Supplemental Material for "Testing Slater determinants with limited quantum memory"

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S1. EXACT VARIANCE OF OCCUPATION COLLISIONS

With the notation of Appendix B, Eq. (B2) gives the variance from three orbital averages. We evaluate their coefficients below.

Let Γ_2 act on the normalized two-particle wedge, with $\operatorname{Tr}\Gamma_2 = \binom{k}{2}$ and $\langle u\wedge v, \Gamma_2(u\wedge v)\rangle = \langle n_u n_v\rangle$ for orthonormal u, v. Define $g_s = \operatorname{Tr}\gamma^s$, $A = \gamma - kI/n$, $t_2 = \operatorname{Tr}A^2$, $t_4 = \operatorname{Tr}A^4$, and let $a = u^\dagger\gamma u$ for a complex Haar vector u. In the following display $\mu = \mathbb{E}\theta_U$, $\vartheta = \mathbb{E}\theta_U^2$, $V_{\rm b} = \operatorname{Var}\theta_U$, $K = \mathbb{E}\operatorname{Tr}{M_U}^2$, and $H = \mathbb{E}\bar{x}_U^{\mathsf{T}} M_U \bar{x}_U$. Then

$$\begin{aligned}
&\mu = (k^2+g_2)/(n+1), \qquad \vartheta = \mu^2 + V_{\rm b},\\
&V_{\rm b} = \frac{{t_2}^2}{n(n+1)^2} + \frac{{t_2}^2+2t_4}{(n+1)(n+2)(n+3)},\\
&\mathbb{E}a^2 = \mu/n, \qquad \mathbb{E}a^3 = \frac{k^3+3kg_2+2g_3}{n(n+1)(n+2)},\\
&K = [n+(k-1)^2]\,\mathbb{E}a^2 + \frac{(k-1)^2 g_2 + 4\operatorname{Tr}{\Gamma_2}^2}{n(n+1)},\\
&H = (n-k+1)\,\mathbb{E}a^3 + k(k-1)\,\mathbb{E}a^2\\
&\qquad + \frac{k(k-1)g_2 + 2\operatorname{Tr}(\Gamma_2\wedge^2\gamma)}{n(n+1)}.
\end{aligned} \tag{S1}$$

For the derivation put $d_u = u^\dagger A u$ and $z = u^\dagger A^2 u$. The Haar identity $\mathbb{E}(uu^\dagger)^{\otimes l} = (\sum_{\pi\in S_l} W_\pi)/n^{\overline{l}}$, where W_π permutes the l tensor factors according to π and $n^{\overline{l}}$ is a rising factorial, follows from the normalized projector onto the symmetric power. It gives the two moments of a above and

$$\mathbb{E}d_u^2 = t_2/n^{\overline{2}}, \qquad \mathbb{E}d_u^4 = (3{t_2}^2+6t_4)/n^{\overline{4}}, \tag{S2}$$

$$\mathbb{E}d_u^2 z = ({t_2}^2+2t_4)/n^{\overline{3}}. \tag{S3}$$

For a second row v uniform in $u^\perp$, compression of A has trace $-d_u$ and square trace $t_2 - 2z + d_u^2$. Its second diagonal moment is $(t_2 - 2z + 2d_u^2)/[(n-1)n]$. For $L = \theta_U - k^2/n$ this gives $\mathbb{E}L = t_2/(n+1)$ and $\mathbb{E}L^2 = (n+2)\mathbb{E}d_u^4 + t_2\mathbb{E}d_u^2 - 2\mathbb{E}d_u^2 z$. Subtraction gives $V_{\rm b}$.

Set $T_u v = u\wedge v$, $\Gamma_u = T_u^\dagger \Gamma_2 T_u$. Then $\Gamma_u u = 0$, $\operatorname{Tr}\Gamma_u = (k-1)a$ and $0 \le \Gamma_u \le aI$. The trace identity counts the other $k-1$ particles whenever u is occupied. The order bound follows from $n_u n_v \le n_u$. A diagonal entry of M_U is a and an off-diagonal one is $v^\dagger\Gamma_u v$. The second Haar moment in $u^\perp$, summed over $n(n-1)$ ordered pairs, gives

$$K = [n+(k-1)^2]\mathbb{E}a^2 + \mathbb{E}\operatorname{Tr}\Gamma_u^2, \tag{S4}$$

$$H = n\mathbb{E}a^3 + \mathbb{E}\{a[(k-1)a(k-a) + \operatorname{Tr}(\Gamma_u\gamma)]\}. \tag{S5}$$

Write $\mathcal{C}_{iv} = T_i^\dagger \Gamma_2 T_v$. In expanding $\Gamma_u = \sum \bar{u}_i u_v \mathcal{C}_{iv}$, the two pairings give $n(n+1)\mathbb{E}\operatorname{Tr}\Gamma_u^2 = (k-1)^2 g_2 + 4\operatorname{Tr}{\Gamma_2}^2$ since $\sum \mathcal{C}_{ii} = (k-1)\gamma$ and $\sum T_i T_i^\dagger = 2I$. For $a\operatorname{Tr}(\Gamma_u\gamma)$ the first pairing gives $k(k-1)g_2$. In natural orbitals the second gives $\sum_{iv}\lambda_i\lambda_v\langle i\wedge v, \Gamma_2(i\wedge v)\rangle = 2\operatorname{Tr}(\Gamma_2\wedge^2\gamma)$. This establishes the closed form for any density operator.

S2. FINITE BOUNDS WITH A CLEARED REGISTER

Here $D = n-k+1$ and $d = \lfloor (n-k+2)/2\rfloor$ are the numbers of active orbitals of the two embeddings.

Proposition S1 (Explicit bounds for a cleared register). A test with a cleared register, r levels and error at most $1/3$ on each side requires

$$\begin{aligned}
&R > \min\{\sqrt{D}/6, D/(4r)\} \qquad \text{(mixed allowed)},\\
&R > \min\{\sqrt{d}/8, d/(25r)\} \qquad \text{(pure)}.
\end{aligned} \tag{S6}$$

The first bound holds for $k \ge 1$, $0 < \varepsilon \le 1 - 1/D$, and the second for $k \ge 2$, $0 < \varepsilon < 1$ and $d \ge \max\{64, 40/(1-\varepsilon)^2\}$. With $r = 1$, adaptive measurements on individual copies also satisfy $T(T-1) \ge (D-1)/(3\varepsilon)$ in the first case.

Lemma S2.1 (Comparing repeated and redrawn Haar sources). Write $\mathcal{S}_l = \sum_{\pi\in S_l} W_\pi$ and $D^{\overline{l}} = D(D+1)\cdots(D+l-1)$, where W_π permutes the l tensor factors according to π. The l-copy Haar moment is $\mathcal{S}_l/D^{\overline{l}}$. For two independent factors there is a permutation sum for each factor. On a particular record of a test with a cleared register the effect is a tensor product of pair effects even when the test is adaptive. To compare the same draw across pairs to independent draws per pair, the unnormalized permutation contraction for two consecutive blocks of positive effects dominates the product of their separate contractions.

