Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Dynamic Behavior of Functionally Graded Beams Supported by Three-Parameter Viscoelastic Foundations

Khayra Draouche
University of Tiaret
Sihame Ait Yahia
University of M'Hamed Bougara Boumerdes
Lazreg Hadji
University of Tiaret
Nafissa Zouatnia
nafissa.zouatnia@univ-tiaret.dz

University of Tiaret
Hassen Ait Atmane
University of Chlef
Royal Madan
Graphic Era (Deemed to be University)
Riadh Bennai
University of Chlef

Research Article

Keywords: Functionally graded beams, Free vibration, viscoelastic foundation, Natural frequencies

Posted Date: November 27th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11822020/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

Dynamic Behavior of Functionally Graded Beams Supported by Three-Parameter Viscoelastic Foundations

Khayra Draouche[1], Sihame Ait Yahia[2], Lazreg Hadji[1], Nafissa Zouatnia[1*], Hassen Ait Atmane[3], Royal Madan[4], Riadh Bennai[3]

[1] Department of Civil Engineering, University of Tiaret, BP 78 Zaaroura, Tiaret, 14000, Algeria.
[2] Department of Civil Engineering, Faculty of Technology, University of M'Hamed Bougara Boumerdes. Algeria.
[3] Laboratory of Structures, Geotechnics and Risks, Department of Civil Engineering, University of Chlef, Chlef, Algeria.
[4] Department of Mechanical Engineering, Graphic Era (Deemed to be University), Dehradun, India.

Email address: nafissa.zouatnia@univ-tiaret.dz

Abstract

This paper presents two refined shear deformation theories for investigating the free vibration behavior of functionally graded (FG) beams resting on three-parameter viscoelastic foundations (VEFs). The proposed theories employ sinusoidal and parabolic distributions of transverse shear strains through the beam thickness and satisfy the zero-shear-stress conditions at the upper and lower surfaces without requiring shear correction factors. The formulations also reduce the number of unknowns and governing equations, simplifying the computational process. The material properties are assumed to vary continuously across the beam thickness according to a power-law distribution. The foundation is modeled using Winkler and Pasternak parameters, together with a damping coefficient. The governing equations are derived using Hamilton's principle, and analytical solutions are obtained through the Navier method. The accuracy of the proposed models is validated against existing solutions. Finally, a comprehensive parametric study examines the effects of material gradation, slenderness ratio, foundation stiffness, shear interaction, and damping on the natural frequencies of FG beams.

Keywords: Functionally graded beams; Free vibration; viscoelastic foundation; Natural frequencies.

1. Introduction

Functionally graded materials (FGMs) have garnered immense interest across modern aerospace, civil, and mechanical engineering sectors over recent decades [1]. Initially introduced to mitigate severe thermal and mechanical stresses in high-temperature environments, FGMs feature continuous and gradual spatial variations in material composition, typically transitioning smoothly from a heat-resistant ceramic phase to a load-bearing metallic phase. Unlike conventional laminated composite structures, which are susceptible to abrupt interfacial stress concentrations, delamination, and matrix cracking, functionally graded

architectures provide superior structural continuity, optimized weight-to-stiffness ratios, and improved dynamic characteristics.

To reliably capture the static, dynamic, and stability behavior of functionally graded structural components, establishing precise kinematic models is paramount. While classical beam theory (Euler–Bernoulli) neglects transverse shear deformations—rendering it inadequate for thick or moderately thick geometries—first-order shear deformation theories (FSDTs) rely on arbitrary shear correction factors that are difficult to evaluate accurately for non-homogeneous cross-sections. Consequently, higher-order shear deformation theories (HSDTs) and refined kinematic frameworks have been developed to reflect realistic, non-linear shear strain distributions across the beam thickness. Pioneering plate and beam models by Reddy [2], followed by sophisticated four-variable shear formulations by Meksi et al. [3], demonstrated that parabolic or sinusoidal distributions can naturally enforce traction-free conditions on the upper and lower surfaces without requiring correction factors. Parallel developments in advanced structures have explored dynamic, buckling, and wave characteristics across related composite, smart, and graded systems—including axially graded beams investigated by Selmi [4], fiber-reinforced composite elements under moving loads by the Ritz Method treated by Akbaş et al. [5], and sigmoid functionally graded sandwich beams studied by Avcar et al. [6]. Furthermore, refined kinematic theories have been successfully adapted to micro- and nanoscale structural analysis, as evidenced by the work of Akgöz and Civalek [7] on tapered microbeams, Ghazwani et al. [8] on the high-frequency analysis of exponentially graded nanobeams resting on Winkler–Pasternak foundations, and Turan and Akgöz [9] on non-uniform beams within the framework of modified couple stress theory.

In a broad array of operational environments, structural elements interact directly with supporting elastic or viscoelastic subgrades. Foundational investigations often relied on classical one-parameter Winkler or two-parameter Pasternak models, which capture normal spring stiffness and shear layer continuity, respectively [9, 10–11]. For instance, Ait Atmane et al. [10] assessed the simultaneous effects of porosity and thickness stretching on FG beams supported by elastic foundations, while Nebab et al. [12] developed fundamental frequencies of cracked FGM beams with influence of porosity and Winkler/Pasternak/Kerr foundation support using a new quasi-3D HSDT. Similar refined models have been established for graded plates by Thai and Choi [11], exponentially graded sandwich plates under various boundary conditions by Sobhy [13], higher-order plate formulations by Xiang et al. [14], and imperfect sandwich plates on Winkler–Pasternak foundations by Emad et al. [15].

Nevertheless, purely elastic foundation models omit the physical reality of energy dissipation in the surrounding medium. Real subgrades—such as saturated soils, biological matrices, or viscoelastic damping layers—exhibit time-dependent dissipation alongside restoring stiffness. Incorporating viscous damping leads to the three-parameter viscoelastic foundation (VEF) model, which unites the Winkler normal stiffness, the Pasternak shear interaction, and a continuous damping coefficient. Recently, Safarpour et al. [16] highlighted the necessity of accounting for viscoelastic substrates in wave propagation analysis of functionally graded bio-composite circular plates using an improved sinusoidal shear deformation theory, while Calim et al. [17] analyzed dynamic of viscoelastic FGM shells with porosities on elastic foundation. Alnujaie et al. [18] investigated the size-dependent free vibration behavior of coated functionally graded graphene-reinforced nanoplates resting on a viscoelastic medium, taking into account the effects of material gradation and nanoscale size dependency. Ragb et al. [19] investigated the nonlinear vibration behavior of laminated piezoelectric layered plates resting on nonlinear viscoelastic supports, employing different differential quadrature method (DQM) techniques to analyze their dynamic response. Recently, Calim et al. [20] proposed dynamic analysis of viscoelastic FGM shells with porosities on elastic foundation. Tahir et al. [21]

investigatd waves dispersion in an imperfect functionally graded beam resting on visco-Pasternak foundation.
In free vibration analysis, the presence of damping transforms the governing eigenvalue formulation into a complex-frequency domain problem, wherein the real component governs the oscillatory natural frequency and the imaginary component dictates the structural decay rate.
Despite substantial research efforts, a unified analytical framework that combines computationally economical refined shear kinematics (reducing both field variables and governing equations) with the complete three-parameter viscoelastic foundation interaction for FG beams remains insufficiently documented. Many existing formulations either demand computationally burdensome numerical discretizations or fail to address the coupled effects of material gradient index, beam slenderness, shear interaction, and foundation damping simultaneously.
To address these limitations, the present study investigates the free vibration behavior of simply supported functionally graded beams resting on three-parameter viscoelastic foundations. Two refined shear deformation models—incorporating sinusoidal and parabolic transverse shear variations—are established, naturally fulfilling traction-free boundary conditions on the beam surfaces without shear correction factors. The material properties vary across the thickness according to a continuous power-law rule. Hamilton's principle is applied to formulate the equations of motion, which are subsequently resolved analytically in closed form via the Navier procedure. A comprehensive parametric study is conducted to examine how the material index, beam slenderness ratio, foundation Winkler and Pasternak constants, and damping coefficient interact to influence complex natural frequencies and damping ratios.

