Supplementary Material: Reduced-Order Quadratic Representation of Higher-Order Nonlinear Dynamical Systems for Phononic Frequency Comb Generation

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Supplementary Section S1: Expressions for the nonlinear terms in terms of x_1, x_2 and x_1x_2

The equations of motion corresponding to two coupled modes can be written as

$$\ddot{x}_1 + 2\gamma_1\dot{x}_1 + \omega_1^2 x_1 + N.I. = Fcos\,(\omega_D t) \tag{S1-1}$$

$$\ddot{x}_2 + 2\gamma_2\dot{x}_2 + \omega_2^2 x_2 + N.I. = 0 \tag{S1-2}$$

Here,

$$\begin{aligned}\mathbf{N.I.} = {} & \alpha_{2,1}x_1^2 + \alpha_{2,2}x_1x_2 + \alpha_{2,3}x_2^2 + \\ & \alpha_{3,1}x_1^3 + \alpha_{3,2}x_1^2x_2 + \alpha_{3,3}x_1x_2^2 + \alpha_{3,4}x_2^3 + \\ & \alpha_{4,1}x_1^4 + \alpha_{4,2}x_1^3x_2 + \alpha_{4,3}x_1^2x_2^2 \\ & + \alpha_{4,4}x_1x_2^3 + \alpha_{4,5}x_2^4 + \\ & \alpha_{5,1}x_1^5 + \alpha_{5,2}x_1^4x_2 + \alpha_{5,3}x_1^3x_2^2 \\ & + \alpha_{5,4}x_1^2x_2^3 + \alpha_{5,5}x_1x_2^4 + \alpha_{5,6}x_2^5 + \\ & \alpha_{6,1}x_1^6 + \alpha_{6,2}x_1^5x_2 + \alpha_{6,3}x_1^4x_2^2 \\ & + \alpha_{6,4}x_1^3x_2^3 + \alpha_{6,5}x_1^2x_2^4 \\ & + \alpha_{6,6}x_1x_2^5 + \alpha_{6,7}x_2^6 + \\ & \dots\end{aligned}$$

The modes are written as

$$x_1 = \frac{1}{2}\left(u_1 e^{i\omega_D t} + u_1^* e^{-i\omega_D t}\right) \tag{S1-3}$$

$$x_2 = \frac{1}{2}\left(u_2 e^{i\frac{\omega_D}{2}t} + u_2^* e^{-i\frac{\omega_D}{2}t}\right) \tag{S1-4}$$

The derivatives $\dot{\mathbf{x}}_1$, $\dot{\mathbf{x}}_2$, $\ddot{\mathbf{x}}_1$ and $\ddot{\mathbf{x}}_2$ are obtained as

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$$\dot{x}_1 = \frac{1}{2}\begin{pmatrix} i\omega_D u_1 e^{i\omega_D t} + \dot{u}_1 e^{i\omega_D t} - i\omega_D u_1 e^{-i\omega_D t} \\ +\dot{u}_1^* e^{-i\omega_D t} \end{pmatrix} \tag{S1-5}$$

$$\dot{x}_2 = \frac{1}{2}\begin{pmatrix} i\frac{\omega_D}{2} u_2 e^{i\frac{\omega_D}{2}t} + \dot{u}_2 e^{i\frac{\omega_D}{2}t} \\ -i\frac{\omega_D}{2} u_2 e^{-i\frac{\omega_D}{2}t} + \dot{u}_2^* e^{-i\frac{\omega_D}{2}t} \end{pmatrix} \tag{S1-6}$$

$$\ddot{x}_1 = \frac{1}{2}\begin{pmatrix} \ddot{u}_1 e^{i\omega_D t} + 2i\omega_D \dot{u}_1 e^{i\omega_D t} - \omega_D^2 u_1 e^{i\omega_D t} \\ +\ddot{u}_1^* e^{-i\omega_D t} - 2i\omega_D \dot{u}_1^* e^{-i\omega_D t} \\ -\omega_D^2 u_1^* e^{-i\omega_D t} \end{pmatrix} \tag{S1-7}$$

$$\ddot{x}_2 = \frac{1}{2}\begin{pmatrix} \ddot{u}_2 e^{i\frac{\omega_D}{2}t} + i\omega_D \dot{u}_2 e^{i\frac{\omega_D}{2}t} \\ -\frac{\omega_D^2}{4} u_2 e^{i\frac{\omega_D}{2}t} + \ddot{u}_2^* e^{-i\frac{\omega_D}{2}t} \\ -i\omega_D \dot{u}_2^* e^{-i\frac{\omega_D}{2}t} - \frac{\omega_D^2}{4} u_2^* e^{-i\frac{\omega_D}{2}t} \end{pmatrix} \tag{S1-8}$$

