Supplemental Material for "Optical connectivity governs the loss cost of multiphoton parity correlations"

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This Supplemental Material applies the survivor thresholds of the main text to a conditional optical task on the same source (d groups of m photons), with task probability determined by the mirror-label parity bias h. We give finite-sample calibration and resource bounds. The application assumes stable pure phase encoders, known population p, a calibrated internal-probe reference and identical source states in calibration and inference. Sections, equations, figures, tables, theorems and lemmas of this document carry the prefix S; numbers without this prefix refer to the main text.

OVERVIEW OF THE APPLICATION

In each group of pure states, a balanced two-layer cyclic network sends input j toward stations j and $j+1$ and then combines the two rails arriving at each station into output ports $z = 0, 1$, with an additional phase $\pi/2$ on the rail that closes the cycle. Let $\mathcal{A}$ be the event that every station of every group detects exactly one photon, and on $\mathcal{A}$ define the output bit $b_{\text{out}} = \sum_{\ell,j} z_{\ell j} \bmod 2$. Let $B_\ell = \prod_j \langle \phi_{\ell j,+} | \phi_{\ell,j+1,+} \rangle$, $v_\ell = \operatorname{Im} B_\ell$ and $V = \prod_\ell v_\ell$. Then

$$p_{\mathcal{A}} = \Pr(\mathcal{A}) = 2^{d-n}\eta_t^n, \qquad P_{\text{task}} = \Pr_h(b_{\text{out}} = 0 \mid \mathcal{A}) = \tfrac{1}{2}(1 + hV). \tag{S1}$$

The transmission η_t of the task network may differ from that of the calibration and readout runs. Within each group, only two routings give one photon per station: all photons stay at their stations, or all advance. Their interference gives $2^{1-2m_\ell}[1 + (-1)^{\sum_j z_j} s_\ell v_\ell]$ for each accepted pattern of group ℓ, before loss. Summing proves Eq. (S1). These interferometer geometries and collective-phase observables are known [1, 2]. The maximal imaginary part of a Bargmann invariant and the optimal population parameter are given in Refs. [3–5]; we use the fixed value $p = 1/4$.

Controlled-label calibration estimates each reading's uniform-label mean and coefficient; independent inference gives a Bernstein confidence interval for h. Separate source probes give an interval for V and thus for P_{task}. Sections S2 and S3 give the finite-sample certificate, phase-error bound and full resource accounting under the assumptions above.

At $n = 16, m = 8, d = 2$, $\eta = 0.01$, per-photon emission probability $p_{\text{em}} = 0.99$, accuracy 0.02 and total risk 0.05, the HOM protocol uses 10^9 inference shots, four calibration configurations of 10^9 shots each, and 16×10^6 detected source probes. Its certified radius is below 0.015972. Including the preparation test and configuration changes, an upper bound on the expected number of attempted photons is $\mathcal{C}_{\text{HOM}} < 9.3975320692 \times 10^{10}$. The test points $\pm h_0$, with $h_0 = 0.08/V$, have task probabilities that differ by $0.08 > 2(0.02)$. Theorem 4 and Proposition 2 of the main text give the separate-group lower bound $\mathcal{C}_{\text{sep}} > 4.2736556202 \times 10^{13}$ for product POVMs on the separate groups, even with free calibration. Directed interval arithmetic encloses the ratio of these bounds: $454.7636 < \mathcal{C}_{\text{sep}}/\mathcal{C}_{\text{HOM}} < 454.7637$.

Section S5 quantifies the effects of phase drift between calibration and inference and of internal-state mixing.

