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An Innovative Perspective on the $3x + 1$ Problem through Probabilistic Number Theory

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Abstract

In this paper, we recommend a proof for the $3x + 1$ problem or Collatz conjecture utilizing concepts from probabilistic number theory and probability theory. Here we try on the Collatz conjecture via the probabilistic methods especially expected value of random variables over the binary representation. At first, we introduce an innovative representation for odd integers and then by using the binary representation of them, we present a probability space that have an infinite σ-algebra that we present it. Based on all these probabilistic concepts, we introduce some lemmas and theorems to support our proof.

Based on the principle of random selection, we consider a subsequence of the odd numbers appearing in the Collatz sequence, chosen entirely at random. Following this approach, we examine various random variables and derive notable results concerning the odd numbers within the Collatz sequence. Furthermore, the appendixes provide numerical findings that serve to empirically validate the theoretical results. We conduct the numerical computations in Mathematica owing to its practical and versatile commands, particularly those designed for manipulating lists and sequences.

Keywords: Collatz conjecture, $3x + 1$ problem, number theory, probablistic methods, binary representation, random variables, natural density
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1. Introduction

1.1. Introduction of Collatz conjecture

The Collatz conjecture, also called the $3x+1$ problem, Syracuse problem, Ulams problem, is one of the toughest math problems that still hasnt been solved. It was first suggested by a German mathematician named Lothar Collatz back in 1937 [5]. The problem is based on a pretty simple rule that keep repeating. Definition 1 is a simple explain of Collatz conjecture. Of course, we try to prove $3x + 1$ problem that is equivalent by the Collatz conjecture. We name the sequence of numbers that generated using 2 rules in definition 1 as **Collatz Sequence** (**CS**).

Definition 1. (**Collatz Conjecture**)
For every positive number x that we keep doing below simple rules over and over, we will eventually end up at 1.

i :If the number x is even, just divide it by 2.

ii :If the number x is odd, multiply it by 3 and then add 1.

1.2. History and early works

The problem admits a short statement, and its early reputation for accessibility rested on the fact that individual cases are easy to verify. General verification has proved another matter. Extensive computation now supports the conjecture, but no proof is known, and it is fair to say the obstruction has been more productive of research than the evidence has been. The resulting literature is large and split among several approaches, none of which has yet been folded into a unified account. Among these studies and publications, we can highlight over 1000 papers regarding the Collatz conjecture that have appeared on *ResearchGate* [32].
The Collatz conjecture holds a distinctive place in both pure and applied mathematics, establishing significant links across various disciplines, including number theory, that we can refer to the 2010 survey by Marc Chamberland [4], a collaborative effort involving Erdos and Lagarias [8, 18], essential insight regarding every complete arithmetic series of Everett in [7], along with several other studies, are referenced as [9, 10, 11, 25, 30, 31].
Extensive computation supports the conjecture over a range of initial values far beyond hand calculation. It has not produced a proof, and the standard techniques for problems of this type do not appear to apply. Iteration of

a short arithmetic rule evidently generates behaviour that current methods cannot account for, and it is the gap between what can be checked and what can be proved that keeps the problem active.
Collatz conjecture also studied in the field of probablistic and stochastic models. Advances in computational power have enabled tests on unprecedented scales. For instance, Jeffrey.C Lagarias (2010) verified the conjecture for all integers up to 5.4×10^{18} [19]. More recently, Samrat Dutta (2025) extended this bound to 2^{71} through the implementation of optimized sieve algorithms and parallel computing architectures, which significantly accelerate the step-by-step simulation of the so-called Collatz process [6]. Kontorovich and Lagarias presented a stochastic model corresponding to random tree growth, e.g. a branching random walk [14].
This paper argues that exhaustive verification is the wrong instrument for this problem, and that the statistics of Collatz trajectories can do work that case-by-case checking cannot. We describe a framework for that analysis and use it to argue for a probabilistic proof of the conjecture, in the sense that the claim is established for trajectories in the aggregate, through their convergence behaviour, rather than integer by integer. Aperiodicity is a property of individual orbits; taken together, the trajectories have enough regularity to be analyzed.

1.3. About the sections

We need to introduce the theoretical framework underpinning our probabilistic approach to the Collatz conjecture. So we investigate a congruence-based representation of positive odd integers and its probabilistic interpretation. A maximum congruence index and an associated quotient are defined through a sequence of a congruence conditions in lemma 2.2.
We first derive the distribution of the index using relative natural density. In the section 2, some definitions and lemmas establish the number theory foundation for our proof. In the section 3, we state a precise definition for probability space that we need and use in our proof. Since natural density does not, in general, define a countably additive probability measure on the entire power set of the positive odd integers, we construct a σ-algebra generated by the joint events specifying the index. We prove that every event in this σ-algebra has a relative natural density and that the restriction of natural density to this σ-algebra is a countably additive probability measure.
In the section 4, we then derive the joint distribution of the index and the residue variable, determine the distributions of several related random

variables, and obtain a transition matrix for the quotient modulo 3. Also we introduce alternative discrete probability models with arbitrary residue distributions and compare their independence properties with those of the natural-density model. Connections with the measure-theoretic approach to density and related work on density measures are discussed. In fact, at first we guess that using the congruence modulo 3, the uniformity of every class in the natural-density model can be transfered to the next term of sequence, so we try to find a transition matrix between 2 variables modulo 3 and we calculate it at section 5.
In the section 5, we disscus about transition matrix that introduce in the section 4.2.2 and calculate it. However, the transition matrix between X mod 3 and κ mod 3 is not sufficient. Because the distribution of the variables X mod 3 and κ mod 3 in *CONS* changes from the third term onwards. Every class of X mod 3, for the first and second term of the sequence, has a uniform distribution by natural density model, but every class of κ mod 3, in the second term has not uniform distribution and so for the third term, X mod 3 has not uniform distribution by natural density model. So in section 6, we introduce a transition between X_n and κ_n that solve this problem. So in the sections 2, 3, 4, 5 and 6, we present some preliminary concepts.
For the first point in our paper, we use a unique representation for odd numbers that in sections 2 and 3 we state and prove number-theoretic and probabilistic aspects of this representation. But we need more probabilistic aspects. Therefore, in section 4, we present models that can be used for probability measure and probability calculation to strengthen the foundations for our probabilistic approach. To do this, we need to find and prove the existence of a uniformity and its preservation throughout the *CONS*.
So we first examine a transition from X mod 3 to κ mod 3 in section 5 and after it, in section 6, we succeed to find a suitable transformation to solve the problem.
In this section, we identify transformations from X_n mod 9 to κ_n mod 6 and from κ_n mod 6 to X_{n+1} mod 9 that can be used to preserve and transfer the uniformity we desire.
The distributional relation between X_n and κ_n is governed by two successive mechanisms. First, the conditional geometric distribution of A_n within each admissible residue class of X_n mod 9, together with the congruence relation (54), determines the stochastic matrix K. Second, the iteration rule determines the deterministic matrix T. Their product $M = KT$ gives the evolution of the marginal distribution of X_n mod 9. In particular, the distribution

reaches the vector $\boldsymbol{\pi}_X$ after two applications of the combined transition, and the corresponding distribution of $\kappa_n \bmod 6$ is $\boldsymbol{\pi}_\kappa$ from the first iterate onward. This uniformity preserve distribution of A_n for all n in *CONS* and we need it to solve $3x+1$ problem.
In the section 7, we state a paradox about the Collatz odd number sequence between the standard form and inverse form of the sequence. Standard form is usuall CONS and inverse form is a sequence when start from 1 and after it one of the numbers in CI_1 and this continues for other terms that we point it in section 7.
In the section 8, we use two approaches that similar in plan but different in indexes and numbers that pointed in two lemmas 2.4 and 2.5 and sections 2, 3, 4, 5 and 6. So in this section, we state and prove two theorems based on the concepts of number theory and probability theory in sections 2, 3, 4, 5 and 6, by proving which we conclude that the $3x+1$ problem is solved and the Collatz conjecture is correct.
In the section 9 we state a problem that is usefull in the probability theory and set theory and also 2 problems that are new generalized forms of $3x+1$ problem as the $Qx+1$ problem.
At the end, in the appendix A, we have some programs and expriments about the probabilistic results and our problems. In the experiments and programming for this section, we demonstrate what occurs during observations of random samples, which leads us to introduce new problems in section 9. Also in the appendix B, we create a random *CONS* in an experiment involving 100,000 initial samples and demonstrate that the proven results from sections 5 and 6 are approximately replicated in this experiment.
Appendix C, located at the end, contains functions, code, and programs written in Mathematica14.2. We use the output from this code to conduct our experiments, and we present our observations in appendixes A and B.

1.4. How to use AI while keeping it under our control

We employ a novel approach to utilizing artificial intelligence for the purpose of logical deduction. First, we conceive a patha line of reasoningto serve as the basis for a proof. This path is constructed upon numerical experiments and the results derived from them. Consequently, we maintain full oversight of the entire trajectory, the obstacles encountered, and the tools available. For instance, when we randomly select a positive odd integer, we generate a corresponding sequence. We can then analyze each term within this sequence, examining the parameters of individual terms as well as the various

relationships between them. We design the path based on our own observations, while tasking the artificial intelligence solely with organizing the entire set of relationships into a coherent classification. In effect, once the mathematical rules and the intended path are established, the process is akin to traveling between two points by car; in this analogy, the artificial intelligence serves as the vehicle. This is much like the impact that computers have had on the evolution of chessan influence clearly visible today.
For some parts of our paper to translate or improve the texts or collect and categorize our ideas, we use the AI (Google translator and Chat-GPT). When we retain full control over the proofs trajectory and the objectives we are pursuing, artificial intelligence functions as a powerful engine for inference and reasoning. We leverage it to construct a proof, while the foundational concept, the strategic path, and the numerical experiments remain of our own design.
Based on our findings in sections 2 and 3, we guess some theoritical relations between parameters in positive odd integers representations. In additions we design some numerical expriments that show us our theoritical assumptions are true. So we need to collect and categorize our ideas as theoritical framework. So we design a structure to the next sections 4, 5 and 6. For doing it better, we use the free version of Chat-GPT as AI help and by asking some questions based on the first our ideas, we can create some parts of structures of lemmas and theorems in sections 4, 5 and 6. It is necessary to emphasize that we just use the AI Chat-GPT in these 3 sections to improvements the structure of our paper. In fact we have the main ideas of lemmas and theorems in these sections and just need a better categorization and structure for them.

2. Number theory

In this section we introduce a new representation for positive odd numbers that we use of it to solve $3x + 1$ problem.
In the paper [29] 2025, Bulent Sukusu has introduced **Collatz Numbers**. In this paper, we use our own terminology and concepts to maintain our own punctuation and definition, so we name and define it as the sequence α_n in definition 2.

Definition 2. For $n \geq 1$ we define α_n as a recursive sequence as follows:

$$\alpha_{n+1} = 1 + 4\alpha_n, \alpha_1 = 1.$$

So we have

$$\alpha_n = \sum_{j=0}^{n-1} 4^j = \frac{4^n - 1}{3}, \qquad \alpha_k \equiv k \pmod 3. \tag{1}$$

2.1. Representation of positive odd numbers

Considering the binary representation of $\alpha_n, n \geq 1$, that is one of the terms in the below sequence. Now, via a standard induction on n, we can prove for every n, α_n has exactly $2n-1$ bits.

$$1, 5, 21, 85, \cdots \qquad (Sequence \quad \alpha_n),$$

$$1, 101, 10101, 1010101, \cdots (Binary \quad representation \quad of \quad Sequence \quad \alpha_n).$$

Now for every positive integer a, let

$$c_a = \alpha_{\lfloor \frac{a+1}{2} \rfloor}.$$

The first terms are

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 5, \quad c_4 = 5, \quad c_5 = 21, \quad c_6 = 21.$$

The sequence satisfies

$$c_{a+1} = \begin{cases} c_a + 2^a, & a \text{ even}, \\ c_a, & a \text{ odd}. \end{cases}$$

Moreover,

$$0 < c_a < 2^a, \qquad a \geq 1.$$

Definition 3. For every positive odd integer x, we define its maximum congruence index by

$$A(x) = \max \left\{ a \in \mathbb{N} : a \geq 1, \quad x \equiv c_a \pmod{2^a} \right\}.$$

and the associated quotient by

$$\kappa(x) = \frac{x - c_{A(x)}}{2^{A(x)}}.$$

Thus

$$x = 2^{A(x)} \kappa(x) + c_{A(x)}.$$

Lemma 2.1. *The maximum congruence index $A(x)$ exists and is finite for every positive odd integer $x < \infty$.*

Proof. Since $c_1 = 1$ and x is odd,

$$x \equiv c_1 \pmod 2.$$

Hence the set of admissible indexes is nonempty. Since $c_a \to \infty$, for all sufficiently large a we have $c_a > x$. Together with $c_a < 2^a$, this gives $0 < c_a - x < 2^a$ whenever $c_a > x$. Therefore $x \not\equiv c_a \pmod{2^a}$ for all such a. The set of admissible indexes is finite and consequently has a maximum. □

Now we define

$$\rho_a = \begin{cases} c_a, & a \text{ even}, \\ c_a + 2^a, & a \text{ odd}. \end{cases} \tag{2}$$

and use it in the lemma 2.2.

Lemma 2.2. *For every positive odd integer x,*

$$A(x) = a \quad \Longleftrightarrow \quad x \equiv \rho_a \pmod{2^{a+1}}.$$

Proof. The condition $A(x) = a$ requires

$$x \equiv c_a \pmod{2^a},$$

but

$$x \not\equiv c_{a+1} \pmod{2^{a+1}}.$$

The two residue classes modulo 2^{a+1} satisfying the first congruence are c_a and $c_a + 2^a$.
If a is even, then $c_{a+1} = c_a + 2^a$, so the admissible class is c_a. If a is odd, then $c_{a+1} = c_a$, so the admissible class is $c_a + 2^a$. These are precisely the classes specified by ρ_a. □

Lemma 2.3. *The index and quotient have the same parity:*

$$A(x) \equiv \kappa(x) \pmod 2.$$

Proof. Write $a = A(x)$ and $\kappa = \kappa(x)$. By definition,

$$x = c_a + 2^a \kappa.$$

If a is even, then $c_{a+1} = c_a + 2^a$. Maximality of a requires κ to be even, since an odd κ would imply

$$x \equiv c_a + 2^a = c_{a+1} \pmod{2^{a+1}}.$$

If a is odd, then $c_{a+1} = c_a$. Maximality of a requires κ to be odd, since an even κ would imply

$$x \equiv c_a = c_{a+1} \pmod{2^{a+1}}.$$

Thus $A(x)$ and $\kappa(x)$ have the same parity. □

In the lemmas 2.4 and 2.6, we state and prove all the stated points from definitions 2 to lemma 2.3 using the binary representation of odd number x.

Lemma 2.4. *For every arbitrary odd number $x < \infty$, $\exists!\quad a \in \mathbb{N}, \kappa \in \mathbb{Z}$: $x = 2^a\kappa + \alpha_{[\frac{a+1}{2}]}$.*

Proof. To prove the lemma 2.4, we pay attention to the rightmost bits of the x and prove $\exists n \geq 1$ that $2n-1$ rightmost bits of every odd number x is exactly as α_n.
So we let $\mu_{x,n}$ as $2n-1$ rightmost bits of x and

$$\nu_x = \{n \in \mathbb{N} | \mu_{x,n} = \alpha_n\}$$

and try to find $\max \nu_x$. Considering that the right bit of every odd number x is equal to 1, there is at least number $n = 1 \in \nu_x$, and the largest value of n must be found.
Because

$$b = \text{size}(x) \leq x < \infty,$$

where size(x) is the number of bits of x in binary representation, so $|\nu_x| < \infty$ and ν_x has a maximum. Now assume

$$n = j = A(x) = \max \nu_x.$$

A critical observation involves the $2n^{th}$ bit. Its value, 0 or 1, directly indicates whether the number κ is even or odd.

- i : Bit $2j$ of x is equal to 1. since the $2j-1$ rightmost bits of x are exactly equivalent to α_j, so

$$\exists \kappa \in \mathbb{Z} : x - \alpha_j = 2^{2j-1}\kappa,$$

κ is an odd number (bit $2j$ of x is the first bit of κ), so

$$\exists! a = 2j-1, \kappa \in \mathbb{Z} : x = 2^a \kappa + \alpha_{[\frac{(a)+1}{2}]},$$

also in this case, all of $b-2j$ leftmost bits of x can be 0 or 1 with the same probability.

ii : Bit $2j$ of x is equal to 0, since the $2j-1$ rightmost bits of x are exactly equivalent to α_j. Because of $n = j$ is the largest value of n, so bit $2j+1$ of x can not be 1, otherwise $2j+1 = 2(j+1)-1$ rightmost bits of x is equivalent to α_{j+1}, so $n = j$ will not be the largest n. So bit $2j+1$ of x is equal to 0. So

$$\exists \kappa \in \mathbb{Z} : x - \alpha_j = 2^{2j} \kappa,$$

κ is an even number (bit $2j+1$ of x is the first bit of κ), so

$$\exists! a = 2j, \kappa \in \mathbb{Z} : x = 2^a \kappa + \alpha_{[\frac{(a)+1}{2}]},$$

also in this case, all of $b-2j-1$ leftmost bits of m can be 0 or 1 with the same probability.

□

In fact in the binary representation of m, if bit $2j$ of m is 1, then m is divided by 2^{2j-1} and the reminder is equal to $\alpha_j = \alpha_{[\frac{(2j-1)+1}{2}]}$. If bit $2j$ of m is 0, then m is divided by 2^{2j} and the reminder is equal to $\alpha_j = \alpha_{[\frac{(2j)+1}{2}]}$. In this way the lemma 2.4 is proven.