We collect only the terms containing $e^{i\omega_D t}$ and $e^{i\frac{\omega_D}{2}t}$.

$$\dot{x}_1 = \frac{1}{2}\left(i\omega_D u_1 e^{i\omega_D t} + \dot{u}_1 e^{i\omega_D t}\right) \tag{S1-9}$$

$$\dot{x}_2 = \frac{1}{2}\left(i\frac{\omega_D}{2} u_2 e^{i\frac{\omega_D}{2}t} + \dot{u}_2 e^{i\frac{\omega_D}{2}t}\right) \tag{S1-10}$$

$$\ddot{x}_1 = \frac{1}{2}\left(\ddot{u}_1 e^{i\omega_D t} + 2i\omega_D \dot{u}_1 e^{i\omega_D t} - \omega_D^2 u_1 e^{i\omega_D t}\right) \tag{S1-11}$$

$$\ddot{x}_2 = \frac{1}{2}\begin{pmatrix} \ddot{u}_2 e^{i\frac{\omega_D}{2}t} + i\omega_D \dot{u}_2 e^{i\frac{\omega_D}{2}t} \\ -\frac{\omega_D^2}{4} u_2 e^{i\frac{\omega_D}{2}t} \end{pmatrix} \tag{S1-12}$$

Similarly, the coupling terms can also be estimated as follows.

2nd Order Nonlinearity

$$x_1^2 = \frac{1}{4}\left(u_1^2 e^{2i\omega_D t} + u_1^{*2} e^{-2i\omega_D t} + 2|u_1|^2\right) \sim 0 \tag{S1-13}$$

$$\begin{aligned} x_1 x_2 &= \frac{1}{4}\begin{pmatrix} u_1^* u_2^* e^{-3i\frac{\omega_D}{2}t} + u_1^* u_2 e^{-i\frac{\omega_D}{2}t} \\ +u_1 u_2^* e^{i\frac{\omega_D}{2}t} + u_1 u_2 e^{3i\frac{\omega_D}{2}t} \end{pmatrix} \\ &\sim \frac{1}{4} u_1 u_2^* e^{i\frac{\omega_D}{2}t} \end{aligned} \tag{S1-14}$$

$$\begin{aligned} x_2^2 &= \frac{1}{4}\left(u_2^2 e^{i\omega_D t} + u_2^{*2} e^{-i\omega_D t} + 2|u_2|^2\right) \\ &\sim \frac{1}{4} u_2^2 e^{i\omega_D t} \end{aligned} \tag{S1-15}$$

3rd Order Nonlinearity

$$\begin{aligned} x_1^3 &= \frac{1}{8}\begin{pmatrix} u_1^3 e^{3i\omega_D t} + 3|u_1|^2 u_1 e^{i\omega_D t} \\ +3|u_1|^2 u_1^* e^{-i\omega_D t} + u_1^{*3} e^{-3i\omega_D t} \end{pmatrix} \\ &\sim \frac{3}{8}|u_1|^2 u_1 e^{i\omega_D t} \end{aligned} \tag{S1-16}$$

$$x_1^2 x_2 = \frac{1}{8}\begin{pmatrix} u_1^2 u_2 e^{i\frac{5\omega_D}{2}t} + u_1^{*2} u_2 e^{-i\frac{3\omega_D}{2}t} \\ +2\left|u_1\right|^2 u_2 e^{i\frac{\omega_D}{2}t} + u_1^2 u_2^* e^{i\frac{3\omega_D}{2}t} \\ +u_1^{*2} u_2^* e^{-i\frac{5\omega_D}{2}t} + 2\left|u_1\right|^2 u_2^* e^{-i\frac{\omega_D}{2}t} \end{pmatrix} \tag{S1-17}$$
$$\sim \frac{1}{4}\left|u_1\right|^2 u_2 e^{i\frac{\omega_D}{2}t}$$

