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Continuation Value and Termination in a Three-Player Differential Game with a State-Dependent Random Horizon: An Application to the Strategic Triangle

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Abstract

This article develops a three-player, continuous-time, non-zero-sum differential game of the strategic triangle. The state comprises the three bilateral relations, which evolve under the players' efforts, together with cumulative hazard, and players use Markov feedback strategies conditioned on this state. Interaction ends at a random time whose intensity depends on the state, or when cumulative hazard reaches an institutional closure boundary. Augmenting the state with cumulative hazard yields a deterministic equivalent and coupled Hamilton-Jacobi-Bellman equations whose solutions, under stated verification conditions, are Markov perfect Nash equilibria. The induced dynamics are classified into four operational regimes: stable (SSS), saddle (USS), repelling (SUS), and absorbing closure or termination (UUS). In a linear-quadratic specialization in which each relation decays naturally and carries its own moderation weight, the regimes are characterized in closed form. A relation destabilizes only if its decay is slower than the effective discount rate and its moderation weight lies in an explicit interval, so that declining moderation generically moves the system from SSS through USS to SUS. Interaction can also end through two endogenous forms of collapse. Dynamic collapse occurs when a relation diverges to rupture and is necessarily slow relative to discounting. Equilibrium collapse occurs at a fold beyond which no equilibrium exists. Institutional closure can preempt it, and when relational decay is faster than the effective discount rate it arrives without prior instability. Neither form coincides with the failure of continuation to dominate termination, which can occur while the equilibrium persists.

Keywords Differential games · Random time horizon · Markov perfect Nash equilibrium · Linear-quadratic games · Continuation value · Strategic triangle

JEL Classification C73 · C61 · D74

1 Introduction

The strategic triangle, first identified in the relationship among Washington, Moscow, and Beijing (Tatu 1970) and reformulated as a game of positional advantage among three actors with interdependent bilateral relations (Dittmer 1981, 1987), has subsequently been examined through coalition-theoretic analogies (Hsiung 1987) and dyadic conflict series (Rummel 1987), yet these accounts remain largely static or descriptive, classifying configurations or tracing observed conflict without deriving trajectories from the actors' optimizing behavior. This limitation reflects a broader modeling problem. Many strategic environments involve several players whose interaction persists under an accumulating risk of termination that no individual player can trigger directly. In such environments three questions are logically distinct: whether each player's strategy is a best response, whether the dynamics induced by equilibrium strategies are stable, and whether continued interaction yields a higher payoff than termination. Standard differential-game analysis addresses the first two questions through coupled Hamilton-Jacobi-Bellman (HJB) equations and stability criteria for the induced dynamics. The third requires a payoff attached to termination and a horizon whose distribution responds to the evolving state.

This article develops a three-player, continuous-time, non-zero-sum differential game designed to separate these questions. Following the N-player framework of Case (1969) and Starr (1969), players use Markov feedback strategies conditioned on an observable state. Interaction ends at a random time generated by a state-dependent hazard or, in a closed version of the game, when cumulative hazard reaches a prescribed boundary. Players receive running payoffs while interaction continues and a state-dependent terminal payoff when it ends, and ordinary time preference is distinguished from termination risk throughout.

The article makes six contributions. First, augmenting the state with cumulative hazard yields a deterministic equivalent (Proposition 1) that combines ordinary discounting, hazard-weighted terminal payoffs, and, under closure, an endpoint term carrying the probability mass of paths that survive to the boundary. The associated HJB system contains a transport term in cumulative hazard, and Proposition 2 states the verification conditions under which its solutions constitute a Markov perfect Nash equilibrium (MPNE). Second, a continuation-margin identity expresses the difference between the value of continuation and the terminal payoff as the discounted, survival-weighted integral of an adjusted net flow (Lemma 1). Third, under a linear erosion of payoffs in cumulative hazard and a uniform bound on the remaining flow, this identity yields a cumulative-hazard ceiling beyond which no feasible continuation, whether or not it is an equilibrium, improves any player's payoff relative to immediate termination

(Proposition 4). Fourth, a tractable three-player specialization yields exact individual and common continuation thresholds, establishes that the common threshold cannot exceed the uniform ceiling, and derives a critical hazard rate that exists only for an intermediate range of payoff parameters (Proposition 5). Fifth, the specialization shows constructively that all players' continuation margins can be negative while the interior feedback profile remains an MPNE of the forced-continuation game (Corollary 1). The failure of continuation dominance is therefore distinct from the nonexistence of equilibrium, and both comparisons are distinct from the stability classification of the induced dynamics, which rests on the spectrum of the feedback-induced Jacobian (Proposition 3). Sixth, a linear-quadratic specialization in which three bilateral relations enter payoffs and dynamics characterizes regime transitions in closed form. A relation destabilizes only if its natural decay is slower than the effective discount rate and its moderation weight lies in an explicit interval (Proposition 6), the regime of the triangle follows from the signs of three scalar rates (Proposition 7), and declining moderation generically moves the system from SSS through USS to SUS (Proposition 8). The same specialization distinguishes institutional closure and random termination from two endogenous exits: dynamic collapse, which is necessarily slow relative to discounting (Propositions 9 and 10), and equilibrium collapse through a fold, which institutional closure can preempt and which occurs without prior instability when relational decay is faster than the effective discount rate (Propositions 11 and 12).

The article proceeds as follows. Section 2 situates the framework within the literatures on random-horizon differential games and feedback equilibria in N-player games. Section 3 specifies players, controls, information, state dynamics, payoffs, and operational regimes. Section 4 characterizes MPNE verification, boundary conditions, stability criteria, the uniform ceiling, and the exact thresholds of the three-player specialization. Section 5 develops the linear-quadratic specialization and derives the conditions for regime transitions and collapse. After a brief treatment of time consistency in Section 6, Section 7 applies the results to the strategic triangle, and Section 8 identifies directions for further work.

2 Related literature

The analysis belongs to the literature on optimal control and differential games with a random terminal time. Yaari (1965) introduced optimal control with an uncertain lifetime, and Petrosyan and Shevkoplyas (2003) studied cooperative solutions of differential games with random duration, a class whose game-theoretic formulation is traced to Petrosjan and Mursov (1966). For a single controller, Boukas et al. (1990) characterized a random stopping time by

an intensity that depends on the state and the control, and reformulated the problem as an equivalent deterministic infinite-horizon problem amenable to dynamic programming and the minimum principle. Marín-Solano and Shevkoplyas (2011) extended dynamic programming for random-duration games to nonconstant discounting and examined both cooperative and noncooperative solutions. Tur and Magnitskaya (2020) constructed feedback and open-loop Nash equilibria when termination can occur only within a given finite interval, and Balas and Tur (2023) allowed the distribution of the terminal time to be composite, deriving the corresponding HJB equation with value matching at switching dates. Balas and Tur (2023) characterize cooperative strategies that maximize the players' total payoff and leave the noncooperative case for further work. Proposition 1 and the matching condition in Eq. (10) follow their line of argument, whereas the coupled system (8) and its verification in Proposition 2 extend their dynamic-programming decomposition to Markov perfect Nash equilibrium.

Recent work has carried random horizons into games with many agents. Liang et al. (2026) study equilibrium strategies for competing mean-variance investors over a random horizon whose hazard rate depends on calendar time and which is independent of the Brownian motions driving wealth; under a constant hazard their problem becomes time-homogeneous. Because their criterion is time-inconsistent, their equilibrium is defined through an extended HJB system and differs from the MPNE examined here. The present model carries a state-dependent intensity, of the kind analyzed by Boukas et al. (1990) for a single controller, into a three-player game. Cumulative hazard enters the augmented state, so that payoffs depending on accumulated crisis remain Markovian and the HJB system acquires a transport term in that state.

A second difference concerns the value of termination. The baseline functional of Balas and Tur (2023) contains running payoffs only, and attenuation arises solely from the distribution of the terminal time. The present model adds ordinary discounting, a state-dependent terminal payoff, and an institutional closure boundary defined as the hitting time of a cumulative-hazard level. Under closure, the deterministic equivalent contains an endpoint term that the hazard-density representation alone would omit. These elements permit a comparison, at every surviving state, between the value of continued interaction and the value of termination, and the continuation-margin identity converts that comparison into the sign of an integrated adjusted flow. Propositions 4 and 5 rest on this identity.

A further body of work characterizes feedback Nash equilibria in N-player differential games. Standard treatments define Markov feedback strategies and verification conditions for coupled

HJB systems (Başar and Olsder 1999; Dockner et al. 2000; Haurie et al. 2012), and Maskin and Tirole (2001) provide the rationale for conditioning strategies on payoff-relevant states. Linear-quadratic structures yield the sharpest existence results. Bardi and Priuli (2014) gave necessary and sufficient conditions for affine Nash equilibria in N-person games with linear Gaussian dynamics and long-run average quadratic costs, expressed through algebraic Riccati and Sylvester equations, and established uniqueness under an additional condition. Priuli (2015) extended the analysis to discounted costs, showing that affine Nash equilibria exist and are unique for nearly identical players and small discount rates. Neither formulation includes termination risk, so neither addresses whether continuation dominates exit.

The present article adopts the verification framework of this literature and uses the linear-quadratic structure in two ways. The specialization underlying Proposition 5 is deliberately simple, because effort affects neither the relational state nor the hazard, so that equilibrium reduces to pointwise maximization and the erosion of continuation value can be isolated in closed form. The specialization of Section 5 lets relational states enter payoffs and dynamics, so that Riccati equations of the kind studied by Bardi and Priuli (2014) and Priuli (2015), here with a constant termination hazard, determine the stability classification of Proposition 3. Haurie et al. (2012, Sect. 7.8) show that feedback equilibria of infinite-horizon linear-quadratic games need not be unique; Proposition 6 identifies the parameter region in which two linear feedback equilibria satisfy the transversality condition and selects the one obtained as the limit of the closed game.

3 Model setup

The model describes three players with partially conflicting objectives who adjust their strategies under uncertainty about the survival of their interaction. Its non-zero-sum formulation includes zero-sum interaction as a limiting case. The following assumptions specify players, controls, information, state dynamics, payoffs, and equilibrium.

3.1 Assumptions

Assumption 1 (Players and termination). Three players, $N = \{1,2,3\}$, maximize expected discounted payoffs. A nonnegative hazard h governs random termination. The random clock and the institutional closure boundary introduced below are distinct: the forced-continuation benchmark continues until random termination, whereas the closed game ends at the earlier of random termination and the prescribed crisis boundary. No player has an unmodeled unilateral stopping action.

Assumption 2 (Controls). For each $j \in N$, the control $u_j(t)$ lies in an admissible set $U_j \subseteq \mathbb{R}^{n_j}$. Because $u_j(t)$ may be multidimensional, its overall vector sign should not be interpreted directly as friendliness or hostility. The sign of its strategic-orientation component indicates whether the action is friendly or hostile, while the remaining components may represent resource allocation, diplomatic intensity, military pressure, or other feasible policy instruments.

