $C = J(J+1)$. Summation of $1/[v(v+1)]$ in Eq. (8) gives

$$1 - e^{-C\tau_{j'}} \le 1 - a_J(k,j') \le \min(1, C\tau_{j'}).$$

Use the exponential and union bounds before any zero factor. If the product contains zero, $J \ge k - j' + 1$ implies $C\tau_{j'} \ge 1$. Concavity of $1-e^{-z}$ now implies $p_{j'} \ge (1-e^{-c})p_j$ if $\tau_{j'} \ge c\tau_j$, $0 < c \le 1$.

Let $\sigma^2 = j(k-j)/k$. For $j \ge 256$ and $\sigma \ge 1$, expanding the centered fourth moment of the binomial gives $\mathbb{E}(j'-j)^4 = 3\sigma^4 + \sigma^2[1 - 6y(1-y)] \le 4\sigma^4$. The same interpolation gives $\mathbb{E}(j-j')_+ \ge \sigma/4$. Cauchy–Schwarz yields $\Pr(j' < j) \ge 1/16$. Removing $j' < j - 8\sigma$ by the variance inequality leaves mass at least $3/64$. There $j' \ge j/2$ by $\sigma \le \sqrt{j}$, and $k - j' + 1 \le 5(k-j+1)$ by $\sigma \le \sqrt{k-j}$. Thus $\tau_{j'} \ge \tau_j/10$. The probability averaged over the retained counts, with their binomial weights, is at least $(3/64)(1-e^{-1/10})p_j > p_j/300$.

Otherwise $\sigma < 16$. For $\sigma > 0$ at least $3/4$ of the mass lies within 2σ of j, on at most 65 integers. The ratio of successive binomial masses makes its integer mean j a mode. Hence $j' = j$ alone has mass at least $3/260$. For $\sigma = 0$ it has mass one. Combining either case with the conditional efficiency and using $p_{j'} \le p_j$ for every retained count gives

$$p_j/300\,000 \le p_j^{\text{count}} \le p_j. \tag{A7}$$

Lemma A6 (One stored particle and two complete copies). Let $j' = 1$, so that the storage layer holds one particle in the state γ/k, and couple the two layers with tunneling probability x. The count N_s lies outside $[1,k]$ only if all $k+1$ particles end in the fresh layer or all in the storage layer. In the first case the stored particle tunnels and the k fresh particles stay, with amplitude $\sqrt{x(1-x)^k}$, and the unnormalized output is proportional to $v \wedge \psi = W(v\otimes\psi)$. In the second case the roles of the layers are interchanged and the amplitude is $\sqrt{(1-x)x^k}$. Since $W^\dagger W = (k+1)\Pi_1$, the two events have the effects $\eta_f\Pi_1$ and $\eta_s\Pi_1$ of Sec. IV. Multiplying their sum by $\Pr(j'=1) = ky(1-y)^{k-1}$ at $y = 1/k$ and $x = 1/(k+1)$ gives Eq. (15). For $k > 1$, $(k-1)\ln(1 + 1/(k-1)) < 1$ and $k\ln(1+1/k) < 1$, so the extraction probability and the first term in brackets in Eq. (15) both exceed e^{-1}. Both decrease to e^{-1} because $x\ln(1+1/x)$ increases to one. The other term $k/(k+1)^k$ decreases to zero. For $k=1$ extraction is certain and $\alpha_1 = 1$. For $j = k$ one has $y = 1$, $h = 0$ and $x_k = 1/2$. Layer pseudospin J at $J_z = 0$ returns to $J_z = 0$ with probability $[d^J_{00}(\pi/2)]^2 = \mathcal{L}_J(0)^2$, where d^J is the Wigner small-d matrix and $\mathcal{L}_J$ the Legendre polynomial. For odd J the return probability vanishes. For $J = 2l$, $|\mathcal{L}_{2l}(0)| = \binom{2l}{l}4^{-l}$ decreases with l, so the return probability is at most $1/4$ for even $J \ge 2$. As $p_k = P_{>0}$, it follows that $p_k^{\text{count}} \ge \frac{3}{4}\,p_k$.

Appendix B: Variance bounds for occupation measurements

For a basis U, set $\bar{x}_U = \mathbb{E}[X \mid U]$, $M_U = \mathbb{E}[XX^{\mathsf{T}} \mid U]$, $C_U = M_U - \bar{x}_U \bar{x}_U^{\mathsf{T}}$ and $\theta_U = \bar{x}_U^{\mathsf{T}} \bar{x}_U$. Center each copy as $Y_c = X_c - \bar{x}_U$, $c = 1, \dots, m$. At fixed basis,

$$s - \theta_U = \frac{2}{m} \sum_c \bar{x}_U^{\mathsf{T}} Y_c + \binom{m}{2}^{-1} \sum_{c<c'} Y_c^{\mathsf{T}} Y_{c'}. \tag{B1}$$

Given the basis, distinct terms are uncorrelated. Their conditional variances sum to $4\bar{x}_U^{\mathsf{T}} C_U \bar{x}_U/m + 2\operatorname{Tr} {C_U}^2/[m(m-1)]$. Averaging and adding the variance across bases, $V_{\mathrm{b}} = \operatorname{Var}_U \theta_U$, gives

$$\operatorname{Var} s = \operatorname{Var}_U \theta_U + \frac{4}{m} \mathbb{E}(\bar{x}_U^{\mathsf{T}} C_U \bar{x}_U) + \frac{2\mathbb{E} \operatorname{Tr} C_U^2}{m(m-1)}. \tag{B2}$$

Write $a = u^\dagger \gamma u$, $A = \gamma - kI/n$, $d_u = u^\dagger A u$, $t_2 = \operatorname{Tr} A^2$, $t_4 = \operatorname{Tr} A^4$ and Γ_u for the contraction of the unnormalized two-body density with orbital u. Then $0 \le \Gamma_u \le aI$, $\Gamma_u u = 0$ and $\operatorname{Tr} \Gamma_u = (k-1)a$ by counting the other particles. For u uniform and v uniform in $u^\perp$, the diagonal and off-diagonal occupation moments are a and $v^\dagger \Gamma_u v$. The symmetric-power moments of u give $\mathbb{E} d_u^2 = t_2/n^{\overline{2}}$, $\mathbb{E} d_u^4 = (3t_2^2 + 6t_4)/n^{\overline{4}}$ and $\mathbb{E}[d_u^2 u^\dagger A^2 u] = (t_2^2 + 2t_4)/n^{\overline{3}}$, with rising factorials. The compression of A to $u^\perp$ has trace $-d_u$ and squared Frobenius norm $t_2 - 2u^\dagger A^2 u + d_u^2$. Evaluating the conditional second moment and subtracting the squared mean gives

$$V_{\mathrm{b}} = \frac{t_2^2}{n(n+1)^2} + \frac{t_2^2 + 2t_4}{(n+1)(n+2)(n+3)}.$$