$$x_1 x_2^2 = \frac{1}{8}\begin{pmatrix} u_1 u_2^2 e^{2i\omega_D t} + u_1^* u_2^2 + u_1 u_2^{*2} \\ +u_1^* u_2^{*2} e^{-2i\omega_D t} + 2\left|u_2\right|^2 u_1 e^{i\omega_D t} \\ +2\left|u_2\right|^2 u_1^* e^{-i\omega_D t} \end{pmatrix} \tag{S1-18}$$
$$\sim \frac{1}{4}\left|u_2\right|^2 u_1 e^{i\omega_D t}$$

$$x_2^3 = \frac{1}{8}\begin{pmatrix} u_2^3 e^{i\frac{3\omega_D}{2}t} + 3\left|u_2\right|^2 u_2 e^{i\frac{\omega_D}{2}t} \\ +3\left|u_2\right|^2 u_2^* e^{-i\frac{\omega_D}{2}t} + u_2^{*3} e^{-i\frac{3\omega_D}{2}t} \end{pmatrix} \tag{S1-19}$$
$$\sim \frac{3}{8}\left|u_2\right|^2 u_2 e^{i\frac{\omega_D}{2}t}$$

4th Order Nonlinearity

$$x_1^4 = \frac{1}{16}\begin{pmatrix} u_1^4 e^{4i\omega_D t} + 4\left|u_1\right|^2 u_1^2 e^{2i\omega_D t} + 6\left|u_1\right|^4 \\ +4\left|u_1\right|^2 u_1^{*2} e^{-2i\omega_D t} + u_1^{*4} e^{-4i\omega_D t} \end{pmatrix} \tag{S1-20}$$
$$\sim 0$$

$$x_1^3 x_2 = \frac{1}{16}\begin{pmatrix} u_1^3 e^{3i\omega_D t} + 3\left|u_1\right|^2 u_1 e^{i\omega_D t} \\ +3\left|u_1\right|^2 u_1^* e^{-i\omega_D t} + u_1^{*3} e^{-3i\omega_D t} \end{pmatrix} \left(u_2 e^{i\frac{\omega_D}{2}t} + u_2^* e^{-i\frac{\omega_D}{2}t}\right) \tag{S1-21}$$
$$\sim \frac{3}{16}\left|u_1\right|^2 \left(u_1 u_2^*\right) e^{i\frac{\omega_D}{2}t}$$

$$x_1^2 x_2^2 = \frac{1}{16}\begin{pmatrix} u_1^* u_2^* e^{-3i\frac{\omega_D}{2}t} + u_1^* u_2 e^{-i\frac{\omega_D}{2}t} \\ +u_1 u_2^* e^{i\frac{\omega_D}{2}t} + u_1 u_2 e^{3i\frac{\omega_D}{2}t} \end{pmatrix} \begin{pmatrix} u_1^* u_2^* e^{-3i\frac{\omega_D}{2}t} + u_1^* u_2 e^{-i\frac{\omega_D}{2}t} \\ +u_1 u_2^* e^{i\frac{\omega_D}{2}t} + u_1 u_2 e^{3i\frac{\omega_D}{2}t} \end{pmatrix} \tag{S1-22}$$
$$\sim \frac{1}{8}\left|u_1\right|^2 u_2^2 e^{i\omega_D t} + \frac{1}{16}\left(u_1^2 u_2^{*2}\right) e^{i\omega_D t}$$

$$x_1 x_2^3 = \frac{1}{16}\left(u_1 e^{i\omega_D t} + u_1^* e^{-i\omega_D t}\right) \begin{pmatrix} u_2^3 e^{i\frac{3\omega_D}{2}t} + 3\left|u_2\right|^2 u_2 e^{i\frac{\omega_D}{2}t} \\ +3\left|u_2\right|^2 u_2^* e^{-i\frac{\omega_D}{2}t} + u_2^{*3} e^{-i\frac{3\omega_D}{2}t} \end{pmatrix} \tag{S1-23}$$
$$\sim \frac{1}{16}\left(u_1^* u_2^3\right) e^{i\frac{\omega_D}{2}t} + \frac{3}{16}\left|u_2\right|^2 \left(u_1 u_2^*\right) e^{i\frac{\omega_D}{2}t}$$