To prove the domination, take a rank-one effect from a vector v on l copies of the bipartite system and put $p_v(z,w) = \langle v|(z\otimes w)^{\otimes l}\rangle$, with z, w independent standard complex Gaussians. Wick contraction gives $\|p_v\|^2 = \langle v|\mathcal{S}_l^A \mathcal{S}_l^B|v\rangle$. In the polynomial inner product monomials have squared norm $\beta!$, and tensoring effects multiplies the polynomials. If M_p denotes multiplication by a homogeneous p, the product rule and $(M_{z_i})^\dagger = \partial_i$ give

$$M_p^\dagger M_p = \sum_\beta M_{\partial^\beta p} M_{\partial^\beta p}^\dagger/\beta!.$$

The terms of order $\deg p$ sum to $\|p\|^2 I$ and all remaining terms are positive. Therefore $\|pf\|^2 \ge \|p\|^2\|f\|^2$, the

Fock normalization of the polynomial product inequality [1]. Spectral decomposition extends this to arbitrary positive effects. Dropping w proves its one-system version.

For one factor, restoring the Haar denominators shows that every true record density dominates its redrawn density times

$$C_D = \frac{[D(D+1)]^R}{D^{\overline{2R}}}, \qquad 1 - C_D \le \sum_{i=0}^{R-1} \frac{4i}{D} = \frac{2R(R-1)}{D}.$$

The product source costs two factors of C_d. Pointwise domination $f \ge cg$ for normalized densities implies $\mathrm{TV}(f,g) \le 1-c$. The redrawing costs are thus at most $4R(R-1)/d$ for the product source and $2R(R-1)/d^2$ for the global Haar source.

Lemma S2.2 (Partial swaps in one pair). Let $D = d_A d_B$. Refine the first instrument to Kraus matrices $K_x : \mathbb{C}^D \to \mathbb{C}^r$ with $\sum_x K_x^\dagger K_x = I$. For the second positive operator-valued measure (POVM) $F_{y|x}$ write $M = (K_x^\dagger \otimes I) F_{y|x} (K_x \otimes I)$. For the full and partial swaps define $B(\mathsf{S}) = \sum_M (\operatorname{Tr} M\mathsf{S})^2 / \operatorname{Tr} M$, with zero terms omitted. Refining F to rank one increases this sum by Cauchy–Schwarz. For a fixed K put $A = KK^\dagger \otimes I$, $Q = (K \otimes I)\mathsf{S}(K^\dagger \otimes I)$, and $P = \operatorname{supp}(K^\dagger K)$ as a projector. For a refined vector f,

$$|\langle f|Q|f\rangle|^2 \le \langle f|A|f\rangle \, \langle f|QA^+Q|f\rangle,$$

where A^+ is the pseudoinverse. Summing the POVM gives at most $\operatorname{Tr}[\mathsf{S}(P\otimes I)\mathsf{S}(K^\dagger K \otimes I)]$. For the full swap this is $\operatorname{Tr} P \operatorname{Tr} K^\dagger K$. For S_A it is $d_B \operatorname{Tr}[(\operatorname{Tr}_A P)(\operatorname{Tr}_A K^\dagger K)]$ and $\operatorname{Tr}_A P \le \min(r, d_A) I$. Summing over K proves

$$B(\mathsf{S}_{AB}) \le rD,$$
$$B(\mathsf{S}_A) \le d_B \min(r, d_A) D, \quad B(\mathsf{S}_B) \le d_A \min(r, d_B) D.$$

Lemma S2.3 (Distance between record distributions). Set $d_A = d_B = d$. Under the independent maximally mixed source each conditional outcome has probability $\operatorname{Tr} M/d^4$. For $H \in \{A, B, AB\}$ write $s_H = \operatorname{Tr}(M\mathsf{S}_H)/\operatorname{Tr} M$. The one-pair likelihood ratios for the redrawn product and global sources are $(d/(d+1))^2(1+s_A+s_B+s_{AB})$ and $[d^2/(d^2+1)](1+s_{AB})$. Each has conditional mean one. The swap bounds and the L^2 triangle inequality give conditional variance at most $v_p = (2+d^{-1/2})^2 r/d$ and $v_h = r/d^2$, respectively. In particular the three second moments of the s_H under that source are $B(\mathsf{S}_H)/d^4$. Iterating the conditional second moments bounds the second moment of the product likelihood ratio by $(1+v)^R$. Thus the distance to that source is at most $\frac{1}{2}\sqrt{\exp(Rv)-1}$ in each case. Combining the two redrawing costs with these two distances gives, when $R \le \min(\sqrt{d}/8, d/(25r))$,

$$\mathrm{TV} \le \frac{1}{16} + \frac{1}{32d} + \frac{1}{2}\sqrt{\exp((2+d^{-1/2})^2/25) - 1} + \frac{1}{2}\sqrt{\exp(1/(25d)) - 1} < 0.299.$$

The maximum of this upper expression is attained at $d = 64$. The single-system proof uses the full swap, giving the pure versus I/D distance at most $2R(R-1)/D + \frac{1}{2}\sqrt{\exp(Rr/D) - 1}$. At $R \le \min(\sqrt{D}/6, D/(4r))$ this is at most $1/18 + \frac{1}{2}\sqrt{\exp(1/4) - 1} < 1/3$. Continuous outcomes in these arguments use densities against the trace of the conditional effects. Positivity, conditional expectations and the inequalities above then hold almost everywhere.

Lemma S2.4 (Distance from product states). For a global Haar vector let ζ be its largest squared Schmidt coefficient. Contracting the two-copy Haar moment with the local swap gives $\mathbb{E} \operatorname{Tr} \rho_A^2 = 2d/(d^2+1)$. Since $\zeta^2 \le \operatorname{Tr} \rho_A^2$, $\Pr(\zeta > t) \le 2d/[(d^2+1)t^2]$. For $t = 1-\varepsilon$ the probability π_{near} that the state has $e < \varepsilon$ is at most $1/20$ under the size hypothesis. Under the product source acceptance is at least $2/3$. Under the global source it is at most $\pi_{\mathrm{near}} + (1-\pi_{\mathrm{near}})/3$, so $\mathrm{TV} \ge (1 - 2\pi_{\mathrm{near}})/3 \ge 0.300$. This contradiction gives the pure line of Eq. (S6). The one-particle embedding gives the mixed line directly. At fixed ε all size conditions hold for large n.

Lemma S2.5 (Mixed alternatives with separate measurements). In the one-particle embedding use the pure state P_u and $\sigma = (1-z)P_u + zI/D$, $z = \varepsilon/(1-1/D)$, with u Haar. The far infidelity is exactly ε. Each adaptive effect on one copy and its complement remain positive under partial transposition of any subset of whole copies, since the effect of each record is a tensor product of positive one-copy effects. The telescoping estimate in Proposition 2 of Ref. [2] (Appendix, Eqs. 64–71) bounds the bias by $\mathbb{E}[V(2T-V-1)]/(2D)$ with $V \sim \mathrm{Bin}(T, z)$, which equals $T(T-1)(2z-z^2)/(2D)$. Bias at least $1/3$ is necessary. Using $2z - z^2 \le 2z$ proves the last line of Proposition S1.