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S1. THE CONDITIONAL TASK AND QUBIT EXAMPLES

Restricted to the occupied inputs of one group, the task network has the transfer matrix

$$U_{(k,z),j} = \frac{1}{2}\left[\delta_{k,j} + (-1)^z e^{i\pi\delta_{k,0}/2}\delta_{k,j+1 \bmod m}\right], \tag{S2}$$

where k labels the station and z its output port. Its columns are orthonormal, so it extends to a unitary with vacuum in the other inputs. Let $x_j = 0$ if photon j stays at station j and $x_j = 1$ if it advances. Station j then holds $1 - x_j + x_{j-1}$ photons, and $\mathcal{A}$ forces all x_j to be equal. The internal states of the two accepted routings have overlap $\overline{B}$ and relative phase $(-1)^{\sum_j z_j} i$, which proves Eq. (S1).

When all m photons of a regular group pass through F_m and are detected, define the charge $Q = \sum_a a N_a \bmod m$ and the reading $f = -\sin(2\pi Q/m)$. The Fourier identity of Appendix B of the main text gives $\mathbb{E}[f \mid s] = s \operatorname{Im} B$. The product of the readings of all groups has mean hV. A bit that is set to zero with probability $(1 + \prod_\ell f_\ell)/2$ therefore has the same distribution as b_{out} given $\mathcal{A}$, without the task network. This requires all n photons to survive.

For regular groups of any size $m \geq 3$, with $p = 1/4$ and phases $2\pi j/m$, the pair of inputs j from two groups has coefficient $\beta_j = (3/16)\sin^2(2\pi j/m)$, which is positive for some j. Three inputs with distinct phases in cyclic order, separated by gaps $\vartheta_1 + \vartheta_2 + \vartheta_3 = 2\pi$, have $\operatorname{Im} B_3 = (3/32)\sum_i \sin\vartheta_i = (3/8)\prod_i \sin(\vartheta_i/2) > 0$. Informative pairs and local triples therefore exist for every $m \geq 3$. Choosing the pair input nearest a quarter turn and the triple inputs nearest thirds of a turn keeps both coefficients away from zero as m grows. For regular groups the task coefficient obeys $V = \exp[-3\pi^2 n/(8m^2) + o(n/m^2)]$ as m grows with n/m^2 bounded, so groups of size of order $\sqrt{n}$ keep it away from zero. Each attempt prepares n photons. The resource comparisons consider $|V| > 2\epsilon$; when $|V| \leq 2\epsilon$ is known in advance, the fixed estimate $P_{\text{task}} = 1/2$ already meets accuracy ϵ.

S2. CALIBRATION AND FINITE-SAMPLE CERTIFICATE

Partition the groups into disjoint subsets g of one, two or three groups, one subset for each reading. A subset may be smaller than the readout set that contains it, so the subsets can refine the readout partition $\mathcal{P}$. Throughout, w with a subscript denotes the half-width of a confidence interval and $[x]$ an interval that contains x. In calibration the labels $s_g \in \{-1, +1\}^{|g|}$ of a subset are set by the experimenter. For each of the $2^{|g|}$ label configurations take N_g independent shots in which all $m|g|$ photons are delivered. Let f_g be the reading of subset g, and define its uniform-label mean and its coefficient,

$$c_g = 2^{-|g|}\sum_{s_g} \mathbb{E}[f_g \mid s_g], \qquad b_g = 2^{-|g|}\sum_{s_g}\Big(\prod_{\ell\in g} s_\ell\Big)\mathbb{E}[f_g \mid s_g]. \tag{S3}$$

The estimates $\widehat{c}_g$ and $\widehat{b}_g$ are obtained by replacing each conditional mean in Eq. (S3) with its calibration sample mean. For each label configuration, we apply the following bound separately to f_g and f_g^2, then combine the interval endpoints with the signs in Eq. (S3).

Lemma S1 (Confidence intervals for the calibration)**.** For $N \geq 2$ independent samples of a variable in $[x_-, x_+]$, with empirical mean $\widehat{\mu}$ and unbiased sample variance $\widehat{\sigma}^2$, let

$$w_N = \sqrt{\frac{2\widehat{\sigma}^2 \log(4/\alpha_0)}{N}} + \frac{7(x_+ - x_-)\log(4/\alpha_0)}{3(N-1)}. \tag{S4}$$

The interval $[\widehat{\mu} - w_N, \widehat{\mu} + w_N] \cap [x_-, x_+]$ contains the mean except with probability at most α_0 [6]. A union bound over the $2\sum_g 2^{|g|}$ intervals gives simultaneous bounds on all means and second moments with total risk α_r, and hence intervals for $a_g = c_g - \widehat{c}_g$ and b_g.