Lemma 2.5. *For every arbitrary odd number $x < \infty$,*

$$\forall t \in \mathbb{N} \quad \exists! \quad a \in \{1, 2, 3, \cdots, 2t\}, \kappa \in \mathbb{Z} : x = 2^a \kappa + \alpha_{[\frac{a+1}{2}]}.$$

Proof. To prove the lemma for any odd number x, pay attention to the rightmost bits of the x and prove $\exists n \geq 1$ that $2n-1$ rightmost bits of every odd number x is exactly as α_n.
Let $\mu_{x,n}$ as $2n-1$ rightmost bits of x and

$$\nu_x = \{n \in \mathbb{N} | \mu_{x,n} = \alpha_n\}.$$

Now try to find $\max \nu_x$. Considering that the right bit of every odd number m is equal to 1, there is at least number $n = 1 \in \nu_x$, and the largest value of n must be found.
Because

$$b = \text{size}(x) \leq x < \infty,$$

where size(x) is the number of bits of x in binary representation, so $|\nu_x| < \infty$ and ν_x has a maximum. Now we assume

$$n = j = \min\{\max \nu_x, t\}.$$

A critical observation involves the $2n^{th}$ bit. Its value, 0 or 1, directly indicates whether the number κ is even or odd.
If $t \geq \max \nu_x$ then $j = \max \nu_x \leq t$ and all conditions to prove are same as lemma 2.4, because $a = 2j, a = 2j - 1 \leq 2t$, so

$$\exists! \quad a \in \{1, 2, 3, \cdots, 2t\}, \kappa \in \mathbb{Z} : x = 2^a \kappa + \alpha_{[\frac{a+1}{2}]}.$$

But for $t < \max \nu_x$ we have just one case $j = t < \max \nu_x$. In this case bit $2j$ must be 0, because $2j = 2t < 2 \max \nu_x - 1$ and $2 \max \nu_x - 1$ rightmost bits of x is equal to $\alpha_{\max \nu_x}$ and the even bit positions of the $\alpha_{\max \nu_x}$ are zero. So

$$\exists \kappa \in \mathbb{Z} \text{s.t.} x - \alpha_j = 2^{2j-1} \kappa$$

and κ is an even number (bit $2j$ of x is the first bit of κ). Now we have $2j - 1 = 2t - 1 < 2t$ so

$$\exists! \quad a = 2j - 1 = 2t - 1 \in \{1, 2, 3, \cdots, 2t\}, \kappa \in \mathbb{Z} : x = 2^a \kappa + \alpha_{[\frac{a+1}{2}]}.$$

Also in this case, the bits $2a$ to $2a + \max \nu_x - t$ repeat in the pattern 10. Moreover, all of $b - 2 \max \nu_x + 1$ leftmost bits of x can be 0 or 1 with the same probability. □

2.2. Collatz odd number sequence

In the lemma 2.6, we point to a sequence of odd numbers that obtains from $3x + 1$ problem.

Lemma 2.6. *For a given random odd number x, let α_n be the rightmost bits of x. Using x as the initial seed, a sequence of odd numbers is generated by the iterative application of the following 2 rules:*

(i) If bit $2n$ of x is 1, the next odd number after x in sequence is $\frac{3x+1}{2^{2n-1}}$.

(ii) If bit $2n$ of x is 0, the next odd number after x in sequence is $\frac{3x+1}{2^{2n}}$.

Proof. (i)

$$\frac{3x+1}{2^{2n-1}} = \frac{3(2^{2n-1}\kappa + \alpha_n) + 1}{2^{2n-1}} = \frac{2^{2n-1}3\kappa + (3\alpha_n + 1)}{2^{2n-1}} =$$

$$\frac{2^{2n-1}3\kappa + 2^{2n}}{2^{2n-1}} = \frac{2^{2n-1}(3\kappa + 2)}{2^{2n-1}} = (3\kappa + 2).$$

To start the proof, it must be shown that $3\kappa + 2, \kappa$ are odd numbers. We proceed as follows:
Since the $2n-1$ bits on the right side of x are exactly the same as the α_n and are somewhere between 1 and 0, when the $2n^{th}$ bit of the number x is 1, this means that the number 2^{2n-1} is multiplied by the odd number κ and added to the α_n to recieve number x (the first bit of κ is $2n^{th}$ bit of the number x). So $3\kappa+2$, the next term in the sequence and κ are odd numbers.

(ii)

$$\frac{3x+1}{2^{2n}} = \frac{3(2^{2n}\kappa + \alpha_n) + 1}{2^{2n}} = \frac{2^{2n}3\kappa + (3\alpha_n + 1)}{2^{2n}} =$$

$$\frac{2^{2n}3\kappa + 2^{2n}}{2^{2n}} = \frac{2^{2n}(3\kappa + 1)}{2^{2n}} = (3\kappa + 1).$$

To continue the proof, it must be shown that $3\kappa + 1$ is an odd number and κ is an even number. We proceed as follows:
Since the $2n-1$ bits on the right side of x are exactly the same as the α_n so they are somewhere 1 or 0. Because in this case the $2n^{th}, 2n+1^{th}$ bits of the number x is 0, this means that the number 2^{2n} is multiplied by the even number κ and added to the α_n to recieve number x (the first bit of κ is $2n+1^{th}$ bit of the number x). So $3\kappa+1$, the next term in the sequence is odd number and κ is even number.

To complete the verification of lemma, it must be shown that $(3\alpha_n+1) = 2^{2n}$. We proceed as follows:
Clearly, we have

$$3\alpha_n = 2\alpha_n + \alpha_n.$$

Applying this identity simplifies the proof of $(3\alpha_n + 1) = 2^{2n}$. $2\alpha_n$ in binary representation is equivalent to one shift of α_n to the left and the right bit of

$2\alpha_n$ is 0. Therefore in binary representation, $3\alpha_n = 2\alpha_n + \alpha_n$ is equivalent to a $2n$ bits number where all bits are 1. So

$$3\alpha_n = 2^0 + 2^1 + ... + 2^{2n-1} = 2^{2n} - 1.$$

Therefore $3\alpha_n + 1 = 2^{2n}$. □

Definition 4. (**CONS and RCONS**) For summarizing, we name the generated sequence of odd numbrs using 2 rules of lemma 2.6 as the **Collatz Odd Number Sequence** (**CONS**). Also we name 2 rules of lemma 2.6 as the **Rules of Collatz Odd Number Sequence** (**RCONS**). We know that every **CONS** is a subsequence of a **CS**.

3. Probability

At first of this section we use an foundamental axiom about numbers (*Arbitrary Random Selection*). It help us to prove $3x + 1$ problem based on choice of first number in the problem.

> **Axiom (Arbitrary Selection)**. For the purpose of our probabilistic method, the initial value x from definition 1 x is an arbitrary integer is treated as an arbitrary positive integer.

Natural density is finitely additive on disjoint sets whenever the relevant densities exist. However, the family of all sets having natural density is not, in general, a σ-algebra, and natural density does not define a countably additive probability measure on 2^{Ω}. These distinctions are part of the general measure-theoretic study of density [3, 26, 16].
The probabilistic interpretation requires a distinction between natural density and a probability measure. Natural density is defined through limiting proportions and is a classical tool in number theory. The natural density or asymptotic density of a set A of positive integers, if it exists, is

$$d(A) = \lim_{n \longrightarrow \infty} \frac{|A \cap \{1, 2, \cdots, n\}|}{n}.$$

However, it is not defined for every subset of the positive integers, and it does not generally provide a countably additive probability measure on the entire power set. But it can be used as a model to calculate probability measure.

Definition 5. We let

$$\Omega = \{1, 3, 5, \ldots\}$$

be the set of positive odd integers, and $\forall N \in \mathbb{N}$

$$\Omega_N = \{x \in \Omega : x \leq N\}.$$

Then

$$|\Omega_N| = \frac{N}{2} + O(1).$$

For $S \subseteq \Omega$, whenever the limit exists, we define the relative natural density

$$d_{\text{odd}}(S) = \lim_{N \to \infty} \frac{|S \cap \Omega_N|}{|\Omega_N|}.$$

We do not seek to extend natural density to every subset of Ω. Rather, we restrict our attention to a σ-algebra generated by the congruence events relevant to the representation studied in this paper.

The use of measure-theoretic ideas in the study of density goes back to Buck [3]. Related developments include investigations of Buck measure density [26, 27, 28] and the additivity properties of density measures [16]. Finitely additive probability constructions on subsets of the natural numbers have also been studied by Kerkvliet and Meester [12]. The present work addresses a different question: whether relative natural density can serve as a countably additive probability measure on a σ-algebra adapted to a particular congruence representation.

To address this question, we consider the joint events determined by the maximum congruence index and the residue class modulo three. These events generate the σ-algebra used throughout the paper. We establish that relative natural density is countably additive on this σ-algebra and consequently defines a probability measure there.

The main contributions of this paper are as follows:

(i) We establish a congruence representation for every positive odd integer;

(ii) We construct a σ-algebra on which relative natural density defines a countably additive probability measure;

(iii) We derive exact distributions for the congruence index and the associated residue variables;

(iv) We obtain a transition matrix describing the evolution of the quotient modulo three;

(v) We compare the resulting natural-density probability model with alternative probability models that allow nonuniform residue distributions.

3.1. Probability space

Probability theory begins with a mathematical structure called a *probability space*. A probability space provides a rigorous framework for describing random phenomena, specifying which events are measurable, and assigning probabilities to those events.

Definition 6 (Probability Space)**.** A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where:

(i) Ω is the *sample space*, is the set of possible outcomes;

(ii) $\mathcal{F}$ is a σ-algebra of subsets of Ω and determines which collections of outcomes are events at all;

(iii) $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$ and assigns probabilities to the events, $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$.

In the next sections, we explain the three components based on definition 4 in our paper to prove $3x + 1$ problem. Here for every term of **CONS**, we need to define random variable for a (the power of 2) in the representatin $x = 2^a\kappa + \alpha_{[\frac{(a)+1}{2}]}$. So in the sections 3.2 , 3.3 , 3.4 , 3.5 we define sample space and introduce our basic idea for σ-algebra and probability measure that be used in two discrete probability models in our paper.

3.2. Sample space, σ-algebra, probability measure

Definition 7 (Sample Space)**.** The *sample space*, denoted by $\Omega = \mathbb{Z}^+_{odd}$, is the set of all positive odd numbers those are all possible elementary outcomes in our experiment. Here in $3x + 1$ problem, select an arbitrary positive odd number is our experiment.

Definition 8 (σ-Algebra)**.** We let $\Omega = \mathbb{Z}^+_{odd}$. $\forall a \in \mathbb{N}$ we define subset $F_a \subseteq \Omega$ such that :

$$F_a = \{x = 2^i\kappa + \alpha_{[\frac{i+1}{2}]} | i = a\} = \{2^a\kappa + \alpha_{[\frac{a+1}{2}]}\},$$

and call $\mathcal{F} = \{\cup_{a \in N \subseteq \mathbb{N}} F_a\} \subseteq 2^\Omega$, as a σ-algebra on Ω based on lemma 2.4 . Because it satisfies below three conditions:

(i) we can write $\Omega = \cup_{a\in N=\mathbb{N}} F_a \in \mathcal{F}$.

(ii) if $F = \cup_{a\in N\subseteq\mathbb{N}} F_a \in \mathcal{F}$, then $F^c = \Omega \setminus F = \cup_{a'\in\mathbb{N}\setminus N\subseteq\mathbb{N}} F_{a'} \in \mathcal{F}$.

(iii) if $E_n = \cup_{a\in N_n\subseteq\mathbb{N}} F_a \in \mathcal{F}$, $n \in \mathbb{N}$, then

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \cup_{a\in N_n\subseteq\mathbb{N}} F_a = \cup_{a\in\bigcup_{n=1}^{\infty} N_n\subseteq\mathbb{N}} F_a \in \mathcal{F}.$$

The elements of $\mathcal{F}$ are called *events*. From properties 1 , 2 , it follows that $\varnothing \in \mathcal{F}$. Also $\mathcal{F}$ is closed under countable intersections:

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} \cup_{a\in N_n\subseteq\mathbb{N}} F_a = \cup_{a\in\bigcap_{n=1}^{\infty} N_n\subseteq\mathbb{N}} F_a \in \mathcal{F}.$$

Also because of uniqe representation of positive odd numbers based on lemma 2.4 we have:

$$\forall a, b \in \mathbb{N}, a \neq b \to F_a \cap F_b = \emptyset \Longrightarrow \cap_{a\in N\subseteq\mathbb{N}} F_a = \emptyset.$$

Corollary 1. $F, E \in \mathcal{F}$ are disjoint events if and only if $\exists N, M \subseteq \mathbb{N} : N \cap M = \emptyset$ and $F = \cup_{a\in N} F_a, E = \cup_{a\in M} F_a$.

By the lemma 3.1, based on natural density we can calculate the probability of an event $F_a \in \mathcal{F}$.

Lemma 3.1. *Based on lemma 2.4, we have:*

$$\forall a \in \mathbb{N} : \mathbb{P}(F_a) = \frac{1}{2^a}.$$

Proof. At first we know that

$$\forall a \in \mathbb{N}, \exists b \in \mathbb{N}(b < \infty) \quad \text{such that} \quad a < b.$$

So

$$\forall x \in F_a, \text{size}(x) \leq b.$$

Here we use of asymptotic density over all positive odd numbers $x \in \Omega_b$ that $\text{size}(x) \leq b$, so $|\Omega_b| = 2^{b-1}$. For an arbitrary $a \in \mathbb{N}, a < b$, we need to count all odd numbers

$$x \in F_a = \{2^a\kappa + \alpha_{[\frac{a+1}{2}]}\}, \text{size}(x) \leq b.$$

If $a = 2n-1$, $\alpha_{[\frac{a+1}{2}]} = \alpha_n$ has $a = 2n-1$ bits, so it fix on a rightmost bits of x in exactly one way and bit $a+1 = 2n$ is equal to 1 in exactly one way. Now the number of positive odd numbers x satisfying these conditions can be determined by a straightforward combinatorial count. All bits more significant than position $a+1 = 2n$ of number x are independent and identically distributed Bernoulli($\frac{1}{2}$) random variables and each of them can take on 2 values: 0 and 1. In this condition all of possible states of x are equal to 2^{b-a-1} that are all bits more significant than position $2n-1$ (from $a+2 = 2n+1$ to b), because size$(x) \leq b$.
Now the probability of this occurring for m is equal to

$$\mathbb{P}(F_a) = d(F_a) = \lim_{b\longrightarrow\infty} \frac{|\{x|x = 2^a\kappa + \alpha_{[\frac{a+1}{2}]}\} \cap \Omega_b|}{|\Omega_b|} = \lim_{b\longrightarrow\infty} \frac{1 * 1 * 2^{b-a-1}}{2^{b-1}} = 2^{-a} = \frac{1}{2^a}.$$

If $a = 2n$, $\alpha_{[\frac{a+1}{2}]} = \alpha_n$ has $a-1 = 2n-1$ bits, so it fix on $a-1$ rightmost bits of x in exactly one way and each of bits $a = 2n, a+1 = 2n+1$ are equal to 0 in exactly one way. Now the number of positive odd numbers x satisfying these conditions can be determined by a straightforward combinatorial count. All bits more significant than position $a+1 = 2n+1$ of number x are independent and identically distributed Bernoulli($\frac{1}{2}$) random variables and each of them can take on 2 values: 0 and 1. In this condition all of possible states of x are equal to 2^{b-a-1} that is all bits more significant than position $2n+1$ (from $a+2 = 2n+2$ to b), because size$(x) \leq b$.
Now the probability of this occurring for x is equal to

$$\mathbb{P}(F_a) = d(F_a) = \lim_{b\longrightarrow\infty} \frac{|\{x|x = 2^a\kappa + \alpha_{[\frac{a+1}{2}]}\} \cap \Omega_b|}{|\Omega_b|} = \lim_{b\longrightarrow\infty} \frac{1 * 1 * 1 * 2^{b-a-1}}{2^{b-1}} = 2^{-a} = \frac{1}{2^a}.$$

□

Here, we find a clear result as the corollary 2 from proof of lemma 3.1.

Corollary 2. Because of all $b-a-1$ leftmost bits of m are independent and identically distributed Bernoulli($\frac{1}{2}$) random variables, so the probability of occurring every κ is equal to $\frac{1}{2^{b-a-1}}$ where $\kappa \in [0, 2^{b-a}-1]$. It is correct for a finite b.

Now based on lemma 3.1 we define probability measure as definition 9.

Definition 9 (Probability Measure)**.** We define a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ as a probability measure with rule :

$$\forall N \subseteq \mathbb{N} : \mathbb{P}((\cup_{a\in N} F_a) \in \mathcal{F}) = \sum_{a\in N} \frac{1}{2^a},$$

because of satisfies as following:

(P1) For every $F_a \in \mathcal{F}$, we set $N = \{a\}$, so $F_a = \cup_{i\in N} F_i$,

$$\mathbb{P}(F_a) = \sum_{i\in N} \frac{1}{2^i} = \frac{1}{2^a} \geq 0.$$

(P2) Because $\forall i, j(i \neq j)F_i \cap F_j = \emptyset$ and $\forall F \neq \emptyset \in \mathcal{F}, \exists N \subseteq \mathbb{N} : F = \cup_{a\in N} F_a$, so

$$\mathbb{P}(F) = \mathbb{P}(\cup_{a\in N} F_a) = \sum_{a\in N} \mathbb{P}(F_a) = \sum_{a\in N} \frac{1}{2^a} \geq 0.$$

(P3) Also we have,

$$\mathbb{P}(\Omega = \cup_{a\in N=\mathbb{N}} F_a) = \sum_{a\in\mathbb{N}} \mathbb{P}(F_a) = \sum_{a\in\mathbb{N}} \frac{1}{2^a} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

(P4) Based on corollary 1 for every sequence of pairwise disjoint events $E_1, E_2, \ldots \in \mathcal{F}$, $\exists N_1, N_2, \ldots \subset \mathbb{N} \quad s.t : \forall i, j(i \neq j)N_i \cap N_j = \emptyset$. So

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} \cup_{a\in N_n} F_a\right) = \mathbb{P}\left(\cup_{a\in\bigcup_{n=1}^{\infty} N_n} F_a\right) =$$

$$\sum_{a\in\bigcup_{n=1}^{\infty} N_n} \mathbb{P}(F_a) = \sum_{n=1}^{\infty} \sum_{a\in N_n} \mathbb{P}(F_a) = \sum_{n=1}^{\infty} \mathbb{P}(E_n).$$

The last property is called *countable additivity* or σ-additivity.

3.3. Random variable

A random variable is a mathematical function that assigns a numerical value to each outcome of a random experiment.

Definition 10 (Random Variable)**.** We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A real-valued random variable is a measurable function

$$A : \Omega \longrightarrow \mathbb{R},$$

where $\mathbb{R}$ is equipped with its Borel σ-algebra $\mathcal{B}(\mathbb{R})$. Measurability means that for every Borel set $B \in \mathcal{B}(\mathbb{R})$, we have

$$A^{-1}(B) = \{\omega \in \Omega : A(\omega) \in B\} \in \mathcal{F}.$$

An equivalent characterization is

$$\{\omega \in \Omega : A(\omega) \leq a\} \in \mathcal{F}, \qquad \forall a \in \mathbb{R}.$$

This condition give us a confidence that mathematically, probabilities such as $\mathbb{P}(A \leq a)$ are well-defined. As we define in section 3.2, we have

$$\mathbb{P}(A \leq a) = \sum_{i=1}^{a} \frac{1}{2^i}.$$

In our problem, we have a natural-value random variable that we define in section 3.4.

3.4. Discrete random variable

The concept of a discrete random variable is particularly important because its probability distribution can be described by assigning probabilities to individual values. In section 3.3 we are defined a real-value random variable with Borel $\mathcal{B}(\mathbb{R})$.