Assumption 3 (State and clocks). The relational state x lies in X. The cumulative-hazard state q is added to make crisis-dependent payoffs Markovian. On a nonterminated path the augmented dynamics are

$$\dot{x} = f(x, q, u, R, t), \qquad \dot{q} = h(x, q, R, t), \qquad q(t_0) = 0. \qquad (1)$$

The augmented state is (x, q, R), where R is the current mode. The locally Lipschitz induced dynamics have unique feasible trajectories up to termination or a specified switch. A steady relational reference satisfies $f = 0$, whereas q need not be stationary. Stability therefore concerns the relational subsystem or tracking error, not convergence of the entire augmented state when $h > 0$. The intensity does not depend directly on the controls, so strategies affect termination risk only through the relational state; this keeps the hazard terms outside the own-control maximization in the HJB system (8). Boukas et al. (1990) allow the intensity to depend on the control for a single controller, and extending the present game in that direction is left for further work. On every bounded time interval a surviving trajectory undergoes at most finitely many mode switches. Let E be an independent unit exponential random variable, sampled conditionally on survival at date t. The clocks are

$$\begin{aligned} T_h &= \inf\{s \geq t:\ q(s) - q(t) \geq E\}, \qquad E \sim \mathrm{Exp}(1), \\ T_c &= \min\{T_h, \tau_c\}, \qquad \tau_c = \inf\{s \geq t:\ q(s) \geq \bar{q}\}. \end{aligned} \qquad (2)$$

Assumption 4 (Payoffs). From any surviving state at date t, player j evaluates a running payoff g_j and a terminal payoff $S_j(x)$. Ordinary discounting satisfies $\rho > 0$. The terminal time is $T = T_h$ in the forced-continuation benchmark and $T = T_c$ in the closed game. Both payoff components are absolutely integrable, and S_j is continuously differentiable and does not depend directly on q. The conditional objective is

$$J_j = \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}\, g_j(s)\, ds + e^{-\rho(T-t)} S_j(x(T))\right]. \qquad (3)$$

Balas and Tur (2023) provide the random-horizon baseline for a cooperative problem in which players maximize their total payoff. Their functional contains no separate ordinary discount factor, so attenuation comes from the terminal-time distribution alone. The present model adds positive ordinary discounting, so the effective discount rate combines time

preference with state-dependent termination intensity. This extension combines optimal control with a random stopping time (Boukas et al. 1990) with the verification theory for discounted feedback games over a prespecified horizon (Haurie et al. 2012, Theorem 7.2 and Proposition 7.3). The two sources of discounting have different substantive meanings and should not be conflated.

Proposition 1 (Random-horizon transformation). Fix an admissible feedback profile with no additional stochastic state drivers, so that the nonterminated path is deterministic conditional on that profile, and let ordinary discounting multiplied by conditional survival be

$$D_t(s) = e^{-\rho(s-t)-\int_t^s h(v)\,dv} = e^{-\rho(s-t)-[q(s)-q(t)]}. \qquad (4)$$

(i) In the forced-continuation benchmark, under absolute integrability and a vanishing discounted terminal remainder when T_h can be infinite, the conditional objective (3) equals

$$W_j = \int_t^{\infty} D_t\,(s)\big[g_j(s) + h(s)\,S_j\big(x(s)\big)\big]ds. \qquad (5)$$

(ii) Under institutional closure, the conditional objective equals

$$W_j^c = \int_t^{\tau_c} D_t\,(s)\big[g_j(s) + h(s)\,S_j\big(x(s)\big)\big]ds + D_t(\tau_c)\,S_j\big(x(\tau_c)\big), \qquad (6)$$

whose final term carries the probability mass of paths surviving to the closure boundary. If the boundary is not reached, the infinite-horizon formula (5) applies with the vanishing-remainder convention.

Here $h(s)$ is evaluated along the induced augmented trajectory, and conditional survival is $\exp\big[-\big(q(s)-q(t)\big)\big]$. The symbol f denotes the state drift only, and the terminal-time density is denoted f_T. Appendix A.1 and A.2 establish the running-payoff and terminal-payoff transformations separately. The final term in Eq. (6) is essential, because surviving probability is concentrated at the prescribed closure time and the continuous hazard-density term alone would omit this terminal payoff.

The terminal payoff is a further extension of the baseline running-payoff functional of Balas and Tur (2023). It represents the state-dependent outcome at termination: positive values indicate residual salvage, whereas negative values indicate a breakdown penalty.

On each mode interval the hazard is $h_R(x, q, t)$. Switches are prescribed dates or exits from specified regions, not necessarily exits from invariant basins of attraction. Following the composite-distribution logic of Balas and Tur (2023), continuation values are matched at transparent switching boundaries. This matching does not itself derive a switch or verify equilibrium.

Assumption 5 (Information and strategies). Players observe x, q, R, and the current termination intensity. Markov feedback laws condition on the augmented state, and rivals' future actions are determined by their laws. For the closed game, strategies are defined only before closure:

$$\sigma_j : [t_0, \infty) \times X \times [0, \bar{q}) \times \mathcal{R} \to U_j, \qquad \sigma^* = (\sigma_1^*, \sigma_2^*, \sigma_3^*). \qquad (7)$$

Assumption 6 (Admissibility and equilibrium). Admissible Markov strategies are measurable and locally Lipschitz in the state, and control sets are compact and convex. Strict own-control concavity yields a unique maximizer when the relevant interior conditions hold. An MPNE requires each player's law to be a best response to the other two players at every admissible state. Section 5 relaxes compactness for a linear-quadratic specialization, in which existence is established directly.

3.2 Operational regimes

The labels SSS, USS, SUS, and UUS organize four operational regimes. SSS and SUS concern local attraction or repulsion around a relational reference, and USS denotes conditional attraction along the stable manifold of a saddle. UUS denotes terminal institutional closure, not failure of an interior first-order condition. These labels are not a literal steady/unsteady by stable/unstable partition and are not mapped to Dittmer's relational configurations.

Table 1 Terminology

Term	Mathematical criterion	Implications
Steady	Time-invariant relational reference, normally a fixed point x^*; the cumulative crisis state need not be stationary	Equilibrium or recurrent pattern
Unsteady	Transient, path-dependent dynamics toward regime exit, structural reconfiguration, or conditional stability along a restricted manifold	Non-equilibrium, time-varying trajectories
Stable	Perturbations decay; the equilibrium or orbit is attracting	Robust against disturbances
Unstable	Perturbations have a growing component; persistence alone can indicate neutrality rather than repulsion	Sensitive to disturbances

USS is not Lyapunov stable in a full neighborhood, because generic off-manifold perturbations can depart from the saddle, and stable and unstable manifolds need not be orthogonal. UUS is absorbing only because the rules explicitly terminate interaction at the closure boundary. UUS comprises the exogenous exits, institutional closure and random termination. Section 5.4 distinguishes them from two endogenous exits, dynamic collapse and equilibrium collapse, which the four regimes do not classify.

Table 2 Four operational regimes

	Stable	Unstable
Steady	SSS: locally attracting relational reference under a negative Lyapunov derivative or a Hurwitz induced-field Jacobian; the classification is conditional on verified feedback and maintained dynamics	SUS: repelling relational reference; node-type motion is directional and a focus illustrates outward spiraling; neither implies finite-time collapse without a closure rule; in the specialization of Section 5 only the node type arises
Unsteady	USS: saddle, conditionally attracting on its stable manifold and repelling in unstable directions; not Lyapunov stable in a full neighborhood, and the manifolds need not be orthogonal	UUS: absorbing institutional closure at the prescribed crisis ceiling, or earlier random termination; controls cease after termination, and negative continuation margins do not imply that an interior MPNE fails

4 Dynamic analysis

4.1 Identifying MPNE through the HJB system

Let $V_j(t, x, q, R)$ be continuously differentiable within each mode. Holding rivals' equilibrium feedback fixed, each player faces the following HJB equation before closure, in which the term $h\ \partial_q V_j$ accounts for the accumulation of crisis along surviving paths:

$$(\rho + h)V_j = \partial_t V_j + h\ \partial_q V_j + h\, S_j + \max_{u_j \in U_j}\{g_j + \nabla_x V_j \cdot f\}. \qquad (8)$$

The maximization includes all own-control-dependent terms. Because $h = h(x, q, R, t)$ is not directly controlled, hS_j and $h\ \partial_q V_j$ lie outside the maximization, although players influence future hazard through x. Ordinary discounting and random killing are conceptually distinct, although both enter $\rho + h$. With all functions evaluated at (t, x, q, R) and rivals' feedback held fixed, the maximizing law satisfies

$$\sigma_j^* \in \arg\max_{u_j \in U_j}\{g_j + \nabla_x V_j \cdot f\}. \qquad (9)$$

Strict own-control concavity can ensure a unique individual maximizer, but it does not guarantee existence or uniqueness of the coupled equilibrium.

Proposition 2 (Conditional MPNE verification). Suppose that V_j, $j \in N$, are continuously differentiable within each mode and satisfy Eq. (8) at every (t, x, q, R) with $q < \bar{q}$ in each mode, with maximizing laws given by Eq. (9). If the maximizing laws are admissible, the induced trajectories are feasible, values match at mode interfaces as in Eq. (10), and the closure condition (11) or the transversality condition (12) holds along every admissible unilateral deviation, then the profile of maximizing laws is an MPNE and V_j is player j's equilibrium value.

Appendix A.3 proves the payoff comparison. A formal HJB solution without these conditions is not an equilibrium existence theorem.

4.2 Boundary conditions and regime transitions

At a transparent interface with no instantaneous payment, state reset, or atom of random termination, continuation values satisfy

$$V_j^R(\tau_n^-, x, q) = V_j^{R'}(\tau_n^+, x, q). \qquad (10)$$

This condition matches continuation values across regimes. It does not require continuous controls or equal derivatives at the interface. The closed game has the terminal boundary condition

$$V_j^c(t, x, \bar{q}, R) = S_j(x), \qquad (11)$$

whereas the forced-continuation benchmark requires the discounted-survival transversality condition

$$\lim_{s\to\infty} D_t(s)\, V_j\big(s, x(s), q(s), R(s)\big) = 0. \qquad (12)$$

Since $\rho > 0$, bounded values satisfy Eq. (12) even if cumulative termination probability does not approach one, and polynomial value growth also suffices when x and q grow at most polynomially along admissible paths. The condition must hold along unilateral deviations, not merely along the proposed equilibrium path.

Closure and infinite-end transversality are alternative endpoint conditions for different games. Value matching does not impose smooth pasting or continuity of controls, and it does not by itself establish time consistency. The verification argument follows discounted feedback-game reasoning (Haurie et al. 2012, Theorem 7.2 and Proposition 7.3).

4.3 Stability analysis and Lyapunov criteria

Stability is evaluated for the relational trajectory induced by a verified feedback profile. For a fixed autonomous mode, let $e = x - x^*$. If q or time enters the induced field, the calculation either freezes those quantities or uses a moving reference with a uniform tracking argument; a frozen-mode calculation alone does not prove stability of the full nonautonomous game. Consider

$$L(e) = 1/2\, e^\top P e, \qquad P = P^\top > 0, \qquad \dot{L} = e^\top P \dot{e}. \qquad (13)$$

For a fixed autonomous relational system, a positive-definite Lyapunov function with a strictly negative derivative in a punctured neighborhood establishes local asymptotic attraction. For a moving reference, the same conclusion requires uniform positive-definite bounds and a uniform negative derivative for the tracking error. An indefinite Lyapunov derivative alone does not prove a saddle. For a differentiable autonomous field with a hyperbolic reference, the total Jacobian of the feedback-induced field determines local stable and unstable directions:

$$A = \frac{\partial f}{\partial x} + \sum_{k \in N} \frac{\partial f}{\partial u_k} \frac{\partial \sigma_k^*}{\partial x},$$

$$\text{SSS: } \mathrm{Reeig}(A) < 0; \quad \text{USS: mixed signs;} \quad \text{SUS: } \mathrm{Reeig}(A) > 0. \qquad (14)$$

The derivatives of the feedback laws therefore enter the stability classification (cf. Haurie et al. 2012, Eq. 7.16).