Theorem S1 (Certificate for the task probability)**.** Fix the calibration data. On every inference shot form $X = \prod_g (f_g - \widehat{c}_g)$, setting each f_g to zero whenever its own selected-photon counts are incomplete. Other readings keep their values; losses of unselected inputs do not affect acceptance. Then

$$\mathbb{E}_h X = a + hb, \qquad a = \prod_g a_g, \quad b = \prod_g b_g, \tag{S5}$$

$$\mathbb{E}_h X^2 = \prod_g M_g + h \prod_g H_g, \tag{S6}$$

where

$$\begin{aligned} M_g &= 2^{-|g|} \sum_{s_g} \mathbb{E}[(f_g - \widehat{c}_g)^2 \mid s_g], \\ H_g &= 2^{-|g|} \sum_{s_g} \Big(\prod_{\ell \in g} s_\ell \Big) \mathbb{E}[(f_g - \widehat{c}_g)^2 \mid s_g]. \end{aligned} \tag{S7}$$

Suppose the calibration gives intervals $a \in [a]$ and $b \in [b]$ with $\inf[b] > 0$, a bound $\mathbb{E}_h X^2 \le \sigma_+^2$ for all $|h| \le 1$, and $|X| \le X_{\max}$. For an independent inference stage of N shots, where N may be chosen after calibration but is fixed during the stage, let $\bar{X}$ be the sample mean of X and

$$w_X = \sqrt{2\sigma_+^2 \log(2/\alpha_i)/N} + 4X_{\max} \log(2/\alpha_i)/(3N). \tag{S8}$$

Write $\partial[\,\cdot\,]$ for the two endpoints of a closed interval and hull for the smallest closed interval containing a set. Then

$$[h] = \operatorname{hull} \left\{ \frac{\bar{X} \pm w_X - a'}{b'} : a' \in \partial[a],\ b' \in \partial[b] \right\} \cap [-1, 1] \tag{S9}$$

contains h except with probability at most $\alpha_r + \alpha_i$. Given an independent interval $V \in [V]$ of risk α_s, the hull of $\{(1 + h'v')/2 : h' \in \partial[h], v' \in \partial[V]\}$ contains P_{task} except with probability at most $\alpha_s + \alpha_r + \alpha_i$. If an intersection is empty or $\inf[b] \le 0$, the procedure reports that no certificate was obtained.

Proof. The label distribution of Eq. (1) of the main text is the uniform distribution, under which the labels of different groups are independent, plus h times the parity term. Given the labels, the readings of different subsets are independent. The uniform term therefore contributes the product of the uniform-label averages and the parity term the product of the parity-weighted averages. This gives Eqs. (S5) and (S6). Upper bounds on $\mathbb{E}[(f_g - \widehat{c}_g)^2 \mid s_g]$ follow from the simultaneous intervals for $\mathbb{E} f_g$ and $\mathbb{E} f_g^2$ and the observed $\widehat{c}_g$. Because $\mathbb{E}_h X^2$ is linear in h, its maximum over $|h| \le 1$ is attained at $h = \pm 1$, where the label distribution has nonnegative weights; this gives σ_+^2. Bernstein's inequality [7] for a variable bounded by $X_{\max}$, applied conditionally on the calibration data, gives the interval for $\mathbb{E}_h X$; the inference stage is independent of the calibration, so the bound also holds unconditionally. The ratio in Eq. (S9) is continuous and monotone in each argument for $b' > 0$, so its extrema over a box are attained at the corners. Interval arithmetic and a union bound complete the proof. □