Set $\mu = \mathbb{E}\theta_U = (k^2 + \operatorname{Tr} \gamma^2)/(n+1)$, $\vartheta = \mathbb{E}\theta_U^2$, $H = \mathbb{E}\bar{x}_U^{\mathsf{T}} M_U \bar{x}_U$ and $K = \mathbb{E} \operatorname{Tr} M_U^2$. The conditional second moment of $v^\dagger \Gamma_u v$ gives $K = [n + (k-1)^2]\mathbb{E}a^2 + \mathbb{E} \operatorname{Tr} \Gamma_u^2$, with $\mathbb{E}a^2 = \mu/n$. For the first uniform bound, $0 \le \gamma \le I$ implies $t_4 \le t_2 \le k$, so $(n+1)^2 V_{\mathrm{b}} \le (2{t_2}^2 + 2t_4)/n \le 4k^2/n$. For the second let $b_U = \bar{x}_U - (k/n)\mathbf{1}$. Particle-number conservation gives $C_U \mathbf{1} = 0$ and $\bar{x}_U^{\mathsf{T}} C_U \bar{x}_U = {b_U}^{\mathsf{T}} C_U b_U \le {b_U}^{\mathsf{T}} M_U b_U$. Integrating first over the second row gives

$$\mathbb{E} {b_U}^{\mathsf{T}} M_U b_U = (n-k+1)\mathbb{E}[a d_u^2] + \mathbb{E}[d_u \operatorname{Tr}(\Gamma_u A)]. \tag{B3}$$

Indeed the two compressed traces for its off-diagonal term are $-d_u$ and $(k-1)a$. The spherical third moment gives $n(n+1)\mathbb{E}[a d_u^2] = (k/n)t_2 + 2\operatorname{Tr} A^3/(n+2)$. Write $\operatorname{Tr}(\Gamma_u A) = u^\dagger H_A u$. The bound $|\operatorname{Tr}(\Gamma_u A)| \le (k-1)a$ yields $-(k-1)\gamma \le H_A \le (k-1)\gamma$. The last expectation multiplied by $n(n+1)$ is $\operatorname{Tr}(AH_A)$, bounded above in the natural basis by $(k-1)\sum_i |\lambda_i - k/n| \lambda_i \le k(k-1)$. Since $|\operatorname{Tr} A^3| \le t_2$, this gives $\mathbb{E}\bar{x}_U^{\mathsf{T}} C_U \bar{x}_U \le k(2k+1)/[n(n+1)]$.

For the third bound, $\operatorname{Tr} \Gamma_u^2 \le (k-1)a^2$, and $H \ge \vartheta \ge \mu^2$. Hence

$$\begin{aligned} \mathbb{E} \operatorname{Tr} {C_U}^2 &\le K - \mu^2 && \text{(B4)} \\ &\le \mu[1 + k(k-1)/n - \mu] && \text{(B5)} \\ &\le k(k+1)/(n+1). && \text{(B6)} \end{aligned}$$

The last inequality uses $k(k-1)/n \le k^2/(n+1)$ and $\operatorname{Tr} \gamma^2 \le k$. For $n \ge 4$, $2 \le m \le n$ these three terms give

$$\operatorname{Var} \widehat{\Delta} \le \frac{4k^2}{n} + \frac{16k^2}{m} + \frac{4k^2(n+1)}{m(m-1)} \le \frac{32k^2 n}{m^2}. \tag{B7}$$

Here $m - 1 \ge m/2$ and $n + 1 \le 5n/4$. For $n = 2, 3$ one has $k = 1$ and $0 \le s \le 1$, which gives the same final variance bound. Set $z = k\sqrt{n}/\varepsilon$ and choose

$$m = \min(n, \lceil \sqrt{768}\, z \rceil), \qquad B = \lceil 768 z^2/m^2 \rceil. \tag{B8}$$

Independent batches then have average variance at most $\varepsilon^2/24$. The second-moment bound at distance $\varepsilon/2$ gives per-side error at most $1/6$. If $m < n$, one batch suffices. If $m = n$, then $Bm \le n + 768k^2/\varepsilon^2$ and $n \le 1 + \sqrt{768}\, k\sqrt{n}/\varepsilon$. This proves Proposition 1.

Appendix C: Prediction and affinity for a coherent register

Let u be a Haar-random unit vector, $P_u = |u\rangle\langle u|$ and $\omega = I/D$. Under the pure hypothesis every copy is in the state P_u. We follow the prediction, transport and affinity method of He [9].

Let $\mathrm{TV}_T(D, r)$ be the largest total variation distance between the distributions of the two classical records, including a final measurement of the register, over all tests with T copies and r levels. Write du for normalized sphere measure. For positive operators use squared Bures distance $B^2(X, Y) = \operatorname{Tr}(X + Y) - 2\|\sqrt{X}\sqrt{Y}\|_1$ and Petz affinity $a(X, Y) = \operatorname{Tr}(\sqrt{X}\sqrt{Y})$. The squared distance is convex, homogeneous and additive on classical blocks. Affinity is monotone under a common channel.

Lemma C1 (Uniform prediction, including singular memory). For any measurable, pointwise normalized memory state $\eta(u)$ on at most r levels, let $\tau = \int \eta\, du$, $R = \int \eta \otimes P_u\, du$ and $S = \tau \otimes \omega$. Then

$$B^2(R, S) \le (r-1)/(D+1). \tag{C1}$$

Indeed, the map $Lf = \int f(u)(P_u - \omega)\, du$ satisfies $LL^\dagger A = [A - \operatorname{Tr}(A) I/D]/[D(D+1)]$ by the second Haar moment. Thus $\|Lf\|_F^2 \le \|f\|_{L^2}^2/[D(D+1)]$. Each vector in $\ker \tau$ has zero quadratic form against η almost everywhere by positivity. Finite dimensionality allows restriction to $\operatorname{supp} \tau$ on one common full-measure set. Diagonalize τ there with eigenvalues $t_i > 0$. For $\Xi = R - S$ apply this inequality entrywise to $f_{iv} = \eta_{iv} - \tau_{iv}$. It gives

$$\begin{aligned} \operatorname{Tr} \Xi S^{-1} \Xi &\le \frac{\sum_{iv} {t_i}^{-1} \int |f_{iv}|^2 du}{D+1} \\ &= \frac{\int \operatorname{Tr}(\tau^{-1}\eta^2) du - 1}{D+1} \le \frac{\operatorname{rank} \tau - 1}{D+1}. \end{aligned}$$

The last step uses $\eta^2 \le \eta$.

For the metric comparison put $\mathcal{J}_S(A) = (SA + AS)/2$ and $Q_S(\Xi) = \operatorname{Tr}[\Xi \mathcal{J}_S^{-1}(\Xi)]$. In an eigenbasis $2/(s_i +$

$s_v) \le (s_i^{-1} + s_v^{-1})/2$, so $Q_S(\Xi) \le \operatorname{Tr} \Xi S^{-1} \Xi$. Along $S_z = (1-z)S + zR$ lift to amplitudes A_z satisfying $A_z A_z^\dagger = S_z$ by solving $A'_z = \mathcal{J}_{S_z}^{-1}(\Xi) A_z/2$ with $A_0 = \sqrt{S}$. Differentiation verifies the lift and $\|A'_z\|_F^2 = Q_{S_z}(\Xi)/4 \le Q_S(\Xi)/[4(1-z)]$ by inverse order of the positive superoperators. Integrating the amplitude length bounds the Bures distance between R and S by $\sqrt{Q_S(\Xi)}$, also when R is singular by the limit $z \uparrow 1$.