Definition 11 (Discrete Random Variable)**.** We let $A : \Omega \to \mathbb{R}$ be a random variable. We say that A is *discrete* if there exists a finite or countably infinite set $S \subseteq \mathbb{R}$ such that $\mathbb{P}(A \in S) = 1$.
In $3x + 1$-problem $S = \mathbb{N}$ is called a countable support set of A.

Thus, our discrete random variable can take its values in the set

$$S = \mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

And so the probability is concentrated on a countable set $\mathbb{P}(A \in S) = 1$.

3.5. Probability mass function

Definition 12 (Probability Mass Function)**.** Let A be a discrete random variable. Its probability mass function (PMF) is the function $p_A : \mathbb{N} \to [0, 1]$ and defined by

$$p_A(a) = \mathbb{P}(A = a) = \frac{1}{2^a}.$$

The PMF satisfies the fundamental properties

$$p_A(a) \geq 0 \qquad \forall a \in \mathbb{N},$$

and

$$\sum_{a \in \mathbb{N}} p_A(a) = \sum_{a \in N} \frac{1}{2^a} = 1.$$

If $a \notin \mathbb{N}$, then

$$p_A(a) = 0.$$

3.6. Random variable versus random phenomenon

It is important to distinguish between a random phenomenon and the random variable used to mathematically represent it. A random variable is not a physical phenomenon. It is a mathematical function associated with the outcomes of that phenomenon. Here in $3x + 1$-problem we want to select an odd number randomly as our random phenomenon in a random experiment. Then the selected random odd number is random outcome that we can represent it as $2^a\kappa + \alpha_{[\frac{a+1}{2}]}$. Now A is the random variable that take numerical quantity $a \in \mathbb{N}$. We can state that the random variable A take the power of 2 in representation $x = 2^a\kappa + \alpha_{[\frac{a+1}{2}]}$ for every arbitrary odd number x that we select it randomly as a random phenomenon.
In the proof of lemma 2.6, for a selected random odd number x_0, the next term of sequence is as $3\kappa_0 + 2$ or $3\kappa_0 + 1$, when $x_0 = 2^{a_0}\kappa_0 + \alpha_{n_0}$ and $a_0 = 2n_0 - 1$ or $2n_0$ respectively. Moreover, corollary 2 shows that for every fixed a_0, b, κ_0 is uniformly distributed over integer values of $[0, 2^{b-a} - 1]$, so we can select it randomly.
In the sections 4, 5 and 6 we discuss about some discrete random variables that are usefull to solve our problem.

4. σ-algebras for congruence representation

Congruence conditions often provide a useful way to represent integers by means of a discrete index and an associated quotient. The present work studies special representation for positive odd integers as section 2, with particular attention to the distribution of the index and the dependence structure of the associated residue variables.
As we stated and proved in sections 3.2 , 3.3 , 3.4 , 3.5 , in short we assumed a single index event as below:

$$E_a = \{x \in \Omega : A(x) = a\}.$$

By the preceding lemma, E_a is only odd residue class modulo 2^{a+1}. Consequently,

$$d_{\text{odd}}(E_a) = 2^{-a}.$$

This give us the index distribution that we are proved it at lemma 3.1.

$$\mathbb{P}(A = a) = 2^{-a}, \qquad a \geq 1,$$

once the appropriate probability space has been constructed. In particular,

$$\mathbb{P}(A \geq n) = \sum_{a=n}^{\infty} 2^{-a} = 2^{-(n-1)}, \quad \mathbb{P}(A \leq n) = 1 - 2^{-n}.$$

Also we have probabilities of parity A as

$$\mathbb{P}(A \text{ even}) = \sum_{k=1}^{\infty} 2^{-2k} = \frac{1}{3}, \quad \mathbb{P}(A \text{ odd}) = \sum_{k=1}^{\infty} 2^{-(2k-1)} = \frac{2}{3}.$$

The expectation and variance of A are

$$\mathbb{E}[A] = 2, \qquad \text{Var}(A) = 2.$$

4.1. The generated σ-algebra using joint congruence events

When we define

$$R(x) = x \bmod 3, \qquad R(x) \in \{0, 1, 2\}.$$

For $a \geq 1$ and $r \in \{0, 1, 2\}$, we can let

$$E_{a,r} = \{x \in \Omega : A(x) = a,\ R(x) = r\}.$$

The events $E_{a,r}$ are pairwise disjoint and form a countable partition of Ω:

$$\Omega = \bigcup_{a=1}^{\infty} \bigcup_{r=0}^{2} E_{a,r}.$$

The Chinese remainder theorem shows that the following system of congruences has exactly one residue class modulo $3 \cdot 2^{a+1}$.

$$x \equiv \rho_a \pmod{2^{a+1}}, \qquad x \equiv r \pmod{3}$$

This class consists of odd integers and its relative density among the positive odd integers is therefore

$$d_{\text{odd}}(E_{a,r}) = \frac{1}{3 \cdot 2^a}.$$

Since the $E_{a,r}$ form a countable partition, we can generate the σ-algebra $\mathcal{G}$ consists of the arbitrary unions of $E_{a,r}$ for $a \in \mathbb{N}, r \in \{0,1,2\}$. Thus

$$\mathcal{G} = \left\{ \bigcup_{(a,r)\in J} E_{a,r} : J \subseteq \mathbb{Z}^+ \times \{0,1,2\} \right\}.$$

Because each $E_{a,r}$ is infinite, so every nonempty member of $\mathcal{G}$ is infinite. Thus

$$\nexists \quad \text{finite} \quad E \subset \Omega : \quad E \in \mathcal{G}$$

4.2. The probability measure by natural density and joint distribution of A and R

For every $J \subseteq \mathbb{Z}^+ \times \{0,1,2\}$, we define

$$E_J = \bigcup_{(a,r)\in J} E_{a,r}.$$

Theorem 4.1. *For every $E_J \in \mathcal{G}$, the relative natural density exists and satisfies*

$$d_{\text{odd}}(E_J) = \sum_{(a,r)\in J} \frac{1}{3 \cdot 2^a}.$$

We can define a countably additive probability measure on $(\Omega, \mathcal{G})$ as

$$\mathbb{P}(E_J) = \sum_{(a,r)\in J} \frac{1}{3 \cdot 2^a} \longrightarrow \mathbb{P}(E_J) = d_{\text{odd}}(E_J) \quad \textit{for every } E_J \in \mathcal{G}.$$

Proof. For $m \geq 1$, we define the finite union

$$E_J^{(m)} = \bigcup_{\substack{(a,r)\in J \\ a\leq m}} E_{a,r} \quad \xrightarrow{\text{finite union of disjoint events}} \quad d_{\text{odd}}(E_J^{(m)}) = \sum_{\substack{(a,r)\in J \\ a\leq m}} \frac{1}{3\cdot 2^a}.$$

The omitted part $E_J \setminus E_J^{(m)} = \{x \in \Omega : A(x) > m\} = \bigcup_{a>m} E_a$ as an event has density is

$$\sum_{a=m+1}^{\infty} 2^{-a} = 2^{-m}.$$

Therefore this part is bounded above by a quantity tending to zero as $m \to \infty$. It follows that

$$d_{\text{odd}}(E_J) = \lim_{m\to\infty} \sum_{\substack{(a,r)\in J \\ a\leq m}} \frac{1}{3\cdot 2^a} = \sum_{(a,r)\in J} \frac{1}{3\cdot 2^a}, \qquad \sum_{a=1}^{\infty}\sum_{r=0}^{2} \frac{1}{3\cdot 2^a} = \sum_{a=1}^{\infty} 2^{-a} = 1.$$

Thus the formula defines a probability measure on $\mathcal{G}$. For any countable disjoint family of members of $\mathcal{G}$, their index sets are disjoint, so countable additivity follows from countable additivity of sums of nonnegative terms. □

Remark 1. The theorem does not assert that the natural density defines a countably additive measure on 2^{Ω}. It proves that the restriction of relative natural density to the particular σ-algebra $\mathcal{G}$ is a countably additive probability measure. This is the measure-theoretic construction used throughout the paper.

Theorem 4.2. *In the probability space* $(\Omega, \mathcal{G}, \mathbb{P})$*,*

$$\mathbb{P}(A = a, R = r) = \frac{1}{3\cdot 2^a}.$$

Consequently,

$$\mathbb{P}(A = a) = 2^{-a}, \qquad \mathbb{P}(R = r) = \frac{1}{3},$$

and A and R are independent.

Proof. The event $\{A = a, R = r\}$ equals $E_{a,r}$, so its probability is $1/(3\cdot 2^a)$. Now by summing over r, a we have

$$\mathbb{P}(A = a) = \sum_{r=0}^{2} \frac{1}{3\cdot 2^a} = 2^{-a}, \quad \mathbb{P}(R = r) = \sum_{a=1}^{\infty} \frac{1}{3\cdot 2^a} = \frac{1}{3}.$$

So,

$$\mathbb{P}(A = a, R = r) = \mathbb{P}(A = a)\mathbb{P}(R = r),$$

which proves that A and R are independent. □

4.2.1. Related random variables and distributions

In this section we discuss about random variables related by our representation of odd numbers. So we define

$$B = A \bmod 2, \qquad D = A \bmod 3, \qquad E = \kappa \bmod 2, \qquad S = \kappa \bmod 3,$$

and the variable R that remains $R = X \bmod 3$. The variables A and R are measurable with respect to $\mathcal{G}$ by construction. The variables B and D are functions of A. By the parity lemma, $E = B$. As shown below, S is a function of A and R. Thus all variables used in the distribution and independence analysis are measurable on $(\Omega, \mathcal{G})$. Now we have,

$$E = B \longrightarrow \mathbb{P}(B = 0) = \mathbb{P}(E = 0) = \frac{1}{3}, \quad \mathbb{P}(B = 1) = \mathbb{P}(E = 1) = \frac{2}{3}.$$

By using $\mathbb{P}(A = a) = 2^{-a} = \frac{1}{2^a}$, we have

$$\mathbb{P}(D = 0) = \sum_{k=1}^{\infty} \frac{1}{2^{3k}} = \frac{1}{7}, \quad \mathbb{P}(D = 1) = \sum_{k=0}^{\infty} \frac{1}{2^{(3k+1)}} = \frac{4}{7}, \quad \mathbb{P}(D = 2) = \sum_{k=0}^{\infty} \frac{1}{2^{(3k+2)}} = \frac{2}{7}.$$

$A \bmod 6$ has residues ordered as $0, 1, 2, 3, 4, 5$, so we have

$$\mathbb{P}(A \equiv 0 \pmod 6) = \sum_{k=1}^{\infty} 2^{-6k} = \frac{1}{63}, \quad \mathbb{P}(A \equiv 1 \pmod 6) = \sum_{k=0}^{\infty} 2^{-(6k+1)} = \frac{32}{63}, \cdots$$

and the other entries follow by the same geometric-series calculation. So the distribution of $A \bmod 6$ is

$$(\frac{1}{63}, \frac{32}{63}, \frac{16}{63}, \frac{8}{63}, \frac{4}{63}, \frac{2}{63})$$

4.2.2. General construction of the conditional transition matrix

We let

$$R = X \pmod 3, \qquad S = \kappa \pmod 3,$$

Since $X = 2^A\kappa + c_A$ and $2 \equiv -1 \pmod 3$, we obtain

$$S \equiv \begin{cases} R - c_A \pmod 3, & A \text{ is even}, \\ c_A - R \pmod 3, & A \text{ is odd}. \end{cases} \tag{3}$$

For each $r, s \in \{0, 1, 2\}$, we define the conditional transition probability by

$$T_{rs} = P(S = s \mid R = r).$$

Now we let

$$q_{a|r} = P(A = a \mid R = r), \qquad a \geq 1, \quad r \in \{0, 1, 2\}.$$

Using the congruence relation in equation 3, the general formula for the transition matrix is

$$T_{rs} = \sum_{\substack{a \geq 1 \\ a \text{ even}}} q_{a|r}\, \mathbf{1}_{\{s \equiv r - c_a \pmod 3\}} + \sum_{\substack{a \geq 1 \\ a \text{ odd}}} q_{a|r}\, \mathbf{1}_{\{s \equiv c_a - r \pmod 3\}} \tag{4}$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function:

$$\mathbf{1}_{\{s \equiv r - c_a \pmod 3\}} = \begin{cases} 1, & s \equiv r - c_a \pmod 3, \\ 0, & \text{other wise} \end{cases},$$

and

$$\mathbf{1}_{\{s \equiv c_a - r \pmod 3\}} = \begin{cases} 1, & s \equiv c_a - r \pmod 3, \\ 0, & \text{other wise} \end{cases}.$$

Consequently, the conditional transition matrix is

$$T = \begin{pmatrix} T_{00} & T_{01} & T_{02} \\ T_{10} & T_{11} & T_{12} \\ T_{20} & T_{21} & T_{22} \end{pmatrix}.$$

Since $\sum_{a=1}^{\infty} q_{a|r} = 1$, so for every r with $P(R = r) > 0$, the entries of T satisfy

$$T_{rs} \geq 0, \qquad \sum_{s=0}^{2} T_{rs} = 1.$$

Thus, T is a row-stochastic matrix. In the natural-density model, A and R are independent and

$$P(A = a \mid R = r) = P(A = a) = \frac{1}{2^a}.$$

Therefore, formula 4 reduces to

$$T_{rs} = \sum_{\substack{a \geq 1 \\ a \text{ even}}} \frac{1}{2^a} \mathbf{1}_{\{s \equiv r - c_a \pmod 3\}} + \sum_{\substack{a \geq 1 \\ a \text{ odd}}} \frac{1}{2^a} \mathbf{1}_{\{s \equiv c_a - r \pmod 3\}}.$$

More generally, if the marginal distribution of R is changed while the conditional distributions $P(A = a \mid R = r)$ remain unchanged, the transition matrix T does not change. Only the resulting marginal distribution of S changes. In particular, if $p_R = (p_0, p_1, p_2)$ is the row vector representing the distribution of R, then the distribution of S is given by $p_S = p_R T$. However, if the conditional distributions $P(A = a \mid R = r)$ are modified, the entries of T must be recalculated using the general formula 4. Here in our paper, we need to discuss about 2 special vectors

$$p_R = (p_0, p_1, p_2) = \begin{cases} (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), \\ (0, \frac{1}{3}, \frac{2}{3}). \end{cases}$$

In the section 5, we explain how to calculate conditional transition matrix T and use it to obtain $p_S = p_R T$.

4.3. Alternative discrete probability models

The natural-density model forces R to be uniform on $\{0, 1, 2\}$. To consider other residue distributions, we define a separate probability model.
So we let

$$p = (p_0, p_1, p_2), \qquad p_r \geq 0, \qquad p_0 + p_1 + p_2 = 1,$$

and set

$$\Omega_p = \mathbb{Z}^+ \times \{0, 1, 2\}, \qquad \mathcal{H}_p = 2^{\Omega_p}.$$

So we assign a probability to each event. It can be concluded

$$\mathbb{P}_p(\{(a, r)\}) = 2^{-a} p_r \Longrightarrow \sum_{a=1}^{\infty} \sum_{r=0}^{2} 2^{-a} p_r = 1.$$

Now this defines a probability measure on $(\Omega_p, \mathcal{H}_p)$. By defining coordinate random variables

$$A_p(a, r) = a, \qquad R_p(a, r) = r,$$

we have

$$\mathbb{P}_p(A_p = a) = 2^{-a}, \qquad \mathbb{P}_p(R_p = r) = p_r, \qquad \mathbb{P}_p(A_p = a, R_p = r) = 2^{-a} p_r.$$

Therefore, we can conclude,

$$A_p \text{ and } R_p \text{ are independent for every } p.$$

Remark 2. This alternative model is a genuine discrete probability space, but it is not the same construction as the natural-density probability space on $(\Omega, \mathcal{G})$. When $p = (1/3, 1/3, 1/3)$, the two models have the same joint distribution of A and R. A nonuniform p defines an alternative probability model rather than natural-density sampling of positive odd integers.

We can assume every residue distribution for R. If we let uniform residue distribution, we have the residue distribution in the natural-density model. So by affecting transition matrix

$$p^{(1)} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \Longrightarrow p_S = p^{(1)} T = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Thus S is uniform modulo three in this case.
Since A and R are independent, every function of A, including B, D, E, is independent of R. Now we consider a nonuniform residue distribution and affect transition matrix as below:

$$p^{(2)} = \left(0, \frac{1}{3}, \frac{2}{3}\right) \Longrightarrow p_S = p^{(2)} T = \left(\frac{8}{21}, \frac{5}{21}, \frac{8}{21}\right).$$

Here $R = 0$ has probability zero, while

$$\mathbb{P}_p(R = 1) = \frac{1}{3}, \qquad \mathbb{P}_p(R = 2) = \frac{2}{3}.$$

This distribution is not uniform. Moreover, S and R are dependent, since their conditional distribution matrix has different rows on the residue classes with positive probability. As in the uniform model, B, D, E are functions of A and are independent of R.

4.4. Independence results

Theorem 4.3. *In the natural-density probability space* $(\Omega, \mathcal{G}, \mathbb{P})$*:*

(i) A *and* R *are independent;*

(ii) each of B, D, E *is independent of* R*;*

(iii) S *and* R *are dependent.*

Proof. The first assertion follows from

$$\mathbb{P}(A = a, R = r) = \mathbb{P}(A = a)\mathbb{P}(R = r).$$

Since B, D, E are functions of A, their independence from R follows from the independence of A and R. For the final assertion, the transition matrix has different rows. For example,

$$\mathbb{P}(S = 0 \mid R = 0) = \frac{3}{63}, \qquad \mathbb{P}(S = 0 \mid R = 1) = \frac{48}{63}.$$

Both conditioning events have positive probability, and the conditional probabilities differ. Therefore S and R are dependent. □

Theorem 4.4. *In the alternative model* $(\Omega_p, \mathcal{H}_p, \mathbb{P}_p)$*,* A_p *and* R_p *are independent for every probability vector* p*. Every measurable function of* A_p *is consequently independent of* R_p*.*

Proof. For every $a \geq 1$ and $r \in \{0, 1, 2\}$,

$$\mathbb{P}_p(A_p = a, R_p = r) = 2^{-a} p_r = \mathbb{P}_p(A_p = a)\mathbb{P}_p(R_p = r).$$

This proves independence. Independence is preserved when a measurable function is applied to one of the variables. □

Remark 3. If $p_r = 0$, the ordinary conditional probability $\mathbb{P}_p(S = s \mid R_p = r)$ is undefined because its conditioning event has probability zero. The matrix T can still be used as a transition rule specified by the congruence representation and the distribution of the index. Its row for a zero-probability residue does not affect the resulting joint distribution.