Radius of the certificate. Let $\widehat{b} = \prod_g \widehat{b}_g > 0$ and let $\widehat{V}$ estimate V from the source probes. Define auxiliary point estimates $\widehat{h} = \operatorname{clip}(\bar{X}/\widehat{b}, -1, 1)$ and $\widehat{P}_{\text{task}} = (1 + \widehat{V}\widehat{h})/2$. When all calibration and source intervals hold, the calibration contribution to the error in $\widehat{P}_{\text{task}}$ is bounded by

$$\epsilon_{\text{cal}} = \frac{|\widehat{V}| \max_{a' \in [a]} |a'|}{2\widehat{b}} + \frac{1}{2} \max_{b' \in [b],\, v' \in [V]} |\widehat{V} b'/\widehat{b} - v'|. \tag{S10}$$

Before clipping, the task-probability estimation error is the sum of the sampling error of the inference stage, the error from $a \neq 0$, and $h(\widehat{V} b/\widehat{b} - V)/2$. To preserve the cancellation in the last term after

clipping, write $\varkappa = b/\widehat{b} > 0$. Checking the linear regions of the clipping function gives $|\widehat{V}\,\mathrm{clip}(\varkappa h) - Vh| \le \max\{|\widehat{V}\varkappa - V|, |\widehat{V} - V|\}$ for $|h| \le 1$. Since $[b]$ contains $\widehat{b}$, both terms are bounded by the maximum in Eq. (S10). Clipping is 1-Lipschitz, so it does not increase the other two errors. The certificate is the hull of Theorem S1 intersected with the interval of radius $\epsilon_{\mathrm{cal}} + |\widehat{V}|w_X/(2\widehat{b})$ around $\widehat{P}_{\mathrm{task}}$. Requiring radius at most ϵ determines N when $\epsilon_{\mathrm{cal}} < \epsilon$; we require $\inf[V] \ge 8\epsilon$. The reported estimate is the midpoint of the final intersection.

S3. SOURCE CHARACTERIZATION AND RESOURCE ACCOUNTING

Write the phase of photon j of group ℓ for label $+1$ as $\theta_{\ell j} = 2\pi j/m + 2e_{\ell j}$; the conjugate state has the opposite phase. The encoder error $e_{\ell j}$ is thus half the optical phase error, and we assume that it is stable with $|e_{\ell j}| \le \pi/8$. For each encoder, prepare a probe photon with equal populations, remove the nominal phase and measure σ_y with the trusted internal analyzer. One outcome has probability $[1+\sin(2e_{\ell j})]/2$. With $\widehat{p}$ its observed frequency in N_s detected probes, Hoeffding's inequality [7] with risk α_s/n per encoder gives

$$\widehat{e} = \tfrac{1}{2}\arcsin(\mathrm{clip}(2\widehat{p}-1, -1/\sqrt{2}, 1/\sqrt{2})), \qquad |\widehat{e} - e| \le w_e \equiv \sqrt{\log(2n/\alpha_s)/N_s}. \tag{S11}$$

The bound uses the derivative of the arcsine on this interval and the Hoeffding radius of $\widehat{p}$. The signal photons keep $p = 1/4$; the probe photons are counted as diagnostic resources, separately from the signal photons. Within one group the overlaps are $\langle\phi_j|\phi_{j+1}\rangle = 1 - p + pe^{i(\theta_{j+1}-\theta_j)}$ and B is their product. Define $B^{(j)} = ipe^{i(\theta_{j+1}-\theta_j)}\prod_{k\ne j}\langle\phi_k|\phi_{k+1}\rangle$, so that $\partial B/\partial\theta_j = B^{(j-1)} - B^{(j)}$. For the certificate, evaluate B and $B^{(j)}$ at the estimated phases $\widehat{\theta}_{\ell j} = 2\pi j/m + 2\widehat{e}_{\ell j}$. Within a group, $|\widehat{\theta}_j - \theta_j| \le 2w_{e,j}$, where $w_{e,j}$ bounds the error in e_j. With $\xi_j = 2(w_{e,j} + w_{e,j+1})$, the error in B is at most

$$\sum_j |B^{(j-1)} - B^{(j)}|\, 2w_{e,j} + \frac{p-p^2}{2}\sum_j \xi_j^2 + \frac{p^2}{2}\left(\sum_j \xi_j\right)^2. \tag{S12}$$