$$x_2^4 = \frac{1}{16}\begin{pmatrix} u_2^4 e^{2i\omega_D t} + 4\left|u_2\right|^2 u_2^2 e^{i\omega_D t} + 6\left|u_2\right|^4 \\ +4\left|u_2\right|^2 u_2^{*2} e^{-i\omega_D t} + u_1^{*4} e^{-2i\omega_D t} \end{pmatrix} \tag{S1-24}$$
$$\sim \frac{1}{4}\left|u_2\right|^2 u_2^2 e^{i\omega_D t}$$

5th Order Nonlinearity

$$x_1^5 = \frac{1}{32}\begin{pmatrix} u_1^5 e^{5i\omega_D t} + 5u_1^4 u_1^* e^{3i\omega_D t} \\ +10u_1^3 u_1^{*2} e^{i\omega_D t} + 10u_1^2 u_1^{*3} e^{-i\omega_D t} \\ + u_1^{*5} e^{-5\omega_D t} \end{pmatrix} \tag{S1-25}$$
$$\sim \frac{5}{16}\left|u_1\right|^4 u_1 e^{i\omega_D t}$$

$$x_1^4 x_2 = \frac{1}{32}\begin{pmatrix} u_1^4 e^{4i\omega_D t} + 4\left|u_1\right|^2 u_1^2 e^{2i\omega_D t} + 6\left|u_1\right|^4 \\ +4\left|u_1\right|^2 u_1^{*2} e^{-2i\omega_D t} + u_1^{*4} e^{-4i\omega_D t} \end{pmatrix} \left(u_2 e^{i\frac{\omega_D}{2}t} + u_2^* e^{-i\frac{\omega_D}{2}t}\right) \tag{S1-26}$$
$$\sim \frac{3}{16}\left|u_1\right|^4 u_2 e^{i\frac{\omega_D}{2}t}$$

$$x_1^3 x_2^2 = \frac{1}{32}\begin{pmatrix} u_1^3 e^{3i\omega_D t} + 3\left|u_1\right|^2 u_1 e^{i\omega_D t} \\ +3\left|u_1\right|^2 u_1^* e^{-i\omega_D t} + u_1^{*3} e^{-3i\omega_D t} \end{pmatrix} \left(u_2^2 e^{i\omega_D t} + u_2^{*2} e^{-i\omega_D t} + 2\left|u_2\right|^2\right) \tag{S1-27}$$
$$\sim \frac{3}{16}\left|u_1\right|^2\left|u_2\right|^2 u_1 e^{i\omega_D t}$$

$$x_1^2 x_2^3 = \frac{1}{32}\begin{pmatrix} u_2^3 e^{i\frac{3\omega_D}{2}t} + 3\left|u_2\right|^2 u_2 e^{i\frac{\omega_D}{2}t} \\ +3\left|u_2\right|^2 u_2^* e^{-i\frac{\omega_D}{2}t} + u_2^{*3} e^{-i\frac{3\omega_D}{2}t} \end{pmatrix} \left(u_1^2 e^{2i\omega_D t} + u_1^{*2} e^{-2i\omega_D t} + 2\left|u_1\right|^2\right) \tag{S1-28}$$
$$\sim \frac{1}{32}u_1^2 u_2^{*3} e^{i\frac{\omega_D}{2}t} + \frac{3}{16}\left|u_1\right|^2\left|u_2\right|^2 u_2 e^{i\frac{\omega_D}{2}t}$$

$$x_1 x_2^4 = \frac{1}{32}\begin{pmatrix} u_2^4 e^{2i\omega_D t} + 4\left|u_2\right|^2 u_2^2 e^{i\omega_D t} + 6\left|u_2\right|^4 \\ +4\left|u_2\right|^2 u_2^{*2} e^{-i\omega_D t} + u_1^{*4} e^{-2i\omega_D t} \end{pmatrix} \left(u_1 e^{i\omega_D t} + u_1^* e^{-i\omega_D t}\right) \tag{S1-29}$$
$$\sim \frac{1}{32}u_2^4 u_1^* e^{i\omega_D t} + \frac{3}{16}\left|u_2\right|^4 u_1 e^{i\omega_D t}$$