2. The formulation of problem

2.1. Geometrical configuration

As shown in Figure 1, consider an FG beam that is resting on the VEFs of length L, width b, and thickness h. The cartesian coordinate system (x, y, and z) is used as the foundation for mathematical calculations when the x and y axes are positioned in the center of the beam. The FG beam is simply supported at both ends, as can be seen show.

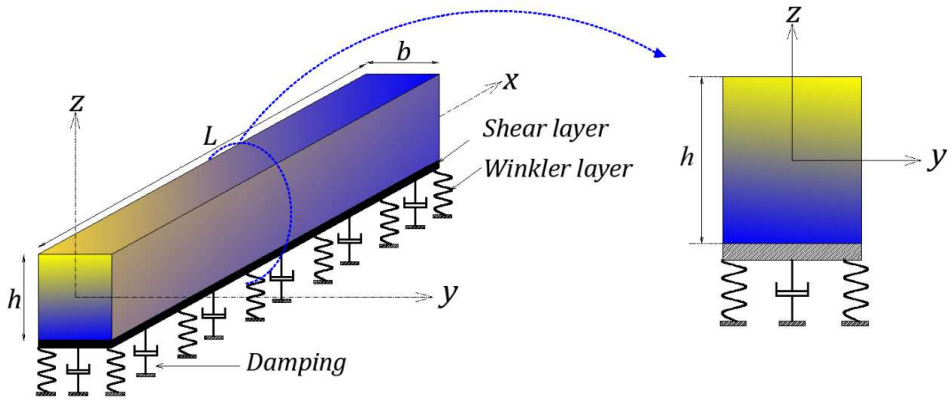

Figure 1. Configuration of an FG beam resting on the VEFs.

2.2. Material properties

The FG beam consists of a combination of two metallic and ceramic materials whose properties vary with thickness according to a power law distribution of the volume fractions of the two materials constituents by the following formulation :

$$P = (P_c - P_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + P_m \tag{1}$$

Where the power-law index p characterizes the material change across the thickness of the FG beam, such as the Young's modulus E and the mass density ρ, and the symbols c and m stand for ceramic and metal, respectively. Note that when p equals 0, the beam is made entirely of ceramic; yet, when k equals an infinite number, the beam is made entirely of metal. Since Young's modulus has a greater influence on the mechanical behaviors of FG beams than Poisson's ratio does, the Poisson's ratio is supposed constant in this work. It is assumed constant in this work, the total volume fraction of metal and ceramic is V_m +Vc=1, and the power law of the volume fraction of ceramic is described as :

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^p \tag{2}$$

Consequently, the FG beam's density (ρ) and Young's modulus (E) may be expressed as follows :

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + E_m \tag{3}$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + \rho_m \tag{4}$$

2.3. Viscoelastic foundation

The FG beam is designed to rest on a multi-layer elastic base. The second layer is connected to the first by a Pasternak shear layer, while the first layer consists of Winkler springs without coupling effect in addition to the dampers. Winkler-Pasternak viscoelastic foundations are described in general terms as follows:

$$R_e = K_w w - k_p\left(\frac{\partial^2 w}{\partial x^2}\right) + c_d \frac{\partial w}{\partial t} \tag{5}$$

where Re is the reaction of the VE foundations, kw is the constant of proportionality that is referred to as the modulus of medium reaction (so-called spring constant), K_p is shear layer is constant and cd is the damping coefficient.

2.4 Basic assumptions

The assumptions of the present theory are as follows:

- The origin of the Cartesian coordinate system is taken at the median surface of the FG beam.

- The displacements are small in comparison with the height of the beam and, therefore, strains involved are infinitesimal.
- The transverse displacement w includes two components of bending w_b, and shear w_s. These components are functions of coordinates x, y only.

$$W(x,z,t) = w_b(x,t) + w_s(x,t) \tag{6}$$

- The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x.
- The axial displacement u in x-direction, consists of extension, bending, and shear components.

$$u = u_0 + u_b + u_s \tag{7}$$

- The bending component u_b is assumed to be similar to the displacements given by the classical beam theory. Therefore, the expression for ub can be given as

$$u_b = -z\frac{\partial w_b}{\partial x} \tag{8}$$

- The shear component us gives rise, in conjunction with ws , to the parabolic variation of shear strain γ_{xz} and hence to shear stress τ_{xz} through the thickness of the beam in such a way that shear stress τ_{xz} is zero at the top and bottom faces of the beam. Consequently, the expression for us can be given as

$$u_s = -f(z)\frac{\partial w_s}{\partial x} \tag{9}$$

2.5 Kinematics and constitutive equations

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (6)-(9) as

$$U(x,z,t) = u_0(x,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{10a}$$

$$W(x,z,t) = w_b(x,t) + w_s(x,t) \tag{10b}$$

$$f(z) = z - \frac{1}{2}z\left(\frac{1}{4}h^2 - \frac{1}{3}z^2\right) \tag{11a}$$

$$f(z) = z - \frac{h}{\pi}\sin(\frac{\pi z}{h}) \tag{11b}$$

The strains associated with the displacements in Eq. (10) are

$$\varepsilon_x = \varepsilon_x^0 + zk_x^b + f(z)k_x^s \tag{12a}$$

$$\gamma_{xz} = g(z)\gamma_{xz}^s \tag{12b}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x},\quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2},\quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2},\quad \gamma_{xz}^s = \frac{\partial w_s}{\partial x} \tag{12c}$$

$$g(z)=1-f^{'}(z) \quad \text{and} \quad f^{'}(z)=\frac{\partial f(z)}{\partial z} \tag{12d}$$

The transverse shear and normal deformations are considered in the stress and strain relationships as follows

$$\sigma_x = Q_{11}(z)\varepsilon_x \quad \text{and} \quad \tau_{xz} = Q_{55}(z)\gamma_{xz} \tag{13a}$$

where

$$Q_{11}(z) = E(z) \quad \text{and} \quad Q_{55} = \frac{E(z)}{2(1+\nu)} \tag{13b}$$

2.6. Equations of motion

Hamilton's principle is used here to derive the equations of motion for the free vibration of FG beams resting on VEFs. The principle can be stated in the analytical form [22]

$$\int_0^t (\delta\ U + \delta\ U_{ef} - \delta\ K)\ dt = 0 \tag{14}$$

Where δU is the variation of strain energy; δU_{ef} is the variation of strain energy of elastic foundation and δK is the variation of kinetic energy.

2.6.1. Strain energy

The variation of strain energy of the beam is given as

$$\begin{aligned} \delta\ U &= \int_{-h/2}^{h/2}\int_0^L \left[\sigma_x \delta\ \varepsilon_x + \tau_{xz}\delta\ \gamma_{xz}\right] dx dz \\ &= \int_0^L \left[N_x \delta\ \varepsilon_x^0 + M_x^b \delta\ k_x^b + M_x^s \delta\ k_x^s + S_{xz}^s \delta\ \gamma_{xz}^s \right] dA = 0 \end{aligned} \tag{15}$$

where N_x, M_x^b , M_x^s and S_{xz}^s are the stress resultants defined as

$$\left(N_x, M_x^b, M_x^s\right) = \int_{-h/2}^{h/2} (1, z, f)\,\sigma_x dz \tag{16a}$$

$$S_{xz}^s = \int_{-h/2}^{h/2} g\,\tau_{xz} dz \tag{16b}$$

2.6.2. Strain energy of VEFs

The variation of the deformation energy of the viscoelastic foundations is indicated as :

$$\delta U_{ef} = \int_A \left(K_w w - k_p \frac{\partial^2}{\partial x^2} + c_d \frac{\partial}{\partial t} \right) w dA \tag{17}$$