The σ-algebra $\mathcal{G}$ is adapted to the variables A and R. Its elements are arbitrary unions of the cells $E_{a,r}$, and the natural density of every such union exists. The density formula is a sum of the cell densities, and this representation yields countable additivity.
This construction differs from approaches that seek finitely additive extensions of density to larger families of sets [3, 16, 12]. The present paper does not claim to resolve the general problem of extending natural density to all subsets of the positive integers. Instead, it provides a concrete probability space suited to the congruence variables studied here and use it to solve $3x+1$ problem.
The uniform residue model agrees with natural density at the level of the joint distribution of A and R. Nonuniform residue distributions belong to a separate family of probability models. Keeping these models distinct is necessary for interpreting the independence results correctly.
We have defined a σ-algebra generated by the joint congruence events specifying the maximum congruence index and the residue modulo three. Relative natural density restricted to this σ-algebra is a countably additive probability measure.
The index has distribution

$$\mathbb{P}(A = a) = 2^{-a}, \qquad a \geq 1,$$

and the residue R is uniform modulo three and independent of A. The variables B, D, E are independent of R, whereas $S = \kappa \bmod 3$ and R are dependent, as demonstrated by the transition matrix.
Finally, we have defined alternative discrete probability models with arbitrary residue distributions. These models preserve the index distribution and the independence of the index and residue coordinates, but nonuniform residue distributions should not be interpreted as natural-density sampling of positive odd integers.

5. Calculation of the conditional transition matrix

In this subsection, we derive the conditional transition matrix connecting the residue classes of the odd integer X and its corresponding quotient κ modulo 3. Throughout this subsection, we use the representation

$$X = 2^A \kappa + c_A, \qquad c_a = \alpha_{\lfloor \frac{a+1}{2} \rfloor}, \qquad \alpha_k = \frac{4^k - 1}{3} \tag{5}$$

Now we define the random variables

$$R = X \bmod 3, \qquad S = \kappa \bmod 3, \qquad C = c_A \bmod 3. \tag{6}$$

All residue classes in this subsection are represented by elements of $\{0, 1, 2\}$. Now by reducing (5) modulo 3, we obtain

$$R \equiv 2^A S + C \pmod 3. \tag{7}$$

Since $2 \equiv -1 \pmod 3$, solving (7) for S yields

$$\begin{cases} 2^A \equiv 1 \pmod 3 \longrightarrow S \equiv R - C \pmod 3, & A \text{ is even}, \\ 2^A \equiv -1 \pmod 3 \longrightarrow S \equiv C - R \pmod 3, & A \text{ is odd}. \end{cases} \tag{8}$$

This relation is the key identity used in the derivation of the transition matrix.

5.1. Distribution of the index A

Recall that the exact congruence characterization of the event $A = a$ is

$$A = a \iff X \equiv \rho_a \pmod{2^{a+1}}, \tag{9}$$

where ρ_a is defined in the relation 2. Thus, with respect to the natural-density probability measure introduced in the previous section,

$$\mathbb{P}(A = a) = \frac{1}{2^a}, \qquad a \geq 1. \tag{10}$$

Moreover, for each $a \geq 1$ and $r \in \{0, 1, 2\}$, the Chinese remainder theorem guarantees exactly one residue class modulo $3 \cdot 2^{a+1}$ satisfying

$$\begin{cases} X \equiv \rho_a \pmod{2^{a+1}}, \\ X \equiv r \pmod 3. \end{cases}$$

Therefore,

$$\mathbb{P}(A = a, R = r) = \frac{1}{3 \cdot 2^a} \Longrightarrow \mathbb{P}(A = a \mid R = r) = \frac{1}{2^a}. \tag{11}$$

and hence A and R are independent. In particular,

$$\mathbb{P}(A \text{ is even}) = \sum_{k=1}^{\infty} \frac{1}{2^{2k}} = \frac{1}{3}, \qquad \mathbb{P}(A \text{ is odd}) = \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} = \frac{2}{3} \tag{12}$$

5.2. Distribution of $C = c_A \bmod 3$

The powers of 4 modulo 9 are periodic with period 3:

$$4^1 \equiv 4, \qquad 4^2 \equiv 7, \qquad 4^3 \equiv 1 \pmod 9.$$

Consequently, from the definition of α in equation 1, we have

$$\alpha_k \bmod 3 = \begin{cases} 1, & k \equiv 1 \pmod 3, \\ 2, & k \equiv 2 \pmod 3, \\ 0, & k \equiv 0 \pmod 3. \end{cases} \tag{13}$$

We consider separately the even and odd values of A and compute probability distribution $C = \alpha_{\lfloor \frac{A+1}{2} \rfloor} \bmod 3$. For even indexes $A = 2k$, we can write

$$\mathbb{P}(A = 2k) = \frac{1}{4^k}, \qquad k \geq 1,$$

and $C = \alpha_k \bmod 3$. Therefore,

$$\mathbb{P}(A \text{ even}, C = 0) = \sum_{j=1}^{\infty} \frac{1}{4^{3j}} = \frac{1}{63}, \tag{14}$$

$$\mathbb{P}(A \text{ even}, C = 1) = \sum_{j=0}^{\infty} \frac{1}{4^{3j+1}} = \frac{16}{63}, \tag{15}$$

$$\mathbb{P}(A \text{ even}, C = 2) = \sum_{j=0}^{\infty} \frac{1}{4^{3j+2}} = \frac{4}{63}. \tag{16}$$

Similarly, for odd indexes $A = 2k - 1$, we have

$$\mathbb{P}(A = 2k - 1) = \frac{2}{4^k}, \qquad k \geq 1,$$

and again $C = \alpha_k \bmod 3$. Thus,

$$\mathbb{P}(A \text{ odd}, C = 0) = 2\sum_{j=1}^{\infty} \frac{1}{4^{3j}} = \frac{2}{63}, \tag{17}$$

$$\mathbb{P}(A \text{ odd}, C = 1) = 2\sum_{j=0}^{\infty} \frac{1}{4^{3j+1}} = \frac{32}{63}, \tag{18}$$

$$\mathbb{P}(A \text{ odd}, C = 2) = 2\sum_{j=0}^{\infty} \frac{1}{4^{3j+2}} = \frac{8}{63}. \tag{19}$$

Adding the even and odd contributions, we obtain the marginal distribution of C:

$$\big(\mathbb{P}(C=0),\mathbb{P}(C=1),\mathbb{P}(C=2)\big)=\left(\frac{1}{21},\frac{16}{21},\frac{4}{21}\right). \tag{20}$$

For the transition matrix, however, the joint distributions of C with the parity of A must be retained, since (8) has different forms according to the parity of A.

5.3. Definition and computation of the transition probabilities

We define the conditional transition probabilities by

$$T_{rs}=\mathbb{P}(S=s\mid R=r),\qquad r,s\in\{0,1,2\}. \tag{21}$$

Using the independence of A and R, together with (8), we obtain

$$T_{rs}=\sum_{\substack{a\geq 1\\ a\ \text{even}}}2^{-a}\mathbf{1}_{\{s\equiv r-c_a \pmod 3\}}+\sum_{\substack{a\geq 1\\ a\ \text{odd}}}2^{-a}\mathbf{1}_{\{s\equiv c_a-r \pmod 3\}},$$

So we now compute the three rows of T.

5.3.1. Case $R=0$

By (8), when A is even, $S\equiv -C \pmod 3$, whereas when A is odd, $S\equiv C \pmod 3$. Hence,

$$T_{00}=\mathbb{P}(A\text{ even},C=0)+\mathbb{P}(A\text{ odd},C=0)=\frac{1}{63}+\frac{2}{63}=\frac{3}{63}, \tag{22}$$

$$T_{01}=\mathbb{P}(A\text{ even},C=2)+\mathbb{P}(A\text{ odd},C=1)=\frac{4}{63}+\frac{32}{63}=\frac{36}{63}, \tag{23}$$

$$T_{02}=\mathbb{P}(A\text{ even},C=1)+\mathbb{P}(A\text{ odd},C=2)=\frac{16}{63}+\frac{8}{63}=\frac{24}{63}. \tag{24}$$

Thus, the first row is

$$(T_{00},T_{01},T_{02})=\frac{1}{63}(3,36,24). \tag{25}$$

5.3.2. Case $R = 1$

When A is even, $S \equiv 1 - C \pmod 3$, while for odd A, $S \equiv C - 1 \pmod 3$. Therefore,

$$T_{10} = \mathbb{P}(A \text{ even}, C = 1) + \mathbb{P}(A \text{ odd}, C = 1) = \frac{16}{63} + \frac{32}{63} = \frac{48}{63}, \tag{26}$$

$$T_{11} = \mathbb{P}(A \text{ even}, C = 0) + \mathbb{P}(A \text{ odd}, C = 2) = \frac{1}{63} + \frac{8}{63} = \frac{9}{63}, \tag{27}$$

$$T_{12} = \mathbb{P}(A \text{ even}, C = 2) + \mathbb{P}(A \text{ odd}, C = 0) = \frac{4}{63} + \frac{2}{63} = \frac{6}{63}. \tag{28}$$

Consequently, the second row is

$$(T_{10}, T_{11}, T_{12}) = \frac{1}{63}(48, 9, 6). \tag{29}$$

5.3.3. Case $R = 2$

When A is even, $S \equiv 2 - C \pmod 3$, whereas for odd A, $S \equiv C - 2 \pmod 3$. It follows that

$$T_{20} = \mathbb{P}(A \text{ even}, C = 2) + \mathbb{P}(A \text{ odd}, C = 2) = \frac{4}{63} + \frac{8}{63} = \frac{12}{63}, \tag{30}$$

$$T_{21} = \mathbb{P}(A \text{ even}, C = 1) + \mathbb{P}(A \text{ odd}, C = 0) = \frac{16}{63} + \frac{2}{63} = \frac{18}{63}, \tag{31}$$

$$T_{22} = \mathbb{P}(A \text{ even}, C = 0) + \mathbb{P}(A \text{ odd}, C = 1) = \frac{1}{63} + \frac{32}{63} = \frac{33}{63}. \tag{32}$$

Hence, the third row is

$$(T_{20}, T_{21}, T_{22}) = \frac{1}{63}(12, 18, 33). \tag{33}$$

Combining (25), (29), and (33), we arrive at the following result.

Theorem 5.1 (Conditional transition matrix)**.** *The conditional transition matrix from $R = X \bmod 3$ to $S = \kappa \bmod 3$, with rows and columns ordered as $0, 1, 2$, is given by*

$$T = \frac{1}{63}\begin{pmatrix} 3 & 36 & 24 \\ 48 & 9 & 6 \\ 12 & 18 & 33 \end{pmatrix}, \qquad \sum_{s=0}^{2} T_{rs} = 1, \quad r = 0, 1, 2, \qquad \sum_{r=0}^{2} T_{rs} = 1, \quad s = 0, 1, 2. \tag{34}$$

So, T is a double stochastic matrix.

5.4. Transformation of the marginal distribution

Let the marginal distribution of R be denoted by the row vector

$$p_R = (p_0, p_1, p_2), \qquad p_0 + p_1 + p_2 = 1. \tag{35}$$

By the law of total probability and the definition of T, the marginal distribution of S is

$$p_S = p_R T. \tag{36}$$

Explicitly,

$$\mathbb{P}(S = 0) = \frac{3p_0 + 48p_1 + 12p_2}{63}, \tag{37}$$

$$\mathbb{P}(S = 1) = \frac{36p_0 + 9p_1 + 18p_2}{63}, \tag{38}$$

$$\mathbb{P}(S = 2) = \frac{24p_0 + 6p_1 + 33p_2}{63}. \tag{39}$$

5.4.1. Two examples of vector p_R

Uniform distribution.. If R is uniformly distributed over $\{0, 1, 2\}$, then

$$p_R = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Using (36), we obtain

$$p_S = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{40}$$

A non-uniform distribution.. Consider the distribution

$$p_R = \left(0, \frac{1}{3}, \frac{2}{3}\right).$$

Then

$$p_S = \left(\frac{8}{21}, \frac{5}{21}, \frac{8}{21}\right). \tag{41}$$

These examples show how the conditional transition matrix determines the distribution of S from a prescribed distribution of R.

The random variables R and S are independent if and only if the conditional distribution of S given $R = r$ is the same for every r with positive probability.

Since the rows of the matrix in (34) are not identical, R and S are not independent under the uniform natural-density model, where

$$\mathbb{P}(R=0)=\mathbb{P}(R=1)=\mathbb{P}(R=2)=\frac{1}{3}.$$

More generally, the non-uniform example above also yields dependence, because both $R=1$ and $R=2$ have positive probability and the corresponding rows of T are different.
Thus, the transition matrix provides an explicit description of the dependence structure between the residue classes of X and κ modulo 3.

6. Transitions between X_n and κ_n

6.1. Distribution of the exponent

Throughout this section, the initial value X_0 is selected according to this natural-density model. All subsequent variables are regarded as functions of X_0. Consequently, every probability and conditional probability below is understood with respect to the initial cutoff $X_0 \le N$, followed by the limit as $N \to \infty$. For each positive odd integer X_n, consider its unique representation

$$X_n = 2^{A_n}\kappa_n + c_{A_n}, \quad A_n \ge 1, \quad c_a = \alpha_{\lfloor \frac{a+1}{2} \rfloor}, \quad \alpha_k = \frac{4^k-1}{3}, \quad A_n \equiv \kappa_n \pmod 2. \tag{42}$$

Where the sequence α_k is defined in section 2. So we have

$$X_{n+1} = \begin{cases} 3\kappa_n + 1, & \kappa_n \equiv 0 \pmod 2, \\ 3\kappa_n + 2, & \kappa_n \equiv 1 \pmod 2. \end{cases} \tag{43}$$

In view of (42), this can equivalently be written as

$$X_{n+1} = 3\kappa_n + \varepsilon(A_n), \qquad \varepsilon(a) = \begin{cases} 1, & a \equiv 0 \pmod 2, \\ 2, & a \equiv 1 \pmod 2. \end{cases} \tag{44}$$

The purpose of this section is to establish the distributional relation between $X_n \bmod 9$ and $\kappa_n \bmod 6$, and then to describe the resulting distribution of $X_{n+1} \bmod 9$.
We first record the congruence characterization of the event $A_n = a$. From (42), this event is equivalent to

$$X_n \equiv c_a + 2^a (a \bmod 2) \pmod{2^{a+1}}. \tag{45}$$

In particular, for the initial variable X_0, this is a single admissible residue class among the odd residue classes modulo 2^{a+1}. Hence

$$\mathbb{P}(A_0 = a) = 2^{-a}, \qquad a \geq 1. \tag{46}$$

The main result needed for the subsequent transition matrices is the corresponding conditional statement at every iteration.

Theorem 6.1 (Conditional distribution of the exponent)**.** *For every $n \geq 0$, every admissible residue class $t \pmod 9$, and every integer $a \geq 1$,*

$$\mathbb{P}\left(A_n = a \mid X_n \equiv t \pmod 9\right) = 2^{-a}. \tag{47}$$

Here, admissibility means that the conditioning event has positive natural density.

Proof. We use the arithmetic-progression structure generated by fixing a finite history of exponents.
Fix a history

$$A_0 = a_0,\ A_1 = a_1,\ \ldots,\ A_{n-1} = a_{n-1}.$$

On this history, the possible values of L_n, apart from finitely many initial boundary restrictions, form an arithmetic progression of the form

$$\kappa_n = r_n + 2 \cdot 3^n q, \qquad q \in \mathbb{Z}, \tag{48}$$

where r_n depends on the fixed history but not on q.
This assertion follows by induction. For $n = 0$, fixing A_0 in (42) fixes the parity of L_0, so $\kappa_0 = r_0 + 2q$. Suppose the assertion holds at stage $n-1$. By (44),

$$X_n = 3\kappa_{n-1} + \varepsilon(a_{n-1}),$$

and therefore

$$X_n = r'_n + 2 \cdot 3^n q.$$

Imposing $A_n = a_n$ is, by (45), one congruence condition modulo 2^{a_n+1}. Since the coefficient of q is $2 \cdot 3^n$, and 3^n is odd, this condition restricts q to one residue class modulo 2^{a_n}. Writing $q = q_0 + 2^{a_n} k$ gives

$$\kappa_n = r_n + 2 \cdot 3^n k,$$

which proves the induction step.We now fix such a history and examine the conditional density of $A_n = a$.

For $n = 0$, the assertion follows directly from (46) and the Chinese remainder theorem, because the moduli 9 and 2^{a+1} are relatively prime.
For $n = 1$, the fixed history gives

$$X_1 = r'_1 + 6q.$$

The condition $X_1 \equiv t \pmod 9$ restricts q to one residue class modulo 3, whenever the condition is compatible. On the other hand, the condition $A_1 = a$ restricts q to one residue class modulo 2^a. Since 2^a and 3 are coprime, the Chinese remainder theorem shows that the proportion of the admissible q satisfying $A_1 = a$ is exactly 2^{-a}.
For $n \geq 2$, equation (48) gives

$$X_n = 3\kappa_{n-1} + \varepsilon(a_{n-1}) = r'_n + 2 \cdot 3^n q.$$

Because $9 \mid 2 \cdot 3^n$ for $n \geq 2$, the residue X_n mod 9 is constant on the fixed history. Thus, if the prescribed residue t is compatible with that history, conditioning on $X_n \equiv t \pmod 9$ imposes no additional restriction on q. The condition $A_n = a$ still selects one residue class modulo 2^a, and hence has relative density 2^{-a}.
It remains to remove the conditioning on a fixed history. For a fixed R, restrict attention to histories satisfying $a_i \leq R$ for all $0 \leq i < n$. There are only finitely many such histories, and the preceding argument applies to each compatible history. The contribution of the discarded histories tends to zero as $R \to \infty$: indeed, the same congruence-counting argument gives

$$\mathbb{P}(A_i > R) = 2^{-R},$$

and therefore

$$\mathbb{P}\left(\max_{0 \leq i < n} A_i > R\right) \leq n\, 2^{-R}.$$

Consequently, the finite-history calculation passes to the full natural-density model. This proves (47). □

Remark 4. The conditional distribution in Theorem 6.1 is geometric, rather than uniform over the possible values of A_n. In particular,

$$\mathbb{P}(A_n = a \mid X_n \equiv t \pmod 9) = 2^{-a}.$$

It follows that the distribution of A_n mod 6 is independent of the admissible residue class t.