Proposition 3 (Conditional relational stability). Let a verified feedback profile induce, in a fixed mode, a differentiable autonomous relational field with a hyperbolic reference x^*, and let A be the total Jacobian in Eq. (14), including feedback derivatives. If every eigenvalue of A has negative real part, x^* is locally asymptotically attracting (SSS). If A has eigenvalues with real parts of both signs, x^* is a saddle that attracts only along its stable manifold (USS). If every eigenvalue has positive real part, x^* is repelling (SUS).

Appendix B.1 gives the proof. SSS denotes attraction around a relational reference, so that feedback adjustment restores sufficiently small perturbations; cooperative relations may motivate such a benchmark but do not, by themselves, establish attraction. USS denotes a hyperbolic saddle in which convergence is conditional on the stable manifold and generic perturbations with an unstable component grow; neither an allied dyad nor an excluded actor automatically generates this geometry. SUS contains two repelling benchmarks, a node with directional divergence and a focus with outward spiraling. Unlike a bounded limit cycle, a repelling focus amplifies deviations, and these benchmarks distinguish directional from oscillatory instability without requiring the same realized path to collapse.

UUS denotes closure of the strategic environment at the prescribed crisis boundary, or earlier termination by the random clock. The label identifies a terminal institutional condition, not a further attracting or repelling equilibrium field. After closure there are no additional controls or continuation payoffs in this game. Reopening interaction would require a new model with an explicit reset or re-entry rule, because monotone cumulative hazard cannot reverse across the closure boundary.

4.4 Transitional dynamics and regime changes

Active-mode switches occur when a surviving trajectory reaches a specified interface. The regions must be supplied by model primitives or institutional rules, because a trajectory cannot simply exit the invariant basin of an unchanged autonomous attractor. Reverse switches may be allowed before closure if their rules and dynamics are specified. The exit time is

$$\tau_n = \inf\{s \geq \tau_{n-1} : \big(x(s), q(s)\big) \notin \Omega_R\}. \qquad (15)$$

The exit time specifies where the model changes mode; it does not prove the stability of a particular destination. The following mechanisms are conditional hypotheses for the full alignment model, distinct from the thresholds derived below.

First, from SSS to USS, reduced cooperative intensity can motivate a bilateral-alignment switch. A saddle destination requires a feedback-induced Jacobian with both stable and unstable directions, and no universal scalar cutoff on a multidimensional control establishes this result.

Second, from SSS to SUS, a pivot-oriented control may be selected when its maximized Hamiltonian exceeds that of cooperative controls, holding rivals' laws fixed. All feasible alternatives must be compared. An equilibrium player has no profitable admissible deviation from its maximizing law, so the restricted comparison is not evidence of such a deviation, and repulsion must be checked separately.

Third, from USS to SUS, increased exposure can change a player's continuation margin, but a hazard crossing does not universally select pivot behavior or create a repelling focus. Such a transition requires specified payoff effects, feasible controls, and an induced-field stability change. Eq. (26) derives an individual exposure crossing in a restricted benchmark only.

Fourth, from an active regime to UUS, the adopted closure rule ends the surviving environment when q reaches the sufficient ceiling $\bar{q}$. The argument below derives that ceiling from a uniform bound and separately derives an exact common continuation-value crossing. Neither crossing establishes that an MPNE ceases to exist.

In the general model these mechanisms remain conditional. Section 5 shows, for a linear-quadratic specialization, that the first and third transitions occur when the moderation weight of a bilateral relation crosses an explicit critical value, that a direct transition from SSS to SUS is nongeneric, and that interaction can also end through dynamic or equilibrium collapse, which are distinct from UUS.

4.5 Collapse ceiling and exact continuation thresholds

Assumption 7 (Crisis erosion). On the full forced-continuation domain, including crisis levels beyond the proposed closure ceiling, each running payoff decomposes into a crisis-free return and a linear cumulative-hazard cost:

$$g_j(x, q, u, R, t) = g_j^0(x, u, R, t) - \beta_j q, \qquad \beta_j > 0. \qquad (16)$$

The relevant comparison with immediate termination uses the adjusted net flow, not the running payoff alone:

$$r_j = g_j^0 - \rho S_j + \nabla_x S_j \cdot f - \beta_j q \;\le\; \alpha_j - \beta_j q, \qquad (17)$$

where a finite upper bound holds uniformly over all feasible states, dates, active modes, and joint controls, with the same terminal function S_j at mode interfaces:

$$g_j^0 - \rho S_j + \nabla_x S_j \cdot f \;\le\; \alpha_j < \infty. \qquad (18)$$

Lemma 1 (Continuation-margin identity). For any admissible profile and any endpoint L paying the terminal function S_j, with the surviving endpoint payment included when L is finite and its discounted remainder vanishing when $L = \infty$,

$$W_j(t, x, q; L) - S_j(x) = \int_t^L D_t(s)\, r_j(s)\, ds. \qquad (19)$$

Appendix B.2 gives the proof.

Proposition 4 (Sufficient uniform ceiling). Under Assumption 7, integrability, and a vanishing discounted salvage remainder for an infinite endpoint, define

$$\bar{q} = \max\left\{0,\ \max_{j \in N} \frac{\alpha_j}{\beta_j}\right\}. \qquad (20)$$

If $q(t) \ge \bar{q}$, every feasible positive-duration continuation has a nonpositive margin relative to the immediate terminal benchmark for all three players. If $q(t) > \bar{q}$, the margin is strictly negative.

This is a sufficient uniform bound, not an exact equilibrium threshold. The closure time τ_c in Eq. (2) is adopted as an explicit institutional rule; it terminates the environment rather than

invalidating an equilibrium first-order condition. If the hazard is bounded below by $h_{\min} > 0$ until the ceiling, arrival on a surviving path satisfies

$$\tau_c - t \leq \frac{\bar{q} - q(t)}{h_{\min}}, \qquad q(t) < \bar{q}. \qquad (21)$$

Appendix B.2 derives the bound. Closure is not inferred from a circular definition involving unexplained value functions, because both the sufficient level and the decision to adopt it are stated explicitly. Section 5.4 shows that the bound (18) can fail precisely where relations destabilize, in which case the ceiling depends on the admissible relational range.

4.6 Three-player specialization

A tractable specialization separates the sufficient ceiling from an exact continuation-value crossing. Let $h = \lambda > 0$, so that $\dot{q} = \lambda$, and let $S_j = s_j$ be constant. Each player chooses one bounded effort u_j; hazard and crisis accumulation do not respond to effort, and relational states do not enter payoffs. The running return is

$$g_j = a_j + b_j u_j + \sum_{k \neq j} \eta_{jk}\, u_k - \frac{d_j}{2} u_j^2 - \beta_j q, \qquad (22)$$

where $d_j > 0$, $\beta_j > 0$, and η_{jk} permits different cross-player interests. The ratio b_j/d_j lies strictly inside the control interval. This example isolates the collapse mechanism and is not a calibrated solution for the full alignment model. Constant controls are admissible degenerate Markov feedback laws. Let $\delta = \rho + \lambda$ and define

$$a_j^* = a_j + \frac{b_j^2}{2d_j} + \sum_{k \neq j} \eta_{jk} \frac{b_k}{d_k}, \qquad B_j = a_j^* - \rho s_j. \qquad (23)$$

Because the state enters payoffs linearly while neither the state dynamics nor the hazard depend on the controls, the specialization is a linear-state differential game in the sense of Haurie et al. (2012, Sect. 7.6). Equilibrium strategies are therefore constant, open-loop and feedback Nash equilibria coincide, and the value function is affine in q, as Eq. (24) confirms. Corollary 1 accordingly concerns the separation between equilibrium and continuation dominance rather than the structure of feedback strategies.

Proposition 5 (Exact benchmark thresholds). In the specialization (22) with constant hazard λ, the following hold in the forced-continuation game without institutional closure.

(i) The profile $u_j^* = b_j/d_j$ is the unique own best response at every state and constitutes an MPNE. The exact continuation margin relative to immediate termination is

$$M_j^{\infty}(q) = V_j^{\infty}(q) - s_j = \frac{B_j - \beta_j q}{\delta} - \frac{\beta_j \lambda}{\delta^2}, \qquad (24)$$

and the individual crossings and the first nonnegative crisis level at which all three margins are nonpositive are

$$q_j^* = \frac{B_j}{\beta_j} - \frac{\lambda}{\delta}, \qquad q^{\dagger} = \max\left\{0, \max_{j \in N} q_j^*\right\}. \qquad (25)$$

(ii) The common threshold satisfies $q^{\dagger} \leq \bar{q}$.

(iii) Holding q fixed and setting $K_j = B_j - \beta_j q$, a finite positive hazard at which M_j^{∞} vanishes exists if and only if $0 < K_j < \beta_j$, and it is given by

$$\lambda_j^{\text{crit}} = \frac{\rho K_j}{\beta_j - K_j}, \qquad 0 < K_j < \beta_j. \qquad (26)$$

Appendix B.3 gives the proof. If all margins are positive at $q = 0$, each decreases strictly and crosses zero once. The maximum, not the minimum, is required for simultaneous failure of continuation dominance. A player with high β_j or low B_j may become exposed earlier, so individual exposure is distinct from common continuation failure. In this specialization α_j is the maximum of g_j^0 over all admissible joint controls minus ρs_j. It may exceed B_j because the control profile most beneficial to one player need not be an equilibrium profile, which is why part (ii) holds with possible strict inequality. Appendix B.4 evaluates the different game in which closure is already imposed at $\bar{q}$.

For $K_j \geq \beta_j$ there is no finite positive hazard crossing, and for $K_j \leq 0$ the margin is already negative at positive hazard. In this comparison λ changes both random killing and crisis growth. Higher hazard is therefore not a universal cause of collapse, pivot selection, or focus instability.

Table 3 and Fig. 1 use chosen parameters rather than observed data: $\rho = 0.05$, $\lambda = 0.10$, $b_j = d_j = 1$, $u_j \in [-2,2]$, $\eta_{jk} = 0.10$ for each cross-player term, and $s_j = 0$. The intercepts for players 1, 2, and 3 are 0.50, 0.65, and 0.74. Equilibrium effort is one for each player, and maximizing the crisis-free return over all admissible joint controls gives $\alpha_j = B_j + 0.20$.

Table 3 Chosen parameters and derived thresholds

Player	B_j	β_j	q_j^*	α_j/β_j
1	1.20	0.12	9.333	11.667
2	1.35	0.15	8.333	10.333

Player	B_j	β_j	q_j^*	α_j/β_j
3	1.44	0.18	7.333	9.111

Note: $\delta = 0.15$. The exact common benchmark crossing is $q^\dagger = 9.333$ and the separate sufficient ceiling is $\bar{q} = 11.667$. From $q = 0$, their dates on surviving paths are 93.333 and 116.667 model time units, and the corresponding conditional survival probabilities are approximately 0.00884% and 0.000857%. These high cumulative-hazard levels therefore describe very rare surviving paths, not empirical forecasts.