Lemma C2 (The posterior after any classical record). After t copies consider a record with positive Haar likelihood density. Its posterior g relative to du satisfies

$$\int g^2 du \le s_t^2/s_{2t} \le \binom{2t}{t},$$
$$\operatorname{Ent}(g) = \int g \log g \, du \le t \log 4, \tag{C2}$$

where $s_t = \binom{D+t-1}{t}$. To see this, regard the whole adaptive instrument up to the record, with the memory traced out, as a positive operator-valued measure (POVM) on t input copies. Compress its positive effect E to $\operatorname{Sym}^t \mathbb{C}^D$. Its Haar likelihood is $\ell = \operatorname{Tr} E/s_t$, and $g(u) = \langle u^{\otimes t}|E|u^{\otimes t}\rangle/\ell$. If Π_s^{sym} projects onto Sym^s, the $2t$ moment gives

$$\int g^2 du = \frac{\operatorname{Tr}[(E \otimes E)\Pi_{2t}^{\mathrm{sym}}]}{\ell^2 s_{2t}} \le \frac{(\operatorname{Tr} E)^2}{\ell^2 s_{2t}}.$$

Use $\Pi_{2t}^{\mathrm{sym}} \le \Pi_t^{\mathrm{sym}} \otimes \Pi_t^{\mathrm{sym}}$. Also $s_t^2/s_{2t} = \binom{2t}{t} D^{\bar{t}}/(D+t)^{\bar{t}} \le 4^t$. Jensen's inequality under $g\,du$ gives the entropy estimate.

Lemma C3 (Continuous records). Use the trace measure of the effective POVM on the classical record, which dominates the record laws for all inputs. Positivity in finite dimension gives positive matrix densities by entrywise Radon–Nikodym differentiation. Compress the effect density as in Lemma C2. The instrument has a state density whose trace gives the likelihood, so division gives a measurable conditional encoding, extended arbitrarily at zero likelihood. Fubini now permits integration over records. For two experiments the sum of their record measures dominates both.

Lemma C4 (Transporting a posterior with its encoding). For a normalized posterior g and pointwise normalized encoding $\eta(u)$ put $R_g = \int g\eta \otimes P_u \, du$ and $\tau_g = \int g\eta \, du$. On the real sphere S^{2D-1} the geodesic transport inequality $W_2^2(g\,du, du) \le \operatorname{Ent}(g)/(D-1)$ follows from its Ricci curvature $2(D-1)$ (He, Eq. G14). Couple a posterior U and uniform V at this cost. With the encoding $\eta(U)$ fixed, first replace P_U by P_V in the mean state. Joint convexity and $B^2(P_U, P_V) \le \|U - V\|^2 \le \operatorname{dist}(U,V)^2$ bound the squared change by the transport cost. Conditional on V the mean encoding is normalized and has the same marginal τ_g. Lemma C1 applies to this uniform V. By the triangle inequality and $(a+b)^2 \le 2a^2 + 2b^2$, the state before the next copy obeys

$$B^2(R_g, \tau_g \otimes \omega) \le b_t, \tag{C3}$$
$$b_t = \frac{2(r-1)}{D+1} + \frac{2t \log 4}{D-1} \qquad (t \ge 1).$$

This is the coupling of He's Lemma 9. Initially the new copy has marginal ω under both labels, so take $b_0 = 0$. For the maximally mixed source the prediction error is zero on each copy. Integrating true record weights preserves the bound.

Lemma C5 (Affinity loss on unnormalized branches). Suppose $R_i \ge 0$ on memory and fresh input have memory marginals τ_i, $i = 0, 1$. They are subnormalized, with label-dependent weights. Assume $\tau_i \le \kappa \tau_{1-i}$ and set $S_i = \tau_i \otimes \omega$. Then

$$a(\tau_0, \tau_1) - a(R_0, R_1) \le (1 + \sqrt{\kappa}) \sum_{i=0}^{1} B^2(R_i, S_i). \tag{C4}$$

If one marginal is zero, both branches vanish. Otherwise restrict to their common support. Put $\chi_i = \sqrt{S_i}$ and $\delta_i = \sqrt{R_i} - \chi_i$. The common marginal of each pair gives $\operatorname{Tr}_{\mathrm{fresh}}(\{\chi_i, \delta_i\} + {\delta_i}^2) = 0$. Solve $\{G, \sqrt{\tau_1}\} = 2\sqrt{\tau_0}$ there by

$$G = 2 \int_0^\infty e^{-s\sqrt{\tau_1}} \sqrt{\tau_0}\, e^{-s\sqrt{\tau_1}} ds.$$

Square-root order gives $0 \le G \le \sqrt{\kappa} I$. Multiplying the marginal identity by G gives $\operatorname{Tr}(\chi_0 \delta_1) = -\operatorname{Tr}[(G \otimes I){\delta_1}^2]/2$, and exchanging indices deals with $\operatorname{Tr}(\chi_1 \delta_0)$. Also $|\operatorname{Tr}(\delta_0 \delta_1)| \le (\|\delta_0\|_F^2 + \|\delta_1\|_F^2)/2$. For R, S of the same mass w, write $f = \|\sqrt{R}\sqrt{S}\|_1$. Polar decomposition and Hilbert–Schmidt Cauchy–Schwarz give $f^2 \le w\, a(R,S)$. Factor the optimizing trace into $R^{1/4} S^{1/4}$ and $S^{1/4} U R^{1/4}$, whose squared norms are $a(R,S)$ and at most w. As $f \le w$, this implies $\|\sqrt{R} - \sqrt{S}\|_F^2 \le 2B^2(R,S)$. Expanding the affinity and combining the three estimates proves Eq. (C4).

Lemma C6 (Accumulating prediction costs). Mix the two hypotheses for the whole experiment with probabilities $(3/4, 1/4)$ and $(1/4, 3/4)$, respectively. The conditional operations are common channels controlled by the same classical record. For every record the two unnormalized mixed marginals are within factor 3 in operator order. Their total squared prediction error, integrated over records, is at most b_t by convexity and homogeneity. Apply Lemma C5 before each common channel and use monotonicity of the affinity under the instrument including its outcome. The initial affinity is one, hence after the final measurement $1 - a \le (1 + \sqrt{3}) \sum_{t<T} b_t$. The total variation distance of the mixed label distributions is half the original one. Classical $\mathrm{TV}^2 \le 2(1-a)$ therefore yields

$$\mathrm{TV}_T(D, r)^2 \le 8(1 + \sqrt{3}) \sum_{t=1}^{T-1} b_t \le C\, \frac{(r-1)T + T^2}{D} \tag{C5}$$

with an absolute C and any T, by $D-1 \geq D/2$. Errors of at most $1/3$ on each side require total variation at least $1/3$, which proves the lower bound for purity.

Lemma C7 (Fidelity-preserving embeddings). Adding $k-1$ fixed occupied orbitals to one active orbital in dimension $D = n-k+1$ defines an isometry V. For any density matrix σ on the active space, the fidelity $1-e$ of $V\sigma V^\dagger$ is $\lambda_{\max}(\sigma)$. Compression of a determinant has norm at most one, and adding the occupied spectator orbitals to a top eigenvector attains equality. For $k \geq 2$, use $k-2$ filled spectators and map $|i\rangle|u\rangle$ to $a_i \wedge b_u$ in two orthogonal spaces of d orbitals. For coefficient matrix C, the active skew matrix has off-diagonal blocks $C, -C^{\mathsf{T}}$. Its operator norm bounds the overlap with every normalized two-particle determinant, and a leading Schmidt pair attains the bound. Contraction against a filled spectator leaves a scalar multiple of a determinant, since a basis for its occupied subspace can be chosen with all but one vector in the contracting covector's kernel. Projecting off an empty spectator likewise leaves a scalar multiple of a determinant and cannot increase its norm. Thus the largest squared overlap with a determinant is $\lambda_{\max}(CC^\dagger)$. These isometries act separately on each copy and preserve both the classical record and r. For every $k \leq n/2$, $D = n-k+1 > n/2$ and $d = \lfloor (n-k+2)/2 \rfloor \geq n/4$. Thus the large-dimension requirement below depends only on ε, uniformly over k.