6.2. The distribution of A_n mod 6

For $h \in \{0,1,2,3,4,5\}$, define

$$\hat{a}_h = \mathbb{P}(A_n \equiv h \pmod 6).$$

By Theorem 6.1,

$$\hat{a}_h = \sum_{k=1}^{\infty} 2^{-(h+6k)} = \frac{2^{-(h+6)}}{1-2^{-6}} = \frac{2^{-6}}{1-2^{-6}} = \frac{1}{63} \quad (h=0), \tag{49}$$

$$\hat{a}_h = \sum_{k=0}^{\infty} 2^{-(h+6k)} = \frac{2^{-h}}{1-2^{-6}} \quad (1 \le h \le 5). \tag{50}$$

Thus, in the order $h = 0, 1, \ldots, 5$,

$$\hat{\boldsymbol{a}} = \frac{1}{63}(1, 32, 16, 8, 4, 2). \tag{51}$$

In particular,

$$\mathbb{P}(A_n \text{ is even}) = \frac{1}{3}, \qquad \mathbb{P}(A_n \text{ is odd}) = \frac{2}{3}. \tag{52}$$

Since $A_n \equiv \kappa_n \pmod 2$, the same parity probabilities hold for κ_n.

6.3. The congruence relation between X_n mod 9 and κ_n mod 6

We now derive the exact relation between the two residue classes. From (42) and (1),

$$c_a \equiv \left\lfloor \frac{a+1}{2} \right\rfloor \pmod 3 \longrightarrow \kappa_n \equiv 2^{-A_n} \left(X_n - \left\lceil \frac{A_n}{2} \right\rceil \right) \pmod 3. \tag{53}$$

Here, 2^{-A_n} denotes the multiplicative inverse of 2^{A_n} modulo 3. Now we let

$$h = A_n \bmod 6, \qquad r = X_n \bmod 3.$$

Combining (53) with $\kappa_n \equiv A_n \pmod 2$ gives the following deterministic map from (h, r) to κ_n mod 6:

	$r=0$	$r=1$	$r=2$
$h=0$	0	4	2
$h=1$	1	3	5
$h=2$	2	0	4
$h=3$	5	1	3
$h=4$	4	2	0
$h=5$	3	5	1

(54)

For clarity, this table is obtained directly from (53); its rows correspond to $A_n \bmod 6 = 0, \ldots, 5$, and its columns correspond to $X_n \bmod 3 = 0, 1, 2$.

6.4. The Transition Matrix from $X_n \bmod 9$ to $\kappa_n \bmod 6$

For $n \geq 1$, the iteration (44) implies that X_n is not divisible by 3. Hence the admissible residue classes modulo 9 are

$$\mathcal{R}_X = \{1, 2, 4, 5, 7, 8\}.$$

We order these classes as

$$(1, 2, 4, 5, 7, 8).$$

The conditional distribution of $A_n \bmod 6$ is given by (51). Combining this distribution with the deterministic table (54) yields the conditional distribution of $\kappa_n \bmod 6$. More precisely, if $X_n \equiv 1 \pmod 9$, then $X_n \equiv 1 \pmod 3$, and the resulting distribution of $\kappa_n \bmod 6$ is

$$\frac{1}{63}(16, 8, 4, 32, 1, 2).$$

If $X_n \equiv 2 \pmod 9$, then $X_n \equiv 2 \pmod 3$, and the corresponding distribution is

$$\frac{1}{63}(4, 2, 1, 8, 16, 32).$$

The same two rows repeat according to the residue modulo 3. Consequently, the transition matrix from $X_n \bmod 9$ to $\kappa_n \bmod 6$ is

$$K = \frac{1}{63} \begin{pmatrix} 16 & 8 & 4 & 32 & 1 & 2 \\ 4 & 2 & 1 & 8 & 16 & 32 \\ 16 & 8 & 4 & 32 & 1 & 2 \\ 4 & 2 & 1 & 8 & 16 & 32 \\ 16 & 8 & 4 & 32 & 1 & 2 \\ 4 & 2 & 1 & 8 & 16 & 32 \end{pmatrix}. \tag{55}$$

The rows are indexed by $X_n \bmod 9 = (1, 2, 4, 5, 7, 8)$, and the columns by $\kappa_n \bmod 6 = (0, 1, 2, 3, 4, 5)$. Every row of K sums to one. Thus, if

$$\boldsymbol{p}_{X_n} = \big(P(X_n \equiv 1 \pmod 9), \ldots, P(X_n \equiv 8 \pmod 9)\big)$$

is restricted to the six admissible classes in the specified order, and

$$\boldsymbol{p}_{\kappa_n} = \big(P(\kappa_n \equiv 0 \pmod 6), \ldots, P(\kappa_n \equiv 5 \pmod 6)\big),$$

then

$$\boldsymbol{p}_{\kappa_n} = \boldsymbol{p}_{X_n} K. \tag{56}$$

6.5. The Deterministic Transition from $\kappa_n \bmod 6$ to $X_{n+1} \bmod 9$

The second transition is entirely deterministic. If

$$\kappa_n = 6v + j, \qquad j \in \{0, 1, 2, 3, 4, 5\},$$

then (43) gives

$$X_{n+1} = \begin{cases} 18v + 3j + 1, & j \text{ even}, \\ 18v + 3j + 2, & j \text{ odd}. \end{cases}$$

It follows that

$\kappa_n \bmod 6$	0	1	2	3	4	5
$X_{n+1} \bmod 9$	1	5	7	2	4	8

(57)

Let T denote the corresponding 6×6 matrix, whose rows are indexed by $\kappa_n \bmod 6 = 0, \ldots, 5$ and whose columns are indexed by $X_{n+1} \bmod 9 = (1, 2, 4, 5, 7, 8)$. Then

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{58}$$

Accordingly, the marginal distributions satisfy

$$\boldsymbol{p}_{X_{n+1}} = \boldsymbol{p}_{\kappa_n} T. \tag{59}$$

Combining (56) and (59), we obtain

$$\boldsymbol{p}_{X_{n+1}} = \boldsymbol{p}_{X_n} M, \qquad M = KT. \tag{60}$$

6.6. The combined six-state transition matrix

Using (55) and (58), the combined matrix is

$$M = \frac{1}{63} \begin{pmatrix} 16 & 32 & 1 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 & 1 & 32 \\ 16 & 32 & 1 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 & 1 & 32 \\ 16 & 32 & 1 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 & 1 & 32 \end{pmatrix}. \tag{61}$$

The row and column order is

$$(1, 2, 4, 5, 7, 8)$$

for the residue classes of X_n and X_{n+1}, respectively.
The matrix M is a stochastic matrix: all its entries are nonnegative and every row sums to one. It describes the evolution of the one-dimensional marginal distributions of the residue classes. The identity (60), by itself, does not assert that the full residue process is a Markov chain.

6.7. Initial distributions

Since the odd integers are uniformly distributed among the nine residue classes modulo 9, we have

$$P(X_0 \equiv t \pmod 9) = \frac{1}{9}, \qquad t = 0, 1, \ldots, 8.$$

Using the same congruence table, including the row corresponding to $X_0 \equiv 0 \pmod 3$, gives

$$\boldsymbol{p}_{\kappa_0} = \frac{1}{9}(1, 2, 1, 2, 1, 2) \longrightarrow \boldsymbol{p}_{X_1} = \frac{1}{9}(1, 2, 1, 2, 1, 2). \tag{62}$$

where the entries are ordered according to $(1, 2, 4, 5, 7, 8)$.
For subsequent iterations, the combined transition matrix (61) gives

$$\boldsymbol{p}_{X_{n+1}} = \boldsymbol{p}_{X_n} M.$$

A direct multiplication shows that every row of M^2 is identical. In particular,

$$M^2 = \begin{pmatrix} \boldsymbol{\pi}_X \\ \boldsymbol{\pi}_X \\ \boldsymbol{\pi}_X \\ \boldsymbol{\pi}_X \\ \boldsymbol{\pi}_X \\ \boldsymbol{\pi}_X \end{pmatrix}, \tag{63}$$

where

$$\boldsymbol{\pi}_X = \frac{1}{189}(24, 48, 33, 12, 6, 66) \longrightarrow \boldsymbol{p}_{X_n} = \boldsymbol{\pi}_X, \qquad n \geq 2. \tag{64}$$

The corresponding distribution of κ_n mod 6 is

$$\boldsymbol{\pi}_\kappa = \frac{1}{189}(24, 12, 6, 48, 33, 66) \longrightarrow \boldsymbol{p}_{\kappa_n} = \boldsymbol{\pi}_\kappa, \qquad n \geq 1.. \tag{65}$$

Notice that $\boldsymbol{\pi}_X$ and $\boldsymbol{\pi}_\kappa$ are probability vectors: their entries are nonnegative and each vector sums to one. Moreover,

$$\boldsymbol{\pi}_X K = \boldsymbol{\pi}_\kappa, \qquad \boldsymbol{\pi}_\kappa T = \boldsymbol{\pi}_X.$$

These identities verify the consistency of the two transition steps and the resulting marginal distributions.

6.8. Preservation and transfer of conditional distributions

The preceding results establish a conditional distributional property of the exponent A_n with respect to the residue classes of X_n modulo 9. We now show how this property is transferred to the residue classes of κ_n modulo 6, and subsequently to those of X_{n+1} modulo 9. It is important to emphasize that the term "preservation of uniformity" in this context does not mean that all residue classes have equal probabilities. Rather, it means that the same conditional distribution of the exponent is preserved across all admissible residue classes of X_n.

Theorem 6.2 (Preservation of the conditional exponent distribution)**.** *For every admissible residue class $t \pmod 9$ and every integer $a \geq 1$, suppose that*

$$\mathbb{P}(A_n = a \mid X_n \equiv t \pmod 9) = 2^{-a}.$$

Then the conditional distribution of $A_n \bmod 6$ is independent of the particular admissible residue class t. More precisely, for $h = 0, 1, \ldots, 5$,

$$\mathbb{P}(A_n \equiv h \pmod 6 \mid X_n \equiv t \pmod 9) = \hat{a}_h, \tag{66}$$

where

$$\hat{\boldsymbol{a}} = \frac{1}{63}(1, 32, 16, 8, 4, 2). \tag{67}$$

Proof. For each $h \in \{0, 1, \ldots, 5\}$, the event $A_n \equiv h \pmod 6$ is the disjoint union of the events $A_n = h + 6k$, with the restriction $A_n \geq 1$. Consequently,

$$\begin{aligned} &\mathbb{P}(A_n \equiv h \pmod 6 \mid X_n \equiv t \pmod 9) \\ &\quad = \sum_{\substack{k \geq 0 \\ h+6k \geq 1}} \mathbb{P}(A_n = h + 6k \mid X_n \equiv t \pmod 9). \end{aligned}$$

Using the assumed conditional geometric distribution, we obtain

$$\sum_{\substack{k\geq 0\\ h+6k\geq 1}} 2^{-(h+6k)}.$$

Evaluating this geometric series gives

$$\hat{a}_h = \begin{cases} \displaystyle\sum_{k=1}^{\infty} 2^{-6k} = \frac{1}{63}, & h = 0, \\ \displaystyle\sum_{k=0}^{\infty} 2^{-(h+6k)} = \frac{2^{6-h}}{63}, & 1 \leq h \leq 5. \end{cases}$$

Thus,

$$\hat{\boldsymbol{a}} = \frac{1}{63}(1, 32, 16, 8, 4, 2),$$

which is independent of t. This proves the assertion. □

Theorem 6.3 (Transfer from $X_n \bmod 9$ to $\kappa_n \bmod 6$)**.** *Under the assumptions of Theorem 6.2, the conditional distribution of $\kappa_n \bmod 6$ is determined by the residue class of $X_n \bmod 3$.*
For $n \geq 1$, the admissible residue classes of X_n modulo 9 are

$$\mathcal{R}_X = \{1, 2, 4, 5, 7, 8\}.$$

With rows indexed by these six classes and columns indexed by $\kappa_n \bmod 6 = 0, 1, \ldots, 5$, the conditional transition matrix is

$$K = \frac{1}{63}\begin{pmatrix} 16 & 8 & 4 & 32 & 1 & 2 \\ 4 & 2 & 1 & 8 & 16 & 32 \\ 16 & 8 & 4 & 32 & 1 & 2 \\ 4 & 2 & 1 & 8 & 16 & 32 \\ 16 & 8 & 4 & 32 & 1 & 2 \\ 4 & 2 & 1 & 8 & 16 & 32 \end{pmatrix}. \tag{68}$$

In particular, the conditional distribution of κ_n is identical for residue classes of X_n that are congruent modulo 3.

Proof. By the decomposition of X_n and the parity relation $A_n \equiv \kappa_n \pmod 2$, the residue $\kappa_n \bmod 6$ is completely determined by the pair

$$(A_n \bmod 6,\ X_n \bmod 3).$$

Table 1: Mapping from ($A_n \bmod 6, X_n \bmod 3$) to $kappa_n \bmod 6$

$A_n \bmod 6$	$X_n \equiv 0 \pmod 3$	$X_n \equiv 1 \pmod 3$	$X_n \equiv 2 \pmod 3$
0	0	4	2
1	1	3	5
2	2	0	4
3	5	1	3
4	4	2	0
5	3	5	1

The corresponding deterministic relation is given in Table 1. By Theorem 6.2, the probabilities assigned to the six possible values of $A_n \bmod 6$ are given by

$$\frac{1}{63}(1, 32, 16, 8, 4, 2),$$

independently of the admissible residue class of $X_n \bmod 9$. Applying the deterministic map in Table 1 to these probabilities gives

$$\mathbb{P}(\kappa_n \bmod 6 = \cdot \mid X_n \bmod 3 = 1) = \frac{1}{63}(16, 8, 4, 32, 1, 2),$$

and

$$\mathbb{P}(\kappa_n \bmod 6 = \cdot \mid X_n \bmod 3 = 2) = \frac{1}{63}(4, 2, 1, 8, 16, 32).$$

Since the residue modulo 9 determines the residue modulo 3, the same two conditional distributions repeat among the six admissible classes modulo 9. This yields the matrix K in (68). □

Theorem 6.4 (Transfer from $\kappa_n \bmod 6$ to $X_{n+1} \bmod 9$)**.** *The iteration rule induces the deterministic map*

$\kappa_n \bmod 6$	0	1	2	3	4	5
$X_{n+1} \bmod 9$	1	5	7	2	4	8.

Consequently, if T denotes the corresponding transition matrix, with columns

ordered as $(1,2,4,5,7,8)$*, then*

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{69}$$

Thus, the distribution of $\kappa_n \bmod 6$ *is transferred to the distribution of* $X_{n+1} \bmod 9$ *by*

$$\boldsymbol{p}_{X_{n+1}} = \boldsymbol{p}_{\kappa_n} T. \tag{70}$$

Proof. Write $\kappa_n = 6v + j$, where $j \in \{0,1,2,3,4,5\}$. By the iteration rule,

$$X_{n+1} = \begin{cases} 18v + 3j + 1, & j \text{ even}, \\ 18v + 3j + 2, & j \text{ odd}. \end{cases}$$

Reducing modulo 9 gives

$$j = 0,1,2,3,4,5 \quad \longmapsto \quad 1,5,7,2,4,8,$$

respectively. This proves the deterministic map and the matrix representation in (69). The distributional identity (70) follows immediately. □

Corollary 3 (Combined distributional transfer). Let $M = KT$. Then the marginal distributions satisfy

$$\boldsymbol{p}_{X_{n+1}} = \boldsymbol{p}_{X_n} M. \tag{71}$$

Here, K describes the conditional distributional transfer from $X_n \bmod 9$ to $\kappa_n \bmod 6$, whereas T describes the deterministic transfer from $\kappa_n \bmod 6$ to $X_{n+1} \bmod 9$. The resulting identity concerns marginal distributions and does not, by itself, imply that the residue process is a Markov chain.

7. A new paradox

In this section, we peresent a new paradox and answer it. At the beginning of this section, we define two new concepts about of recursive sequences.

Definition 13. Let x_n be a recursive sequence with the relation $x_{n+1} = f(x_1, x_2, x_3, \cdots, x_n)$. Suppose that a part of the terms of this sequence is $x_1, x_2, x_3, \cdots$, and it is possible to insert a term $x_0 = x$ before x_1 into the sequence such that the new sequence $x_0, x_1, x_2, x_3, \cdots$ is still obtained recursively via the relation $x_n = f(x_0, x_1, x_2, \cdots, x_{n-1})$.
Then x_n is called a **Back-side Extensive Sequence (BES)**. Otherwise, it is called a **Back-side non-Extensive Sequence (BnES)**.

Definition 14. Consider a recursive sequence x_n that is backward-extensible with the relation $x_{n+1} = f(x_1, x_2, x_3, \cdots, x_n)$, where a part of the sequence is $x_1, x_2, x_3, \cdots$.
Take the smallest integer $a < 1$ such that the sequence has been extended backward up to x_a and no term before x_a can be added as an extension. Then x_a is called a **Zero Term of the Sequence (ZTS)**.

Example 1 (Fibonacci Sequence). Consider the Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$ with initial terms $F_0 = 0, F_1 = 1$. The first few terms are:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \cdots$$

According to definition 13, the Fibonacci sequence is backward-extensible. We have $F_{n+1} = F_n + F_{n-1}$; i.e., the difference of any two consecutive terms gives the previous term. The extended Fibonacci sequence is:

$$\cdots,\ -21,\ 13,\ -8,\ 5,\ -3,\ 2,\ -1,\ 1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \cdots$$

Example 2 (Collatz Odd Number Sequence (CONS)). Consider the Collatz Odd Number Sequence (CONS) with initial term 19:

$$19,\ 29,\ 11,\ 17,\ 13,\ 5,\ 1$$

The backward extension can be obtained in several ways (the extension is not necessarily unique). One such form is:

$$33,\ 25,\ 19,\ 29,\ 11,\ 17,\ 13,\ 5,\ 1$$

The latter sequence is non-extensible backward because before 33 no term can be added. Hence 33 is a zero term for the sequence $19, 29, 11, 17, 13, 5, 1$.

Consequently, lemma 2.4 provides a specific representation of odd numbers, which we use to prove the Collatz conjecture, supported by the stated lemmas and theorems.

7.1. The paradox of distribution of odd numbers modulo 3 on CONS and inverse CONS

If an odd number is chosen randomly from the set of odd natural numbers, the probability that it is represented modulo 3 in each of the forms $3t, 3t+1, 3t+2$ is exactly $\frac{1}{3}$.

In the Collatz Odd Number Sequence (CONS), the next odd number is always of the form $3t+1$ or $3t+2$; the form $3t$ never appears. Numbers of the form $3t$ always act as a zero term (**ZTS**) in CONS. Before them, no other odd number can be added as an extension. In the representation of an odd number $m = 2^a\kappa + \alpha_{[\frac{a+1}{2}]}$ when a is odd, κ is odd and 2 bits $(2[\frac{a+1}{2}], 2[\frac{a+1}{2}]+1)^{th}$ are either 01 or 11. When a is even, κ is even and 2 bits $(2[\frac{a+1}{2}], 2[\frac{a+1}{2}]+1)^{th}$ are either 00 or 10. The case 10 occurs when $\alpha_{[\frac{a+1}{2}]+1}$ lies to the right of m, but in the given representation only $\alpha_{[\frac{a+1}{2}]}$ is present; hence only 00 is counted.