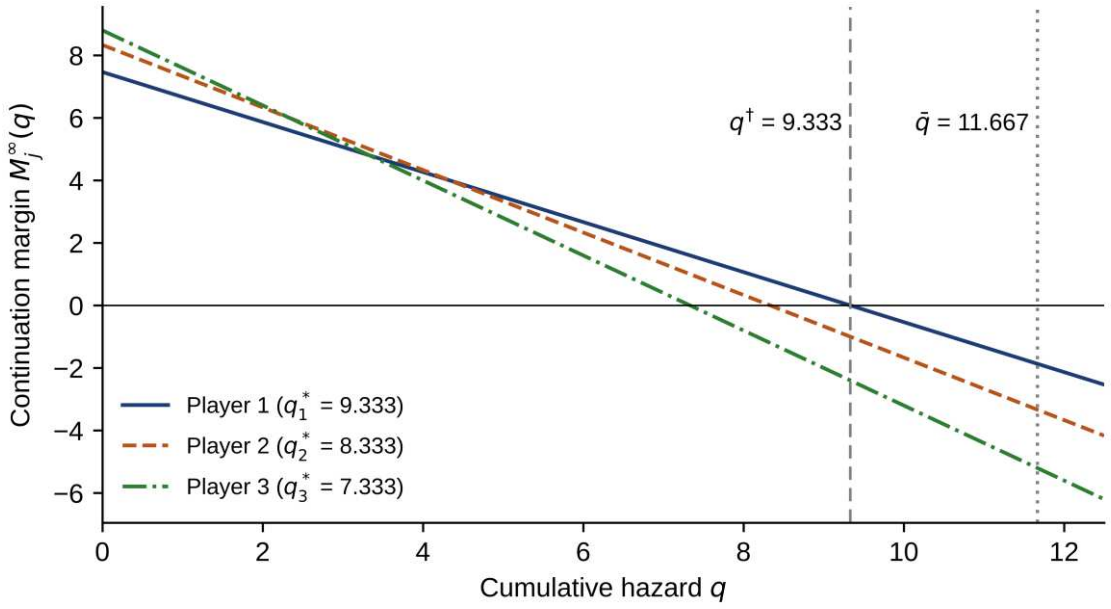

Fig. 1 Exact forced-continuation margins for the parameters in Table 3. Each line plots $M_j^\infty(q)$ for one player; the individual crossings are 7.333, 8.333, and 9.333. All three margins are nonpositive at the common crossing $q^\dagger = 9.333$ (dashed vertical line). The separate uniform ceiling $\bar{q} = 11.667$ (dotted vertical line) follows from a bound over all admissible controls. Neither line solves a free-boundary stopping game

Corollary 1 (Negative continuation margins under equilibrium). In the specialization of Proposition 5, for every $q > q^\dagger$ the continuation margin $M_j^\infty(q)$ is strictly negative for every player, while the profile $u_j^* = b_j/d_j$ remains an MPNE of the forced-continuation game. The failure of continuation dominance therefore does not entail the nonexistence of a Markov perfect Nash equilibrium.

Proof. By Proposition 5(i), each M_j^∞ is strictly decreasing in q and vanishes at $q_j^* \le q^\dagger$, so it is strictly negative for $q > q^\dagger$. The equilibrium property in Proposition 5(i) holds at every state and is independent of the level of q. ▫

What fails beyond $q^\dagger$ is collective continuation dominance relative to the terminal benchmark, not the existence of equilibrium. Institutional closure ends the environment only through the explicit boundary rule.

5 A linear-quadratic specialization: regime transitions and collapse

The specialization of Section 4.6 isolates the erosion of continuation value, but it removes strategic interaction through the relational state, because effort affects neither the state nor the hazard. This section develops a specialization in which three bilateral relations enter payoffs and dynamics, so that the regimes of Section 3.2 and the exits from the active regimes can be characterized algebraically. The results are stated as conditions on parameters rather than as outcomes for chosen parameter values, and numerical computations serve only to verify them.

5.1 Setup

Let $x = (x_{12}, x_{13}, x_{23}) \in \mathbb{R}^3$ collect the three bilateral relations, with positive values denoting amity and negative values hostility. Player i chooses an effort $u_{ij} \in \mathbb{R}$ toward each partner j, and relation $\{i, j\}$ evolves according to

$$\dot{x}_{ij} = -\varphi\, x_{ij} + u_{ij} + u_{ji}, \qquad \varphi > 0, \qquad (27)$$

where φ is the natural rate of relational decay. In this section i denotes a generic player and j and k the other two players, so that x_{ij} is the relation between i and j; Sections 3 and 4 use j for a generic player. With $\{j, k\} = N\backslash\{i\}$, player i's running payoff is

$$g_i = \sum_{j \neq i} \left[-\frac{\mu_{ij}}{2}(x_{ij} - \theta_{ij})^2 - \frac{w}{2} u_{ij}^2 \right] - \psi_i\, x_{jk}\big(x_{ij} + x_{ik}\big) - \beta_i q. \qquad (28)$$

The moderation weight $\mu_d = \mu_{ij} = \mu_{ji}$ is shared by the two members of relation $d = \{i, j\}$. A positive weight means that both members prefer to keep the relation near its target θ_d, and a negative weight means that they value polarization of the relation. The parameter $w > 0$ is the cost of effort, and $\psi_i \geq 0$ measures a positional incentive: player i gains more from amity with both partners when the partners are hostile to each other. The hazard is constant, $h = \lambda$, and terminal payoffs are constants $S_i = s_i$. Strategies therefore depend on ρ and λ only through $\delta = \rho + \lambda$, and the crisis cost $\beta_i q$ affects only the intercept of the value function, so the continuation analysis of Section 4.5 applies unchanged. Compactness of control sets in Assumption 6 is relaxed in this section, and existence is established directly. Admissible profiles are those along which $e^{-\delta t} \parallel x(t) \parallel^2 \to 0$, which makes the transversality condition (12) hold for quadratic values (Haurie et al. 2012, Proposition 7.3).

Let e_d denote the unit vector of relation d, let $G_1 = [e_{12}\ e_{13}]$, $G_2 = [e_{12}\ e_{23}]$, and $G_3 = [e_{13}\ e_{23}]$, and let $M_i = G_i G_i^{\top}/w$. A quadratic value $V_i = 1/2\, x^{\top} \Pi_i x + p_i^{\top} x + v_i(q)$ yields the maximizing law $u_i^* = w^{-1} G_i^{\top}(\Pi_i x + p_i)$ in Eq. (9). Writing the crisis-free part of Eq. (28) as

$-1/2\, x^{\top} Q_i x + l_i^{\top} x$ plus a constant, the quadratic terms of Eq. (8) give the coupled algebraic Riccati equations

$$\delta \Pi_i = -Q_i - 2\varphi \Pi_i + \Pi_i M_i \Pi_i + \sum_{k \neq i} (\Pi_i M_k \Pi_k + \Pi_k M_k \Pi_i), \qquad i \in N. \tag{29}$$

The induced relational field is affine,

$$\dot{x} = A_{\mathrm{cl}}\, x + b, \qquad A_{\mathrm{cl}} = -\varphi I + \sum_{k \in N} M_k\, \Pi_k, \qquad b = \sum_{k \in N} M_k\, p_k, \tag{30}$$

and A_{cl} is the total Jacobian of Eq. (14) for this specialization. Appendix C.1 derives Eqs. (29) and (30).

5.2 Regimes without positional incentives

When $\psi_i = 0$ for every player, the game separates into the three relations. Each member of relation d has the value component $-1/2\, \chi_d x_d^2 + m_d x_d$ plus a constant, where

$$\frac{3\chi_d^2}{w} + \gamma \chi_d - \mu_d = 0, \qquad m_d \left(\delta + \varphi + \frac{3\chi_d}{w}\right) = \mu_d \theta_d, \qquad \gamma = 2\varphi + \delta. \tag{31}$$

Three critical moderation weights organize the results:

$$\mu_{\min} = -\frac{w\gamma^2}{12}, \qquad \mu_u = -\frac{w\gamma^2}{16}, \qquad \mu_c = -\frac{w\varphi(\varphi + 2\delta)}{4}. \tag{32}$$

They satisfy $\mu_{\min} < \mu_u$ and $\mu_{\min} \leq \mu_c$, because $\mu_c - \mu_{\min} = w(\varphi - \delta)^2/12$.

Proposition 6 (Existence, selection, and stability of a relation). Let $\psi_i = 0$ for all $i \in N$.

(i) A quadratic-value MPNE of the forced-continuation game exists if and only if $\mu_d \geq \mu_{\min}$ for every relation d.

The larger root χ_d^+ of Eq. (31) is the limit of the equilibrium of the closed game with constant terminal payoffs as the remaining horizon tends to infinity, and it satisfies Eq. (12). The smaller root also satisfies Eq. (12) if and only if $\mu_d < \mu_u$, so linear feedback equilibria are not unique when $\mu_{\min} \leq \mu_d < \mu_u$.

(ii) Under χ_d^+, the closed-loop eigenvalue of relation d is $-\kappa_d$, where

$$\kappa_d = \frac{1}{3}\left[\varphi - \delta + \sqrt{\gamma^2 + 12\mu_d/w}\right], \tag{33}$$

and $\kappa_d < 0$ if and only if $\varphi < \delta$ and $\mu_{\min} \leq \mu_d < \mu_c$.

The equilibrium selected in part (ii) is used in what follows. Appendix C.2 gives the proof.

Corollary 2 (Fast decay excludes instability). If $\varphi \geq \delta$, the selected equilibrium satisfies $\kappa_d \geq (\varphi - \delta)/3 \geq 0$ for every relation, with equality only when $\varphi = \delta$ and $\mu_d = \mu_{\min}$.

Destabilization of a relation therefore requires relational decay slower than the effective discount rate.

Proposition 7 (Regime classification). Let $\psi_i = 0$ and $\mu_d \geq \mu_{\min}$ for all i and d. Then $A_{\mathrm{cl}} = -\mathrm{diag}(\kappa_{12}, \kappa_{13}, \kappa_{23})$, and the induced field is SSS if $\kappa_d > 0$ for every relation, USS if the κ_d take both signs, and SUS if $\kappa_d < 0$ for every relation. Every eigenvalue is real, so SUS arises only as a repelling node.

Figure 2 displays the classification of a single relation in the plane $\left(\varphi/\delta,\ \mu_d/(w\delta^2)\right)$, in which Eqs. (32) and (33) contain no further parameters. The triangle is SSS when all three relations lie in the stable region, USS when the relations lie on both sides of the curve μ_c, and SUS when all three lie in the unstable band. The unstable band is narrow and closes at $\varphi = \delta$, which is the content of Corollary 2.