For Fig. S1(a) the two states being compared are a determinant and the pure equal superposition of two disjoint two-particle configurations. The latter has $e = 1/2$ and $\Delta = 1$. Separate measurements take 6597 copies by the calibration in Sec. S5. The mean orbital collision counts for this batch are 124.49 and 103.74. A Poisson approximation with rejection at 113 or fewer collisions gives errors 0.162 and 0.169. The block test at 14, 16, 18 and 20 qubits takes 6550, 1638, 410 and 104 copies. The optimal test with one stored particle takes 20 copies, as does the joint test on two full copies. Coordinate blocks, Eq. (S18), take 344, 86 and 20 copies at 18, 19 and 20 qubits and are cheaper than the block test there. The $(k+1)$-level instrument of Proposition S2(iii), here with three levels, takes 11 272 754. Panel (b) uses Proposition S1 with error at most $1/3$ on each side.

S3. PARTICLE EXTRACTION AND COUNTING PROBABILITIES

Let j be the largest number of stored particles, j' the number actually stored, and $h = k - j'$. The initial value

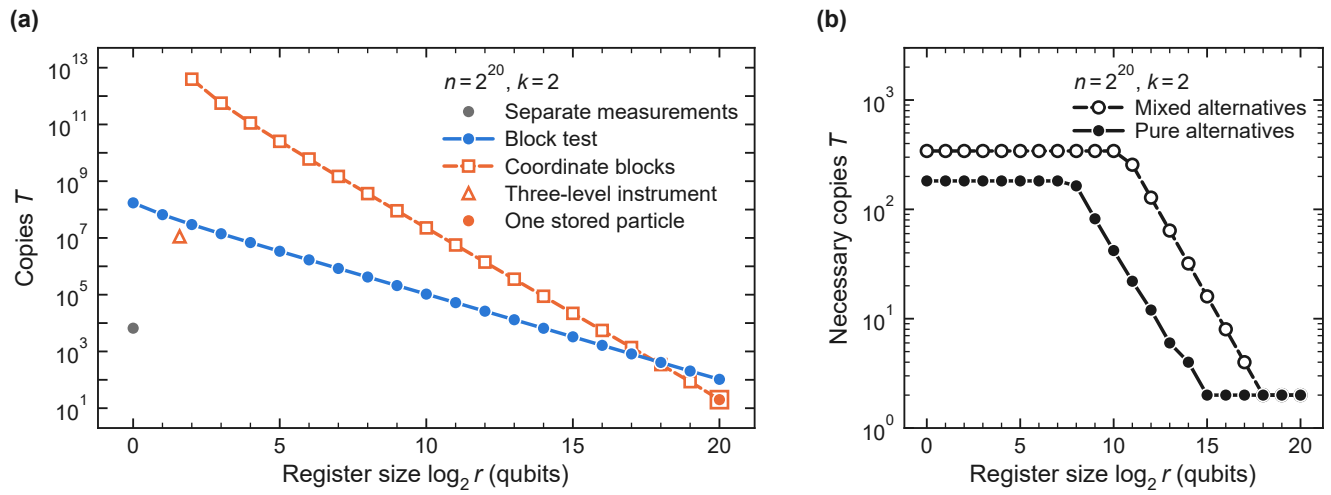

FIG. S1. The example $n = 2^{20}$, $k = 2$ of Fig. 2(a). (a) Copies needed by each test to distinguish a determinant from the pair state $(|12\rangle + |34\rangle)/\sqrt{2}$ at target error 1/6 on each side: separate measurements, which use no register and appear at 0 qubits, the block test, the optimal test with one stored particle, and two perfectly complete tests with a cleared register, namely coordinate blocks, Eq. (S18), and the $(k+1)$-level instrument of Proposition S2(iii), which has three levels here and is placed at $\log_2 3$ qubits. At 20 qubits the coordinate blocks coincide with the optimal test with one stored particle. (b) Copies necessary for every test with a cleared register and error at most 1/3 on each side for all inputs, from the two bounds of Proposition S1, for mixed and for pure alternatives, at $r = 2^q$ with integer q. Lines join the points.

of J_z is $m = -h/2$, and $\bar{x} = 1 - x$ is the probability of staying in the same layer for tunneling probability x.

Lemma S3.1 (Extraction and counting). Split each creation operator of the first copy into its components in the layer of the first copy and in the storage layer, with amplitudes $\sqrt{1-y}$ and $\sqrt{y}$. Selecting the terms with j' stored particles gives the map $y^{j'/2}(1-y)^{(k-j')/2}V^{(j')}$ up to layer ordering. The count j' occurs with probability $\binom{k}{j'}y^{j'}(1-y)^{k-j'}$, and conditional on it tracing out the layer of the first copy leaves $E_{j'}(\rho)$ in the storage layer. On a determinant the two layers have layer pseudospin $h/2$ by Appendix A. The second coupling rotates this pseudospin, so its output J_z lies in $[-h/2, h/2]$. Adding $(k+j')/2$ gives $j' \le N_s \le k$ for every count. This proves the interval rule for any mixing angle.

For a retained count $j' = 1$, Lemma A6 of the article gives the two count effects and the efficiency in Proposition 2. A pair with $j' \notin [1, j]$ contributes zero to the rejection probability.

Lemma S3.2 (Pseudospin distribution and counting probability). Write $\mu_\ell = \mathrm{Tr}[(\Gamma^{(\ell)})^2]$ for the purity of the unnormalized ℓ-body reduction of ρ, with $\mathrm{Tr}\,\Gamma^{(\ell)} = \binom{k}{\ell}$ and $\mu_0 = 1$. Let $w_i^{(j')}$ be the mass at layer pseudospin $S = h/2 + i$. For $\bar{x} = \cos^2(\theta/2)$,

$$\sum_{\ell=0}^{j'} \frac{\binom{j'}{\ell}}{\binom{k}{\ell}} \mu_\ell \bar{x}^{j'-\ell}(1-\bar{x})^\ell = \sum_{i=0}^{j'} w_i^{(j')} P_i^{(0,h)}(2\bar{x}-1). \tag{S7}$$

To see this, take the expectation of the common layer rotation on $E_{j'}(\rho) \otimes \rho$. Terms returning to the initial layer counts exchange ℓ particles in each direction. The reverse rotation and fermionic exchange signs cancel. Their contraction is the Hilbert–Schmidt inner product of the two unnormalized ℓ-body reductions, equal to $\binom{j'}{\ell}\mu_\ell/\binom{k}{\ell}$ by Eq. (A2). The untouched particles contribute $\bar{x}^{h/2+j'-\ell}$. The diagonal entry of the Wigner small-d rotation matrix is $d^S_{m,m} = \bar{x}^{h/2} P_i^{(0,h)}(2\bar{x}-1)$. Cancel the common factor for $\bar{x} > 0$ and extend polynomially to zero. The Jacobi basis determines all the weights.