Thus, when a is odd, the next odd number is $3\kappa+2$ with twice the probability compared to when a is even, where the next odd number is $3\kappa+1$. When a is even, in counting the two consecutive bits $2[\frac{a+1}{2}]$ and $2[\frac{a+1}{2}]+1$, we neglect case 10 because of $\alpha_{[\frac{a+1}{2}]+1}$. Moreover, it is uncertain whether a term with $\alpha_{[\frac{a+1}{2}]+1}$ on its right will ever appear. Counting this case is indefinitely postponed, causing unequal probabilities.

Based on what was observed in the sections concerning probabilistic rules for *CONS*, we saw that:

$$\mathbb{P}(A \text{ is even}) = \mathbb{P}(\kappa \text{ is even}) = \mathbb{P}(X \bmod 3 = 1) = \sum_{k=1}^{\infty} \frac{1}{2^{2k}} = \frac{1}{3},$$

$$\mathbb{P}(A \text{ is odd}) = \mathbb{P}(\kappa \text{ is odd}) = \mathbb{P}(X \bmod 3 = 2) = \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} = \frac{2}{3}.$$

In the paper [29] 2025, Bulent Sukusu has introduced Collatz inverse operation (*CIO*). The main idea of Sukusu for Collatz inverse operation is as follow. With this idea Sukusu stated that every odd number generate in *CIO* in a finite chain of sets CI_r but didn't prove it , in fact it is a new explaination of $3x+1$ problem.

At first assume the set $CI_0 = \{1\}$ and after compute all integers $ci_1 \in CI_1$ as follow:

$$ci_1 = \frac{2^{c_0}-1}{3}, \quad c_0 > 2 \quad \text{is even.}$$

So we have

$$CI_1 = \{5, 21, 85, 341, \cdots\}.$$

After it compute all integers $ci_{ci_1} \in CI_{ci_1}, ci_1 \in CI_1$ as follow:

$$ci_{ci_1} = \frac{2^{c_1} ci_1 - 1}{3}, \quad \text{for some} \quad c_1.$$

So we have

$$CI_5 = \{3, 13, 53, \cdots\}, 13 = 4 \times 3 + 1, 53 = 4 \times 13 + 1, \cdots,$$

$$CI_{21} = \emptyset,$$

$$CI_{85} = \{113, 453, 1813, \cdots\}, 453 = 4 \times 113 + 1, 1813 = 4 \times 453 + 1, \cdots,$$

$$CI_{341} = \{227, 909, 3637, \cdots\}, 909 = 4 \times 227 + 1, 3637 = 4 \times 909 + 1, \cdots,$$

$$\vdots$$

After we can compute

$$CI_{13}, CI_{53}, CI_{213}, \cdots, CI_{113}, CI_{453}, CI_{1813}, \cdots, CI_{227}, CI_{909}, CI_{3637}, \cdots,$$

also $CI_{3t} = \emptyset$ and so we continue this chains.
Now If we examine the aforementioned sets CI_rexcluding the empty sets and CI_0we observe that the elements of each set are distributed modulo 3 with uniform probabilities of 1/3, 1/3, and 1/3. In other words, when the Collatz inverse operator is applied and inverse CONS created starting from 1 all congruence classes modulo 3 appear with a uniform distribution (1/3, 1/3, 1/3) in every next terms of the sequence. In contrast, within the standard CONS, the congruence classes 1 and 2 appear with a distribution of 1/3 and 2/3, respectively; this constitutes the strange paradox arising from the $3x+1$ problem and Collatz conjecture.

8. Proof

In this section we try to use all number theory and probabilistic concepts that are stated in before sections. At first we state 2 main theorems to prove the $3x+1$ problem and furthermore, while proving two theorems, we employ two approachs that are similar in plan but differ in their indexes.
The following theorem allows us to prove that the sequence defined in lemma 2.6 is cycle-free.

Theorem 8.1. *Let m_1 be an arbitrary odd number. By using **RCONS** generate **CONS** as the sequence $m_1, m_2, m_3, m_4, \cdots, m_l, \cdots$. Then $\nexists k \geq 2 : m_k = m_1$ excepted of $m_1 = 1$.*

Proof. By contradiction, we assume $\exists k \geq 2 : m_k = m_1$ in the sequence. $k = 2 \longrightarrow m_2 = m_1$, so $2^{a_1} * m_1 = 3m_1 + 1 \longrightarrow (2^{a_1} - 3) * m_1 = 1$, because of $m_1, 2^{a_1} - 3 \in \mathbb{Z}$, $m_1 = 1, 2^{a_1} - 3 = 1$, and $a_1 = 2$. For $k \geq 3$, it must be $m_l > 1$ and because m_l's are odd numbers so $m_l \geq 3$. Now the set of equations 72 is obtained using lemma 2.6 as follows:

$$\begin{aligned} m_1 = 2^{a_1}\kappa_1 + \alpha_{[\frac{(a_1+1)}{2}]} &\Longrightarrow m_2 = \tfrac{3m_1+1}{2^{a_1}} \\ m_2 = 2^{a_2}\kappa_2 + \alpha_{[\frac{(a_2+1)}{2}]} &\Longrightarrow m_3 = \tfrac{3m_2+1}{2^{a_2}} \\ &\vdots \\ m_{k-1} = 2^{a_{k-1}}\kappa_{k-1} + \alpha_{[\frac{(a_{k-1}+1)}{2}]} &\Longrightarrow m_k = \tfrac{3m_{k-1}+1}{2^{a_{k-1}}} = m_1 \end{aligned} \tag{72}$$

On the other hand, $\prod_{l=1}^{k-1} \frac{m_{l+1}}{m_l}$ can be computed as the following:

$$\prod_{l=1}^{k-1} \frac{m_{l+1}}{m_l} = \frac{m_2}{m_1} * \frac{m_3}{m_2} * \cdots * \frac{m_{k-1}}{m_{k-2}} * \frac{m_k}{m_{k-1}} = \frac{m_k}{m_1} = 1. \tag{73}$$

Because $\frac{m_{l+1}}{m_l} = (\frac{3}{2^{a_l}} + \frac{1}{2^{a_l} * m_l}), 1 \leq l \leq k-1$, it can be written as $\frac{m_{l+1}}{m_l} \leq \frac{3}{2^{a_l}} + \frac{1}{3*2^{a_l}} = \frac{10}{3*2^{a_l}}$ using equations 72. So

$$\prod_{l=1}^{k-1} \frac{m_{l+1}}{m_l} \leq \prod_{l=1}^{k-1} \frac{10}{3 * 2^{a_l}} \tag{74}$$

From equations 73 , 74 we have

$$1 \leq \prod_{l=1}^{k-1} \frac{10}{3 * 2^{a_l}} = \prod_{l=1}^{k-1} \frac{10}{3 * 2^{A_l}}. \tag{75}$$

Now we can affect natural logarithm function $\ln = \log_e$ on inequality 75. So

$$\ln(1) \leq \ln(\prod_{l=1}^{k-1} \frac{10}{3 * 2^{A_l}}) \longrightarrow 0 \leq \sum_{l=1}^{k-1} \ln(\frac{10}{3 * 2^{A_l}}) = \sum_{l=1}^{k-1} (\ln(\frac{10}{3}) - A_l \ln 2) \tag{76}$$

As we point in before sections, $\forall l, A_l$ is a random variable. Specially as sections 3, 4 and 6 we able to compute probability measure of random variable

A_l as $\mathbb{P}(A_l = a_l) = \frac{1}{2^{a_l}} = 2^{-a_l}$. So we can compute the expected value for random variable A_l for all index values l to approximate an upper bound for $\frac{m_{l+1}}{m_l}$. The expected value of random variable A_l that is the arithmetic mean of all possible values a_l is equal to

$$\forall l \quad \mathbb{E}(A_l) = \sum_{j=1}^{\infty}(j * \mathbb{P}(A_l = j)) = \sum_{j=1}^{\infty}(j * \frac{1}{2^j}) = 2.$$

Now we can affect expected value on the latest inequality 76, so because of linearity of expectation we have

$$0 = \mathbb{E}(0) \leq \mathbb{E}(\sum_{l=1}^{k-1}(\ln(\frac{10}{3}) - A_l \ln 2)) = \sum_{l=1}^{k-1}\mathbb{E}(\ln(\frac{10}{3}) - A_l \ln 2) \longrightarrow \quad (77)$$

$$0 \leq \sum_{l=1}^{k-1}(\ln(\frac{10}{3}) - \mathbb{E}(A_l) \ln 2) = \sum_{l=1}^{k-1}(\ln(\frac{10}{3}) - \ln 4) = \sum_{l=1}^{k-1}(\ln(\frac{10}{12})) < 0 \quad (78)$$

Inequations 78 show us : $0 < 0$ and this is a contradiction stemming from the initial assumption and this prove the theorm 8.1. So there isn't any cycle in our odd numbers sequence except for odd number $m_1 = 1$.
Before in the section 1.3, we pointed to the two approaches that similar in plan but different in indexes that pointed in lemmas 2.4 and 2.5.
The proof of theorm 8.1 is presented based on lemma 2.4 that shows itself in inequation 78 when we use $\mathbb{E}(A_l) = 2$. Based on lemma 2.5 we can select $t = 2$ in the below relation:

$$\forall t \in \mathbb{N} \quad \exists! \quad a \in \{1, 2, 3, \cdots, 2t\}, \kappa \in \mathbb{Z} : x = 2^a \kappa + \alpha_{[\frac{a+1}{2}]}.$$

and based on probability we show

$$\mathbb{E}(A_l) \geq \frac{7}{4}.$$

For $t = 2$, we have

$$\mathbb{E}(A_l) = \sum_{\forall a_l} a_l \times \mathbb{P}(A_l = a_l) \geq 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{8} \geq 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} = \frac{7}{4}.$$

Because $\mathbb{P}(A_l \geq 4) = \frac{1}{8}, \quad \mathbb{P}(A_l \geq 3) = \frac{1}{4}$, we can use the relation $\mathbb{E}(A_l) \geq \frac{7}{4}$ in the inequation 78, so we have

$$0 \leq \sum_{l=1}^{k-1}(\ln(\frac{10}{3}) - \mathbb{E}(A_l) \ln 2) \leq \sum_{l=1}^{k-1}(\ln(\frac{10}{3}) - \ln 2^{\frac{7}{4}}) < \sum_{l=1}^{k-1}(\ln(\frac{3.33}{3.36})) < 0 \quad (79)$$

In fact we demonstrated that the required relationships can be derived even when a different computational approach is used for the indexes. □

Note 1. If **CONS** in the lemma 2.6 reaches a term that consecutively repeats, stop the sequence there. In this way, a finite sequence will be available. According to the theorem 8.1, only the number 1 meets this condition. Because the process described in lemma 2.6 can continue indefinitely and the sequence contains no cycles, according to the theorem 8.1, this implies that for a finite sequence, it stop at the number 1.

Theorem 8.2. *For an arbitrarily chosen initial term m_0, prove that **CONS** in lemma 2.6 must be finite under the stated condition in note 1.*

Proof. By contradiction, assume the sequence is infinite. Then, according to the note 1, sequence necessarily contains odd integers greater than 1. Also we can prove:

$$\exists N \in \mathbb{N} \quad s.t. \quad \forall N' \in \mathbb{N}, N' > N : m_N < m_{N'}.$$

Consequently, from some point onward, every term of the sequence are greater than another one. To prove, at first consider the subsequence η as follows:

$$\eta = \{N' \in \mathbb{N} | N' > 1, m_0 > m_{N'}\}.$$

Considering that the first term $m_0 < \infty$, there are finitely odd numbers smaller than this value that may occur later in the subsequence η, so $|\eta| < \infty$. Now set $\varepsilon = \min \eta$. Considering that ε and none of terms in the main sequence and the subsequence η are not repeated (otherwise a cycle is created) its subsequent terms in the main sequence and the subsequence η are greater than it, it is sufficient to let N denote the index at which ε appears in the main sequence. so $m_N = \varepsilon$. Therefore:

$$\exists N \in \mathbb{N} \quad s.t. \quad \forall N' \in \mathbb{N}, N' > N : m_N = \varepsilon < m_{N'}.$$

So

$$\forall n > N : \frac{m_n}{m_N} > 1$$

We know that

$$\frac{m_n}{m_N} = \prod_{l=N}^{n-1} \frac{m_{l+1}}{m_l},$$

so we use the same technique as in the theorem 8.1 and all equations and relations from 75 to 79 are repeated.

$$1 < \frac{m_n}{m_N} \leq \prod_{l=1}^{k-1} \frac{10}{3 * 2^{a_l}} = \prod_{l=1}^{k-1} \frac{10}{3 * 2^{A_l}}. \tag{80}$$

Now we can affect natural logarithm function $\ln = \log_e$ on inequality 80 and all the conclusions and all equations and relations from 75 to 79 are repeated here. So we have $0 < 0$ in both two different computational approachs. It show a contradiction with the first assumption that sequence is infinite. Therefore sequence is finite and considering the note 1, $\exists N'' \in \mathbb{N} : m_{N''} = 1$ appears in the sequence and sequence stops due to the repetition of the number 1. □

9. Some new problems

9.1. Distributivity of Infinite Sets

Based on the above, we propose the following problem about the set theory, probability measure preservation, countable set coverings, weighted empirical distributions, natural density and overlapping partitions and coverings as the **Distributivity of Infinite Sets Problem** which is considered the opposite of the Hilbert's Hotel paradox:

Problem 9.1. *Let S be a well-define countably infinite set partitioned into $s(< \infty)$ well-define and well-order infinite subsets in collection $\Sigma = \{\Sigma_i\}_{i=1}^{s}$ such that the probability of occurrence of each Σ_i in S is PR_i and we have $PR_1 + \cdots + PR_s = 1$. Now consider a collection $\{\sigma_j\}_{j=1}^{\infty}$ consisting of non-trivial subsets of S (not necessary disjoint). If the elements of each σ_j are chosen from $\{\Sigma_1, \cdots, \Sigma_s\}$ according to an existance distribution function $Pr_1, \cdots, Pr_s$ that is same for all j and $Pr_1 + \cdots + Pr_s = 1$, then the probability distribution of the partition $\{\Sigma_1, \cdots, \Sigma_s\}$ over $\bigcup_j \sigma_j = S$ can be remained as $PR_1, \cdots, PR_s$, **only when** each σ_j is finite.*

Proof of the $3x+1$-problem (second proof). Assuming this problem is proved and accepted as the **Distributivity of Infinite Sets Theory (DIST)**, we present a third proof of the Collatz conjecture as following.
We wish to prove that the CONS sequence is finite, i.e., that it terminates at 1. Suppose we start from a ZTS and form the CONS sequence, then continue it using the rules of RCONS. If this sequence continues indefinitely (does not

terminate at 1), then, according to the TDIS, an infinite subset with a non-uniform distribution based on the partition $3x,\ 3x+1,\ 3x+2$ would be found. This would cause the subset of odd numbers, under the same partition, to have a non-uniform distribution (not equal to $\frac{1}{3}$), which is a contradiction. Therefore, all CONS sequences are finite; i.e., they terminate at 1.

9.2. Generalized $Qx+1$ problem for prime number Q

In many articles, different problems introduced as a generalizing of $3x+1$ problem. In 1963, Murray Klamkin proposed a question about function $U(n)$ that defines a permutation of the integers [13]:

$$U(2n) = 3n, \quad U(4n+1) = 3n+1, \quad U(4n+3) = 3n+2.$$

Lagarias in [20] presented a taxonomy of several classes of functions which represent successive generalizations of the $3n+1$ function. The simplest generalization of the $3x+1$ function is the $3x+k$ function, which is defined for $k \equiv 1$ or $5 \pmod 6$, by $T_{3,k}(x) = \frac{3x+k}{2} \quad or \quad \frac{x}{2}$ when $x \equiv 1$ or $0 \pmod 2$.
Kontorovich and Lagarias in [14] and Matthews in [23] worked about an interesting open problem that is to classify all invariant measures for generalized $3n+1$ functions on the d-adic integers. Some other works about it are [21, 17, 22, 24].
Here we present new generalizations of $3n+1$ problem in the form of 2 problems 9.2 and 9.3 that try to work with prime numbers in the problem and propose new vision about $3n+1$ problem. (The appendix A presents observations on numbers that led to these generalizations).

Problem 9.2 (Generalized $Qx+1$ problem 1 (GQX1))**.** *For prime p_m (the m^{th} prime number), consider $\Pi_m = \{p_1, p_2, \cdots, p_{m-1}\}$. Now for which of prime numbers p_m, there exist $\Lambda_l = \{p_{i_1}, p_{i_2}, \cdots, p_{i_l}\} \subseteq \Pi_m$ such that every recursive sequence O_n starting from an arbitrary odd number O converges to 1, where O_n is as following:*

$$O_{n+1} = \frac{p_m \cdot O_n + 1}{\prod_{j=1}^{l} p_{i_j}^{a_{i_j}}}, \qquad \forall j \, \gcd(O_{n+1}, p_{i_j}) = 1.$$

Problem 9.3 (Generalized $Qx+1$ problem 2 (GQX2))**.** *Some prime $p_m > 3$ and subset $\Lambda_l \subseteq \Pi_m$ exist that some recursive sequence O_n pointed in the problem 9.2 (**GQX1**), reaches the loop. Which of the numbers O can create a loop together in the sequence O_n?*

Appendix A What happens in DISP, GQX1 and GQX2

By using the Wolfram Mathematica, we write code 3 to show some sequence of odd numbers pointed in the problem 9.2 (**GQX1**) and code 4 to show what happens in the problem 9.3 (**GQX2**) and the problem 9.1 (**DISP**). Also we define the function 1 (CC) in mathematica to use in both codes.
Function 1 ($CC[OddNumber, Pm, \Lambda_l]$) recieves an arbitrary odd number O and prime p_m and subset λ_l and output the next term of sequence.
Now by running the code 3, we obtain 10 sequences of odd number. So we test the **GQX1** on 10 odd numbers these are multiple by p_m.
After any sequence, we must have a term as Null or $Aborted that show us after maxtime seconds, for every odd number, the sequence convarge to 1 or create a loop or continue. At the end, a real number as absolute timing of program and a term Null are printed.
We set $m = 3, p_m = 5, \Lambda_l = \{2, 3\}$ and run code 3 for 10 random odd numbers multipled by p_m.