2.6.3. Kinetic energy

The variation of kinetic energy of the beam can be expressed as :

$$\begin{aligned}
&\delta K = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\dot{u}\delta \dot{u} + \dot{w}\delta \dot{w} \right] \rho(z)\, dx\, dz = \\
&\int_0^L \Big\{ I_1 \left[\dot{u}_0 \delta \dot{u}_0 + \left(\dot{w}_b + \dot{w}_s \right) \left(\delta \dot{w}_b + \delta \dot{w}_s \right) \right] - \\
&I_2 \left(\dot{u}_0 \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial x} \delta \dot{u}_0 \right) - I_3 \left(\dot{u}_0 \frac{\partial \delta \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x} \delta \dot{u}_0 \right) \\
&+ I_4 \left(\frac{\partial \dot{w}_b}{\partial x} \frac{\partial \delta \dot{w}_b}{\partial x} \right) + I_6 \left(\frac{\partial \dot{w}_s}{\partial x} \frac{\partial \delta \dot{w}_s}{\partial x} \right) + I_5 \left(\frac{\partial \dot{w}_b}{\partial x} \frac{\partial \delta \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x} \frac{\partial \delta \dot{w}_b}{\partial x} \right) \Big\} dA
\end{aligned} \tag{18}$$

Where dot-superscript convention indicates the differentiation with respect to the time variable t, ρ(z) is the mass density given by Eq. (4) ; and (I_1, I_2, I_3, I_4, I_5, I_6) are mass inertias expressed as :

$$\left(I_1, I_2, I_3, I_4, I_5, I_6 \right) = \int_{-h/2}^{h/2} \left(1, z, f, z^2, zf, f^2 \right) \rho(z) dz \tag{19}$$

Substituting the expressions for δU and δT from Eqs. (15), (17) and (18) into Eq. (14) and integrating by parts versus both space and times variables, and collecting the coefficients of δu_0, δw_b, and δw_s, the following equations of motion of the functionally graded beam are obtained :

$$\begin{aligned}
&\delta u_0 : \quad \frac{\partial N}{\partial x} = I_1 \ddot{u}_0 - I_2 \frac{\partial \ddot{w}_b}{\partial x} - I_3 \frac{\partial \ddot{w}_s}{\partial x} \\
&\delta w_b : \quad \frac{\partial^2 M_b}{\partial x^2} + c_d \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) + k_w \left(w_b + w_s \right) - k_p \left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_s}{\partial x^2} \right) \\
&= I_0 \left(\ddot{w}_b + \ddot{w}_s \right) + I_2 \frac{\partial \ddot{u}_0}{\partial x} - I_4 \frac{\partial^2 \ddot{w}_b}{\partial x^2} - I_5 \frac{\partial^2 \ddot{w}_5}{\partial x^2} \\
&\delta w_s : \quad \frac{\partial^2 M_s}{\partial x^2} + \frac{\partial Q}{\partial x} + c_d \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) + k_w \left(w_b + w_s \right) - k_p \left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_s}{\partial x^2} \right) \\
&= I_1 \left(\ddot{w}_b + \ddot{w}_s \right) + I_3 \frac{\partial \ddot{u}_0}{\partial x} - I_5 \frac{\partial^2 \ddot{w}_b}{\partial x^2} - I_6 \frac{\partial^2 \ddot{w}_s}{\partial x^2}
\end{aligned} \tag{20}$$

The stiffness components are given as :

$$\left\{A_{11}, B_{11}, D_{11}, B_{11}^{s}, D_{11}^{s}, H_{11}^{s}\right\}=\int_{-h/2}^{h/2} Q_{11}\left(1, z, z^{2}, f(z), z\, f(z), f^{2}(z)\right)dz$$

(21a)

$$A_{55}^{s}=\int_{-h/2}^{h/2} Q_{55}\left[g(z)\right]^{2} dz$$

(21b)

The equations of motion can be written in terms of displacements (u_0, v_0, w_b and w_s) by introducing Eq. (21) into Eq. (20). The corresponding equations have the following form :

$$A_{11}\frac{\partial^{2}u_{0}}{\partial x^{2}}-B_{11}\frac{\partial^{3}w_{b}}{\partial x^{3}}-B_{11}^{s}\frac{\partial^{3}w_{s}}{\partial x^{3}}=I_{1}\ddot{u}_{0}-I_{2}\frac{\partial\ddot{w}_{b}}{\partial x}-I_{3}\frac{\partial\ddot{w}_{s}}{\partial x} \tag{22a}$$

$$\begin{aligned}&B_{11}\frac{\partial^{3}u_{0}}{\partial x^{3}}-D_{11}\frac{\partial^{4}w_{b}}{\partial x^{4}}-D_{11}^{s}\frac{\partial^{4}w_{s}}{\partial x^{4}}+c_{d}\left(\frac{\partial^{2}w_{b}}{\partial t^{2}}+\frac{\partial^{2}w_{s}}{\partial t^{2}}\right)+k_{w}\left(w_{b}+w_{s}\right)-k_{p}\left(\frac{\partial^{2}w_{b}}{\partial x^{2}}+\frac{\partial^{2}w_{s}}{\partial x^{2}}\right)\\&=I_{1}\left(\ddot{w}_{b}+\ddot{w}_{s}\right)+I_{2}\frac{\partial\ddot{u}_{0}}{\partial x}-I_{4}\frac{\partial^{2}\ddot{w}_{b}}{\partial x^{2}}-I_{5}\frac{\partial^{2}\ddot{w}_{s}}{\partial x^{2}}\end{aligned} \tag{22b}$$

$$\begin{aligned}&B_{11}^{s}\frac{\partial^{3}u_{0}}{\partial x^{3}}-D_{11}^{s}\frac{\partial^{4}w_{b}}{\partial x^{4}}-H_{11}^{s}\frac{\partial^{4}w_{s}}{\partial x^{4}}+A_{55}^{s}\frac{\partial^{2}w_{s}}{\partial x^{2}}+c_{d}\left(\frac{\partial^{2}w_{b}}{\partial t^{2}}+\frac{\partial^{2}w_{s}}{\partial t^{2}}\right)+k_{w}\left(w_{b}+w_{s}\right)-k_{p}\left(\frac{\partial^{2}w_{b}}{\partial x^{2}}+\frac{\partial^{2}w_{s}}{\partial x^{2}}\right)\\&=I_{1}\left(\ddot{w}_{b}+\ddot{w}_{s}\right)+I_{3}\frac{\partial\ddot{u}_{0}}{\partial x}-I_{5}\frac{\partial^{2}\ddot{w}_{b}}{\partial x^{2}}-I_{6}\frac{\partial^{2}\ddot{w}_{s}}{\partial x^{2}}\end{aligned} \tag{22c}$$

3. Analytical solution

A differential equation for the free vibration issue of the FG beam resting on viscoelastic foundations is to be solved using the Navier approach from the categories of analytical solutions. On both sides, the FG beams are thought to have been simply supported. A Fourier series is thought to be the source of the following solution :

$$u_{0}=\sum_{m=1}^{\infty}U_{m}\cos(\lambda x)e^{i\omega t} \tag{23a}$$

$$W_{b}=\sum_{m=1}^{\infty}W_{bm}\sin(\lambda x)e^{i\omega t} \tag{23b}$$

$$W_{s}=\sum_{m=1}^{\infty}W_{sm}\sin(\lambda x)e^{i\omega t} \tag{23c}$$

where U_m , W_{bm}, and W_{sm} are arbitrary parameters to be determined, ω is the natural frequency.$\sqrt{i}=-1$. After inserting Eq. (23) into the equations of motion (22), which define the free vibration of FG beams resting on VEFs, the following eigenvalue equations may be obtained for any fixed value of m :

$$([K]-\omega^2[M]+\omega[C]\{\Delta\}=\{0\} \tag{24}$$

Matrix forms are :

$$(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} + \omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & -ic_d & -ic_d \\ 0 & -ic_d & -ic_d \end{bmatrix}) \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{25}$$