Lemma C8 (Replacements and distance). For two independent D-dimensional Haar factors, fix one and supply it locally on each input. Replacing the other repeated factor by independent maximally mixed copies costs at most $\mathrm{TV}_T(D, r)$ in final total variation. Average the fixed factor and then replace it at the same cost. The global repeated Haar vector costs $\mathrm{TV}_T(D^2, r)$ against the same maximally mixed source. Its largest squared Schmidt coefficient ζ satisfies $\zeta^2 \leq \operatorname{Tr} \rho_A^2$ and $\mathbb{E} \operatorname{Tr} \rho_A^2 = 2D/(D^2+1)$ by the second Haar moment contracted with the local swap. Thus $\Pr(\zeta > 1-\varepsilon) \leq 2D/[(D^2+1)(1-\varepsilon)^2]$. Errors of at most $1/3$ on each side imply a gap at least $1/3 - o(1)$ against the product ensemble. Equation (C5) precludes that gap below a sufficiently small constant times the claimed scale.

Appendix D: A lower bound from the half-filled sector

In the half-filled estimates below $k \geq 2$, $\mathcal{H} = \bigwedge^k \mathbb{C}^{2k}$, $\mathcal{N} = \binom{2k}{k}$ and Q_J projects onto layer pseudospin J at $J_z = 0$. The orbital representation at $J_z = m_z \geq 0$ is $\bigwedge^{k+m_z} \otimes \bigwedge^{k-m_z}$. The exterior Pieri rule lists once each diagram $(2^{k-J}, 1^{2J})$ for $m_z \leq J \leq k$. Subtracting adjacent weights isolates pseudospin J, of orbital multiplicity

$$d_J = \binom{2k}{k+J}^2 - \binom{2k}{k+J+1}^2. \tag{D1}$$

Out-of-range binomials are zero. The state $\phi \otimes \phi$ for a coordinate determinant ϕ is a singlet and its orbital orbit spans the irreducible range of Q_0. The swap S acts as $(-1)^J$ at weight zero. Indeed a layer rotation by π contributes the zero-weight rotation matrix element $(-1)^J$, and the layer phase $(-1)^k$ cancels the exterior ordering sign $(-1)^{k^2}$.

1. One pair with a general first measurement

Refine the classical outcome of the first instrument by its Kraus index. For a resulting operator K let $\Gamma = K^\dagger K$. Its rank is at most r, and the sum (or integral) of these effects is the identity. The pulled-back effect M of an outcome impossible on every determinant satisfies $0 \leq M \leq \Gamma \otimes I$ and $MQ_0 = 0$ by positivity. For $k \geq 32768$ we prove

$$\operatorname{Tr} M \leq \min\{1, 1225(1+\ln r)/k\}\, \mathcal{N} \operatorname{Tr} \Gamma. \tag{D2}$$

Lemma D1 (Weighted trace). If $0 \leq M \leq W$ and H is Hermitian with $\operatorname{Tr} HM = \operatorname{Tr} M$, then $\operatorname{Tr} M \leq \|W^{1/4} H W^{1/4}\|_F^2$. On the support write $M = W^{1/2} E W^{1/2}$, $0 \leq E \leq I$. In an eigenbasis of W with eigenvalues v_a, $\sum_{ab} \sqrt{v_a v_b} |E_{ab}|^2 \leq \operatorname{Tr} WE^2 \leq \operatorname{Tr} M$ by the arithmetic–geometric mean inequality. Hilbert–Schmidt Cauchy–Schwarz applied to $W^{1/4} H W^{1/4}$ and $W^{1/4} E W^{1/4}$ proves the claim, including zero effects.

Lemma D2 (Orbital average). Set $H = I - (1 + k/2)Q_0$, $W = \Gamma \otimes I$ and $X = \sqrt{\Gamma}$. The trace hypothesis now follows from $MQ_0 = 0$. Define the self-adjoint map

$$\mathcal{T}(Y) = \mathcal{N}^{-1} \operatorname{Tr}_2[H(Y \otimes I)H].$$

Cyclicity in the preceding squared norm bounds $\operatorname{Tr} M/(\mathcal{N} \operatorname{Tr} \Gamma)$ by $\langle X, \mathcal{T}(X) \rangle_{\mathrm{HS}} / \|X\|_F^2$ for $\Gamma \neq 0$.

Lemma D3 (Johnson spectrum). Vectorize operators in $\mathcal{H} \otimes \overline{\mathcal{H}}$ and send $\overline{|A\rangle}$ to its signed orbital complement $|A^c\rangle$. Exterior pairing intertwines orbital conjugation up to a determinant character. Thus covariance and multiplicity one make $\mathcal{T}$ a scalar Λ_{k-J} on each pseudospin component. Diagonal operators map to the block with one particle per orbital. In interleaved layer order the complement signs cancel the ordering signs up to a common phase. Here $J_- J_+ = kI + A_1$, with A_1 the adjacency at distance one between k-subsets. All pseudospins occur on this block.

For $\ell = |A \setminus A'|$, the normalized distance-ℓ adjacency has eigenvalue

$$\phi_i(\ell) = \binom{k}{\ell}^{-2} \sum_v (-1)^{\ell - v} \binom{i}{\ell - v} \binom{k-i}{v}^2 \tag{D3}$$

at Johnson level i. To see this, evaluate the adjacency on the product of i disjoint differences of orbital indicator functions at a set containing the first member of each pair. For a nonzero contribution $\ell - v$ pairs exchange

members, while v occupied and v empty unpaired sites change. This alternating function generates the level $(2k-i,i)$ by permuting the sites (the two-row Young symmetrizer). At $\ell=1$, $k+k^2\phi_i(1)=(k-i)(k-i+1)$, identifying $i=k-J$.

On configurations (A,B) with $2v$ singly occupied orbitals, the singlet projection between two choices, at distance ℓ, of the v singly occupied orbitals lying in the storage layer is $(-1)^\ell\int_0^1 z^{v-\ell}(1-z)^\ell\,dz=(-1)^\ell/[(v+1)\binom{v}{\ell}]$ in interleaved order. This is the common $SU(2)$ rotation average, with z the squared diagonal amplitude, uniform on $[0,1]$. For given A,A' there are $\binom{k-\ell}{v-\ell}^2$ possible second configurations at distance v from A; the second output is then fixed by total orbital occupations. Squaring the matrix entries gives a diagonal contribution δ_0 and a multiple β_ℓ of each normalized distance adjacency. Explicitly, with $\Pr(V=v)=\binom{k}{v}^2/\mathcal{N}$,

$$\delta_0=\mathbb{E}\Big(1-\frac{1+k/2}{V+1}\Big)^2,$$
$$\beta_\ell=\mathbb{E}\Big[\Big(\frac{1+k/2}{V+1}\Big)^2\mathbf{1}_{V\ge\ell}\Big],\qquad \Lambda_i=\delta_0+\sum_{\ell=1}^{k}\beta_\ell\phi_i(\ell). \tag{D4}$$

Indeed $\binom{k-\ell}{v-\ell}/\binom{v}{\ell}=\binom{k}{v}/\binom{k}{\ell}$. Orbital phases force diagonal inputs to have diagonal output, so this computation recovers the scalar on every component.