$\{\{1085, 2713, 2261, 5653, 4711, 1963, 409, 341, 853, 79, 11, 7, 1\}, \text{Null}\}$.
$\{\{5855, 7319, 9149, 22873, 19061, 47653, 13237, 3677, 9193, 7661, 19153,$
$15961, 13301, 33253, 3079, 1283, 401, 1003, 209, 523, 109, 91, 19, 1\}, \text{Null}\}$.
$\{\{3135, 3919, 1633, 1361, 3403, 709, 197, 493, 137, 343, 143, 179, 7, 1\}, \text{Null}\}$.
$\{\{6485, 16213, 13511, 16889, 42223, 17593, 181, 151, 7, 1\}, \text{Null}\}$.
$\{\{2825, 7063, 109, 91, 19, 1\}, \text{Null}\}$.
$\{\{1105, 307, 1\}, \text{Null}\}$.
$\{\{3835, 799, 37, 31, 13, 11, 7, 1\}, \text{Null}\}$.
$\{\{9665, 24163, 839, 1049, 2623, 1093, 911, 1139, 89, 223, 31, 13, 11, 7, 1\}, \text{Null}\}$.
$\{\{1215, 1519, 211, 11, 7, 1\}, \text{Null}\}$.
$\{\{1685, 4213, 3511, 1463, 1829, 4573, 3811, 397, 331, 23, 29, 73, 61, 17, 43, 1\}, \text{Null}\}$.
$\{0.0088845, \text{Null}\}$.

We set $m = 4, p_m = 7, \Lambda_l = \{2, 3\}$ and run code 3 for 10 random odd numbers multipled by p_m.

$\{\{2429, 1417, 155, 181, 317, 185, 1\}, \text{Null}\}$.
$\{\{1379, 1609, 11, 13, 23, 1\}, \text{Null}\}$.
$\{\{13923, 48731, 6317, 3685, 6449, 209, 61, 107, 125, 73, 1\}, \text{Null}\}$.
$\{\{2233, 977, 95, 37, 65, 19, 67, 235, 823, 2881, 2521, 1103, 143, 167, 65\}, \text{Null}\}$.
$\{\{13993, 3061, 5357, 3125, 1823, 709, 1241, 181, 317, 185, 1\}, \text{Null}\}$.
$\{\{2863, 10021, 17537, 1705, 373, 653, 127, 445, 779, 101, 59, 23, 1\}, \text{Null}\}$.
$\{\{2359, 8257, 7225, 3161, 461, 269, 157, 275, 107, 125, 73, 1\}, \text{Null}\}$.
$\{\{2639, 3079, 10777, 4715, 5501, 3209, 13, 23, 1\}, \text{Null}\}$.
$\{\{13265, 3869, 2257, 1975, 6913, 6049, 5293, 9263, 10807, 37825, 33097,$
$905, 11, 13, 23, 1\}, \text{Null}\}$.
$\{\{3087, 10805, 2101, 3677, 715, 2503, 8761, 3833, 559, 1957, 3425, 37, 65,$
$19, 67, 235, 823, 2881, 2521, 1103, 143, 167, 65\}, \text{Null}\}$.
$\{0.036188, \text{Null}\}$.

We set $m = 4, p_m = 7, \Lambda_l = \{2, 3, 5\}$ and run code 3 for 10 random odd numbers multipled by p_m.

$\{\{4529, 1321, 289, 253, 443, 517, 181, 317, 37, 13, 23, 1\}, \text{Null}\}$.
$\{\{5761, 5041, 4411, 15439, 54037, 18913, 16549, 28961, 8447, 73, 1\}, \text{Null}\}$.
$\{\{13335, 46673, 13613, 2647, 1853, 1081, 473, 23, 1\}, \text{Null}\}$.
$\{\{3115, 10903, 38161, 33391, 116869, 204521, 1657, 29, 17, 1\}, \text{Null}\}$.
$\{\{7217, 421, 737, 43, 151, 529, 463, 1621, 2837, 331, 1159, 4057, 71,$
$83, 97, 17, 1\}, \text{Null}\}$.
$\{\{11277, 3947, 307, 43, 151, 529, 463, 1621, 2837, 331, 1159, 4057, 71,$
$83, 97, 17, 1\}, \text{Null}\}$.
$\{\{13783, 48241, 42211, 147739, 517087, 361961, 26393, 1283, 499,$
$1747, 1223, 1427, 37, 13, 23, 1\}, \text{Null}\}$.
$\{\{5565, 9739, 34087, 23861, 13919, 5413, 9473, 307, 43, 151, 529, 463,$
$1621, 2837, 331, 1159, 4057, 71, 83, 97, 17, 1\}, \text{Null}\}$.
$\{\{13811, 5371, 18799, 65797, 23029, 40301, 23509, 41141, 23999, 1037,$
$121, 53, 31, 109, 191, 223, 781, 1367, 319, 1117, 391, 1369, 599, 233, 17, 1\}, \text{Null}\}$.
$\{\{6643, 23251, 81379, 284827, 199379, 232609, 203533, 356183, 415547,$
$96961, 84841, 18559, 64957, 4547, 1061, 619, 2167, 1517, 59, 23, 1\}, \text{Null}\}$.
$\{0.0062926, \text{Null}\}$.

We set $m = 5, p_m = 11, \Lambda_l = \{2, 3, 5\}$ and run code 3 for 10 random odd numbers multipled by p_m.

$\{\{15895, 29141, 40069, 3673, 3367, 6173, 1061, 1459, 107, 589, 1\}, \text{Null}\}$.
$\{\{16203, 89117, 15317, 21061, 9653, 13273, 12167, 66919, 8179, 2999,$
$3299, 3629, 499, 61, 7, 13, 1\}, \text{Null}\}$.
$\{\{21659, 953, 2621, 901, 413, 71, 391, 239, 263, 1447, 2653, 19, 7, 13, 1\}, \text{Null}\}$.
$\{\{4037, 5551, 10177, 9329, 5131, 9407, 51739, 18971, 104341, 15941,$
$21919, 893, 307, 563, 3097, 2839, 347, 1909, 7, 13, 1\}, \text{Null}\}$.
$\{\{13299, 14629, 149, 41, 113, 311, 1711, 3137, 8627, 47449, 8699, 9569,$
$5263, 9649, 1769, 973, 223, 409, 1\}, \text{Null}\}$.
$\{\{2959, 217, 199, 73, 67, 41, 113, 311, 1711, 3137, 8627, 47449, 8699, 9569,$
$5263, 9649, 1769, 973, 223, 409, 1\}, \text{Null}\}$.
$\{\{2079, 2287, 4193, 11531, 63421, 7267, 4441, 1357, 311, 1711, 3137, 8627,$
$47449, 8699, 9569, 5263, 9649, 1769, 973, 223, 409, 1\}, \text{Null}\}$.
$\{\{21351, 117431, 645871, 394699, 48241, 44221, 563, 3097, 2839, 347,$
$1909, 7, 13, 1\}, \text{Null}\}$.
$\{\{14157, 9733, 1487, 8179, 2999, 3299, 3629, 499, 61, 7, 13, 1\}, \text{Null}\}$.
$\{\{19547, 107509, 73, 67, 41, 113, 311, 1711, 3137, 8627, 47449, 8699, 9569,$
$5263, 9649, 1769, 973, 223, 409, 1\}, \text{Null}\}$.
$\{0.0088032, \text{Null}\}$.

We set $m = 5, p_m = 11, \Lambda_l = \{2, 3, 5, 7\}$ and run code 3 for 10 random odd numbers multipled by p_m.

$\{\{20867, 114769, 21041, 57863, 318247, 583453, 33427, 61283, 48151, 12611,$
$69361, 9083, 49957, 3271, 1999, 733, 1\}, \text{Null}\}.$
$\{\{19767, 108719, 119591, 657751, 401959, 4211, 23161, 337, 103, 1\}, \text{Null}\}.$
$\{\{8217, 22597, 10357, 4747, 967, 197, 271, 71, 391, 239, 263, 1447, 379, 139,$
$17, 47, 37, 17\}, \text{Null}\}.$
$\{\{7095, 39023, 30661, 14053, 2147, 241, 221, 19, 1\}, \text{Null}\}.$
$\{\{1155, 6353, 17471, 96091, 176167, 46139, 50753, 139571, 109663, 201049,$
$36859, 901, 59, 13, 1\}, \text{Null}\}.$
$\{\{20735, 114043, 23231, 18253, 4183, 7669, 703, 1289, 709, 13, 1\}, \text{Null}\}.$
$\{\{11781, 16199, 17819, 19601, 53903, 296467, 543523, 332153, 913421, 89711,$
$493411, 301529, 165841, 152021, 209029, 2129, 1171, 2147, 241, 221, 19, 1\}, \text{Null}\}.$
$\{\{7447, 1517, 149, 41, 113, 311, 1711, 3137, 8627, 47449, 8699, 1367, 7519,$
$919, 337, 103, 1\}, \text{Null}\}.$
$\{\{20955, 115253, 22639, 2767, 1691, 9301, 29, 1\}, \text{Null}\}.$
$\{\{21329, 11731, 7169, 3943, 7229, 71, 391, 239, 263, 1447, 379, 139,$
$17, 47, 37, 17\}, \text{Null}\}.$
$\{0.0081451, \text{Null}\}.$

We set $m = 6, p_m = 13, \Lambda_l = \{2, 3, 5, 7\}$ and run code 3 for 10 random odd numbers multipled by p_m.

$\{\{19227, 7811, 4231, 13751, 14897, 1537, 9991, 32471, 35177, 76217,$
$23591, 1217, 293, 127, 59, 1\}, \text{Null}\}.$
$\{\{24141, 156917, 113329, 736639, 342011, 2573, 223, 29, 1\}, \text{Null}\}.$
$\{\{3913, 5087, 1837, 11941, 77617, 72073, 2677, 17401, 113107, 3751,$
$12191, 13207, 42923, 31, 101, 73, 19, 31\}, \text{Null}\}.$
$\{\{9347, 5063, 1097, 2377, 15451, 6277, 40801, 265207, 861923, 83, 1\}, \text{Null}\}.$
$\{\{17901, 116357, 252107, 68279, 10567, 34343, 1063, 691, 1123,$
$73, 19, 31, 101, 73\}, \text{Null}\}.$
$\{\{5421, 35237, 8483, 919, 2987, 809, 1753, 2279, 823, 107, 29, 1\}, \text{Null}\}.$
$\{\{5785, 37603, 12221, 26479, 86057, 186457, 1211971, 1969453,$
$2560289, 5547293, 2403827, 1302073, 338539, 275063, 59597,$
$129127, 419663, 10103, 2189, 527, 571, 29, 1\}, \text{Null}\}.$
$\{\{20397, 132581, 13679, 2117, 1529, 3313, 4307, 2333, 337, 313, 407, 1\}, \text{Null}\}.$
$\{\{25363, 8243, 893, 43, 1\}, \text{Null}\}.$
$\{\{21697, 141031, 458351, 496547, 268963, 87413, 37879, 123107, 66683, 43, 1\}, \text{Null}\}.$
$\{0.0231519, \text{Null}\}.$

By using the code 4 we can compute the distributions $O \equiv z \mod p_m$ such that O is an odd number in the sequence and $z \in \mathbb{Z}_{p_m} = \{0, 1, 2, \cdots, p_m - 1\}$. Also which of numbers make a loop in the sequence. And also for how many sequences after maxtime seconds, for every odd number, the sequence convarge to 1 or make a loop or continue otherwise.
We set $m = 3, p_m = 5, \Lambda_l = \{2, 3\}$ and run code 4 for 10000 random odd numbers $\in [10^5, 10^6]$. As we see, in the 3-tuples of the output, the third element is the probability distribution of $O \equiv z \bmod p_m$ for all terms in random sequence. Also we guess $\Lambda_l = \{2, 3\}$ for $p_m = 5$ in **GQX1**.

$\{\{\{0, 2033, 0.0298738\}, \{1, 17077, 0.250937\}, \{2, 11138, 0.163667\},$
$\{3, 23978, 0.352343\}, \{4, 13827, 0.20318\}\}, \{\}, \{\{\text{Null}, 10000\}\}\}.$
$\{10.1444, \text{Null}\}.$

We set $m = 4, p_m = 7, \Lambda_l = \{2, 3\}$ and run code 4 for 10000 random odd numbers $\in [10^5, 10^6]$.
In this case we see the numbers $823, 2881, 2521, 1103, 143, 167, 65, 19, 67, 235$ create a loop, as we guess in GQX2. In addition, 2214 numbers of 10000 selected random numbers reach to this loop.

$\{\{0, 1366, 0.00343715\}, \{1, 46267, 0.116418\}, \{2, 88197, 0.221923\},$
$\{3, 37092, 0.0933315\}, \{4, 142315, 0.358095\}, \{5, 19652, 0.0494487\}, \{6, 62533, 0.157347\}\},$
$\{\{\{823, 2881, 2521, 1103, 143, 167, 65, 19, 67, 235\}, 2214\}\}, \{\{Null, 10000\}\}\}.$
$\{130.29, \text{Null}\}.$

We set $m = 4, p_m = 7, \Lambda_l = \{2, 3, 5\}$ and run code 4 for 10000 random odd numbers $\in [10^5, 10^6]$. In this case, we don't see any loop and we can guess $\Lambda_l = \{2, 3, 5\}$ for $p_m = 7$ in the **GQX1**.

$\{\{\{0, 1390, 0.0205646\}, \{1, 7580, 0.112143\}, \{2, 12186, 0.180288\}, \{3, 7265, 0.107483\},$
$\{4, 19228, 0.284472\}, \{5, 8254, 0.122115\}, \{6, 11689, 0.172935\}\}, \{\}, \{\{\text{Null}, 10000\}\}\}.$
$\{10.4529, \text{Null}\}.$

We set $m = 5, p_m = 11, \Lambda_l = \{2, 3\}$ and run code 4 for 1000 random odd numbers $\in [10^5, 10^6]$. In this case, the growing speed of numbers in the sequence is very high and because of this, we see 571 number after the specified

time $0.5s$ for every sequence dosen't not converge to 1 or any loop.

$\{\{\{0, 96, 0.0000499672\}, \{1, 201810, 0.10504\}, \{2, 402844, 0.209677\}, \{3, 258279, 0.134432\},$
$\{4, 70848, 0.0368758\}, \{5, 38266, 0.0199171\}, \{6, 516418, 0.268791\}, \{7, 142881, 0.0743683\},$
$\{8, 141504, 0.0736516\}, \{9, 71444, 0.037186\}, \{10, 76871, 0.0400107\}\}, \{\},$
$\{\{\$\text{Aborted}, 571\}, \{\text{Null}, 429\}\}\}.$
$\{325.709, \text{Null}\}.$

We set $m = 5, p_m = 11, \Lambda_l = \{2, 3, 5\}$ and run code 4 for 10000 random odd numbers $\in [10^5, 10^6]$. In this case there is not any loop and we can guess $\Lambda_l = \{2, 3, 5\}$ for $p_m = 11$ in **GQX1**.

$\{\{\{0, 959, 0.0082479\}, \{1, 9272, 0.079744\}, \{2, 17545, 0.150896\}, \{3, 14353, 0.123443\},$
$\{4, 4564, 0.0392528\}, \{5, 5326, 0.0458064\}, \{6, 27429, 0.235904\}, \{7, 12342, 0.106148\},$
$\{8, 8391, 0.072167\}, \{9, 6533, 0.0561872\}, \{10, 9558, 0.0822038\}\}, \{\}, \{\{\text{Null}, 10000\}\}\}.$
$\{19.0705, \text{Null}\}.$

We set $m = 5, p_m = 11, \Lambda_l = \{2, 3, 5, 7\}$ and run code 4 for 10000 random odd numbers $\in [10^5, 10^6]$. In this case we see loop $17, 47, 37$ for 2083 numbers of 10000 random numbers.

$\{\{\{0, 913, 0.0133877\}, \{1, 5486, 0.0804434\}, \{2, 9909, 0.1453\}, \{3, 7921, 0.116149\},$
$\{4, 4614, 0.0676569\}, \{5, 4351, 0.0638005\}, \{6, 14193, 0.208118\}, \{7, 6388, 0.0936698\},$
$\{8, 4916, 0.0720853\}, \{9, 3978, 0.058331\}, \{10, 5528, 0.0810593\}\},$
$\{\{\{17, 47, 37\}, 2075\}\}, \{\{\text{Null}, 10000\}\}\}.$
$\{13.2117, \text{Null}\}.$

We set $m = 6, p_m = 13, \Lambda_l = \{2, 3, 5, 7\}$ and run code 4 for 10000 random odd numbers $\in [10^5, 10^6]$. In this case we see loop $101, 73, 19, 31$ for 1794 numbers of 10000 random numbers.

$\{\{\{0, 766, 0.00541048\}, \{1, 8524, 0.0602075\}, \{2, 6137, 0.0433474\}, \{3, 4341, 0.0306618\},$
$\{4, 9083, 0.0641559\}, \{5, 10927, 0.0771806\}, \{6, 6168, 0.0435664\}, \{7, 33686, 0.237934\},$
$\{8, 7116, 0.0502624\}, \{9, 7887, 0.0557082\}, \{10, 20314, 0.143484\}, \{11, 16678, 0.117802\},$
$\{12, 9950, 0.0702798\}\}, \{\{\{101, 73, 19, 31\}, 1757\}\}, \{\{\text{Null}, 10000\}\}\}.$
$\{30.3861, \text{Null}\}.$

We set $m = 6, p_m = 13, \Lambda_l = \{2, 5, 7, 11\}$ and run code 4 for 1000 random odd numbers $\in [10^5, 10^6]$.In this case, the growing speed of numbers in the sequence is very high and because of this, we see 451 number after the specified

time $0.2s$ for every sequence dosen't not converge to 1 or any loop.

$\{\{\{0, 66, 0.0000675117\}, \{1, 76606, 0.0783607\}, \{2, 46875, 0.0479487\}, \{3, 46716, 0.047786\},$
$\{4, 77816, 0.0795984\}, \{5, 85369, 0.0873244\}, \{6, 27329, 0.027955\}, \{7, 325929, 0.333394\},$
$\{8, 31439, 0.0321591\}, \{9, 46611, 0.0476786\}, \{10, 165205, 0.168989\}, \{11, 29907, 0.030592\},$
$\{12, 17740, 0.0181463\}\}, \{\ \}, \{\{\text{Null}, 549\}, \{\$\text{Aborted}, 451\}\}\}.$
$\{104.733, \text{Null}\}.$

After checking some $\Lambda_l \subset \Pi_m$ for $p_m = 13$, we guess that the set $\Lambda_l = \{2, 5, 7, 11\}$ might satisfy the conditions of the problem 9.2 (**GQX1**); however, this cannot be verified. As the sequence generation progresses, the computational load increases significantly and consumes a great deal of time, yet the sequence fails to reach number 1 or a cycle of numbers, leading us to halt the calculations. At present we found that there is not any Λ_l to satisfy in the problem 9.2 (**GQX1**) with confidence and we think and guess $p_m = 13$ is the first unlucky number here! So $p_m = 13$ is an attractive case to study.