[K] is the stiffness matrix, [M] presents the mass matrix in the case of free vibrations, and [C] is the global damping matrix of the system. The elements of the three preceding matrices are given by :

$$\begin{aligned} &a_{11} = A_{11}\lambda^2, a_{12} = -B_{11}\lambda^3, \\ &a_{13} = -B^s_{11}\lambda^3, a_{22} = D_{11}\lambda^4 + k_w + k_p\lambda^2, \\ &a_{23} = D^s_{11}\lambda^4 + k_w + k_p\lambda^2, \\ &a_{33} = H^s_{11}\lambda^4 + A^s_{55}\lambda^2 + k_w + k_p\lambda^2 \end{aligned} \tag{26}$$

The elements of the mass matrix are given by :

$$\begin{aligned} &m_{11} = I_1, m_{12} = -I_2\lambda, m_{13} = -I_3\lambda \\ &m_{22} = I_1 + I_4\lambda^2, m_{23} = I_1 + I_5\lambda^2 \\ &m_{33} = I_1 + I_6\lambda^2 \end{aligned} \tag{27}$$

and the damping matrix is :

$$[C] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -ic_d & -ic_d \\ 0 & -ic_d & -ic_d \end{bmatrix} \tag{28}$$

4. Numerical Results and Discussion

Numerical results are presented for FG beams on Three-Parameter Viscoelastic Foundations composed of aluminum (Al) and alumina (Al_2O_3). The effects of material gradation, slenderness ratio, foundation characteristics, and damping coefficient on the natural frequencies of FG beams resting on VEFs are examined systematically.

The natural frequency and the viscoelastic foundation parameters are expressed in the following nondimensional forms:

$$\hat{\omega} = \sqrt[4]{\frac{\rho_c \omega A L^4}{EI}}, \bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}} \tag{29}$$

$$\overline{k_w} = \frac{k_w L^4}{EI}, \overline{k_p} = \frac{k_p L^2}{EI}, \quad D = \frac{E_c h^3}{12}, C_d = cd\sqrt{\frac{L^4}{\rho_c h D}}, \quad \overline{k_s} = \frac{k_s L^2}{D} \tag{30}$$

Table 1 summarizes the constituent material properties used throughout the numerical study. Alumina (Al_2O_3) represents the ceramic phase, characterized by high stiffness (Young's modulus Ec=380 GPa) and higher density (ρc=3960 kg/m³), while Aluminum (Al) represents the metallic phase with lower stiffness (Em=70 GPa) and lower density (ρm=2702 kg/m³). Poisson's ratio (ν=0.3) is assumed constant for both phases, a standard simplification justified by its relatively minor influence on bending-dominated responses compared to Young's modulus. This significant contrast in elastic properties allows a clear investigation of the effect of material gradation on vibration behavior.

Table 1. Material properties of the functionally graded beams Fahsi et al. [23].

Materiaux	Proprietes		
	$E(GPa)$	ν	$\rho(kg/m^3)$
Alumine Al_2O_3	380	0.3	3960
Aluminum Al	70	0.3	2702

Table 2 presents a validation of the proposed refined shear deformation theories (Présente théorie 1 and 2) against numerous existing higher-order models from the literature for an FG beam without foundation effects. The comparison covers a wide range of power-law indices (p=0 to 10) and two slenderness ratios (L/h=5 and 20). An excellent agreement is observed across all cases, with discrepancies being negligible (typically within 0.02%). For p=0 (fully ceramic), the present results (e.g., 5.1527 for L/h=5) match almost exactly with Nguyen et al. (5.1525) and Simsek (5.1525). As p increases, the natural frequency decreases monotonically due to the increasing metal content, which reduces overall stiffness while increasing mass. For the slender beam (L/h=20), frequencies are higher for all k values compared to the thick beam (L/h=5), as expected due to reduced shear deformation effects. The close alignment with multiple independent studies confirms the accuracy, efficiency, and correctness of the proposed kinematic models.

Table 2. Comparison of the first natural frequency ($\bar{\omega}$) for simply supported FG beam with various gradient indexes.

L/h	Theories	Indice materielle p					
		0	0.5	1	2	5	10
5	Nguyen et al. [24]	5.1525	4.4075	3.9902	3.6344	3.4312	3.3135
	Nebab et al. [25]	5.1616	4.4312	4.0238	3.6689	3.4364	3.3039
	Ibnorachid et al. [26]	5.1527	4.4106	3.9904	3.6264	/	/
	Simsek (2010) [27]	5.1525	4.4083	3.9902	3.6344	3.4312	3.3134
	Mellal et al. [28]	5.1531	4.4109	3.9906	3.6263	3.3997	3.2811
	Kehli et al. [29]	5.1616	4.4312	4.0238	3.6689	3.4364	3.3039
	Present 1	5.1527	4.4107	3.9904	3.6264	3.4012	3.2816
	Present 2	5.1531	4.4110	3.9907	3.6263	3.3998	3.2811
20	Nguyen et al. [24]	5.4603	4.6504	4.2051	3.8368	3.6509	3.5416
	Nebab et al. [25]	5.4609	4.6663	4.2356	3.8781	3.6839	3.5599
	Ibnorachid et al. [26]	5.4603	4.6511	4.2050	3.8361	/	/
	Simsek [27]	5.4603	4.6514	4.2051	3.8368	3.6509	3.5416

Mellal et al. [28]	5.4603	4.6511	4.2051	3.8361	3.6483	3.5389
Kehli et al. [29]	5.4609	4.6663	4.2356	3.8781	3.6839	3.5599
Present 1	5.4606	4.6511	4.2050	3.8362	3.6486	3.5389
Present 2	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389

Table 3 validates the present model's capability to handle viscoelastic foundations by comparing the fundamental frequency of an isotropic beam on a Winkler-Pasternak foundation against benchmark solutions. Three foundation parameter combinations (kp/π^2=0,1,2.5 with k_w=0, 10^2, 10^4) and two slenderness ratios (L/h=5 and 15) are considered. The results for the undamped case (Cd=0) show excellent consistency with previous studies, with maximum deviations less than 1%. More importantly, the table introduces new results for damped foundations (C_d=10 and 15). As expected, increasing the damping coefficient reduces the natural frequency significantly, particularly for lower foundation stiffnesses (e.g., for L/h=5, k_w=0, k_p=0, $\bar{\omega}$ drops from 3.0454 at C_d=0 to 2.3914 at C_d=15). This reduction is attributed to the energy dissipation mechanism of the viscoelastic foundation, which effectively increases the system's inertia. Notably, for very stiff foundations (k_w=10^4), the damping effect on frequency becomes negligible because the dominant restoring force comes from the elastic springs rather than the dashpot.

Table 3. Comparison of the first natural frequency ($\bar{\omega}$) of an isotropic beam placed on an elastic foundation for different thickness ratios.