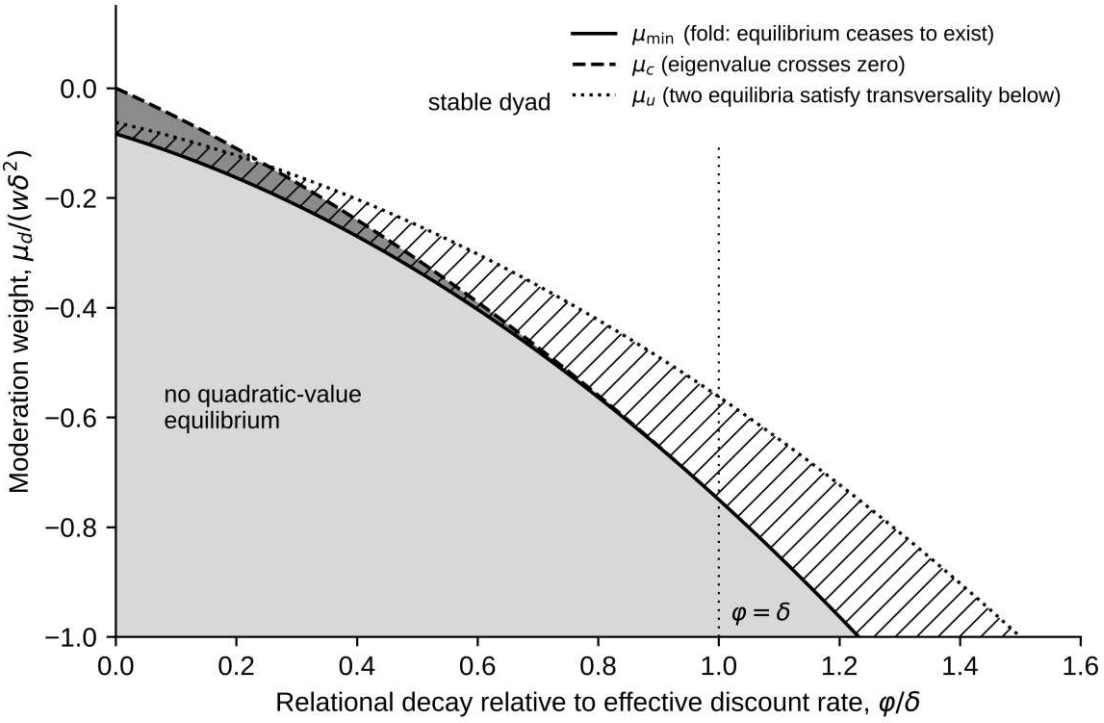

Fig. 2 Classification of a single relation in the dimensionless plane. The solid curve is the fold $\mu_{\min}$, below which no quadratic-value equilibrium exists. The dark band between $\mu_{\min}$ and the dashed curve μ_c, present only for $\varphi < \delta$, is the region in which the relation is unstable. In the hatched band between $\mu_{\min}$ and the dotted curve μ_u, both roots of Eq. (31) satisfy the transversality condition. A triangle is SSS, USS, or SUS according to whether none, some, or all of its three relations lie in the unstable band

Proposition 8 (Crisis-driven sequence of regimes). Let $\varphi < \delta$, and let each moderation weight $\mu_d(q)$ be continuous and nonincreasing in cumulative hazard. For the equilibrium of the game whose parameters are frozen at q, the regime is SSS while $\min_d \mu_d(q) \geq \mu_c$, becomes USS at the first level of q at which some weight falls below μ_c, and becomes SUS at the first level at

which every weight lies below μ_c. A direct transition from SSS to SUS requires all three weights to cross μ_c at the same level of q and is therefore nongeneric. If the weights increase, the sequence reverses. If $\varphi \geq \delta$, no transition among SSS, USS, and SUS occurs while an equilibrium exists.

Appendix C.3 proves Corollary 2 and Propositions 7 and 8. Proposition 8 classifies frozen-parameter equilibria. When parameters change slowly relative to the rates κ_d, the frozen classification approximates the time-varying game, but, as Section 4.3 notes, it does not by itself prove stability of the nonautonomous system. In a closed game, the terminal condition at τ_c also modifies the Riccati solution as closure approaches, so regime changes computed from the time-varying system may be shifted or reversed close to the closure date.

Table 4 compares the eigenvalues predicted by Eq. (33) with numerical solutions of Eq. (29). The parameter values serve only to verify the formulas.

Table 4 Predicted and computed closed-loop eigenvalues

$(\mu_{12}, \mu_{13}, \mu_{23})$	Predicted by Eq. (33)	Computed from Eq. (29)	Regime
(0.1, 0.1, 0.1)	−0.3286 (triple)	−0.3286 (triple)	SSS
(0.1, 0.1, −0.0065)	−0.3286, −0.3286, 0.0135	−0.3286, −0.3286, 0.0135	USS
(−0.0065, −0.0065, 0.1)	−0.3286, 0.0135, 0.0135	−0.3286, 0.0135, 0.0135	USS
(−0.0065, −0.0065, −0.007)	0.0135, 0.0135, 0.0242	0.0135, 0.0135, 0.0242	SUS

Note: $\varphi = 0.05$, $w = 1$, $\rho = \lambda = 0.1$, so that $\delta = 0.2$, $\mu_c = -0.00563$, and $\mu_{\min} = -0.0075$.

5.3 Positional incentives

Positional incentives couple the relations. Let player 1 hold the incentive $\psi_1 > 0$ while $\psi_2 = \psi_3 = 0$, and let $\mu_{12} = \mu_{13} = \mu_P$ and $\mu_{23} = \mu_W$, with roots χ_P and χ_W of Eq. (31). A first-order expansion of Eq. (29) in ψ_1, derived in Appendix C.7, gives

$$A_{\text{cl}} = -\text{diag}(\kappa_P, \kappa_P, \kappa_W) + \psi_1 \begin{pmatrix} 0 & 0 & \zeta_1 \\ 0 & 0 & \zeta_1 \\ \zeta_2 & \zeta_2 & 0 \end{pmatrix} + O(\psi_1^2), \qquad (34)$$

$$\zeta_1 = -\frac{2\chi_P^2 + 6\chi_P\chi_W + 3\chi_W^2 + 3w\gamma\chi_P + 4w\gamma\chi_W + w^2\gamma^2}{(2\chi_P + 2\chi_W + w\gamma)\,E}, \qquad \zeta_2 = \frac{\chi_P\,(2\chi_P + \chi_W + w\gamma)}{(2\chi_P + 2\chi_W + w\gamma)\,E},$$

where $E = 3\chi_P^2 + 8\chi_P\chi_W + 3\chi_W^2 + 4w\gamma(\chi_P + \chi_W) + w^2\gamma^2$. The perturbation has a zero diagonal. When the eigenvalues are separated and bounded away from zero, positional incentives therefore leave their real parts unchanged to first order, and Proposition 7 continues to classify the regime. Near a crossing the coupling matters. On the subspace spanned by $x_{12} + x_{13}$ and x_{23}, the induced field has determinant $\kappa_P\kappa_W - 2\psi_1^2\zeta_1\zeta_2$, which increases with ψ_1 when $\zeta_1\zeta_2 < 0$, as in every case computed. Positional incentives thus stabilize a saddle and, when strong enough, produce a complex pair of eigenvalues, so that convergence becomes oscillatory.

The first-order expression identifies the direction of these effects but not their size near μ_c, where second-order corrections to the diagonal are of comparable magnitude. Table 5 reports the positional incentive at which a USS configuration returns to SSS. The first-order prediction overstates the computed value by a factor of about 2.5, so thresholds must be computed from Eq. (29). The same computations show that an SUS configuration is incompatible with positional incentives: with the parameters of Table 4, the equilibrium approximated by the closed game ceases to exist once ψ_1 exceeds about 0.005, and for smaller values every eigenvalue remains real. The repelling focus listed in Table 2 is therefore not attained in this specialization.

Table 5 Positional incentive at which USS returns to SSS

μ_{23}	First-order prediction	Computed from Eq. (29)
-0.0060	0.0315	0.0126
-0.0065	0.0490	0.0194
-0.0070	0.0633	0.0244

Note: $\mu_{12} = \mu_{13} = 0.1$, $\theta = (1,1,0.1)$, and the remaining parameters as in Table 4.

5.4 Terminal outcomes: closure, random termination, and collapse

The active regimes can end in four ways. UUS comprises the two exogenous exits of Section 3.2, institutional closure at τ_c and random termination at T_h. Two further exits are endogenous and are collected under the term collapse. Dynamic collapse occurs at the first date τ_x at which some relation reaches a rupture level $-\xi$, with $\xi > 0$, so that the relational state leaves the admissible set on which the equilibrium has been verified. Equilibrium collapse occurs at the first date τ_E at which the parameters leave the region in which an equilibrium exists. Interaction ends at

$$T = \min\{T_h,\ \tau_c,\ \tau_x,\ \tau_E\}. \qquad (35)$$

The rupture level is treated as the boundary of the admissible relational set rather than as a stopping rule anticipated in players' strategies. An anticipated rupture boundary would turn the game into an exit-time problem whose values are not quadratic. A negative continuation margin, as in Corollary 1, is none of these exits: the equilibrium persists and interaction continues.

Proposition 9 (Dynamic collapse). Let $\psi_i = 0$, use the selected equilibrium, and let $x_d(0) > -\xi$. Relation d follows

$$x_d(t) = x_d^* + (x_d(0) - x_d^*)e^{-\kappa_d t}, \qquad x_d^* = \frac{2m_d}{\varphi w + 2\chi_d^+}. \qquad (36)$$

(i) If $\kappa_d > 0$, relation d collapses if and only if $x_d^* < -\xi$, and it does so at $t = \kappa_d^{-1}\ln[(x_d(0) - x_d^*)/(-\xi - x_d^*)]$.

(ii) If $\kappa_d < 0$ and $x_d^* > -\xi$, relation d collapses if and only if $x_d(0) < x_d^*$, and it does so at

$$\tau_{x,d} = \frac{1}{|\kappa_d|}\ln R_d, \qquad R_d = \frac{\xi + x_d^*}{x_d^* - x_d(0)}. \qquad (37)$$

If $x_d(0) > x_d^*$ in case (ii), the relation diverges toward amity, which represents saturation rather than collapse.

Proposition 10 (Endogenous divergence is slow). If $\kappa_d < 0$ under the selected equilibrium, then $|\kappa_d| \leq (\delta - \varphi)/3 < \delta/2$. Consequently,

$$\tau_{x,d} \geq \frac{3\ln R_d}{\delta - \varphi}, \qquad e^{-\lambda\tau_{x,d}} \leq R_d^{-3\lambda/(\delta-\varphi)}, \qquad (38)$$

where the second expression bounds the probability that random termination has not occurred by the date of dynamic collapse.

Proposition 10 implies that in a quadratic-value equilibrium, divergence of relations is slower than the discounting that disciplines it. Unless termination risk is small relative to $\delta - \varphi$, the system is more likely to end through random termination or institutional closure than through dynamic collapse.

Proposition 11 (Equilibrium collapse). Let $\psi_i = 0$.

(i) In the forced-continuation game, a quadratic-value MPNE ceases to exist when some weight μ_d falls below $\mu_{\min}$. At $\mu_d = \mu_{\min}$ the two roots of Eq. (31) coincide at $\chi_d = -w\gamma/6$, so strategies remain finite and the equilibrium disappears through a fold.

(ii) In the closed game with constant weights, constant terminal payoffs, and remaining horizon $L = (\bar{q} - q)/\lambda$, a quadratic-value MPNE exists if and only if $L < L_d^{\mathrm{esc}}$ for every relation with $\mu_d < \mu_{\min}$, where

$$L_d^{\mathrm{esc}} = \frac{w}{3\omega_d}\left[\arctan\left(\frac{w\gamma}{6\omega_d}\right) + \frac{\pi}{2}\right], \qquad \omega_d = \frac{1}{6}\sqrt{w(-12\mu_d - w\gamma^2)}. \qquad (39)$$

Part (ii) shows that institutional closure can preempt equilibrium collapse: a closed game whose remaining horizon is shorter than L_d^{esc} retains an equilibrium even when the forced-continuation game has none. Corollary 1 and Proposition 11 together separate two failures that are often conflated. Under Corollary 1 continuation ceases to dominate termination while the equilibrium persists, whereas under Proposition 11 the equilibrium itself disappears. Numerical integration of the Riccati equation reproduces Eq. (39) to four decimal places, for example $L^{\mathrm{esc}} = 4.9884$ for $(\varphi, w, \mu_d, \delta) = (1,1,-0.5,0.2)$ and $L^{\mathrm{esc}} = 11.5149$ for $(0.05,1,-0.02,0.2)$.