Lemma D4 (Spin tail at bounded rank). For a product of two fixed-number states write $J_x=(J_+ + J_-)/2=\frac14\sum_{a=1}^{4k}L_a$. In Majorana coordinates each hopping term contributes two signed bilinears ic_ac_b pairing disjoint Majoranas, one in each layer. These commuting Hermitian involutions have the form $L_a=A_a\otimes b_a$ with the storage-layer parity string absorbed in A_a and distinct fresh-layer Majoranas b_a. Clifford reversion on that layer reverses products and fixes the generators. It is unitarily conjugate to transpose because the transposed generators furnish an equivalent irreducible matrix representation. Its dual therefore sends a state to a state. For $\tau=\tanh(u/4)$, expansion of $\prod_a(I+\tau L_a)$ involves only even lengths in expectation by parity. Reversing the second state and replacing τ by $i\tau$ preserves each such term, since $i^\ell(-1)^{\ell(\ell-1)/2}=1$ for even ℓ. Each transformed factor has norm $\sqrt{1+\tau^2}$. Restoring the hyperbolic cosines gives $\langle e^{uJ_x}\rangle\le\cosh(u/2)^{2k}\le e^{ku^2/4}$. For Schmidt rank at most r, apply the triangle inequality to $e^{uJ_x/2}$ acting on the Schmidt expansion, whose coefficients sum to at most $\sqrt{r}$. Chernoff's inequality gives $\Pr(|J_x|\ge y)\le 2re^{-y^2/k}$. At weight zero within pseudospin $J>0$, $\mathbb{E}J_x^2=J(J+1)/2$ and $|J_x|\le J$, so $\Pr(|J_x|\ge J/2\mid J)\ge 1/3$. For every real $s>0$, conditioning on integer pseudospins $J\ge s$ gives the event $|J_x|\ge s/2$ with conditional probability at least $1/3$. The bound at zero is trivial. Consequently the vector defined in Lemma D3 for $X/\|X\|_F$ satisfies

$$\Pr(J\ge s)\le 6r\,e^{-s^2/(4k)}. \tag{D5}$$

Lemma D5 (Low spins). For integers $0\le s\le k/8$ and $k\ge 16$,

$$\sum_{\ell=1}^{k-1}\max_{0\le u\le s}|\phi_{k-u}(\ell)|\le 20(k^{-1}+s^2/k^2). \tag{D6}$$

For $2\le\ell\le k/2$, factor $\binom{k-u}{\ell-v}\binom{u}{v}/\binom{k}{\ell}$ out of the summand in absolute value in Eq. (D3). These factors sum to one. The other factor is at most $\binom{u+\ell}{\ell}/\binom{k}{\ell}\le[(s+\ell)/k]^\ell$. The last power is at most $2(s^2+\ell^2)(5/8)^{\ell-2}/k^2$. Summing costs at most $(6s^2+100)/k^2$, as the geometric sums with numerator 1 and ℓ^2 are below 3 and 50. Distance one contributes at most $1/k+s^2/k^2$ by the exact eigenvalue $(u^2+u-k)/k^2$. Complementation acts on the indicator differences by $(-1)^i$, hence $|\phi_i(k-\ell)|=|\phi_i(\ell)|$. Doubling the first half gives Eq. (D6); distance k is accounted for separately.

Lemma D6 (Spin-by-spin bound). Suppose $k\ge 32768$. The hypergeometric variable V has mean $k/2$, variance $k^2/[4(2k-1)]\le k/4$, and lower tail $\Pr(V<k/4)\le e^{-k/8}$. Indeed the moment generating function of k draws without replacement from the balanced signs $\pm1/2$ is at most the corresponding power of their mean exponential, by the elementary symmetric mean inequality. Thus it is bounded by $e^{ku^2/8}$ and Chernoff gives the stated tail. For $V\ge k/4$ the squared diagonal term is $(V-k/2)^2/(V+1)^2$ and $(1+k/2)/(V+1)\le 2$. The exceptional contribution is bounded by $k^2e^{-k/8}\le 1/k$. It follows that $\delta_0\le 5/k$ and $\beta_\ell\le 5$. At distance k only $V=k$ contributes, with probability $1/\mathcal{N}$, hence $\beta_k\le 1/\mathcal{N}$. Since the normalized adjacencies have operator norm one, $|\Lambda_i|\le 6k$. Applying Eq. (D6) with threshold J itself yields, for $J\le k/8$,

$$|\Lambda_{k-J}|\le 105/k+100J^2/k^2+1/\mathcal{N}.$$

Put $t_0=4k\ln(6r)$. Integrating Eq. (D5) for $t=J^2$ above t_0 gives $\mathbb{E}J^2\le t_0+24kre^{-t_0/(4k)}=4k[1+\ln(6r)]$. For $r\le e^{k/512}$ the pseudospins above $k/8$ contribute at most $36kre^{-k/256}\le 1/k$, since $36k^2e^{-k/512}\le 1$ on this range of k. Also $\mathcal{N}\ge 2^k$. Consequently the full quadratic form is at most $[107+400(1+\ln(6r))]/k\le 1225(1+\ln r)/k$. For larger r the bound $\operatorname{Tr}M\le\mathcal{N}\operatorname{Tr}\Gamma$ suffices. This proves Eq. (D2).

2. Classical records and the common Haar input

The second Haar moment is $(I+\mathsf{S})/[\mathcal{N}(\mathcal{N}+1)]$. Only the compression to the tensor square of the range of Γ enters $\operatorname{Tr}(M\mathsf{S})$, which is bounded above by the trace of that compression, at most $r\operatorname{Tr}\Gamma$. Summing first effects gives a bound on the probability of an outcome of probability zero for every determinant on a pair in a newly drawn typical state,

$$p_*=\min\Big\{1,\frac{\mathcal{N}\min\{1,1225(1+\ln r)/k\}+r}{\mathcal{N}+1}\Big\}. \tag{D7}$$

For a fixed record all conditional choices are fixed. Pair likelihoods are nonnegative real-analytic functions on the connected Grassmannian. By the domination and Tonelli argument in Appendix E, almost every record that ends in rejection contains an outcome of probability zero for every determinant. Bound the chance of its first occurrence.

Suppose a copies have been measured, and let E be the positive effect of a record of nonzero likelihood, compressed to $\mathrm{Sym}^a \mathcal{H}$. For the next two copies use an effect F compressed to $\mathrm{Sym}^2 \mathcal{H}$. Write $E^{[b]}, F^{[b]}$ for their unnormalized b-copy marginals. The conditional probability is $2/[(\mathcal{N}+a)(\mathcal{N}+a+1)\operatorname{Tr} E]$ times

$$\operatorname{Tr} E \operatorname{Tr} F + 2a \operatorname{Tr}(E^{[1]}F^{[1]}) + \binom{a}{2} \operatorname{Tr}(E^{[2]}F). \tag{D8}$$