L/h	Theories	k_w=0			k_w=10^2			k_w=10^4		
		k_p/π^2			k_p/π^2			k_p/π^2		
		0	1	2.5	0	1	2.5	0	1	2.5
5	Chen et al. [30]	3.0479	3.6580	4.2183	3.6705	4.0663	4.4991	7.3408	7.3409	7.3411
	Chikh [31]	3.1116	3.7010	4.2571	3.7133	4.1052	4.5399	7.8493	7.8493	7.8493
	Fahsi et al. [32]	3.0770	3.6465	4.1824	3.6817	4.0557	4.4700	7.7604	7.7604	7.7604
	Mellal et al. [28]	3.0455	3.6671	4.2395	3.6798	4.0839	4.5279	7.3765	7.3765	7.3765
	Kehli et al. [29]	3.0480	3.6595	4.2246	3.6721	4.0709	4.5094	7.5525	7.5525	7.5525
	Present 1 C_d=0	3.0454	3.6671	4.2395	3.6798	4.0839	4.5279	7.3766	7.3766	7.3766
	Present 1 C_d=10	2.8097	3.5406	4.1596	3.5548	3.9942	4.4629	7.3766	7.3766	7.3766
	Present 1 C_d=15	2.3914	2.3914	4.0529	3.3772	3.8726	4.3773	7.3766	7.3766	7.3766
15	Chen et al. [30]	3.1302	3.7265	4.2880	3.7389	4.1347	4.5734	9.9958	10.0197	10.0552
	Chikh [31]	3.1373	3.7316	4.2923	3.7440	4.1391	4.5775	10.014	10.0386	10.0744
	Fahsi et al. [32]	3.1342	3.7262	4.2851	3.7408	4.1341	4.5708	10.004	10.0283	10.0638
	Mellal et al. [28]	3.1300	3.7274	4.2897	3.7398	4.1361	4.5754	10.015	10.0390	10.0748
	Kehli et al. [29]	3.1302	3.7265	4.2881	3.7389	4.1347	4.5735	10.006	10.0305	10.0662
	Present 1 C_d=0	3.1299	3.7274	4.2896	3.7398	4.1363	4.5758	10.0197	10.0369	10.0670
	Present 1 C_d=10	2.9043	3.6014	4.2088	3.6151	4.0458	4.5096	10.0137	10.0309	10.0611
	Present 1 C_d=15	2.5167	3.4224	4.1005	3.4384	3.9231	4.4224	10.0060	10.0231	10.0534

Figure 2 illustrates the combined effect of material gradation (power-law index p) and Winkler foundation stiffness (k_w) on the dimensionless fundamental frequency. Several key observations can be made. First, for a fixed k_w, increasing p from 0 (full ceramic) to higher values leads to a monotonic decrease in ω. This is physically attributed to the gradual replacement of stiff, lightweight ceramic with softer, denser metal, resulting in a net reduction of the structure's stiffness-to-mass ratio. The most significant drop occurs between p=0 and p=2, after which the frequency asymptotically approaches a

lower bound corresponding to the fully metallic beam. Second, increasing the Winkler parameter kw from 0 to 100 substantially raises the natural frequency for all values of p. Third, the curves for different kw values remain nearly parallel, suggesting that the influence of material gradation and foundation stiffness are additive and independent. Physically, a stiffer foundation provides greater restoring force during vibration, effectively increasing the system's overall stiffness and hence its natural frequency.

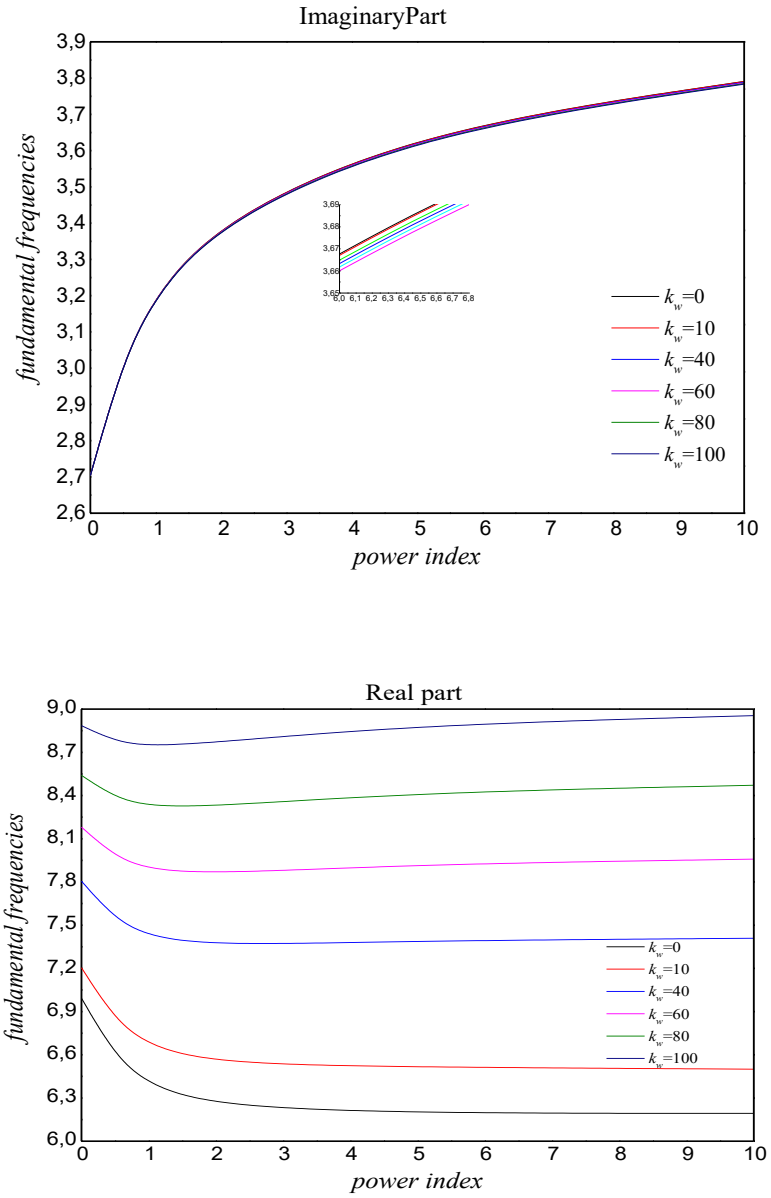

Figure 2. The relationship between the fundamental natural frequency ($\bar{\omega}$) of a beam FG and the volume fraction index k for different values of foundation parameter k_w. (k_s=10, c_d=10, L/h=5).

Figure 3 depicts the relationship between the foundation damping coefficient (c_d) and the fundamental frequency for different power-law indices p. The most prominent feature is the nonlinear, decreasing trend of $\bar{\omega}$ as cd increases from 0 to 20. This reduction is due to the dissipative nature of the dashpot element in the three-parameter VEF model. As damping increases, a larger portion of the vibrational energy is dissipated per cycle, which, in the context of free vibration analysis, manifests as a reduction in the real part of the complex natural

frequency (i.e., the oscillatory component). The rate of frequency reduction is highest for small cd values and gradually diminishes, indicating a saturation effect. Comparing different p values, the beam with p=0 (ceramic) consistently exhibits the highest frequency across all damping levels due to its superior stiffness, while the p=2 beam shows the lowest frequencies. Interestingly, the relative drop in frequency from c_d=0 to cd=20 is similar for all p (approximately 30-35%), suggesting that the damping effect is primarily a foundation property and interacts weakly with the beam's material composition.