The first term is the first-order Taylor term. The other two bound the second derivative along the line segment between the estimated and the true phases, using the diagonal terms p and the off-diagonal terms p^2. Taking imaginary parts gives an interval for each v_ℓ; intersecting with $[-1, 1]$ and multiplying gives $[V]$.

Resource accounting. Each source independently emits with probability p_{em}. Delivered shots are successful preparations of all required photons, assumed heralded before transmission. Failed attempts are discarded and included in the cost. For N delivered inference shots, N_g delivered shots in each of the $2^{|g|}$ calibration configurations of subset g, and N_s^{tot} detected phase probes, the expected number of attempted photons is

$$\mathcal{C} = \frac{nN}{p_{\mathrm{em}}^n} + \sum_g \frac{2^{|g|}m|g|N_g}{p_{\mathrm{em}}^{m|g|}} + \frac{N_s^{\mathrm{tot}}}{\eta_p} + N_e + \mathcal{C}_{\mathrm{switch}}. \tag{S13}$$

Every inference shot prepares the full source, including the photons that the readout discards, and a calibration shot for a pair prepares two complete groups of m photons. η_p is the trusted probability that an attempted probe photon is detected. An independent test of N_e single-source preparation attempts observes heralded success fraction $\widehat{p}_{\mathrm{em}}$, giving $p_- = \widehat{p}_{\mathrm{em}} - \sqrt{\log(2/\alpha_e)/(2N_e)}$. Replacing p_{em} by $p_- > 0$ in the first two terms gives an upper bound on the expected cost that holds with probability at least $1 - \alpha_e$. In the numbers below, $\mathcal{C}_{\mathrm{switch}} = 100n(n + \sum_g 2^{|g|} + 1)$: one hundred full-source attempts are discarded at each change of configuration.

S4. CERTIFIED COMPARISON AT ONE OPERATING POINT

The operating point is two regular groups of pure states, $m = 8, d = 2$, with $\eta = 0.01$, $p_{\mathrm{em}} = 0.99$ and $\eta_p = 0.9$. Use $N = N_c = 10^9$ delivered shots for inference and for each of the four label configurations of the calibrated pair, $N_s = 10^6$ detected phase probes per encoder, and $N_e = 10^6$ preparation-test attempts. In the two configurations with equal labels the coincidence probability is exactly zero; in the other two it is $p_{\mathrm{c}} = 3\eta^2/8$, so $b = p_{\mathrm{c}}/2$. The risks are $(\alpha_s, \alpha_r, \alpha_w, \alpha_i, \alpha_e) = (0.005, 0.01, 0.01, 0.02, 0.005)$, where α_w is the failure-probability budget for the calibration frequency and interval-width bounds used below. The total is 0.05.

Here $w_e^2 = \log(6400)/N_s$. Let $v = \mathrm{Im}[(0.75 + 0.25e^{i\pi/4})^8]$, $V = v^2$, $V_+ = (v + 44w_e^2)^2$ and $V_{++} = (v + 176w_e^2)^2$. At the true regular phases the gradient vanishes, so expansion there bounds each estimated v_ℓ's displacement by $44w_e^2$. At the estimated phases, Eq. (S12) has first-order bound $88w_e^2$ and remainder $44w_e^2$, giving interval radius $132w_e^2$. Thus V_+ bounds $\widehat{V}$ and V_{++} bounds the upper endpoint of $[V]$. Let w_{c} bound the frequency error in each nonzero-coincidence configuration, and w_0, w_1 bound the interval radii in the zero- and nonzero-coincidence configurations, respectively:

$$w_{\mathrm{c}} = \sqrt{2p_{\mathrm{c}} \log(400)/N_c} + 2\log(400)/(3N_c), \tag{S14}$$

$$w_0 = 7\log(3200)/[3(N_c - 1)], \tag{S15}$$

$$w_1 = \sqrt{2(p_{\mathrm{c}} + w_{\mathrm{c}}) \log(3200)/(N_c - 1)} + w_0, \qquad b_- = b - w_{\mathrm{c}}/2. \tag{S16}$$

With probability at least $1 - \alpha_w$, the estimated coefficient obeys $\widehat{b} \geq b_-$, the errors of $\widehat{c}, \widehat{b}$ are at most $w_{\mathrm{c}}/2$, and both reported interval half-widths are at most $(w_0 + w_1)/2$. Since $\widehat{c} = -\widehat{b}$ at this point, we may take

$$\epsilon_{\mathrm{cal}}^{+} = V_+(w_0 + w_1)/(2b_-) + (V_{++} - V_+)/2, \tag{S17}$$

$$\sigma_+^2 = p_{\mathrm{c}} + w_{\mathrm{c}} + w_1 + (b + w_{\mathrm{c}}/2)^2, \tag{S18}$$

$$\epsilon_{\mathrm{tot}} = \epsilon_{\mathrm{cal}}^{+} + \frac{V_+}{2b_-}\left[\sqrt{2\sigma_+^2 \log(100)/N} + 4\log(100)/(3N)\right]. \tag{S19}$$

Directed interval arithmetic proves $\epsilon_{\mathrm{tot}} < 0.015971554 < 0.02$ and $b_- > 0$. The expected number of attempted photons is at most

$$\mathcal{C}_{\mathrm{HOM}} = \frac{16(N + 4N_c)}{0.99^{16}} + \frac{16N_s}{0.9} + 10^6 + 33600 = 93975320691.362768\ldots. \tag{S20}$$

For unknown p_{em}, the preparation test of Sec. S3 gives an upper confidence bound on the expected cost.

For the separate-group receiver, take the test points $\pm h_0$ with $h_0 = 0.08/V = 0.203726902496\ldots$. The two task probabilities differ by $0.08 > 2(0.02)$. Theorem 4 and Proposition 2 of the main text give the lower bound

$$\mathcal{C}_{\mathrm{sep}} = \frac{16(0.9\log 19)}{0.99^{16}\gamma(h_0)C_{\mathrm{an}}(0.01)^2} = 42736556202482.500137\ldots. \tag{S21}$$

The lower bound allows exact knowledge of the source states, readout coefficients and lost-input identities, together with free calibration. It holds for all products of external POVMs on the groups, settings chosen from earlier shots and stopping times of finite mean. The HOM cost in Eq. (S20) includes calibration. Outward-rounded interval arithmetic implemented with `mpmath.iv` (mpmath version 1.3.0), at seventy decimal digits, encloses the ratio of the lower and upper bounds:

$$454.7636 < \mathcal{C}_{\mathrm{sep}}/\mathcal{C}_{\mathrm{HOM}} < 454.7637. \tag{S22}$$

An independent 90-digit calculation using the complex-power expression for v falls within the corresponding interval enclosures. The event-based bound at the end of Appendix E of the main text, which reveals all labels when every group keeps at least three survivors, gives $\mathcal{C}_{\mathrm{sep}}^{\mathrm{ev}} = 203348764369.136695\ldots$ and $\mathcal{C}_{\mathrm{sep}}^{\mathrm{ev}}/\mathcal{C}_{\mathrm{HOM}} = 2.163852837884\ldots$. Both comparisons are certified by directed interval arithmetic.

TABLE S1. Ratio of the separate-group lower bound to the HOM protocol cost bound, $\mathcal{C}_{\rm sep}/\mathcal{C}_{\rm HOM}$, at other transmissions (floating point). Both columns use the same HOM cost bound and the indicated ceiling for the separate-group lower bound. N is rounded for display.