Appendix B Expriment about sections 5 and 6

We know that every odd number m, can be represented as $m = 2^a\kappa + \alpha_n, a = 2n-1$ or $2n$ as well as mention in the lemma 2.4. In this section, we aim to design an experiment that can verify the results established in sections 5 and 6 through statistical sampling of numerical data. To achieve this, we will develop a program that shares certain features with the programs used in appendix A, while also incorporating new code segments.
Essentially, the output of these experiments demonstrates that, within a random sequence, the probability distributions of $X \bmod 3$, $X \bmod 9$, $\kappa \bmod 6$, and $A \bmod 6$ across each congruence class and for every term of the sequence align with the probability distributions derived in sections 5 and 6 via specific mathematical relations and formulas. In the section 6 about the probability distributions of $X \bmod 3$, $X \bmod 9$, $\kappa \bmod 6$, and $A \bmod 6$, we

have:

$$\forall j \in \{1,2\} : \mathbb{P}(X \bmod 3 = j) = \tfrac{1}{3}\{1,2\} = \{0.33333, 0.66667\},$$

$$\forall i \in \{1,2,4,5,7,8\} : \mathbb{P}(X \bmod 9 = i) = \tfrac{1}{189}\{24,48,33,12,6,66\} =$$

$$\{0.126984, 0.253968, 0.174603, 0.0634921, 0.031746, 0.349206\},$$

$$\forall k \in \{0,1,2,3,4,5\} : \mathbb{P}(\kappa \bmod 6 = k) = \tfrac{1}{189}\{24,12,6,48,33,66\} = \quad (81)$$

$$\{0.126984, 0.0634921, 0.031746, 0.253968, 0.174603, 0.349206\},$$

$$\forall h \in \{0,1,2,3,4,5\} : \hat{a}_h = \mathbb{P}(A \bmod 6 = h) = \tfrac{1}{63}\{1,32,16,8,4,2\} =$$

$$\{0.015873, 0.507937, 0.253968, 0.126984, 0.0634921, 0.031746\}.$$

For our expriments we create a random sequence *CONS*. To do this we select a random integer $x \in [10^5, 10^6]$ and we let $x_0 = 2x - 1$ as the first term of our random *CONS*. After based on *RCONS*, we calculate the next terms and the sequence is obtained as

$$x_0, x_1, x_2, \cdots, x_9, \cdots$$

We repeat this expriment 100000 times. For our computations we use the mathematica function 2 and mathematica code 5 and after we save probability distributions of $X \bmod 3$, $X \bmod 9$, $\kappa \bmod 6$, and $A \bmod 6$ for the first 10 terms of the sequence. Furthermore, we observe that the probability distribution of $X \bmod 9$ remains invariant from index 2 onwards, the probability distribution of $\kappa \bmod 6, X \bmod 3$ remain invariant from index 1 onwards and the probability distribution of $A \bmod 6$ remains invariant from index 0 onwards, approximately. Now we see these distribution vectors are

approximately same as relations 81 and are as below:

$$
\begin{pmatrix}
\text{Indexes} & X \bmod 3 \text{ and Probabilities} & X \bmod 9 \text{ and Probabilities} & \kappa \bmod 6 \text{ and Probabilities} & A \bmod 6 \text{ and Probabilities} \\
(\ x_0\ ) & \begin{pmatrix} 0 & 0.33193 \\ 1 & 0.33331 \\ 2 & 0.33476 \end{pmatrix} & \begin{pmatrix} 0 & 0.11066 \\ 1 & 0.11125 \\ 2 & 0.11231 \\ 3 & 0.11172 \\ 4 & 0.11098 \\ 5 & 0.11212 \\ 6 & 0.10955 \\ 7 & 0.11108 \\ 8 & 0.11033 \end{pmatrix} & \begin{pmatrix} 0 & 0.11157 \\ 1 & 0.21995 \\ 2 & 0.11065 \\ 3 & 0.22288 \\ 4 & 0.11226 \\ 5 & 0.22269 \end{pmatrix} & \begin{pmatrix} 0 & 0.0159 \\ 1 & 0.50557 \\ 2 & 0.25453 \\ 3 & 0.1278 \\ 4 & 0.06405 \\ 5 & 0.03215 \end{pmatrix} \\
(\ x_1\ ) & \begin{pmatrix} 1 & 0.33448 \\ 2 & 0.66552 \end{pmatrix} & \begin{pmatrix} 1 & 0.11157 \\ 2 & 0.22288 \\ 4 & 0.11226 \\ 5 & 0.21995 \\ 7 & 0.11065 \\ 8 & 0.22269 \end{pmatrix} & \begin{pmatrix} 0 & 0.12713 \\ 1 & 0.06205 \\ 2 & 0.03121 \\ 3 & 0.25653 \\ 4 & 0.17421 \\ 5 & 0.34887 \end{pmatrix} & \begin{pmatrix} 0 & 0.01587 \\ 1 & 0.50966 \\ 2 & 0.25338 \\ 3 & 0.12665 \\ 4 & 0.0633 \\ 5 & 0.03114 \end{pmatrix} \\
(\ x_2\ ) & \begin{pmatrix} 1 & 0.33255 \\ 2 & 0.66745 \end{pmatrix} & \begin{pmatrix} 1 & 0.12713 \\ 2 & 0.25653 \\ 4 & 0.17421 \\ 5 & 0.06205 \\ 7 & 0.03121 \\ 8 & 0.34887 \end{pmatrix} & \begin{pmatrix} 0 & 0.12626 \\ 1 & 0.06294 \\ 2 & 0.03169 \\ 3 & 0.25377 \\ 4 & 0.17649 \\ 5 & 0.34885 \end{pmatrix} & \begin{pmatrix} 0 & 0.01614 \\ 1 & 0.5075 \\ 2 & 0.25519 \\ 3 & 0.12725 \\ 4 & 0.06311 \\ 5 & 0.03081 \end{pmatrix} \\
(\ x_3\ ) & \begin{pmatrix} 1 & 0.33444 \\ 2 & 0.66556 \end{pmatrix} & \begin{pmatrix} 1 & 0.12626 \\ 2 & 0.25377 \\ 4 & 0.17649 \\ 5 & 0.06294 \\ 7 & 0.03169 \\ 8 & 0.34885 \end{pmatrix} & \begin{pmatrix} 0 & 0.12627 \\ 1 & 0.06403 \\ 2 & 0.03212 \\ 3 & 0.25532 \\ 4 & 0.17422 \\ 5 & 0.34804 \end{pmatrix} & \begin{pmatrix} 0 & 0.01575 \\ 1 & 0.50764 \\ 2 & 0.25276 \\ 3 & 0.12771 \\ 4 & 0.0641 \\ 5 & 0.03204 \end{pmatrix} \\
(\ x_4\ ) & \begin{pmatrix} 1 & 0.33261 \\ 2 & 0.66739 \end{pmatrix} & \begin{pmatrix} 1 & 0.12627 \\ 2 & 0.25532 \\ 4 & 0.17422 \\ 5 & 0.06403 \\ 7 & 0.03212 \\ 8 & 0.34804 \end{pmatrix} & \begin{pmatrix} 0 & 0.12699 \\ 1 & 0.06279 \\ 2 & 0.0317 \\ 3 & 0.25489 \\ 4 & 0.17294 \\ 5 & 0.35069 \end{pmatrix} & \begin{pmatrix} 0 & 0.01577 \\ 1 & 0.50971 \\ 2 & 0.25206 \\ 3 & 0.12719 \\ 4 & 0.0638 \\ 5 & 0.03147 \end{pmatrix}
\end{pmatrix}
$$

Other indexes are in below :

$$
\begin{pmatrix}
\text{Indexes} & \begin{matrix}X \bmod 3 \text{ and} \\ \text{Probabilities}\end{matrix} & \begin{matrix}X \bmod 9 \text{ and} \\ \text{Probabilities}\end{matrix} & \begin{matrix}\kappa \bmod 6 \text{ and} \\ \text{Probabilities}\end{matrix} & \begin{matrix}A \bmod 6 \text{ and} \\ \text{Probabilities}\end{matrix} \\
\begin{pmatrix} x_5 \end{pmatrix} & \begin{pmatrix} 1 & 0.33163 \\ 2 & 0.66837 \end{pmatrix} & \begin{pmatrix} 1 & 0.12699 \\ 2 & 0.25489 \\ 4 & 0.17294 \\ 5 & 0.06279 \\ 7 & 0.0317 \\ 8 & 0.35069 \end{pmatrix} & \begin{pmatrix} 0 & 0.12858 \\ 1 & 0.06306 \\ 2 & 0.03119 \\ 3 & 0.25149 \\ 4 & 0.17656 \\ 5 & 0.34912 \end{pmatrix} & \begin{pmatrix} 0 & 0.01564 \\ 1 & 0.50689 \\ 2 & 0.25659 \\ 3 & 0.12628 \\ 4 & 0.0641 \\ 5 & 0.0305 \end{pmatrix} \\
\begin{pmatrix} x_6 \end{pmatrix} & \begin{pmatrix} 1 & 0.33633 \\ 2 & 0.66367 \end{pmatrix} & \begin{pmatrix} 1 & 0.12858 \\ 2 & 0.25149 \\ 4 & 0.17656 \\ 5 & 0.06306 \\ 7 & 0.03119 \\ 8 & 0.34912 \end{pmatrix} & \begin{pmatrix} 0 & 0.1276 \\ 1 & 0.06282 \\ 2 & 0.03195 \\ 3 & 0.25732 \\ 4 & 0.17328 \\ 5 & 0.34703 \end{pmatrix} & \begin{pmatrix} 0 & 0.01598 \\ 1 & 0.50769 \\ 2 & 0.25421 \\ 3 & 0.12788 \\ 4 & 0.06264 \\ 5 & 0.0316 \end{pmatrix} \\
\begin{pmatrix} x_7 \end{pmatrix} & \begin{pmatrix} 1 & 0.33283 \\ 2 & 0.66717 \end{pmatrix} & \begin{pmatrix} 1 & 0.1276 \\ 2 & 0.25732 \\ 4 & 0.17328 \\ 5 & 0.06282 \\ 7 & 0.03195 \\ 8 & 0.34703 \end{pmatrix} & \begin{pmatrix} 0 & 0.12963 \\ 1 & 0.06358 \\ 2 & 0.03101 \\ 3 & 0.25253 \\ 4 & 0.17358 \\ 5 & 0.34967 \end{pmatrix} & \begin{pmatrix} 0 & 0.01542 \\ 1 & 0.50812 \\ 2 & 0.25362 \\ 3 & 0.12606 \\ 4 & 0.06518 \\ 5 & 0.0316 \end{pmatrix} \\
\begin{pmatrix} x_8 \end{pmatrix} & \begin{pmatrix} 1 & 0.33422 \\ 2 & 0.66578 \end{pmatrix} & \begin{pmatrix} 1 & 0.12963 \\ 2 & 0.25253 \\ 4 & 0.17358 \\ 5 & 0.06358 \\ 7 & 0.03101 \\ 8 & 0.34967 \end{pmatrix} & \begin{pmatrix} 0 & 0.12884 \\ 1 & 0.06375 \\ 2 & 0.03157 \\ 3 & 0.25545 \\ 4 & 0.17306 \\ 5 & 0.34733 \end{pmatrix} & \begin{pmatrix} 0 & 0.01565 \\ 1 & 0.50705 \\ 2 & 0.25315 \\ 3 & 0.12892 \\ 4 & 0.06467 \\ 5 & 0.03056 \end{pmatrix} \\
\begin{pmatrix} x_9 \end{pmatrix} & \begin{pmatrix} 1 & 0.33347 \\ 2 & 0.66653 \end{pmatrix} & \begin{pmatrix} 1 & 0.12884 \\ 2 & 0.25545 \\ 4 & 0.17306 \\ 5 & 0.06375 \\ 7 & 0.03157 \\ 8 & 0.34733 \end{pmatrix} & \begin{pmatrix} 0 & 0.12903 \\ 1 & 0.06448 \\ 2 & 0.03232 \\ 3 & 0.25146 \\ 4 & 0.17409 \\ 5 & 0.34862 \end{pmatrix} & \begin{pmatrix} 0 & 0.01586 \\ 1 & 0.50556 \\ 2 & 0.25624 \\ 3 & 0.12747 \\ 4 & 0.06334 \\ 5 & 0.03153 \end{pmatrix}
\end{pmatrix}
$$

Appendix C Mathematica functions and codes

We use below functions and codes for our test and expriments in appendixes A and B.

```
CC[OddNumber_, Pm_, lambda_] := Module[{},
  (Pm*OddNumber + 1)/
    Product[
      (lambda^
        IntegerExponent[Pm*OddNumber + 1, lambda])[[i]],
      {i, 1, Length[lambda]}
    ] // Floor
]
```

Listing 1: Definition of the function CC in Mathematica

```
f1[x_] := Module[{a},
  a = IntegerExponent[(3*x + 1), 2];
  {Mod[x, 3], Mod[x, 9],
   Mod[(x - (4^Floor[(a + 1)/2] - 1)/3)/2^a, 6],
   Mod[a, 6]}
  ]
```

Listing 2: Definition of the function f1 in Mathematica

```

numSamples = 10;
Pm = 5;
lambda = {2, 3};
maxtime = 1;

AbsoluteTiming[

  Randomsample = RandomSample[Range[1000], numSamples];

  PmMulOdd = Table[
    Pm*(2 Randomsample[[i]] - 1),
    {i, 1, numSamples}
  ];

  j = 1;

  While[j <= Length[PmMulOdd],

    EveryStep = PmMulOdd[[j]];
    CONSOdd = {EveryStep};
```

```
22
23      outoftime = TimeConstrained[
24
25        While[EveryStep != 1,
26
27          EveryStep = CC[EveryStep, Pm, lambda];
28
29          If[MemberQ[CONSOdd, EveryStep] == False,
30
31            AppendTo[CONSOdd, EveryStep],
32
33            AppendTo[CONSOdd, EveryStep];
34            Break[]
35          ]
36        ],
37
38        maxtime
39      ];
40
41      Print[{CONSOdd, outoftime}];
42
43      j = j + 1
44    ]
45  ]
```

Listing 3: TESTCODE1 for numerical iteration to $m = 3$, $P_m = 5$, and $\Lambda_l = \{2, 3\}$

```
1
2   numSamples = 10000;
3   Pm = 5;
4   lambda = {2, 3};
5   maxtime = 1;
6
7   AbsoluteTiming[
8     Randomsample = RandomSample[Range[300000], numSamples];
9
10    PmMulOdd = Table[
11      Pm*(2 Randomsample[[i]] - 1),
12      {i, 1, numSamples}
13    ];
14
```

```
j = 1;
Loopnum = {};
AllOdd = {};
OutTime = {};

While[j <= Length[PmMulOdd],

  i = 0;
  EveryStep = PmMulOdd[[j]];
  CONSOdd = {EveryStep};

  outoftime = TimeConstrained[
    While[EveryStep != 1,

      EveryStep = CC[EveryStep, Pm, lambda];

      If[MemberQ[CONSOdd, EveryStep] == False,

        AppendTo[CONSOdd, EveryStep],

        i = i + 1;

        loopnum = Take[
          CONSOdd,
          {Position[CONSOdd, EveryStep][[1, 1]], -1}
        ];

        Break[]
      ]
    ],
    maxtime
  ];

  AllOdd = Join[AllOdd, CONSOdd];

  AppendTo[OutTime, outoftime];

  If[i > 0,
    AppendTo[Loopnum, loopnum]
  ];
```

```
55
56      j = j + 1
57    ];
58
59    AllOdd = Sort[
60      ((Mod[AllOdd // Union, Pm]) // Tally),
61      #1[[1]] < #2[[1]] &
62    ];
63
64    totalallodd = Table[
65      AllOdd[[i, 2]],
66      {i, 2, Pm}
67    ] // Total;
68
69    Table[
70      AppendTo[
71        AllOdd[[i]],
72        ((Pm - 1)/Pm)*AllOdd[[i, 2]]/totalallodd // N
73      ],
74      {i, 2, Pm}
75    ];
76
77    AppendTo[AllOdd[[1]], 1/Pm];
78
79    looptally = Loopnum // Tally;
80    looptally2 = {};
81
82    For[i = 1, i <= (looptally // Length), i++,
83
84      tally = looptally[[i, 2]];
85
86      For[j = i + 1, j <= (looptally // Length), j++,
87
88        If[
89          MemberQ[looptally[[j, 1]], looptally[[i, 1, 1]]] == True,
90
91          tally = tally + looptally[[j, 2]];
92
93          looptally = DeleteCases[
94            looptally,
```

```
        looptally[[j]]
      ];

      j = j - 1,

      Continue[]
    ]
  ];

  AppendTo[
    looptally2,
    {looptally[[i, 1]], tally}
  ]
 ];

 Print[
  {AllOdd, looptally2, OutTime // Tally}
 ]
]
```

Listing 4: TESTCODE2 for numerical iteration in the case $m = 3$, $P_m = 5$, and $\Lambda_l = \{2, 3\}$

```

numSamples = 100000;
Pm = 3;
lambda = {2};

Randomsample = RandomSample[Range[10^5, 10^6], numSamples];

PmMulOdd = Table[
 (2 Randomsample[[i]] - 1),
 {i, 1, numSamples}
];

j = 1;
AllOddf = {};

While[j <= Length[PmMulOdd],
 EveryStep = PmMulOdd[[j]];
 CONSOddf = {f1[EveryStep]};

```

```
  While[EveryStep != 1,
    EveryStep = CC[EveryStep, Pm, lambda];
    AppendTo[CONSOddf, f1[EveryStep]]
  ];

  AppendTo[AllOddf, CONSOddf];
  j = j + 1
];

r = Max[Map[Length, AllOddf]] - Map[Length, AllOddf];

p = Take[
  Partition[
    Map[Sort,
      Apply[Join,
        Map[Tally,
          Transpose[
            MapThread[
              Join, {
                AllOddf,
                MapThread[
                  ConstantArray,
                  {ConstantArray[{1, 1, 0, 2}, Length[r]], r}
                ]
              }
            ],
            2
          ],
          {2}
        ]
      ]
    ],
    4
  ],
  10
];

For[i = 1, i <= Length[p], i++,
  For[j = 1, j <= Length[p[[i]]], j++,
    For[k = 1, k <= Length[p[[i, j]]], k++,
```

```
      p[[i, j]] = ReplacePart[
        p[[i, j]],
        k -> {
          p[[i, j, k, 1]],
          p[[i, j, k, 2]]/numSamples // N
        }
      ]
    ]
  ]
];

p
```

Listing 5: TESTCODE3 implementation for odd number iteration and statistical analysis

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