For transfer probability $x = \sin^2(\theta/2)$, summing the squared rotation elements over counts outside $[j', k]$ gives $d_i(x) = \sum_{|m'|>h/2} |d^S_{m',m}(\theta)|^2$. Rotations and count projectors preserve the pseudospin. Thus $d_0(x) = 0$ and $d_1(x) = (h+2)[x(1-x)^{h+1} + x^{h+1}(1-x)]$. For $y = j/k$ and $x_{j'} = 1/(k-j'+2)$ the rejection probability per pair is

$$p_j^{\mathrm{count}} = \sum_{j'=1}^{j} \binom{k}{j'} y^{j'}(1-y)^{k-j'} \sum_{i=1}^{j'} w_i^{(j')} d_i(x_{j'}). \tag{S8}$$

Appendix A, Lemma A5, bounds this probability above and below in Eq. (A7).

Lemma S3.3 (Smallest ratio to the optimal test). Both rejection probabilities are linear in the pseudospin distribution of $\rho \otimes \rho$. In particular, $p_j = \sum_J P_J[1 - a_J(k,j)]$ by Eq. (8), and $p_j^{\mathrm{count}} = \sum_J P_J c_J(k,j)$ with

$$c_J(k,j) = \sum_{j'=1}^{j} \binom{k}{j'} y^{j'}(1-y)^{k-j'} \sum_{i=1}^{j'} w_i^{(j')}(J)\, d_i(x_{j'}),$$

where $w^{(j')}(J)$ is the solution of the moment equations (A1) and (A3) for $P_{J'} = \delta_{J'J}$. For $p_j > 0$ their ratio is a weighted average over positive pseudospins, so $p_j^{\mathrm{count}}/p_j \geq \min_{J\geq 1} c_J(k,j)/[1-a_J(k,j)]$ for every input. For $j = 1$ every term of the minimum equals α_k of Eq. (15). For $j = k$ the storage layer holds a complete copy, $x_k = 1/2$, and the ratio for layer pseudospin J is $1 - \mathcal{L}_J(0)^2$, smallest for $k \geq 2$ at $J = 2$ with the value 3/4. Evaluating the minimum for all $2 \leq k \leq 40$ and $1 \leq j \leq k$, with both the weights and the squared rotation amplitudes in exact rational arithmetic, gives a ratio that is nondecreasing in j; its smallest value is α_k, at $j = 1$. For $k = 10$ the minima at $j = 1, \ldots, 10$ are 0.149, 0.164, 0.171, 0.176, 0.181, 0.186, 0.195, 0.222, 0.332 and 0.750.

S4. EXACT VARIANCE OF THE MAJORANA ESTIMATOR

Take $n \geq 2$. Let X be the vector of measured signs and O the orthogonal frame in Sec. VI. In a random Majorana frame write $z_v = (OGO^{\mathsf{T}})_{2v-1,2v}$. The pair average has conditional mean $\theta = \sum z_v^2$ and $\mu = \mathbb{E}\theta = F_2/[2(2n-1)]$, where $F_2 = \|G\|_F^2$. Write $M = \mathbb{E}[XX^{\mathsf{T}} \mid O]$, $C = M - zz^{\mathsf{T}}$, $H = \mathbb{E} z^{\mathsf{T}} M z$ and $K = \mathbb{E}\,\mathrm{Tr}\, M^2$. The centered-pair decomposition from Appendix B gives the variance in a random Majorana frame.

$$\frac{\mathrm{Var}\,\widehat{\Delta}_G}{4(2n-1)^2} = V + \frac{4(H-\vartheta)}{m} + \frac{2(K-2H+\vartheta)}{m(m-1)}, \tag{S9}$$

where $V = \mathrm{Var}\,\theta$ and $\vartheta = \mu^2 + V$. We next compute all three terms explicitly. Put $F_4 = \mathrm{Tr}[(G^{\mathsf{T}}G)^2]$, and $Q_{abcd} = -\langle c_a c_b c_c c_d\rangle$ for increasing distinct indices, extended as a four-form. Let $A_4 = \sum_I Q_I^2$ and $L = \sum_I Q_I \,\mathrm{Pf}(G_I)$ over increasing four-subsets I. Then

$$V = \frac{{F_2}^2}{2(2n-3)(2n-1)^2} + \frac{(2n-8)({F_2}^2 + 2F_4)}{2(2n-3)(2n-1)(2n+1)(2n+2)}, \tag{S10}$$

$$H = \mu + \frac{n(n-1)L}{3\binom{2n}{4}},$$

$$K = n + \frac{n(n-1)A_4}{\binom{2n}{4}}. \tag{S11}$$

These contractions also give the exact variance for non-Gaussian inputs.

To obtain Eq. (S10), take four successive real frame vectors u, v, w, y and set $\widetilde{G}_{12} = u^{\mathsf{T}}Gv$, $\widetilde{G}_{34} = w^{\mathsf{T}}Gy$. Write $\kappa_2 = F_2/[2n(2n-1)]$ and $\kappa_4 = (F_2^2+2F_4)/[2n(2n+2)]$. The sphere moments $\mathbb{E}u_a^4 = 3/[2n(2n+2)]$ and $\mathbb{E}u_a^2u_b^2 = 1/[2n(2n+2)]$ for $a \neq b$ give $\mathbb{E}\|Gu\|^4 = \kappa_4$. For fixed u, $\mathbb{E}[\widetilde{G}_{12}^2 \mid u] = \|Gu\|^2/(2n-1)$ because $Gu \perp u$. Averaging over u gives $\mathbb{E}\widetilde{G}_{12}^2 = \kappa_2$, $\mathbb{E}[\widetilde{G}_{12}^2\|Gu\|^2] = \kappa_4/(2n-1)$ and $\mathbb{E}\widetilde{G}_{12}^4 = 3\kappa_4/[(2n-1)(2n+1)]$. The squared norm of the compression of G off u, v is $F_2 - 2(\|Gu\|^2 + \|Gv\|^2) + 2\widetilde{G}_{12}^2$. Dividing by $(2n-2)(2n-3)$ gives $\mathbb{E}[(\widetilde{G}_{34})^2 \mid u, v]$. Thus

$$\mathbb{E}\widetilde{G}_{12}^2(\widetilde{G}_{34})^2 = \frac{F_2\kappa_2 - 4\kappa_4/(2n-1) + 2\mathbb{E}\widetilde{G}_{12}^4}{(2n-2)(2n-3)}.$$

Now $\vartheta = n\mathbb{E}\widetilde{G}_{12}^4 + n(n-1)\mathbb{E}\widetilde{G}_{12}^2(\widetilde{G}_{34})^2$, which yields Eq. (S10).