Terms with $b > a$ vanish. Indeed the Haar moment of order $a+2$ is the symmetric projector divided by $\binom{\mathcal{N}+a+1}{a+2}$. In its permutation expansion the two new images select zero, one or two old slots with respective multiplicities $1, 2a, \binom{a}{2}$ among $\binom{a+2}{2}$ choices. Separate symmetries within both groups reduce the traces to the displayed marginals. Division by $\operatorname{Tr} E/\binom{\mathcal{N}+a-1}{a}$ gives the formula. The first term is the law of a pair in a newly drawn typical state times $w_a = \mathcal{N}(\mathcal{N}+1)/[(\mathcal{N}+a)(\mathcal{N}+a+1)]$. The other two terms are positive functionals on all positive F and have total mass $1-w_a$ by setting $F = I_{\mathrm{sym}}$. For every record replacing the predictive law by the law of a pair in a newly drawn typical state costs at most $1 - w_a \le 2a/\mathcal{N}$ in total variation. Summing conditional probabilities for $a = 0, 2, \ldots, 2R-2$ gives

$$\Pr(\text{reject}) \le Rp_* + 2R(R-1)/\mathcal{N}. \tag{D9}$$

For fixed $f = 1-\varepsilon > 0$, cover $U(2k)$ in operator norm by an η-net of size at most $(1+2/\eta)^{8k^2}$ by volume packing in the ambient real matrix space. Acting on a determinant is k-Lipschitz by telescoping the exterior power. With $\eta = \sqrt{f}/(2k)$, a squared overlap of at least f with a determinant gives a squared overlap of at least $f/4$ with a determinant of the net. A fixed coordinate of a complex Haar vector has squared-modulus tail $(1-z)^{\mathcal{N}-1}$, so

$$\Pr(e \le \varepsilon) \le (1 + 4k/\sqrt{f})^{8k^2} (1 - f/4)^{\mathcal{N}-1} = o(1).$$

Rejection with probability at least $2/3$ on every input with $e \ge \varepsilon$ implies mean rejection at least $2/3 - o(1)$. In the small-rank range $r/\mathcal{N} \le 1/k$, hence $p_* \le 1226(1 + \ln r)/k$. If $R < k/[10^4(1+\ln r)]$, Eq. (D9) is bounded by $0.1226 + 2k^2/2^k < 1/2$. For sufficiently large k (depending on ε), the required average exceeds $1/2$. For $\ln r > k/512$ the right-hand side of Eq. (10) with $c = 10^{-4}$ is below one, and $R \ge 1$. We may therefore take $c = 10^{-4}$. The isometric restriction in Sec. V extends the worst-case bound to $n \ge 2k$.

Lemma D7 (Concentration at any filling). For the full n-orbital sector put $\mathcal{N} = \binom{n}{k}$. For $j \le k/2$, $\binom{n-j}{k-j} \ge 2^{k-j}$, so the mean in Theorem 3 is at least $j/[4(k+1)]$ for sufficiently large k. The expectation of a contraction on $\psi^{\otimes 2}$ is 4-Lipschitz by telescoping the two tensor factors and the two sides of the expectation. The sphere log-Sobolev inequality (Ricci curvature $2\mathcal{N}-2$) and the Herbst bound give $\Pr(|p_j - \mathbb{E}p_j| > \mathbb{E}p_j/2) \le 2\exp[-(\mathcal{N}-1)j^2/(1024(k+1)^2)]$, using Lipschitz constant four and deviation at least $j/[8(k+1)]$. A union bound handles all $j \le k/2$ on the same input. On the complement $j/[8(k+1)] \le p_j \le 3j/[2(k+1)] \le 3/4$, and Theorem 2(i) gives $\Theta(k/j)$ pairs for a fixed miss probability. For $k/2 < j \le k$, monotonicity in j gives $p_j \ge p_{\lfloor k/2 \rfloor} \ge \lfloor k/2 \rfloor/[8(k+1)]$ on the same event, so a bounded number of pairs suffices there, and $1 \le k/j < 2$.

Appendix E: Records that end in rejection of a perfectly complete test

Let $\mathcal{H}_k = \bigwedge^k \mathbb{C}^n$. Rejection is decided after finitely many copies. Shared randomness can be included in the record.

Lemma E1 (Refining a record that ends in rejection). In one pair refine the first instrument to a single Kraus matrix $K : \mathcal{H}_k \to \mathbb{C}^r$ and the outcome of the measurement on the second copy to a rank-one vector f on $\mathbb{C}^r \otimes \mathcal{H}_k$. Ignoring the refined indices reproduces the original test. For pure identical copies the probability density is $|\mathcal{P}(x)|^2$, where

$$\mathcal{P}(x) = \sum_{m=1}^{r} L_m(x) M_m(x)$$

with $L_m(x) = \langle m|K|x\rangle$ and $M_m(x) = \langle f|(|m\rangle \otimes |x\rangle)$. These are complex-linear forms in the amplitudes of x. Conversely choose K with rows L_m and $\langle f|$ with block rows M_m. Scale K to a contraction and $|f\rangle\langle f|$ to an effect, and complete both measurements with a complementary outcome. This realizes a positive constant multiple of $|\mathcal{P}(x)|^2$ as a rejection probability.

Along a fixed refined record the coefficients in its successive factors are fixed by the recorded outcomes. Clearing the register makes its probability on a pure input the product of the squared moduli of those polynomials. For continuous records take the trace measures of the Choi matrices of the conditional instruments and the trace measures of the conditional POVMs. Their densities have measurable finite Kraus or spectral refinements, for example by factoring their positive square roots into columns. The resulting product of conditional measures is a control measure, that is, a measure that dominates the record laws for all inputs. Integrate the rejection probability, which is zero under every determinant, over the invariant measure on the complex Grassmannian. Tonelli's theorem gives zero average of the density for almost every record that ends in rejection. A nonzero holomorphic section on the connected Grassmannian has zero set of invariant measure zero in its smooth charts.

Thus on almost every record that ends in rejection at least one factor vanishes identically on the Grassmannian. This covers adaptive choices and outcomes with zero individual point mass.

Lemma E2 (Records for the stored-particle test). The first copy supplies $E_j(D_L)$ for the projector D_L on the determinant with occupied subspace L. The channel traces out the other $k-j$ particles. For any fixed record a positive outcome density F_z of the joint measurement has likelihood $\mathrm{Tr}[F_z(E_j(D_L)\otimes D_L)]$, real analytic in L. The same Tonelli argument, using the measure-zero sets of nonzero real-analytic functions, gives an identically vanishing factor on almost every record that ends in rejection. Outcomes with zero integral over occupied subspaces form a measurable set. Their integrated effect annihilates the support span $\ker J_-$, so is at most Π_j. For any record the probability of this measurable set on the independent next pair is at most $p_j = \mathrm{Tr}[\Pi_j(E_j(\rho)\otimes\rho)]$. Let A_t mean that no such outcome has occurred in the first t pairs. Integrating the conditional bound over records in A_{t-1} gives $\Pr(A_t) \geq (1-p_j)\Pr(A_{t-1})$, hence $\Pr(A_R) \geq (1-p_j)^R$. Almost every record that ends in rejection lies outside A_R, so the rejection probability is at most $1-(1-p_j)^R$. Repeating Π_j attains equality by independence.

Appendix F: Pure Gaussian targets

The matrix G is real and antisymmetric, with singular values in pairs $0 \leq \nu_i \leq 1$. Choosing the more likely occupancy in each canonical mode gives a Gaussian vector. A union bound gives $e \leq \sum_i (1-\nu_i)/2 \leq \Delta_G/4$ for every physical density operator. Conversely, in the quasiparticles of a closest Gaussian, the excitation count is at most $n(I - |\phi\rangle\langle\phi|)$. The trace inequality then gives $\sum_i \nu_i \geq n - 2ne$ and $\Delta_G \leq 8ne$.