Proposition 12 (Warning before equilibrium collapse). Let the weight $\mu_d(q)$ decrease continuously and reach $\mu_{\min}$ at q_E, and consider frozen-parameter equilibria. Then κ_d is increasing in μ_d and equals $(\varphi - \delta)/3$ at $\mu_{\min}$.

(i) If $\varphi > \delta$, relation d remains stable for every $q < q_E$, and κ_d stays above $(\varphi - \delta)/3 > 0$. If the other relations are also stable, the system passes from SSS directly to equilibrium collapse, with no unstable phase and with a slowing of adjustment that remains bounded away from zero.

(ii) If $\varphi < \delta$, relation d is unstable for every q at which $\mu_{\min} \le \mu_d(q) < \mu_c$, so equilibrium collapse is preceded by USS or SUS.

For example, with $\varphi = w = 1$ and $\delta = 0.2$, the fold lies at $\mu_{\min} = -0.403$, where $\kappa_d = 0.267$. A relation whose moderation weight declines from one to the fold remains attracting throughout, although its rate of adjustment falls to about one sixth of its initial value of 1.635. Appendix C.4 to C.6 prove Propositions 9 to 12.

The four exits also differ in timing. Institutional closure and dynamic collapse require positive time from any interior state, because q and x move continuously. Random termination has no atom, but it arrives without regard to the state and gives no warning. Equilibrium collapse occurs at the crisis level q_E, after a remaining time $(q_E - q)/\lambda$, and it is the only exit through which the equilibrium structure itself changes discontinuously. By Proposition 12, whether that change is preceded by observable instability depends only on the comparison between φ and δ.

The uniform ceiling of Proposition 4 requires the bound (18). In the specialization (28), the crisis-free return is unbounded above on $\mathbb{R}^3$ whenever some $\mu_d < 0$ or some $\psi_i > 0$, so the bound fails precisely where instability can arise. On a bounded admissible set $[-\xi, \bar{\xi}]^3$ the bound holds, with α_i given by the maximum of the crisis-free return over that set minus ρs_i, and Proposition 4 applies before dynamic collapse. Where relations can destabilize, the ceiling therefore depends on the admissible relational range, and institutional closure remains a maintained rule rather than a consequence of payoffs.

6 Time consistency and subgame perfection

Under the verification conditions of Proposition 2, each player's feedback law is a best response at every admissible subgame state, which makes the verified MPNE subgame perfect (Haurie et al. 2012, Remark 7.6). Current-state dependence alone is not sufficient. Open-loop and feedback solutions differ in general, although their relationship depends on the structure of the game (Haurie et al. 2012, Chap. 7).

7 Application: the strategic triangle

The framework complements strategic-triangle analysis by distinguishing actors' incentives from the dynamics of their relationships and from the value of continued interaction. Dittmer's configurations identify patterns of amity and hostility and the opportunities associated with different positions. A player's best response, the stability of the resulting relational trajectory, and the advantage of continuation over termination are separate analytical questions, and Dittmer's configurations accordingly motivate questions rather than supply answers. Mutual amity in a ménage à trois, a pivot's positional benefit in a romantic triangle, or the cohesion of a stable marriage does not by itself establish compatible incentives, attenuating dynamics, or a positive continuation margin for the triangle as a whole.

The continuation-value analysis adds a further dimension to this interpretation. In the tractable specialization, differences in baseline returns and sensitivity to accumulated crisis produce different individual thresholds, so one actor may lose the advantage of continuation while others retain it. The common threshold $q^{\dagger}$ is reached only when all three equilibrium continuation margins are nonpositive, whereas the sufficient ceiling $\bar{q}$ answers a broader question: under the stated uniform bound, no feasible positive-duration continuation above that ceiling improves any player's payoff relative to immediate termination. Declining continuation value need not produce immediate breakdown. In the forced-continuation benchmark, players

choose their controls but have no unilateral stopping action, so their strategies can remain mutual best responses even when all would receive higher payoffs under termination.

The linear-quadratic specialization of Section 5 also yields one configuration change from optimizing behavior. Let all relations share the moderation weight $\mu > 0$, let player 1 alone hold the positional incentive $\psi_1 > 0$, and let $\theta_{12} = \theta_{13}$ and $\theta_{23} > 0$, so that the triangle without positional incentives converges to a ménage à trois. Let χ_0 denote the larger root of Eq. (31) at μ, and let $D = 14\chi_0^2 + 8w\gamma\chi_0 + w^2\gamma^2$.

Proposition 13 (Positional threshold for a romantic triangle). To first order in ψ_1, the steady state of the wing relation is $x_{23}^* = \Phi\theta_{23} - \Lambda\psi_1\theta_{12} + O(\psi_1^2)$, where

$$\Phi = \frac{2\mu w}{(2\chi_0 + \varphi w)\big(3\chi_0 + w(\varphi + \delta)\big)}, \Lambda = \frac{2\varphi\mu w^3(3\chi_0 + w\gamma)(5\chi_0 + w\gamma)}{(2\chi_0 + \varphi w)^2(3\chi_0 + w(\varphi + \delta))^2(4\chi_0 + w\gamma)\, D}.$$

To the same order, the wing relation becomes hostile in the steady state, so that the triangle becomes a romantic triangle with player 1 as pivot, if and only if

$$\psi_1 > \psi_1^* = \frac{\theta_{23}}{\theta_{12}} \cdot \frac{(2\chi_0 + \varphi w)\big(3\chi_0 + w(\varphi + \delta)\big)(4\chi_0 + w\gamma)\, D}{\varphi w^2(3\chi_0 + w\gamma)(5\chi_0 + w\gamma)}. \qquad (40)$$

Appendix C.7 describes the derivation. The wings' payoffs do not contain ψ_1, so the hostility between them arises entirely from their anticipation of the pivot's feedback law, under which the pivot raises its effort toward a wing when the relation between the wings deteriorates. The threshold depends on the targets only through θ_{23}/θ_{12}, and it rises without bound as φ tends to zero. By Eq. (34) the configuration change leaves the real parts of the eigenvalues unchanged to first order, so the triangle changes its Dittmer configuration while remaining SSS. This illustrates, within a single specification, the distinction between configurations and regimes maintained in Section 3.2. Table 6 compares Eq. (40) with numerical solutions of Eq. (29).

Table 6 Positional threshold relative to the wing target, ψ_1^*/θ_{23}

$(\varphi,\ w,\ \mu,\ \delta)$	Eq. (40)	Computed from Eq. (29)
$(1,\ 1,\ 1,\ 0.2)$	12.099	12.096
$(1,\ 2,\ 1,\ 0.3)$	14.842	14.851
$(1,\ 1,\ 0.5,\ 0.1)$	6.494	6.497
$(2,\ 0.5,\ 1,\ 0.2)$	12.988	12.995
$(0.3,\ 1,\ 1,\ 0.2)$	19.479	17.254

Note: $\theta_{12} = \theta_{13} = 1$ and $\theta_{23} = 0.05$. The first-order approximation loses accuracy when relational decay is slow, because the threshold is then larger.

These results provide conditions and benchmarks rather than an explanation of a particular historical alignment transition. The general framework requires specified relational dynamics, switching rules, and verified feedback strategies before it can explain such a transition, and the linear-quadratic specialization derives one configuration change and the regime transitions of Section 5 under a constant hazard. Empirical application would require identifying observable relational states, estimating the relevant payoff and hazard mechanisms, and specifying what termination and collapse mean in the case under study.

8 Conclusion

This article develops a three-player, continuous-time, non-zero-sum differential game in which termination occurs at a random time with state-dependent intensity or at an institutional boundary on cumulative hazard. Augmenting the state with cumulative hazard yields a deterministic equivalent and a coupled HJB system whose solutions, under stated verification conditions, constitute a Markov perfect Nash equilibrium. The framework distinguishes three objects that are often conflated: individual best responses, the stability of the induced dynamics, and the value of continuation relative to termination.

A continuation-margin identity converts the comparison between continuation and termination into the sign of an integrated adjusted flow. Under linear payoff erosion in cumulative hazard, it yields a sufficient ceiling that holds uniformly over admissible control profiles. A tractable specialization produces exact individual and common thresholds, a critical hazard rate, and a constructive demonstration that negative continuation margins can coexist with a Markov perfect Nash equilibrium. Institutional closure is a separate rule governing when interaction ends.

A linear-quadratic specialization makes the stability classification explicit. A bilateral relation destabilizes only if its natural decay is slower than the effective discount rate and its

moderation weight lies in an explicit interval, the regime of the triangle follows from the signs of three scalar rates, and declining moderation generically moves the system from SSS through USS to SUS. The specialization also separates four exits. Institutional closure and random termination are exogenous. Dynamic collapse is endogenous but necessarily slow relative to discounting, and equilibrium collapse occurs through a fold that institutional closure can preempt and that, when relational decay is faster than the effective discount rate, is not preceded by any unstable phase. Stable dynamics are therefore no guarantee against abrupt breakdown of the strategic structure.

Several extensions follow. A hazard that depends on relational states or controls would require numerical solution of the nonlinear HJB system, because the products of hazard and value destroy the quadratic structure. An anticipated rupture boundary would turn dynamic collapse into an exit-time problem, and negotiated or unilateral exit would require an optimal stopping formulation. These extensions would permit stronger claims about alignment transitions in strategic triangles and in other multilateral relationships.

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Appendix A Random-horizon transformation and HJB derivation

A.1 Lemmas for random time horizons

Lemma A.1 (Expectation over a random horizon). Conditional on survival at t and a fixed admissible profile, let φ be measurable. For nonnegative φ, Tonelli's theorem gives the following identity; for signed φ, the survival-weighted integral of $|\varphi|$ is required to be finite:

$$\mathbb{E}\left[\int_t^T \varphi(s)\, ds \mid T > t\right] = \int_t^\infty e^{-[q(s)-q(t)]}\, \varphi(s)\, ds. \qquad \text{(A1)}$$

The survival function here is conditional on the induced path and the current surviving state. Only the random terminal clock remains stochastic; additional stochastic state drivers would require a further expectation.

Proof. Write the random-horizon integral as the integral over the half-line of the survival indicator multiplied by φ. Tonelli's theorem applies to nonnegative integrands, and for signed integrands absolute integrability permits Fubini's theorem and the same identity:

$$\mathbb{E}\left[\int_t^{T_h} \varphi(s)\, ds\right] = \int_t^{\infty} \varphi(s)\, P(T_h > s \mid T_h > t)\, ds. \qquad \text{(A2)}$$

▫

Lemma A.2 (Conditional hazard and survival). For an absolutely continuous random clock, h is the terminal density divided by conditional survival. Along a fixed profile,

$$P(T_h > s \mid T_h > t) = e^{-\int_t^s h(v)\, dv} = e^{-[q(s)-q(t)]}. \qquad \text{(A3)}$$

Proof. The logarithmic derivative of conditional survival is $-h(s)$. Integrating from t, where conditional survival is one, gives the exponential formula. Because $\dot{q} = h$, the integrated hazard is $q(s) - q(t)$.

A.2 Proof of Proposition 1

Use the conditional objective at date t with $T = T_h$. The absolute-integrability assumptions justify exchanging integrals and expectations:

$$J_j = \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}\, g_j(s)\, ds + e^{-\rho(T-t)} S_j(x(T))\right]. \qquad \text{(A4)}$$

The proposed deterministic equivalent is

$$W_j = \int_t^{\infty} D_t(s)\left[g_j(s) + h(s)\, S_j\big(x(s)\big)\right] ds. \qquad \text{(A5)}$$

Decompose the objective into the running component I_1 and the terminal component I_2.