Write $\omega = u \wedge v \wedge w \wedge y$. Then $(M)_{12} = Q(\omega)$. Within its oriented four-plane the three signed terms in the Pfaffian have equal averages under oriented changes of frame, so $\mathbb{E}[\widetilde{G}_{12}\widetilde{G}_{34} \mid \omega] = \mathrm{Pf}(G|_\omega)/3$. In coordinates the latter Pfaffian is the contraction of the four-form with coefficients $\mathrm{Pf}(G_I)$ against ω. Orthogonal sign flips eliminate off-diagonal covariances of the coordinates $(\omega)_I$. Their squared means agree by permutation invariance and $\sum_I (\omega)_I^2 = 1$. The two contractions in Eq. (S11) follow. The diagonals use $(M)_{vv} = 1$. This argument first uses $O(2n)$. Flipping one output Majorana exchanges the two components of $O(2n)$ and reverses one measured sign on every copy. Products of the same sign on two different copies are unchanged, so the statistic has the same law for $SO(2n)$.

For pure Gaussian inputs substitute $F_2 = F_4 = 2n$, $L = A_4 = \binom{n}{2}$ in Eqs. (S10)–(S11). More generally for a Gaussian with canonical singular values ν_i, $F_2 = 2\sum_i \nu_i^2$, $F_4 = 2\sum_i \nu_i^4$, and Wick's rule gives $L = A_4 = \sum_{i<v} \nu_i^2\nu_v^2$. This gives the three exact coefficients. The uniform bound used for the test appears in Appendix F.

S5. CHAIN AND DOUBLE-WELL CALCULATIONS

Let $v_0(m)$ and $v_1(m)$ be the variances of one batch of m copies on the determinant and on the ground state (Appendix B), and let $g > 0$ be the difference of the mean defects. The threshold lies at $g\sqrt{v_0}/(\sqrt{v_0} + \sqrt{v_1})$ above the mean on the determinant. With $z = \Phi^{-1}(5/6)$, where Φ is the standard normal distribution function, we choose the integer $m \geq 2$ that minimizes mB for

$$B = \max\{1, \lceil z^2(\sqrt{v_0(m)} + \sqrt{v_1(m)})^2/g^2\rceil\}. \tag{S12}$$

For a normally distributed mean of B batches both errors are then at most 1/6.

We simulated the estimator in nine settings with 4096 independent Haar-random orbital bases each. In every basis we rotated the many-body amplitudes, sampled occupation vectors from their Born probabilities and formed the statistic s. Bases were shared across couplings and each state was sampled independently; for $B > 1$ we convolved the single-batch distribution. Table S2 gives

the two error probabilities at the prescribed threshold, with standard errors from a jackknife over 16 groups of bases. The false-rejection probabilities agree with 1/6 within one standard error, and the miss probabilities lie between 0.139 and 0.169. The five couplings $U/t \le 1.5$ use at least 113 batches, for which we use the normal approximation. Cantelli's inequality guarantees both error bounds if z^2 is replaced by 5 in Eq. (S12); the last column of Table S1 gives the resulting numbers of copies. With this calibration the estimator takes 408 times as many copies as counting with at most three stored particles at $U/t = 0.25$, compared with 76 for the normal quantile.

For the block test use Eq. (13). The three outcomes have probabilities

$$p_\pm = (c_I \pm c_S)(1 \pm \mathrm{Tr}[(\gamma/k)^2])/2,$$
$$p_0 = (\bar{n} - b)(\bar{n} - \mathrm{Tr}[(\gamma/k)^2])/(\bar{n}^2 - 1).$$

Compare purities $1/k$ and $(k-g)/k^2$. At each integer R order the triples of counts by likelihood ratio and reject the largest ones up to false-rejection probability 1/6, randomizing at the boundary. Condition on the binomial number of matches to enumerate the distribution. Use the smallest R with miss probability at most 1/6. For Fig. S1 the equal superposition of $|12\rangle$ and $|34\rangle$ has four occupations 1/2 and $e = 1/2$. For it $\sum_{J>0} P_J = \Delta/6$. The pure-input lower bound in panel (b) is $2(\lfloor \min(\sqrt{d}/8, d/(25r)) \rfloor + 1)$ copies from Proposition S1.

For Fig. 3 the Hamiltonian is $-t\sum_{s,\sigma}(c^\dagger_{s\sigma}c_{s+1,\sigma} + \text{h.c.}) + U\sum_s n_{s\uparrow}n_{s\downarrow}$ with open ends. We diagonalized the sector $N_\uparrow = N_\downarrow = N/2$ at each coupling, using $t = 1$. The ten-site sector has $\binom{10}{5}^2 = 63\,504$ basis states. Splitting a ground-state amplitude into stored and complementary occupations with their exterior signs gives every reduction purity. Independently contracting one and two annihilators supplies the exact variance coefficients of Appendix B. For the batch estimator at fixed n, k, the positive batch variance on a determinant gives a positive minimum of $mv_0(m)$, since its basis term is positive. A fixed feasible m also bounds the cost above by a constant times $1/\Delta^2$ as $\Delta \to 0$. Equations (A4) and (A7) give the stored-particle $1/\Delta$ dependence. Tunneling and counting uses the total counts of the two layers, with $N = k$ here. Let μ_ℓ be the squared Frobenius norm of the unnormalized ℓ-particle reduction, whose trace is $\binom{k}{\ell}$. With $\bar{x} = \cos^2(\theta/2)$, the return amplitude of the layer rotation by θ is

$$F(\bar{x}) = \sum_{\ell=0}^{k} \mu_\ell \bar{x}^{k-\ell}(1-\bar{x})^\ell = \sum_{J=0}^{k} P_J\, \mathcal{L}_J(2\bar{x}-1), \quad \text{(S13)}$$

where $\mathcal{L}_J$ is the ordinary Legendre polynomial. In expanding the rotation, equal numbers ℓ of particles move in both directions. The signs $(-1)^{\ell^2}$ from exchange and $(-1)^\ell$ from reverse rotation cancel. Splitting the wedge amplitudes into $k-\ell$ and ℓ indices gives the contraction $\mathrm{Tr}[(A^\dagger_\ell A_\ell)^2] = \mu_\ell$, with A_ℓ the signed coefficient matrix of this split. This gives the first equality. The weight-zero matrix element of a spin-J rotation is $\mathcal{L}_J(\cos\theta)$, giving the second.

Orthogonality gives $P_J = (2J+1)\int_0^1 F(\bar{x})\mathcal{L}_J(2\bar{x}-1)\,d\bar{x}$. The polynomial coefficient of $\bar{x}^v$ in $\mathcal{L}_J(2\bar{x}-1)$ is $(-1)^{J-v}\binom{J}{v}\binom{J+v}{v}$. Each monomial integral follows from

$$\int_0^1 \bar{x}^{k-\ell+v}(1-\bar{x})^\ell\,d\bar{x} = \frac{1}{(k+v+1)\binom{k+v}{\ell}}. \quad \text{(S14)}$$

Inserting these weights in Eq. (8) gives p_j. Equations (A1) and (A3) determine the extracted pseudospin probabilities by descending moment order. Rotating each pseudospin gives the probability of counts outside $[j', k]$ in Eq. (S8). For any such rejection probability p per pair, at most $2\lceil \ln 6/[-\ln(1-p)] \rceil$ copies give miss probability at most 1/6.