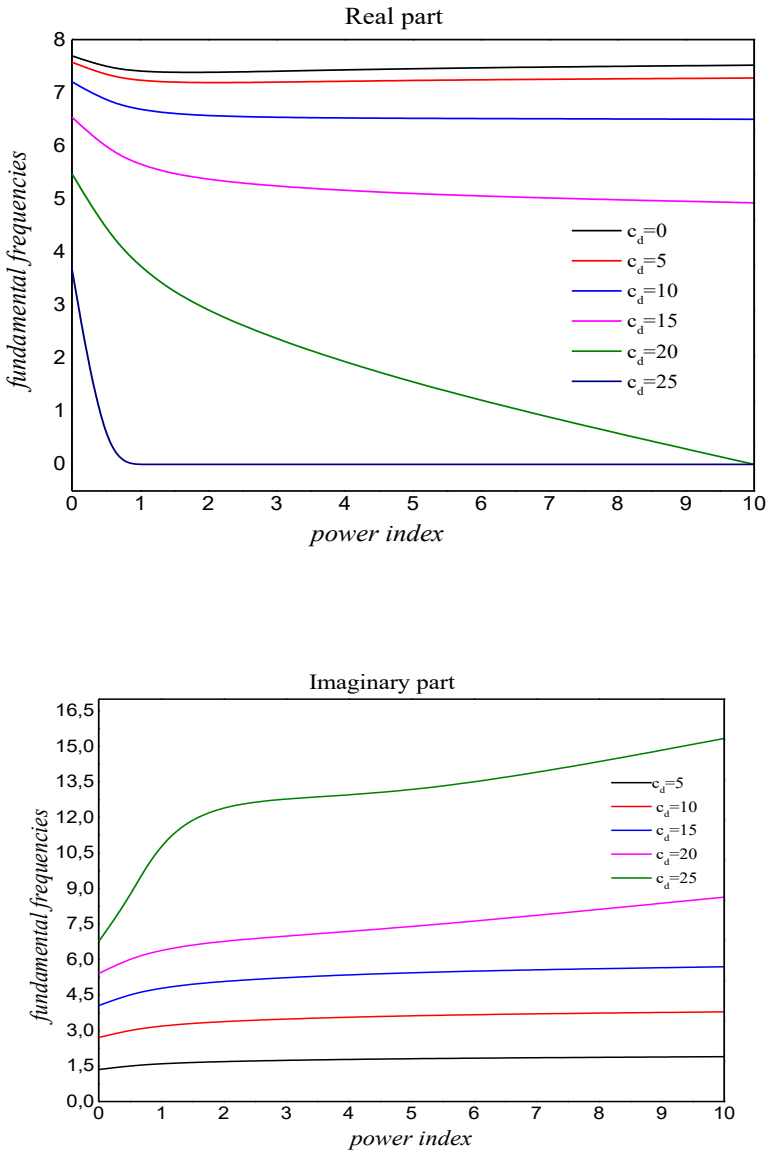

Figure 3. Effect of damping coefficient of the foundation on the variation of the first natural frequency ($\bar{\omega}$) of an FG beam placed on a viscoelastic foundation (k_s=k_w=10, L/h=5).

Figure 4 presents a plot showing the dimensionless frequency ($\bar{\omega}$) as functions of the foundation damping coefficient cd for four slenderness ratios (L/h=5, 10, 15 and 20). The frequency curves demonstrate that slender beams (L/h=20) consistently exhibit higher natural frequencies than thick beams (L/h=5) for the same cd value. This is because shear deformation, which reduces the effective stiffness, is more significant in thick beams. As cd increases, all curves descend,

but the gap between them remains nearly constant. The imaginary part curves show a completely different trend: they increase linearly with cd, as expected from the definition of the damping ratio for a viscoelastic support. Notably, the damping ratio for the thick beam (L/h=5) is substantially higher than that for the slender beam (L/h=20) at the same cd. This indicates that the foundation damping has a more dominant influence on the overall dynamic response of thicker, more flexible beams, where the beam's own structural damping contribution is comparatively smaller.

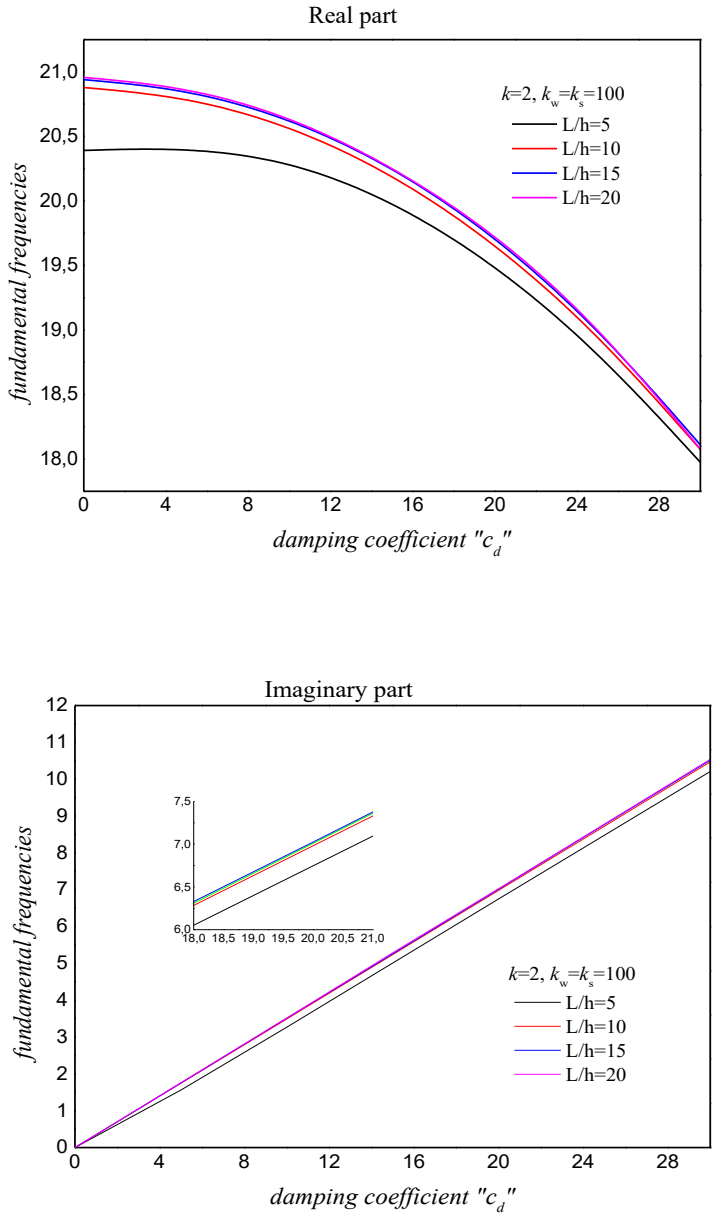

Figure 4. Variation of first natural frequency ($\bar{\omega}$) and damping ratio of an uncracked FG beam placed on a viscoelastic foundation as a function of damping coefficient for different thickness ratio L/h.

Figure 5 presents the influence of the viscoelastic foundation parameters on the dimensionless natural frequency and the damping ratio of an uncracked functionally graded (FG) beam. The foundation is characterized by three parameters: the Winkler stiffness (k_w), the Pasternak shear

parameter (k_s), and the damping coefficient (c_d). The results clearly indicate that increasing the Winkler stiffness parameter (k_w) leads to a significant increase in the natural frequency. This behavior reflects the stiffening effect of the elastic springs, which enhance the overall rigidity of the beam–foundation system and thus increase its resistance to vibration. Similarly, the Pasternak shear parameter (ks) contributes positively to the natural frequency. This parameter accounts for the shear interaction between adjacent foundation elements, providing additional continuity in the supporting medium. As a result, higher values of ks further improve the structural stiffness and lead to higher vibration frequencies. In contrast, the damping coefficient (cd) has a different effect on the system response. While it has a relatively minor influence on the natural frequency, it plays a dominant role in increasing the damping ratio. As cd increases, the system exhibits greater energy dissipation, which enhances vibration attenuation and reduces oscillation amplitudes. Moreover, the figure demonstrates that the combined action of (kw, ks, and cd) governs the dynamic behavior of the FG beam. While kw and ks primarily control the stiffness and frequency characteristics, cd controls the dissipative behavior. This highlights the importance of adopting a three-parameter viscoelastic foundation model, as it provides a more realistic representation of the interaction between the structure and its supporting medium.