Running payoff. Replace the upper random limit by a survival indicator and apply Lemma A.1:

$$I_1 = \int_t^{\infty} e^{-\rho(s-t)}\, g_j(s)\, P(T > s \mid T > t)\, ds. \qquad \text{(A6)}$$

Lemma A.2 then combines ordinary discounting with survival:

$$I_1 = \int_t^{\infty} e^{-\int_t^s [\rho + h(v)] dv}\, g_j(s)\, ds = \int_t^{\infty} D_t(s)\, g_j(s)\, ds. \qquad \text{(A7)}$$

Terminal payoff. The terminal component is the expectation of the discounted salvage or penalty at the random clock. With an absolutely continuous conditional density,

$$I_2 = \int_t^{\infty} e^{-\rho(s-t)}\, S_j\big(x(s)\big)\, f_T(s \mid T_h > t)\, ds, \qquad \text{(A8)}$$

where

$$f_T(s \mid T_h > t) = h(s)\, e^{-[q(s)-q(t)]}. \qquad \text{(A9)}$$

Substituting the density gives

$$I_2 = \int_t^{\infty} e^{-\rho(s-t)}\, S_j\big(x(s)\big)\, h(s)\, e^{-[q(s)-q(t)]} ds$$

$$= \int_t^{\infty} D_t\,(s)\, h(s)\, S_j\big(x(s)\big)\, ds. \qquad \text{(A10)}$$

Combination. The hazard-weighted terminal contribution enters the transformed integral because it is density weighted, not because it is an ordinary additional reward:

$$J_j = I_1 + I_2 = W_j, \qquad \text{(A11)}$$

which proves part (i). For part (ii), truncate both continuous contributions at τ_c and add the discounted probability mass of surviving to the boundary, $D_t(\tau_c)\, S_j\big(x(\tau_c)\big)$. This yields Eq. (6) and explains why the terminal transformation must distinguish random killing from institutional closure.

A.3 Proof of Proposition 2

Over a short interval dt, conditional survival is $1 - h\,dt + o(dt)$, random termination contributes $hS_j\,dt + o(dt)$, and the surviving state advances by $f\,dt$ in x and $h\,dt$ in q. Ordinary discounting contributes $1 - \rho\,dt + o(dt)$. Expansion and own-control maximization, with rivals' feedback fixed, yield Eq. (8):

$$(\rho + h)V_j = \partial_t V_j + h\;\partial_q V_j + h\,S_j + \max_{u_j \in U_j}\big\{g_j + \nabla_x V_j \cdot f\big\}. \qquad \text{(A12)}$$

The coefficient $\rho + h$ discounts continued engagement; it does not mean that termination risk changes intrinsic time preference. The q derivative and the hazard-weighted terminal payoff are separate contributions. For verification, the HJB maximization implies the following inequality along every admissible unilateral deviation:

$$\partial_t V_j + \nabla_x V_j \cdot f + h\;\partial_q V_j - (\rho + h)V_j + g_j + h\,S_j \le 0. \qquad \text{(A13)}$$

Multiplying by $D_t(s)$, whose derivative is $-(\rho + h)D_t(s)$, gives

$$D_t(s)\big[g_j + h\,S_j\big] \le -\frac{d}{ds}\big[D_t(s)\, V_j\big(s, x(s), q(s), R(s)\big)\big]. \qquad \text{(A14)}$$

Integrate to the appropriate endpoint and split at all mode switches. Value matching cancels internal boundary terms, and the endpoint condition implies that the candidate initial value is at least the payoff from the unilateral deviation. Under the maximizing law all inequalities hold

with equality. Repeating this comparison for each player at every surviving subgame state proves that the profile is an MPNE, conditional on the stated regularity and endpoint requirements.

A.4 Continuity conditions under composite distributions

At a transparent switch, values are matched at the same date and augmented state as in Eq. (10), with no instantaneous payoff or state reset. Splitting the verification integral at each interface cancels the internal boundary terms, and neither derivative matching nor continuous controls is required. For the closed game, $V_j = S_j$ at closure; for the benchmark, the discounted-survival remainder must vanish along every admissible deviation. With $\rho > 0$, at-most-polynomial value growth, and at-most-polynomial growth of x and q along admissible paths, this remainder vanishes. Value matching alone establishes neither anticipation nor subgame perfection.

Appendix B Stability, ceiling, and threshold proofs

B.1 Proof of Proposition 3

An MPNE follows from the verification conditions of Proposition 2, not merely from differentiable candidate values or unique own-control maximizers. The active-regime classification additionally requires an induced-field stability argument and is independent of the signs of continuation margins. For a differentiable autonomous frozen-mode relational field with a hyperbolic reference, let A be the total Jacobian, including feedback derivatives. The following are local criteria; nonautonomous tracking instead requires uniform conditions.

SSS. If every eigenvalue of A has negative real part, choose a positive-definite Q and solve the Lyapunov equation

$$A^{\top}P + PA = -Q, \qquad P = P^{\top} > 0, \qquad Q > 0. \qquad \text{(B1)}$$

The resulting positive-definite P yields a quadratic function whose derivative satisfies

$$\dot{L} = -1/2\, e^{\top}Qe + o(\| e \|^2). \qquad \text{(B2)}$$

With $\dot{e} = Ae$ plus a remainder of smaller order than the error norm, this derivative is negative for sufficiently small nonzero e, which proves local asymptotic attraction. For a moving reference, uniform bounds on the reference, tracking field, and Lyapunov derivative are additional requirements.

USS. If A has eigenvalues with both positive and negative real parts and none with zero real part, the reference is a saddle. Its stable manifold attracts only trajectories lying on that

manifold, and a full neighborhood is not Lyapunov stable. An indefinite derivative of one candidate function, without this spectral information, does not establish a saddle.

SUS. If every eigenvalue has positive real part, the reference is a repeller. Real eigenvalues permit a node-type illustration, and a complex conjugate pair gives a focus in a two-dimensional illustration. Neither classification alone proves finite-time institutional closure.

UUS is the terminal institutional condition under the rule $T_c = \min\{T_h, \tau_c\}$. It is not a claim that the coupled HJB maximizers disappear; the specialization of Proposition 5 explicitly retains interior best responses beyond the common benchmark crossing.

B.2 Proofs of Lemma 1 and Proposition 4

Proof of Lemma 1. For finite L the transformed functional includes the surviving endpoint payoff $D_t(L)S_j\big(x(L)\big)$, and for an infinite endpoint this discounted salvage remainder is required to vanish. Differentiation along the path gives

$$\frac{d}{ds}\big[D_t(s)\, S_j\big(x(s)\big)\big] = D_t(s)\big[\nabla_x S_j \cdot f - (\rho + h)S_j\big]. \qquad \text{(B3)}$$

Integrate this derivative from t to L and add the running and hazard-weighted terminal terms. The hS_j contributions cancel, leaving $g_j - \rho S_j$ plus the derivative of S_j along the relational path, which is the adjusted flow in Eq. (17). This establishes Eq. (19).

Proof of Proposition 4. For $q(t) \geq \bar{q}$, monotonicity gives $q(s) \geq q(t)$ at every later surviving date. The uniform bound (18) therefore makes every adjusted flow nonpositive for every feasible joint control profile, and Lemma 1 gives a nonpositive continuation margin for each player. For $q(t) > \bar{q}$, any positive-duration extension has a strictly negative margin. The argument is uniform over future policy changes and does not assume that equilibrium controls jointly maximize all players' returns.

Arrival at the ceiling. If $h \geq h_{\min} > 0$ before closure, integrating $\dot{q} \geq h_{\min}$ gives Eq. (21). This bounds the arrival time on the nonterminated path, and random termination may occur earlier. If q already meets the boundary, institutional closure is immediate. No claim is made that the game survives to the ceiling with probability one.

Limits of the bound. The maximum over players is required for all three bounds to be nonpositive simultaneously. Without positive β_j, finite uniform α_j, and nondecreasing q, the proof does not supply a finite ceiling. The bound is sufficient, not generally sharp, and the closure rule is imposed rather than solved as an optimal stopping boundary. A nonpositive continuation margin provides neither a first-order control condition nor a theorem of MPNE nonexistence; Corollary 1 demonstrates this distinction constructively.

B.3 Proof of Proposition 5

Part (i). Own effort enters the payoff through $b_j u_j - d_j u_j^2/2$, whose derivative is $b_j - d_j u_j$ and whose second derivative is $-d_j < 0$. Effort changes neither the hazard, the growth of q, nor the terminal payoff. Hence the unique interior maximizer $u_j^* = b_j/d_j$ is optimal at every date and state against the other players' laws, and a time-varying deviation cannot improve its pointwise return. The constant profile is therefore an MPNE in the forced-continuation benchmark. Along that profile, future crisis is $q + \lambda s$. The discounted integrals of one and of s are $1/\delta$ and $1/\delta^2$, and substitution into Lemma 1 yields

$$M_j^{\infty}(q) = \int_0^{\infty} e^{-\delta s}\left[B_j - \beta_j(q + \lambda s)\right]ds = \frac{B_j - \beta_j q}{\delta} - \frac{\beta_j \lambda}{\delta^2}. \qquad \text{(B4)}$$

The q derivative is $-\beta_j/\delta < 0$, and solving for zero gives q_j^*. All margins are nonpositive precisely when q is at least the largest individual crossing, subject to $q \geq 0$, which proves Eq. (25). Along a surviving benchmark path the remaining time to the common crossing is $\max\{0, (q^{\dagger} - q)/\lambda\}$.

Part (ii). Because the equilibrium profile is admissible and $S_j = s_j$ is constant, $\alpha_j \geq a_j^* - \rho s_j = B_j$. Hence $\alpha_j/\beta_j \geq B_j/\beta_j > B_j/\beta_j - \lambda/\delta = q_j^*$ for every j, since $\lambda/\delta > 0$. Taking the maximum over players and over zero gives $q^{\dagger} \leq \bar{q}$.

Part (iii). Multiplying Eq. (24) by $(\rho + \lambda)^2$ shows that its sign is the sign of

$$K_j(\rho + \lambda) - \beta_j \lambda = \rho K_j + \lambda\big(K_j - \beta_j\big). \qquad \text{(B5)}$$

Solving for zero gives Eq. (26) when $0 < K_j < \beta_j$. For $K_j \geq \beta_j$ the expression is positive at every finite positive λ, and for $K_j \leq 0$ it is negative at positive λ. The finite crossing therefore depends on payoff parameters and is not a general hazard threshold for strategic realignment.