A two-site calibration is available in closed form. In the covalent and even ionic basis the singlet Hamiltonian has diagonal $(0, U)$ and off-diagonal $-2t$. Its ground energy $(U - \sqrt{U^2+16t^2})/2$ gives natural occupations $(1 \pm 4t/\sqrt{U^2+16t^2})/2$, each twice, from the hopping derivative. Thus $\Delta = U^2/(U^2+16t^2)$. At $U/t = 4$ the optimal test with one stored particle rejects with probability 1/12 and needs 42 copies. With $x_1 = y = 1/2$, extraction succeeds with probability 1/2 and counting has conditional efficiency 3/4, giving $p_1^{\mathrm{count}} = \Delta/16 = 1/32$ and 114 copies (57 pairs). Independent false counts keep the false-rejection probability at most 1/6 if their probability per pair is at most $p_{\mathrm{bg}} = 1 - (5/6)^{1/57} \simeq 0.00319$. Observing zero events in R_0 independent calibration pairs prepared in a determinant gives the one-sided 95% upper confidence limit $1 - 0.05^{1/R_0}$, which first falls below p_{bg} at $R_0 = \lceil 57\ln(0.05)/\ln(5/6) \rceil = 937$. In the dilute chain at $U/t = 4$, balanced tunneling with $j = 1$ takes 88 copies. The prescribed angle takes 98. On the ten-site chain separate measurements become cheaper than counting with up to one stored particle between $U/t = 1$ and 1.5, and cheaper than counting with up to three between $U/t = 2$ and 3. The numbers of copies are 875 versus 730 at $U/t = 2$, and 224 versus 384 at $U/t = 3$. At $U/t = 4$ the ratios of counting to optimal rejection probability for one particle and full storage are $\alpha_{10} \simeq 0.149$ and 0.759 on the half-filled chain, and $\alpha_2 = 1/3$ and 3/4 on the dilute chain, in agreement with Proposition 2.

S6. RECORDING THE ORBITALS OF THE OTHER PARTICLES

Choose a Haar-random orbital basis independently for each pair. Store one particle from the first copy and record the $k-1$ orbitals of the others in that basis. For an orbital set L of size $k-1$ the Kraus operator is $a_L/\sqrt{k}$. Let $U = L^\perp$, identifying a set with its span. Split the second copy according to its intersection C with the recorded set; let $\ell = k - |C|$. In this block the

TABLE S1. Numbers of copies at target error 1/6 on each side for the half-filled ten-site chain and, in the last row, for two particles on 24 sites, where the column $j \le 3$ uses both particles. Counting includes both copies of every pair, discarded pairs too. "Separate" is the separate-measurement estimator calibrated with the normal quantile, Eq. (S12), and "Separate (Cantelli)" the same estimator with both error bounds guaranteed by Cantelli's inequality.

U/t	Δ	Optimal $j=1$	Counting $j \le 1$	Counting $j \le 3$	Optimal $j=k$	Separate	Separate (Cantelli)
0.25	0.01545	25508	170774	42168	1392	3219425	17199515
0.35	0.03026	13026	87212	21538	712	838534	4479756
0.5	0.06163	6396	42822	10578	350	201618	1077044
1	0.2431	1620	10854	2686	90	12760	68103
1.5	0.5315	740	4964	1232	42	2599	13872
2	0.9000	438	2932	730	24	875	4675
3	1.724	228	1530	384	14	224	1170
4	2.464	160	1070	272	10	99	522
6	3.462	114	762	196	8	44	231
6.5	3.631	108	726	188	6	39	210
7	3.775	104	698	180	6	35	185
7.5	3.900	100	676	176	6	32	170
8	4.008	98	658	170	6	30	160
Two particles on 24 sites							
4	0.6465	32	98	44	32	310	1652

TABLE S2. Simulated error probabilities of the separate-measurement estimator at the threshold of Sec. S5, for batches of m copies and B batches, from 4096 Haar-random orbital bases. Parentheses give jackknife standard errors in the last digits.

Chain	U/t	m	B	Copies	False rejection	Miss
Half-filled	2	25	35	875	0.185(29)	0.139(14)
Half-filled	3	28	8	224	0.173(14)	0.156(10)
Half-filled	4	33	3	99	0.170(10)	0.169(8)
Half-filled	6	44	1	44	0.171(5)	0.156(5)
Half-filled	6.5	39	1	39	0.161(6)	0.158(6)
Half-filled	7	35	1	35	0.166(7)	0.167(7)
Half-filled	7.5	32	1	32	0.167(8)	0.153(5)
Half-filled	8	30	1	30	0.170(7)	0.165(6)
Dilute	4	62	5	310	0.171(9)	0.158(8)

largest rejection effect that vanishes on every determinant is $W_\ell^\dagger W_\ell/(\ell+1)$, where $W_\ell : U \otimes \bigwedge^\ell U \to \bigwedge^{\ell+1} U$ is exterior multiplication. Indeed a determinant yielding the record can be written on its nonzero chart as $x \wedge \bigwedge_{i\in L}(e_i + v_i)$, with $x, v_i \in U$. Its contracted first copy is proportional to x. Replacing each v_i by an independent scalar multiple and taking polynomial coefficients separates the C blocks. Within each block the tensors $x \otimes (x \wedge v_1 \wedge \cdots \wedge v_{\ell-1})$ span a nonzero invariant subspace of the wedge kernel. By the exterior Pieri rule that kernel is irreducible. Its complement has projection as above since $W_\ell W_\ell^\dagger = (\ell+1)I$.

For a final pair of orbital sets (L, S), $|S| = k+1$, put $v = k - |L \cap S|$. The wedge row after contraction is the corresponding row of J_- up to an ordering sign. The full effect in this basis is $k^{-1} J_+ D J_-$, where D has diagonal $1/(v+1)$. Orbital averaging acts by a scalar at each pseudospin, and transfer from weight zero has squared amplitude $J(J+1)$. At weight -1 there are $k-v$ doubly occupied orbitals and $2v$ single ones. Counting the placements and the multiplicity $\binom{2v}{v-J} - \binom{2v}{v-J-1}$ of pseudospin $J \ge 1$ gives

$$m_{v,J} = \frac{(2J+1)n!}{(k-v)!(n-k-v)!(v-J)!(v+J+1)!}.$$

Normalizing over v by Vandermonde gives the coefficient of P_J in the rejection probability with the record,

$$p_{\mathrm{rec}} = \sum_J c_J^{\mathrm{rec}} P_J, \qquad c_J^{\mathrm{rec}} = \frac{J(J+1)}{k}\, \mathbb{E}_J \frac{1}{v+1},$$
$$v - J \sim \mathrm{Hypergeom}(n+1, k-J, n-k-J). \qquad \text{(S15)}$$