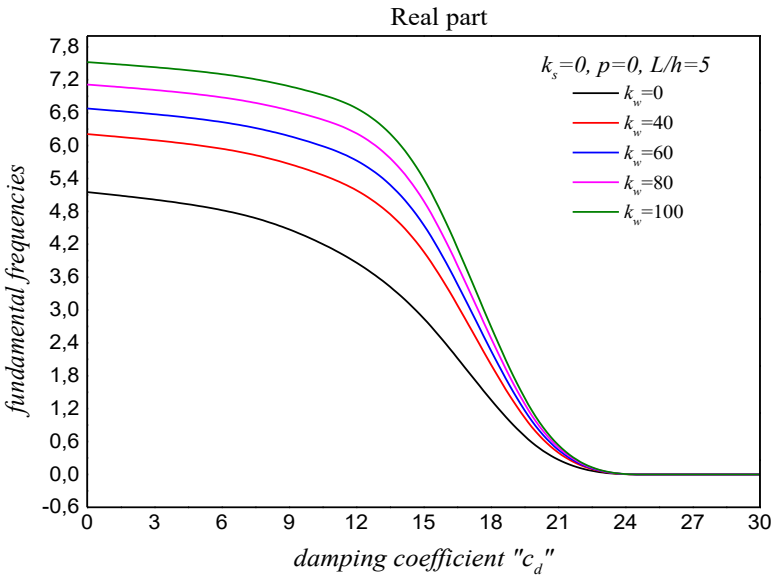

Figure 5. Effect of viscoelastic foundation parameters on dimensionless natural frequency ($\bar{\omega}$) and damping ratio of uncracked FG beams.

Figure 6 illustrates the variation of the first natural frequency and the damping ratio of an uncracked functionally graded (FG) beam resting on a viscoelastic foundation, as a function of the damping coefficient (c_d) for different values of the material gradation index (p).The results show that, for all values of p, the natural frequency decreases as the damping coefficient increases. This behavior is attributed to the dissipative nature of the viscoelastic foundation, where higher damping leads to greater energy dissipation, thereby reducing the vibration frequency of the system. At the same time, the damping ratio increases significantly with increasing damping coefficient, which is consistent with the physical role of the damping parameter in attenuating vibratory motion. This confirms that the viscoelastic foundation not only affects stiffness but also plays a key role in vibration control.

Regarding the influence of the material gradation index, it is observed that higher values of p lead to lower natural frequencies. This is because increasing p corresponds to a higher metallic content in the FG beam, resulting in reduced stiffness compared to ceramic-rich configurations. Consequently, beams with larger p values exhibit greater sensitivity to damping effects.

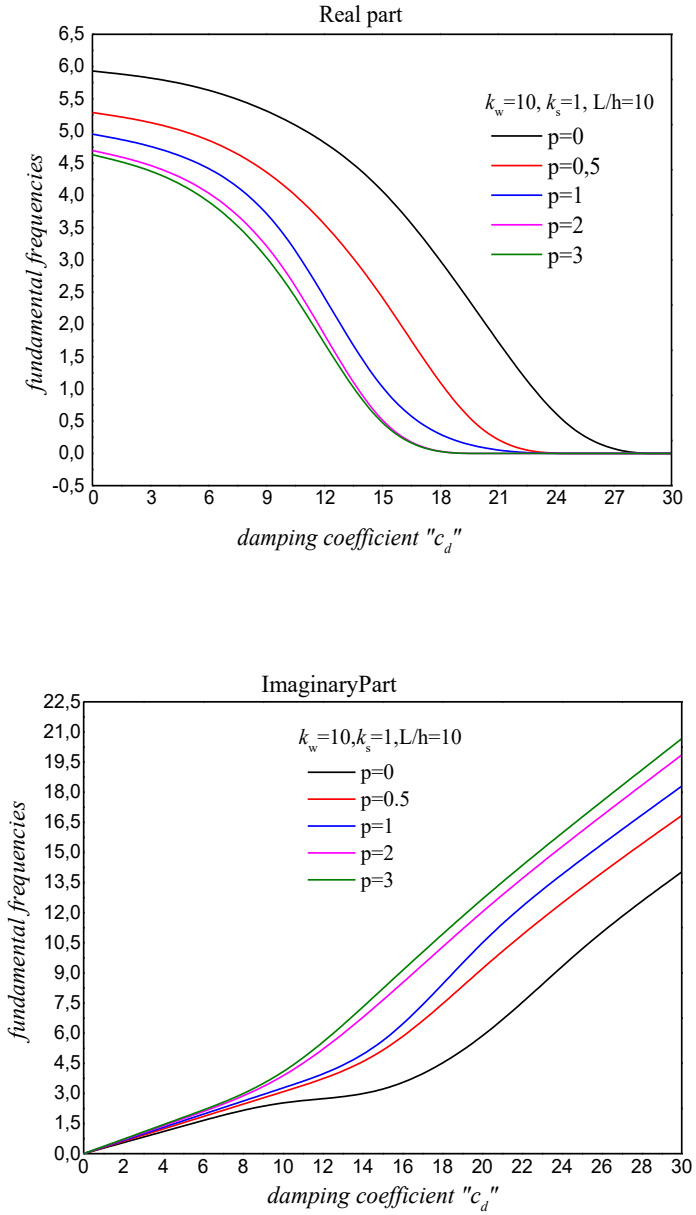

Figure 6. Variation of first natural frequency ($\bar{\omega}$) and damping ratio of an uncracked FG beam placed on a viscoelastic foundation as a function of damping coefficient for different law power index (p).

4. Conclusions

In this work, the free vibration behavior of functionally graded (FG) beams resting on three-parameter viscoelastic foundations has been investigated using two refined higher-order shear deformation theories. The proposed formulations incorporate realistic transverse shear deformation effects without requiring shear correction factors, while reducing the number of unknowns, leading to an efficient and accurate analytical model.

Based on the analytical results and detailed parametric study, the following main conclusions can be drawn:

- The proposed refined theories show excellent agreement with previously published results, confirming their accuracy and reliability for predicting the natural frequencies of FG beams on viscoelastic foundations.
- The material gradation index (k) plays a significant role in the dynamic response. Increasing k (i.e., increasing the metallic volume fraction) leads to a reduction in natural frequencies due to the decrease in overall stiffness of the beam.
- The Winkler stiffness parameter (kw) and Pasternak shear parameter (ks) significantly enhance the natural frequencies, demonstrating the stiffening effect of the viscoelastic foundation on the structural system.
- The damping coefficient (cd) introduces energy dissipation, resulting in a decrease in natural frequencies and an increase in the damping ratio. Its effect is more pronounced in flexible systems and at lower foundation stiffness levels.
- The slenderness ratio (L/h) has a considerable influence on the vibration behavior. Increasing L/h leads to a decrease in natural frequencies, highlighting the reduced bending stiffness of more slender beams.
- The interaction between material gradation, geometric parameters, and foundation characteristics plays a crucial role in the dynamic behavior, indicating that these factors must be carefully considered in design and analysis.
- The use of a three-parameter viscoelastic foundation model provides a more realistic representation of the supporting medium compared to simpler models, as it accounts simultaneously for stiffness, shear interaction, and damping effects.

Future research directions include:

- Extending the proposed refined theories to handle various boundary conditions using numerical methods.
- Incorporating porosity distributions and their effect on vibration and damping characteristics.
- Investigating the nonlinear forced vibration response under harmonic or moving loads.
- Developing a finite element formulation based on the present kinematic assumptions for complex geometries.
- Exploring the use of machine learning surrogate models to rapidly predict natural frequencies for arbitrary combinations of material and foundation parameters.
- Experimental validation of the theoretical predictions using additively manufactured FG beams on controlled viscoelastic supports.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Funding

No funding was received for the research described in the article.

Author contribution

K.D. and S.A.Y. methodology, K.D., S.A.Y. and L.H. software, K.D. and S.A.Y. validation, K.D., S.A.Y., L.H. and H.A.A. formal analysis, K.D., S.A.Y., and L.H. investigation, K.D. and S.A.Y. resources, K.D. and S.A.Y. data curation, K.D. and S.A.Y. writing original draft preparation, K.D., R.M., and R.B. writing review and editing, K.D., S.A.Y. and L.H. visualization, K.D., S.A.Y., L.H., and H.A.A. supervision, All authors have read and agreed to the published version of the manuscript.