B.4 Closure values and absorption

In the same specialization, impose closure at $\bar{q}$ and let $\ell = (\bar{q} - q)/\lambda$ for $q < \bar{q}$. Effort affects neither ℓ nor the hazard, so pointwise optimal efforts remain b_j/d_j. Integrating the adjusted return only up to ℓ gives

$$V_j^c(q) - s_j = \frac{\big(B_j - \beta_j q\big)\big(1 - e^{-\delta \ell}\big)}{\delta} - \frac{\beta_j \lambda}{\delta^2}\big[1 - e^{-\delta \ell}(1 + \delta \ell)\big]. \qquad \text{(B6)}$$

Proof. Integrate $e^{-\delta s}\big[B_j - \beta_j(q + \lambda s)\big]$ from zero to ℓ. The integrals of one and of s are $\big(1 - e^{-\delta \ell}\big)/\delta$ and $\big[1 - e^{-\delta \ell}(1 + \delta \ell)\big]/\delta^2$, and substitution gives Eq. (B6). Equivalently,

$$M_j^c(q) = M_j^{\infty}(q) - e^{-\delta \ell} M_j^{\infty}(\bar{q}). \qquad \text{(B7)}$$

At $q = \bar{q}$, $\ell = 0$ and the closed margin is zero, so the terminal boundary condition holds, and the formula also satisfies the stopped HJB equation. Unlike the infinite-benchmark margin, the closed margin need not decrease monotonically near its boundary, so $q^{\dagger}$ should not be substituted for its crossings.

After closure the original game has ended, and any re-entry requires new rules rather than a reverse crossing of a cumulative state. There is no general strict robustness ordering of SSS, USS, SUS, and UUS, because a saddle is not stable in a full neighborhood and terminal closure is not a local dynamical equilibrium. The common benchmark crossing and the uniform ceiling follow from specified payoff primitives, whereas institutional adoption of the ceiling is a maintained rule. Other active-mode interfaces require explicit primitives or rules and are not endogenous merely because they are written as state thresholds. An optimal or negotiated stopping boundary would require modeling those additional actions and incentives.

Appendix C Proofs for the linear-quadratic specialization

C.1 Derivation of Eqs. (29) to (31)

With constant hazard and constant terminal payoffs, Eq. (8) for player i reads $\delta V_i = \partial_q V_i \, \lambda + \lambda s_i + \max_{u_i}\{g_i + \nabla_x V_i \cdot f\}$ up to the crisis term, which enters only $v_i(q)$. Under a quadratic value, $\nabla_x V_i = \Pi_i x + p_i$, and the own-control part of the maximand is $-w/2 \parallel u_i \parallel^2 + (\Pi_i x + p_i)^{\top} G_i u_i$, which is strictly concave with maximizer $u_i^* = w^{-1} G_i^{\top} (\Pi_i x + p_i)$. Substituting all three maximizing laws, the own-control terms contribute $1/2\,(\Pi_i x + p_i)^{\top} M_i (\Pi_i x + p_i)$ and the rivals' laws contribute $(\Pi_i x + p_i)^{\top} \sum_{k \neq i} M_k \, (\Pi_k x + p_k)$. Matching the coefficients of $1/2\, x^{\top}(\cdot) x$ gives Eq. (29), and the drift under the three laws gives Eq. (30). When $\psi_i = 0$, the matrices Q_i are diagonal and supported on player i's own relations, so Eq. (29) admits a solution in which each Π_i is diagonal and the value separates by relation. For relation d with members i and j, write the common component as $v(x) = -\chi/2\, x^2 + mx$. The scalar HJB equation is $\delta v = -\mu/2\,(x - \theta)^2 + 3/2w\, v'(x)^2 - \varphi x\, v'(x)$ plus a constant, because each member's own effort contributes $v'^2/(2w)$ and the partner's effort contributes v'^2/w. Matching the coefficients of x^2 and x gives Eq. (31). The closed-loop drift of the relation is $-(\varphi + 2\chi/w)x + 2m/w$. ▫

C.2 Proof of Proposition 6

Part (i). Eq. (31) has a real root if and only if its discriminant $\gamma^2 + 12\mu_d/w$ is nonnegative, which is the condition $\mu_d \geq \mu_{\min}$. A quadratic-value equilibrium requires a real coefficient for each relation, so the condition is necessary. Sufficiency follows from part (ii), which shows that the larger root satisfies Eq. (12), and from Proposition 2.

Part (ii). In the closed game with constant terminal payoffs and remaining horizon τ, the coefficient $\chi(\tau)$ solves $d\chi/d\tau = F(\chi)$ with $F(\chi) = \mu_d - \gamma\chi - 3\chi^2/w$ and $\chi(0) = 0$. The function F is a concave parabola whose roots are $\chi^- \leq \chi^+$, with $F'(\chi^+) = -\sqrt{\gamma^2 + 12\mu_d/w} \leq 0$. If $\mu_d > 0$, then $\chi^- < 0 < \chi^+$ and $F(0) = \mu_d > 0$, so χ increases to χ^+. If $\mu_{\min} \leq \mu_d \leq 0$, both roots are nonpositive, $F(0) = \mu_d \leq 0$, and χ decreases to χ^+, because $F < 0$ above χ^+. In both cases $\chi(\tau) \to \chi_d^+$. The linear coefficient converges as well, because $\delta + \varphi + 3\chi_d^+/w \geq \delta/2 > 0$.

Under a root χ, the relation grows at most at the rate $-\kappa$, where $\kappa = \varphi + 2\chi/w$, and the quadratic value grows at twice that rate, so Eq. (12) holds if and only if $\kappa > -\delta/2$. For the larger root, $\kappa^+ = 1/3\left(\varphi - \delta + \sqrt{\gamma^2 + 12\mu_d/w}\right) \geq (\varphi - \delta)/3 > -\delta/2$, since $2\varphi + \delta > 0$. For the smaller root, $\kappa^- = 1/3\left(\varphi - \delta - \sqrt{\gamma^2 + 12\mu_d/w}\right) > -\delta/2$ if and only if $\sqrt{\gamma^2 + 12\mu_d/w} < \gamma/2$, that is, $\mu_d < -w\gamma^2/16 = \mu_u$.

Part (iii). Eq. (33) follows from $\kappa_d = \varphi + 2\chi_d^+/w$. The inequality $\kappa_d < 0$ holds if and only if $\sqrt{\gamma^2 + 12\mu_d/w} < \delta - \varphi$, which requires $\varphi < \delta$ and, after squaring, $12\mu_d/w < (\delta - \varphi)^2 - \gamma^2 = -3\varphi(\varphi + 2\delta)$, that is, $\mu_d < \mu_c$. Existence requires $\mu_d \geq \mu_{\min}$. ▫

C.3 Proofs of Corollary 2 and Propositions 7 and 8

Corollary 2. By Eq. (33), $\kappa_d \geq (\varphi - \delta)/3$, with equality only when the discriminant vanishes, that is, when $\mu_d = \mu_{\min}$. If $\varphi \geq \delta$ the bound is nonnegative.

Proposition 7. With diagonal Π_i, Eq. (30) gives $A_{\text{cl}} = -\text{diag}(\kappa_{12}, \kappa_{13}, \kappa_{23})$, whose eigenvalues are real. The classification follows from Proposition 3.

Proposition 8. By Eq. (33), κ_d is continuous and strictly increasing in μ_d on $[\mu_{\min}, \infty)$, and by Proposition 6(iii) it changes sign exactly at μ_c when $\varphi < \delta$. Composing with continuous nonincreasing weights $\mu_d(q)$, each $\kappa_d(q)$ is continuous and nonincreasing and changes sign at most once, at the level where $\mu_d(q) = \mu_c$. The regime sequence follows from Proposition 7. A direct transition from SSS to SUS requires the three sign changes to occur at the same level of q. If $\varphi \geq \delta$, Corollary 2 excludes sign changes. ▫

C.4 Proofs of Propositions 9 and 10

Proposition 9. Under the selected equilibrium the relation obeys $\dot{x}_d = -\kappa_d x_d + 2m_d/w$, whose solution is Eq. (36) with $x_d^* = 2m_d/(w\kappa_d) = 2m_d/(\varphi w + 2\chi_d^+)$. If $\kappa_d > 0$ the path moves monotonically toward x_d^*, so it reaches $-\xi$ if and only if $x_d^* < -\xi$, and solving $x_d(t) = -\xi$ gives the stated date. If $\kappa_d < 0$, the deviation from x_d^* grows at the rate $|\kappa_d|$ and keeps its initial sign, so the path reaches $-\xi$ if and only if $x_d(0) < x_d^*$, and solving $x_d(t) = -\xi$ gives Eq. (37).

Proposition 10. By Corollary 2's argument, $\kappa_d \geq (\varphi - \delta)/3$, so $\kappa_d < 0$ implies $|\kappa_d| \leq (\delta - \varphi)/3 < \delta/2$. Substituting this bound into Eq. (37) gives the first inequality in Eq. (38), and $e^{-\lambda\tau_{x,d}}$ is decreasing in $\tau_{x,d}$, which gives the second.

C.5 Proof of Proposition 11

Part (i). For $\mu_d < \mu_{\min}$, Eq. (31) has no real root. At $\mu_d = \mu_{\min}$ the discriminant vanishes and the double root is $\chi_d = -w\gamma/6$, at which the maximizing law is finite.

Part (ii). For $\mu_d < \mu_{\min}$, write $-F(\chi) = 3/w\,[(\chi + w\gamma/6)^2 + \omega_d^2]$ with ω_d as in Eq. (39). Since $F < 0$ everywhere, $\chi(\tau)$ decreases from zero and reaches $-\infty$ after the remaining horizon

$$\int_{-\infty}^{0} \frac{w\,d\chi}{3[(\chi + w\gamma/6)^2 + \omega_d^2]} = \frac{w}{3\omega_d}\left[\arctan\left(\frac{w\gamma}{6\omega_d}\right) + \frac{\pi}{2}\right] = L_d^{\mathrm{esc}}. \qquad \text{(C1)}$$

For $L < L_d^{\mathrm{esc}}$ the coefficient remains finite on the remaining horizon, and the finite-horizon verification theorem (Haurie et al. 2012, Theorem 7.2), together with the closure condition (11), establishes the equilibrium. For $L \geq L_d^{\mathrm{esc}}$ no finite quadratic coefficient exists over the whole horizon. Relations with $\mu_d \geq \mu_{\min}$ impose no restriction, by the argument of Appendix C.2.

C.6 Proof of Proposition 12

By Eq. (33), $d\kappa_d/d\mu_d = 2/\left(w\sqrt{\gamma^2 + 12\mu_d/w}\right) > 0$ for $\mu_d > \mu_{\min}$, and $\kappa_d(\mu_{\min}) = (\varphi - \delta)/3$. If $\varphi > \delta$, then $\kappa_d > (\varphi - \delta)/3 > 0$ for every $\mu_d > \mu_{\min}$, so the relation is stable for every $q < q_E$. If $\varphi < \delta$, then $\mu_{\min} < \mu_c$ and Proposition 6(iii) gives $\kappa_d < 0$ on $[\mu_{\min}, \mu_c)$, which a continuous decreasing path $\mu_d(q)$ must traverse before reaching $\mu_{\min}$.

C.7 First-order expansion in positional incentives

Write $\Pi_i = \Pi_i^0 + \psi_1\Pi_i^1 + O(\psi_1^2)$ and $p_i = p_i^0 + \psi_1 p_i^1 + O(\psi_1^2)$, where Π_i^0 and p_i^0 are the diagonal solutions of Appendix C.1. Substituting into Eq. (29) and the corresponding linear

equations, and collecting first-order terms, gives a linear system for Π_i^1 and p_i^1 whose coefficients depend on χ_P, χ_W, φ, δ, and w. The system was solved by symbolic computation. The diagonal of $\sum_k M_k \Pi_k^1$ vanishes, and its off-diagonal entries give ζ_1 and ζ_2 in Eq. (34). The first-order steady state $x^* = -A_{\mathrm{cl}}^{-1} b$ is expanded in the same way and yields Eq. (40) when $\mu_P = \mu_W$. Both results were checked against numerical solutions of Eq. (29), as reported in Tables 5 and 6.