The parameters are population size, number of successes and sample size. The singlet has coefficient zero. Extend the displayed multiplicity formula to $J = 0$ for comparison. The likelihood ratio $m_{v,J+1}/m_{v,J}$ is proportional to $(v - J)/(v + J + 2)$, increasing in v. Thus the expectation of $1/(v + 1)$ is largest at $J = 0$. Using $\binom{k}{v}/(v+1) = \binom{k+1}{v+1}/(k+1)$ and Vandermonde gives $\mathbb{E}_0[1/(v+1)] \le (n+2)/[(k+1)(n-k+1)]$. Since $v \le k$, $\mathbb{E}_J[1/(v+1)] \ge 1/(k+1)$ for every J. Comparison with $p_1 = \sum_J P_J J(J+1)/[k(k+1)]$ yields

$$p_1 \le p_{\mathrm{rec}} \le \frac{n+2}{n-k+1}\, p_1 \le 2p_1. \tag{S16}$$

At $k = 2$, $c_2^{\mathrm{rec}} = 1$ and $c_1^{\mathrm{rec}} = (n+3)/[3(n+1)]$, compared with $1/3$ without the record. Since $P_1 = (1-\mathrm{Tr}\,\rho^2)/2$, the gain is $p_{\mathrm{rec}} - p_1 = (1 - \mathrm{Tr}\,\rho^2)/[3(n+1)]$, which vanishes for pure inputs. The independent basis choice and the hard inputs of Sec. IV give the same worst-case order k^2/ε for this instrument.

S7. SMALLEST REGISTERS AND FURTHER PERFECTLY COMPLETE INSTRUMENTS

Here r counts register levels and R counts pairs. We use the refined records and linear forms in Appendix E of the article.

Proposition S2 (Smallest registers). All tests in this proposition are perfectly complete and use a cleared register. (i) With a classical record alone every such test accepts every state. (ii) For $2 \le k \le n/2$, every such test with two levels accepts every pure input. There is a three-level test that rejects every pure input with $e > 0$ with positive probability, bounded away from zero on the set $e \ge \varepsilon$ at fixed n and k. (iii) For pure or mixed inputs, $k+1$ levels and a classical orbital-subset record give the per-pair rejection probability

$$p_{k+1}^{\mathrm{lev}} = \Delta/[(k+1)(n-k)]; \tag{S17}$$

for $k = 2$ this is a three-level test with $O(n/\varepsilon)$ copies. (iv) For $r \ge 2$ and $\varepsilon \le 1 - 1/(n-k+1)$, rejection with probability at least $2/3$ on every mixed alternative requires $R \ge 4(n-k+1)/[3(r-1)]$.

Lemma S7.1 (One level). With $r = 1$ the refined record factors are products of linear forms on one copy. A form vanishing on all determinants is zero because the coordinate determinants are a basis of $\mathcal{H}_k$. Therefore almost every record that ends in rejection has a zero operator factor, and its probability vanishes on mixed inputs too. The same argument applies to any finite number of copies measured separately.

Lemma S7.2 (Two levels and hyperplane divisors). Let $\mathcal{X} = \mathrm{Gr}(k, n)$ in its Plücker embedding over $\mathbb{C}$. It is smooth, normal, integral and projective. Its Picard group is generated by the hyperplane class H (the complement of the big affine cell is a prime Schubert divisor of degree one on a Schubert line [3]). Smoothness identifies Weil and Cartier classes. The coordinate determinant basis gives nondegeneracy, so every nonzero ambient linear form restricts to a nonzero section with divisor of class H. A nonzero prime divisor of class aH has $a > 0$ by intersection with $H^{\dim \mathcal{X}-1}$. The sum of these positive integers with multiplicities for a hyperplane section is one. Its divisor is therefore prime with coefficient one. The section, an effective Cartier divisor on a smooth variety, is Cohen–Macaulay with no embedded primes, so generic reducedness gives reducedness.

Suppose $L_1M_1 + L_2M_2$ vanishes on $\mathcal{X}$. A zero restricted form vanishes in the ambient space. In that case integrality forces the other product to have a zero factor as well. Otherwise the divisors of the two products agree, so the unordered pairs of prime hyperplane divisors match with multiplicity. Matched sections have ratio with zero divisor. Normality makes ratio and inverse regular, and projectivity makes it a nonzero constant. Nondegeneracy lifts proportionality to ambient linear forms. Evaluating the two products at one point off their divisors fixes the sign and scalar, proving that the quadratic is zero on every $\psi \otimes \psi$. Apply the refined-record conclusion of Appendix E.

Lemma S7.3 (Three levels). Definition 2.1 and Theorem 4.1 of Kasman, Pedings, Reiszl and Shiota [4] give a finite family of rank-six quadrics with common zero set the decomposable cone for $2 \le k \le n-2$. Each is a pullback of the three-product Plücker equation on $\bigwedge^2 \mathbb{C}^4$, so the Kraus construction in Appendix E realizes it with three levels. Give each member positive selection probability. On the compact set of unit pure states with $e \ge \varepsilon$ the sum of the corresponding weighted squared amplitudes has a positive minimum whenever this set is nonempty. Repetition gives miss probability at most $1/3$.

Lemma S7.4 (The $k+1$-level instrument). Fix ordered occupation strings. For every $(k+1)$-subset S define $K_S|S \setminus b\rangle = s(b,S)|\iota_S(b)\rangle/\sqrt{n-k}$ for $b \in S$, and take it zero on the other strings. Here $\iota_S(b)$ is the position of b and $s(b,I) = (-1)^{|\{i \in I : i < b\}|}$ for an ordered set I containing b. The labels for distinct missing orbitals are orthogonal, while each k-string has $n-k$ supersets. Thus $\sum_S K_S^\dagger K_S = I$. Record S classically and its missing-orbital label coherently. For a $(k-1)$-set A form

$$v_{S,A} = \sum_{b \in S \setminus A} s(b, A \cup b)|\iota_S(b)\rangle \otimes |A \cup b\rangle.$$

Here $A \cup b$ abbreviates $A \cup \{b\}$. For fixed S the label and second string determine A, so the supports are disjoint as A varies. Each squared norm is at most $k+1$, making $H_S = \sum_A |v_{S,A}\rangle\langle v_{S,A}|/(k+1)$ an effect. For every input basis pair the row $\langle v_{S,A}|(K_S \otimes I)$ is the row $\langle S, A|J_+/\sqrt{n-k}$ up to a constant fermionic ordering phase. This follows by creating b in the first string and removing it from the second, which gives the two signs above. Summing the rows as outer products gives exactly $J_-J_+/[(k+1)(n-k)]$ as the full effect. Its expectation

on $\rho \otimes \rho$ is Eq. (S17) by Appendix A, also for mixed ρ. The classical register may require $\lceil \log_2 \binom{n}{k+1} \rceil$ bits.