η	N	Ratio, analytic ceiling	Ratio, numerical ceiling
0.005	2.580×10^9	13100	15676
0.01	6.450×10^8	705.02	991.24
0.02	1.612×10^8	34.100	63.392
0.03	7.166×10^7	5.4473	12.806
0.04	4.031×10^7	1.4410	4.1416
0.05	2.580×10^7	0.50648	1.7330

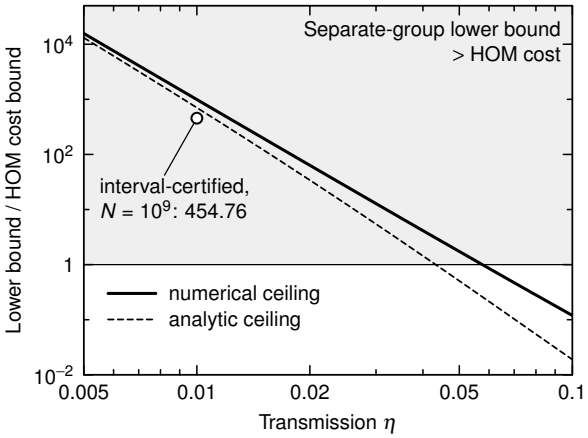


FIG. S1. Ratio of the separate-group lower bound to the HOM protocol cost bound, $\mathcal{C}_{\rm sep}/\mathcal{C}_{\rm HOM}$, without internal-state resolution. Curves: smallest N for accuracy 0.02 at each η (40-digit floating point). Thick line: numerical ceiling [main Eq. (15)]; thin dashed: analytic ceiling [Eq. (D6)]. The ratios cross one at $\eta = 0.0576$ and 0.0433, respectively. Shading marks ratios above one. Circle: interval-certified $\eta = 0.01$, $N = 10^9$ point, ratio 454.76 (analytic ceiling); its larger N places it below the dashed curve.

Other transmissions

Table S1 applies the same formulas at other transmissions. In each row $N = N_c$ is the smallest integer for which $\epsilon_{\rm tot} \leq 0.02$; the numbers of probes are unchanged. The calculations use 40-digit floating-point arithmetic. The analytic-ceiling column uses Eq. (D6) of the main text and the numerical column Eq. (15), with the double-precision sums $F_{\rm Q}^{(k)}$ for $k \geq 4$. The interval-certified comparison at $\eta = 0.01$, $N = 10^9$ is given in Sec. S4. Figure S1 shows both ratios. They cross one near $\eta = 0.043255$ and 0.057596, respectively. At $\eta = 0.01$ and $N = 10^9$, a floating-point evaluation using the numerical ceiling gives a ratio of about 639.39. The numerical data of this section, including those of Table S1 and Fig. S1, are openly available [8].

S5. PHASE DRIFT AND SOURCE MIXING

A stable relative internal phase φ between two groups changes the coefficient of their HOM reading to $(3/16)\eta^2 \cos\varphi$, which vanishes at $\varphi = \pi/2$. Calibration estimates this modified coefficient when the same phase is present in calibration and inference. If the phase is present in inference and absent in calibration, the estimate of $P_{\rm task}$ is biased by $|hV(1 - \cos\varphi)|/2$.

Independent Gaussian jitter of the source phases, with standard deviation σ_θ, multiplies the off-diagonal elements of each single-photon density matrix by $e^{-\sigma_\theta^2/2}$, while the probability of the monitored σ_y outcome can remain $1/2$. For $m = 8, d = 2, h = 0.75, \sigma_\theta = 0.3$, the task probability for the mixed states differs from that for the nominal pure states by approximately 0.0363023. The discrepancy arises because the phase-probe model of Sec. S3 infers V under a pure-state assumption, even when the readout coefficients are accurately calibrated.

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