Lemma F1 (The three-particle embedding). Put $d = \lfloor n/3 \rfloor$ and choose three disjoint groups of d modes. The map $V|i,j,l\rangle = a^\dagger_{1,i} a^\dagger_{2,j} a^\dagger_{3,l}|0\rangle$ is isometric by orthogonality of the occupation strings. It sends every product to a determinant of three orbitals, one in each group. For a fixed unit Gaussian vector ϕ, $\|V^\dagger\phi\| \leq 1$. A Haar vector in the domain therefore satisfies $\Pr(|\langle\phi, V\Psi\rangle|^2 \geq f) \leq (1-f)^{d^3-1}$ for $0 < f < 1$.

Start from the vacuum and a fixed one-particle determinant, the two fiducial vectors for the two $SO(2n)$ orbits. Plane elimination represents every special orthogonal matrix by a fixed ordered list of at most $N_{\mathrm{rot}} = n(2n-1)$ Givens rotations with angles in $[-\pi, \pi]$ (choose positive successive pivots). A lifted rotation is $\exp(\theta c_a c_b/2)$ and changes by at most $|\Delta\theta|/2$ in operator norm. A global phase absorbs the lift sign. Telescoping gives Lipschitz constant $N_{\mathrm{rot}}/2+1$ in the supremum norm on the angles and phase. A grid of spacing $\eta/(N_{\mathrm{rot}}/2+1)$ gives a vector-norm η-net with at most $2[2+2\pi(N_{\mathrm{rot}}/2+1)/\eta]^{N_{\mathrm{rot}}+1}$ points. If the Gaussian fidelity is at least f, some net amplitude has modulus at least $\sqrt{f}/2$ for $\eta = \sqrt{f}/2$. The union bound costs the net cardinality times $(1-f/4)^{d^3-1}$, which tends to zero for every fixed $f = 1-\varepsilon > 0$.

Replace the three independently drawn and repeatedly supplied factors one at a time by maximally mixed inputs, conditioning on the other draws as in Appendix C. A fourth replacement compares the global repeated Haar vector with the same maximally mixed source. The final distance is at most $3\,\mathrm{TV}_T(d,r) + \mathrm{TV}_T(d^3,r)$. A successful test needs a gap $1/3 - o(1)$, proving the pure-input lower bound by Eq. (C5). For mixed alternatives the one-particle sector supplies all pure rays as Gaussian determinants and its maximally mixed state has fidelity at most $1/n$ to every pure target. Apply the same single-source estimate in dimension n.

For the comparison in Sec. VI, take the net at trace-distance scale τ, of log-cardinality $O(n^2 \ln(n/\tau))$. For a pure Gaussian input, hypothesis selection from single-copy measurements [21] returns, with probability at least $11/12$ and from $\widetilde{O}(n^2/\tau^2)$ copies, an element ϕ' of the net within trace distance $O(\tau)$ of the input. Measure $|\phi'\rangle\langle\phi'|$ on $\lceil \ln 12/\varepsilon \rceil$ further copies and accept if no measurement fails. A Gaussian input fails each measurement with probability $O(\tau^2)$, so for τ a sufficiently small multiple of $\sqrt{\varepsilon}$ it is accepted with probability at least $5/6$. An input with $e \geq \varepsilon$ passes each measurement with probability at most $1-\varepsilon$, whichever element was selected, and is accepted with probability at most $1/12$.

Lemma F2 (A half-filled sector). Use $k = \lfloor n/2 \rfloor$ on the first $2k$ modes with the remaining mode, if present, empty. Every determinant is Gaussian. The same net of the full n-mode Gaussian family applies with d^3 replaced by $\binom{2k}{k}$, so a Haar vector in this sector has $e \geq \varepsilon$ with probability $1 - o(1)$. The restricted instrument accepts every determinant and has the same output dimension bound. Its required average rejection is at least $2/3 - o(1)$. Equation (D9) and its constants apply already at average $1/2$, yielding $R = \Omega(n/(1+\ln r))$.

Lemma F3 (Mixed alternatives for the joint test). Let P be the acceptance projector of Ref. [14] and $P_\pm = (I \pm \mathsf{S})/2$. Section 5.1 of that work identifies $\mathrm{ran}\, P$ with the span of $\phi\otimes\phi$ for pure Gaussian ϕ, so $P \leq P_+$. Let λ be the largest eigenvalue of ρ and ψ a corresponding eigenvector of definite parity, with infidelity e_ψ. The P_- component of $I-P$ contributes $(1-\mathrm{Tr}\,\rho^2)/2 \geq (1-\lambda)/2$. On the positive effect $P_+ - P$, the term $\lambda^2(|\psi\rangle\langle\psi|)^{\otimes 2}$ contributes at least $\lambda^2 e_\psi/8$ by Theorem 5.2 of Ref. [14]. The closest Gaussian to ψ gives $e \leq 1-\lambda+\lambda e_\psi$, so the total probability p obeys $8p - e \geq (1-\lambda)(3-\lambda e_\psi) \geq 0$. Independent repetition for $\lceil 8\ln(1/\delta)/\varepsilon \rceil$ pairs gives miss probability at most δ whenever $e \geq \varepsilon$.

Lemma F4 (Majorana measurements). The columns of $W_G/\sqrt{2n}$ are orthonormal since $c_a^2 = I$. Tracing out the Fock space gives register entries $\mathrm{Tr}(c_a\rho c_b)/(2n)$. Measuring $W_G W_G^\dagger/(2n)$ on the register and the next copy gives $(2n)^{-2}\sum_{ab}[\mathrm{Tr}(\rho c_b c_a)]^2$. The di-

agonal contributes $2n$ and each off-diagonal expectation is imaginary with squared modulus $|G_{ab}|^2$. This proves the stated rejection probability and cost. All controlled Majoranas change parity by the same amount.

Lemma F5 (Separate measurements). Put $z_v = (OGO^{\mathsf{T}})_{2v-1,2v}$, $M = \mathbb{E}(XX^{\mathsf{T}} \mid O)$, $C = M - zz^{\mathsf{T}}$, $\theta = z^{\mathsf{T}}z$ and $F_2 = \|G\|_F^2$, $F_4 = \mathrm{Tr}[(G^{\mathsf{T}}G)^2]$. The centered-pair decomposition of Appendix B also holds for sign vectors. For four successive real frame vectors u, v, w, y, write $a = u^{\mathsf{T}}Gv$, $b = w^{\mathsf{T}}Gy$, and $s = 2n$. Spherical moments give $\mathbb{E}a^2 = F_2/[s(s-1)]$, $\mathbb{E}[a^2\|Gu\|^2] = (F_2^2 + 2F_4)/[s(s+2)(s-1)]$ and $\mathbb{E}a^4 = 3\mathbb{E}[a^2\|Gu\|^2]/(s+1)$. The squared norm of G compressed off u, v gives $(s-2)(s-3)\mathbb{E}[b^2 \mid u, v] = F_2 - 2\|Gu\|^2 - 2\|Gv\|^2 + 2a^2$. Subtracting $(n\mathbb{E}a^2)^2$ from $n\mathbb{E}a^4 + n(n-1)\mathbb{E}a^2b^2$ yields

$$\mathrm{Var}\,\theta = \frac{F_2^2}{2(s-3)(s-1)^2} + \frac{(s-8)(F_2^2+2F_4)}{2(s-3)(s-1)(s+1)(s+2)}.$$