Lemma S7.5 (Lower bound for mixed alternatives). First compare pure states to I/D in dimension D, with perfect completeness. Apply the refined-record argument to all pure rays in $\mathbb{P}^{D-1}$. A factor that vanishes on every $|x, x\rangle$ is a positive pair effect supported on $\bigwedge^2 \mathbb{C}^D$, since the vectors $|x, x\rangle$ span the symmetric subspace. For a given record and a fixed first outcome K, sum or integrate the second-outcome effects whose pair effect has that property. The resulting positive pair effect N_K obeys $N_K \le K^\dagger K \otimes I$. With $W = \mathrm{supp}(K^\dagger K)$ it is supported in $(W \otimes \mathbb{C}^D) \cap \bigwedge^2 \mathbb{C}^D = \bigwedge^2 W$. Compressing the domination to that subspace and taking the partial trace of its projector gives

$$\operatorname{Tr} N_K \le \frac{\dim W - 1}{2} \operatorname{Tr} K^\dagger K \le \frac{r-1}{2} \operatorname{Tr} K^\dagger K$$

for a nonzero K. Integrating first outcomes gives $\sum_K \operatorname{Tr} K^\dagger K = D$. Hence, on two copies of I/D, the probability of any such outcome, conditionally on any record, is at most $(r-1)/(2D)$. The chosen outcomes form a measurable subset, since zero trace on the symmetric subspace is a measurable condition. Every record that ends in rejection has one of them almost surely. The union bound yields $R(r-1)/(2D) \ge 2/3$. The one-active-particle embedding with $D = n - k + 1$ and $e(I/D) = 1 - 1/D$ proves part (iv).

Lemma S7.6 (Pure alternatives with a cleared register). Use the embedding of Appendix C with $d = \lfloor (n-k+2)/2 \rfloor$. The product rays form a connected Segre variety. The analytic argument for the record implies that almost every record that ends in rejection has a factor vanishing on that entire variety. Such a factor has $Y = 0$, where Y is the per-pair likelihood ratio for a fresh Haar product draw reused within that pair, relative to the maximally mixed source. Write $\Pr_0$, $\mathbb{E}_0$ and Var_0 for probability, mean and variance under the independent maximally mixed source. For any record $\Pr_0(Y = 0) \le \mathrm{Var}_0 Y \le (2 + d^{-1/2})^2 r/d$ since $\mathbb{E}_0 Y = 1$. Relative to that source the reused global Haar source costs at most $2R(R-1)/d^2 + \sqrt{\exp(Rr/d^2) - 1}/2$ in total variation (Sec. S2). If $R \le d/(10r)$ and $d \ge \max(64, 40/(1-\varepsilon)^2)$, the union bound plus this cost is below 0.492, whereas the far mass 19/20 requires rejection at least 19/30. Thus $R > d/(10r)$.

Lemma S7.7 (An orbital-subset instrument). For $k < \ell \le n$, restrict the first copy to each coordinate ℓ-set S with Kraus factor $\binom{n-k}{\ell-k}^{-1/2}$. Subset multiplicities normalize the instrument. Record S and store $\binom{\ell}{k}$ levels. For $T_S = (P_{\wedge^{k+1} S} \otimes I) J_+$ use $T_S^\dagger T_S / c$, $c = (k+1)\min(k, \ell-k)$. Absolute row and column sums are bounded by $k+1$ and $\min(k, \ell-k)$ on the domain of the stored state. Each final $(k+1)$-set has $\binom{n-k-1}{\ell-k-1}$ supersets, each containing all its predecessors. Summation gives $p_\ell^{\mathrm{sub}} = (\ell-k)\Delta/[(n-k)c]$, equal to Eq. (S17) for $k+1 \le \ell \le 2k$.

Lemma S7.8 (Coordinate blocks). After the one-particle channel record its block P_u and store its b-dimensional state. Set $V = W^\dagger/\sqrt{k+1}$ in the padded space. On block u the second measurement rejects on $V(\bigwedge^{k+1} P_u)V^\dagger$. Its first tensor factor is already in P_u, so the first block projection can be omitted in the joint probability. Under a uniform permutation of coordinates a fixed $(k+1)$-string lies in one block with probability $(\bar{n}/b)\binom{b}{k+1}/\binom{\bar{n}}{k+1}$. The coordinate projectors are diagonal in the wedge basis, so their expected sum is this multiple of the identity there. Multiplying p_1 by the factor gives, including in the zero-padded space,

$$p_b^{\mathrm{blk}} = \frac{\binom{b-1}{k}}{\binom{\bar{n}-1}{k}} \frac{\Delta}{k(k+1)}. \tag{S18}$$

At fixed k and growing $r \le n$, choosing the largest feasible ℓ in the subset instrument and comparing to the coordinate blocks gives $O(\min\{n/r^{1/k}, (n/r)^k\}/\varepsilon)$ copies.

S8. FINITE HALF-FILLED SECTORS

Here k is the number of particles in $2k$ orbitals. Table S3 uses the finite spectrum in Appendix D before the asymptotic estimates. At each pseudospin take the largest value of Λ_{k-J} up to that index. Average it against a distribution with tail $\min(1, 6re^{-J^2/(4k)})$ at positive integer J. Summation by parts bounds the pseudospin average of every distribution satisfying the tail inequality in this way because these largest values are nondecreasing. Add the swap contribution as in Eq. (D7) and take the smaller of the result and one. The R column gives the least integer for which Eq. (D9), evaluated with the unrounded upper bound, permits mean rejection 3/5. For $r = 2k$ the necessary pairs double with k across the four tabulated sizes, from one to eight. The last two columns give $\mathbb{E}p_j$ from Theorem 3, including the swap correction to $p_j(\bar{\rho})$.

TABLE S3. Half-filled sectors. The one-pair upper bound is rounded upward to three decimals. The necessary pair count for mean rejection 3/5 uses the unrounded bound. The last two columns include the swap correction.

k	r	p_* bound	R	$\mathbb{E}p_1$	$\mathbb{E}p_2$
64	128	0.873	1	0.00769231	0.01550575
64	16384	1.000	1	0.00769231	0.01550575
128	256	0.342	2	0.00387597	0.00778234
128	65536	0.741	1	0.00387597	0.00778234
256	512	0.161	4	0.00194553	0.00389867
256	262144	0.308	2	0.00194553	0.00389867
512	1024	0.081	8	0.00097466	0.00195122
512	1048576	0.148	5	0.00097466	0.00195122

Lemma S8.1 (Miss probability on a typical input). For a typical input drawn once and reused in every pair, the mean miss probability of the test in Theorem 2 after R pairs is $\mathbb{E}[(1-p_j)^R]$.

Put $\mu = \mathbb{E}p_j$ and $\mathcal{N} = \binom{n}{k}$. By Appendix D, p_j is a 4-Lipschitz function on the sphere $S^{2\mathcal{N}-1}$, whose Poincaré constant is $1/(2\mathcal{N}-1)$, so $\operatorname{Var} p_j \le 16/(2\mathcal{N}-1)$. Convexity and the bound $R(R-1)$ on the second derivative of $(1-p)^R$ give

$$(1-\mu)^R \le \mathbb{E}[(1-p_j)^R] \le (1-\mu)^R + \frac{4R(R-1)}{\mathcal{N}-1}. \quad \text{(S19)}$$

For $n = 128$, $k = 64$ and miss probability $1/6$ the right-hand side at $R = \lceil \ln(1/6)/\ln(1-\mu) \rceil$ is below $1/6$ and the left-hand side at $R-1$ is above $1/6$, for every $1 \le j \le 64$. The filled circles in Fig. 2(b) are therefore the exact smallest numbers of pairs.

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