References

[1] Koizumi, M., and Niino, M., Overview of FGM research in Japan, MRS Bulletin, 20(1), 19–21 (1995)

[2] Reddy, J. N., Analysis of functionally graded plates, International Journal for Numerical Methods in Engineering, 47(1–3), 663–684 (2000)

[3] Meksi, A., Benyoucef, S., Houari, M.S.A., Tounsi, A., A simple shear deformation theory based on neutral surface position for functionally graded plates resting on Pasternak elastic foundations, Structural Engineering and Mechanics, 53(6), 1215-1240 (2015)

[4] Selmi, A., Free Vibration of Axially Functionally Graded Simply Supported Beams Using Differential Transformation Method, International Journal of Materials and Metallurgical Engineering, 12(8), 368–372 (2018)

[5] Akbaş, Ş. D., Ersoy, H., Akgöz, B., and Civalek, O., Free vibration of imperfect sigmoid and power law functionally graded beams," Mathematics, 9(9), 1048, 567–576 (2021)

[6] Avcar, M., A new higher-order shear deformation theory for free vibration of sigmoid functionally graded sandwich beams, Steel and Composite Structures, 30(6), 603-615 (2019)

[7] Akgöz, B., and Civalek, Ö., Buckling Analysis of Functionally Graded Tapered Microbeams via Rayleigh–Ritz Method, Mathematics, 10(23), 4429, 10(23), 4429 (2022)

[8] Ghazwani, M.H., Alnujaie, A., Tounsi, A., Vinh, P.V., On the high-frequency analysis of exponentially graded nanobeams resting on Winkler–Pasternak foundations, J. Vib. Eng. Technol, 12, 8113–8130 (2024)

[9] Benachi, H., Menasria, A., Bouhadra, A., Chitour, M., Rachedi, M.A., Mamen, B.,Tounsi, A. Elastic instability analysis of functionally graded nanoplates under nonlinear coupling loads on a shear elastic foundation via non-local theory, Geomechanics and Engineering, 41(5), 507-525 (2025)

[10] Ait Atmane, H., Tounsi, A., and Bernard, F., Effect of thickness stretching and porosity on mechanical response of a functionally graded beams resting on elastic foundations, Int J Mech Mater Des, 13, 71–84 (2017)

[11] Thai, H.T., and Choi, D.H., A simple first-order shear deformation theory for laminated composite plates", Composite Structures, 106, 754-763 (2013)

[12] Nebab, M., Dahmane, M., Belqassim, A., Ait Atmane, H., Bernard, F., Benadouda, M., Bennai, R., Hadji, L. Fundamental frequencies of cracked FGM beams with influence of porosity and Winkler/Pasternak/Kerr foundation support using a new quasi-3D HSDT, Mechanics of Advanced Materials and Structures, 31(8) (2024)

[13] Sobhy, M., Buckling and free vibration of exponentially graded sandwich plates resting on elastic foundations under various boundary conditions, Composite Structures, 99, 76–87, 2013 (2013)

[14] Xiang, S., Kang, G.W., and Liu, Y.Q., A nth-order shear deformation theory for natural vibration of functionally graded plates on elastic foundations, Composite Structures, 111, 224-231 (2014)
[15] Njim, E., Bakhy, S., Al-Waily, M., Free vibration analysis of imperfect functionally graded sandwich plates: analytical and experimental investigation, Archives of Materials Science and Engineering. 111(2), 49-65 (2021)
[16] Safarpour, M., SafarPour, H., Civalek, O., Wave propagation analysis of functionally graded bio-composite circular plates using an improved sinusoidal shear deformation theory resting on an advanced viscoelastic foundation, European Journal of Mechanics - A/Solids, 112, 105688 (2025)
[17] Calim, M.H., Çapar, Ö.F., Özbey, M.B., Dynamic analysis of viscoelastic FGM shells with porosities on elastic foundation, Geomechanics and Engineering, 39(1), 55–72 (2024)
[18] Alnujaie, A., Daikh, A.A., Ghazwani, M.H., Assie, A.E., Eltaher, M.A., Size-dependent free vibration of coated functionally graded graphene reinforced nanoplates rested on viscoelastic medium, Advances in Nano Research, 17(2), 181–195 (2024)
[19] Ragb, O., Elkhalek, M.A., Matbuly, M.S., Salah, M., Eltaher, M., Osman, T., Nonlinear vibration of laminated piezoelectric layered plates with nonlinear viscoelastic support using different DQM techniques, Steel and Composite Structures, 53(1), 1–27 (2024)
[20] Calim, M.H., Çapar, Ö.F., Özbey, M.B., Cuma, Y.C., Dynamic analysis of viscoelastic FGM shells with porosities on elastic foundation, Geomechanics and Engineering, 39(1), 55–72 (2024)
[21] Tahir, S.I., Chikh, A., Mudhaffar, I.M., Tounsi, A., Al-Osta, M.A., Waves dispersion in an imperfect functionally graded beam resting on visco-Pasternak foundation, Geomechanics and Engineering, 33(3), 271–277 (2023)
[22] Ould Larbi, L., Kaci, A., Houari, M. S. A., and Tounsi, A., An Efficient Shear Deformation Beam Theory Based on Neutral Surface Position for Bending and Free Vibration of Functionally Graded Beams, Mechanics Based Design of Structures and Machines, 41(4), 421–433 (2013)
[23] Fahsi, B., Bouiadjra, R.B., Mahmoudi, A., Benyoucef, S., and Tounsi, A., Assessing the effects of porosity on the bending, buckling, and vibrations of functionally graded beams resting on an elastic foundation by using a new refined quasi-3D theory", Mech. Compos. Mater., 55(2), 219–230 (2019)
[24] Nguyen, T.K., Vo, T.P., and Thai, H.T., Static and free vibration of axially loaded functionally graded beams based on the first-order shear deformation theory, Compos. B Eng., 55, 147–157 (2013)
[25] Nebab, M., Dahmane, M., Belqassim, A., Ait Atmane, A., Bernard, F., Benadouda, M., Bennai, R., and Hadji, L., Fundamental frequencies of cracked FGM beams with influence of porosity and Winkler/Pasternak/Kerr foundation support using a new quasi-3D HSDT, Mech. Adv. Mater. Struct., 1–13 (2023)
[26] Ibnorachid, Z.,. Boutahar, L., EL Bikri, K., and Benamar, R., Buckling temperature and natural frequencies of thick porous functionally graded beams resting on elastic foundation in a thermal environment, Adv. Acoust. Vib., 1–17 (2019)
[27] Simsek, M., Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories, Nucl. Eng. Des., 240(4), 697–705 (2010)

[28] Mellal, F., Bennai, R., Avcar, M., Nebab, M., and Ait Atmane, H., On the vibration and buckling behaviors of porous FG beams resting on variable elastic foundation utilizing higher-order shear deformation theory", Acta Mech., vol. 234(9), 3955–3977 (2023)
[29] Kehli, A., Nebab, M., Bennai, R., Ait Atmane, H., and Dahmane, M., Dynamic characteristics analysis of functionally graded cracked beams resting on viscoelastic medium using a new quasi-3D HSDT, Mech. Adv. Mater. Struct., 31(30), 12651–12664 (2024)
[30] Chen, W.Q., Lu, C.F., and Bian, Z.G., A mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation, Appl. Math. Modell., 28(10), 877–890 (2004)
[31] Chikh, A., Investigations in static response and free vibration of a functionally graded beam resting on elastic foundations, Frat. Integr. Strut., 14(51), 115–126, 2019 (2019)
[32] Fahsi, B., Bouiadjra, R.B., Mahmoudi, A., Benyoucef, S., and Tounsi, A., Assessing the effects of porosity on the bending, buckling, and vibrations of functionally graded beams resting on an elastic foundation by using a new refined quasi-3D theory, Mech. Compos. Mater., vol. 55(2), 219–230 (2019)