For $s \geq 8$ use $F_2, F_4 \leq s$, and for $s = 4, 6$ use $F_4 \geq F_2^2/s$ in the negative term. This gives $4(s-1)^2\,\mathrm{Var}\,\theta \leq 8n$. For $I = (a, b, c, d)$ with $a < b < c < d$, set $Q_I = -\mathrm{Tr}(\rho c_a c_b c_c c_d)$. The mean squared component of an oriented orthonormal four-frame on each coordinate four-plane is $1/\binom{s}{4}$ by sign and permutation symmetry. Within the four-plane the three signed pairings in a Pfaffian have equal averages. Consequently $\mathbb{E}z^{\mathsf{T}}Mz = \mathbb{E}\theta + n(n-1)\sum Q_I\,\mathrm{Pf}(G_I)/(3\binom{s}{4})$ and $\mathbb{E}\,\mathrm{Tr}\,M^2 = n + n(n-1)\sum Q_I^2/\binom{s}{4}$. In a canonical frame the first contraction has absolute value at most $\sum_{i<l}\nu_i\nu_l \leq n(n-1)/2$. Fixing three indices gives anticommuting quartic observables as the fourth varies, so $4\sum Q_I^2 \leq \binom{s}{3}$. Using $C \leq M$ bounds the coefficients of $1/m$ and $1/[m(m-1)]$ in $\mathrm{Var}\,\widehat{\Delta}_G$ by $128n^2$ and $64n^3$, respectively. For $2 \leq m \leq n$ the variance is at most $264n^3/m^2$. Choosing $m = \min(n, \lceil\sqrt{5280}\,n^{3/2}/g\rceil)$ and $B = \lceil 5280n^3/(m^2g^2)\rceil$ independent batches gives error at most $1/6$ at gap $g = 4\varepsilon$, by Cantelli at the midpoint. This proves the asserted order for $n \geq 2$. For $n = 1$ it suffices to measure the occupation of the mode.

[1] F. Arute *et al.*, Hartree-Fock on a superconducting qubit quantum computer, Science **369**, 1084 (2020).

[2] N. Paul, H. Zhao, and D. D. Dai, Learning the closest Slater determinant (2026), arXiv:2607.20623v1.

[3] M. Yaghubi Rad, R. Puig, S. Hassanandani, L. Coffman, J. Eisert, C. Bravo-Prieto, and A. A. Mele, Testing quantum Gaussianity with constant sample complexity (2026), arXiv:2609.40270v1.

[4] V. Iyer, B. O'Gorman, O. Parekh, K. Thompson, and A. Zhao, Optimal testing of fermionic and bosonic Gaussian states (2026), arXiv:2609.39113v1.

[5] M. West and M. Larocca, Fermionic Gaussianity can be tested with mode-independent sample-complexity (2026), arXiv:2610.02050v1.

[6] S. Arunachalam and L. Schatzki, Optimal stabilizer testing and learning with limited quantum memory (2026), arXiv:2607.02444v2.

[7] S. Chen, W. Gong, and Q. Ye, Optimal tradeoffs for estimating Pauli observables, in *2024 IEEE 65th Annual Symposium on Foundations of Computer Science (FOCS)* (IEEE, 2024) pp. 1086–1105, arXiv:2404.19105v2.

[8] W. Gong, J. Haferkamp, Q. Ye, and Z. Zhang, Sample complexity of purity and inner product estimation, Phys. Rev. Research **8**, 043077 (2026).

[9] Z. He, Memory capacity and lifetime thresholds for testing pure Gaussianity, Zenodo, version 1.0 (2026), doi:10.5281/zenodo.23651220.

[10] R. Rubin, Entangled measurements are necessary for optimal tomography of mixed fermionic Gaussian states and of bosonic Gaussian states near the vacuum (2026), arXiv:2609.23189v2.

[11] T. Haug, X. Turkeshi, and P. Sierant, Practical tests and witnesses of fermionic non-Gaussianity (2026), arXiv:2605.26218v3.

[12] P. S. Tarabunga, B. Jobst, R. Morral-Yepes, M. Langer, B. Kraus, F. Pollmann, and S.-H. Lin, Computable fermionic non-Gaussianity from the covariance matrix (2026), arXiv:2607.02242v2.

[13] J. Beckey, L. Coffman, A. Shlosberg, L. Schatzki, and F. Leditzky, Product testing with single-copy measurements (2026), arXiv:2510.07820v2.

[14] J. Briët, D. Castro-Silva, M. Walter, and F. Witteveen, Testing bosonic and fermionic Gaussian states with a constant number of copies (2026), arXiv:2610.06794v2.

[15] C. Cao, Y. Tang, and J. Eisert, The magic cost of fermionic randomness (2026), arXiv:2610.05093v1.

[16] See Supplemental Material at [URL will be inserted by publisher] for exact variances, explicit finite bounds, counting probabilities, further instruments and numerical details, which includes Refs. [25–28].

[17] A. Anshu, Z. Landau, and Y. Liu, Distributed quantum inner product estimation, in *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (STOC 2022)* (ACM, 2022) pp. 44–51, arXiv:2111.03273v2.

[18] P. S. Tarabunga, Fermionic non-Gaussianity via Bell sampling: Monotones and efficient quantum algorithms, PRX Quantum **7**, 033065 (2026), arXiv:2606.05066.

[19] H. Pichler, L. Bonnes, A. J. Daley, A. M. Läuchli, and P. Zoller, Thermal versus entanglement entropy: a measurement protocol for fermionic atoms with a quantum gas microscope, New J. Phys. **15**, 063003 (2013).

[20] R. Islam, R. Ma, P. M. Preiss, M. E. Tai, A. Lukin, M. Rispoli, and M. Greiner, Measuring entanglement entropy in a quantum many-body system, Nature **528**, 77 (2015).

[21] H. Zhao, L. Lewis, I. Kannan, Y. Quek, H.-Y. Huang, and M. C. Caro, Learning quantum states and unitaries of bounded gate complexity, PRX Quantum **5**, 040306 (2024).

[22] L. Bittel, A. A. Mele, J. Eisert, and L. Leone, Opti-

mal trace-distance bounds for free-fermionic states: Testing and improved tomography, PRX Quantum **6**, 030341 (2025).

[23] T. Chalopin, P. Bojović, D. Bourgund, S. Wang, T. Franz, I. Bloch, and T. Hilker, Optical superlattice for engineering Hubbard couplings in quantum simulation, Phys. Rev. Lett. **134**, 053402 (2025).

[24] A. Chen, Data for "Testing Slater determinants with limited quantum memory", Zenodo, version 1.0 (2026), doi:10.5281/zenodo.24469067.

[25] O. Akresh and J. Beckey, Tight bounds for purity and product testing from partial transposition (2026), arXiv:2608.27217v1.

[26] B. Beauzamy, E. Bombieri, P. Enflo, and H. L. Montgomery, Products of polynomials in many variables, J. Number Theory **36**, 219 (1990).

[27] M. Brion, Lectures on the geometry of flag varieties, in *Topics in Cohomological Studies of Algebraic Varieties*, edited by P. Pragacz (Birkhäuser, Basel, 2005) pp. 33–85, arXiv:math/0410240v1.

[28] A. Kasman, K. Pedings, A. Reiszl, and T. Shiota, Universality of rank 6 Plücker relations and Grassmann cone preserving maps, Proc. Amer. Math. Soc. **136**, 77 (2008).

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