$$x_2^5 = \frac{1}{32}\begin{pmatrix} u_2^5 e^{\frac{5i\omega_D}{2}t} + 5u_2^4 u_2^* e^{\frac{3i\omega_D}{2}t} \\ +10u_2^3 u_2^{*2} e^{\frac{i\omega_D}{2}t} \\ +10u_2^2 u_2^{*3} e^{-\frac{i\omega_D}{2}t} + u_2^{*5} e^{-\frac{5\omega_D}{2}t} \end{pmatrix} \tag{S1-30}$$
$$\sim \frac{5}{16}\left|u_2\right|^4 u_2 e^{\frac{i\omega_D}{2}t}$$

6th Order Nonlinearity

$$x_1^6 = \frac{1}{64}\begin{pmatrix} u_1^6 e^{6i\omega_D t} + 6u_1^5 u_1^* e^{4i\omega_D t} \\ +15u_1^4 u_1^{*2} e^{2i\omega_D t} + 20u_1^3 u_1^{*3} \\ +15u_1^2 u_1^{*4} e^{-2i\omega_D t} + 6u_1 u_1^{*5} e^{-4i\omega_D t} \\ + u_1^{*6} e^{-6\omega_D t} \end{pmatrix} \tag{S1-31}$$
$$\sim 0$$

$$
\begin{aligned}
x_1^5 x_2 &= \frac{1}{64}\begin{pmatrix} u_1^5 e^{5i\omega_D t} + 5u_1^4 u_1^* e^{3i\omega_D t} \\ +10u_1^3 u_1^{*2} e^{i\omega_D t} + 10u_1^2 u_1^{*3} e^{-i\omega_D t} \\ + u_1^{*5} e^{-5\omega_D t} \end{pmatrix} \\
&\quad \left(u_2 e^{i\frac{\omega_D}{2}t} + u_2^* e^{-i\frac{\omega_D}{2}t}\right) \\
&\sim \frac{5}{32}\left|u_1\right|^4 \left(u_1 u_2^*\right) e^{i\frac{\omega_D}{2}t}
\end{aligned}
\tag{S1-32}
$$

$$
\begin{aligned}
x_1^4 x_2^2 &= \frac{1}{64}\begin{pmatrix} u_1^4 e^{4i\omega_D t} + 4\left|u_1\right|^2 u_1^2 e^{2i\omega_D t} + 6\left|u_1\right|^4 \\ +4\left|u_1\right|^2 u_1^{*2} e^{-2i\omega_D t} + u_1^{*4} e^{-4i\omega_D t} \end{pmatrix} \\
&\quad \left(u_2^2 e^{i\omega_D t} + u_2^{*2} e^{-i\omega_D t} + 2\left|u_2\right|^2\right) \\
&\sim \frac{1}{64}\left(6\left|u_1\right|^4 u_2^2 e^{i\omega_D t} + 4\left|u_1\right|^2 u_1^2 u_2^{*2} e^{i\omega_D t}\right)
\end{aligned}
\tag{S1-33}
$$

$$
\begin{aligned}
x_1^3 x_2^3 &= \frac{1}{64}\begin{pmatrix} u_1^3 e^{3i\omega_D t} + 3\left|u_1\right|^2 u_1 e^{i\omega_D t} \\ +3\left|u_1\right|^2 u_1^* e^{-i\omega_D t} + u_1^{*3} e^{-3i\omega_D t} \end{pmatrix} \\
&\quad \begin{pmatrix} u_2^3 e^{i\frac{3\omega_D}{2}t} + 3\left|u_2\right|^2 u_2 e^{i\frac{\omega_D}{2}t} \\ +3\left|u_2\right|^2 u_2^* e^{-i\frac{\omega_D}{2}t} + u_2^{*3} e^{-i\frac{3\omega_D}{2}t} \end{pmatrix} \\
&\sim \frac{1}{64}\left(3\left|u_1\right|^2 u_1^* u_2^3 e^{i\frac{\omega_D}{2}t} + 9\left|u_1\right|^2\left|u_2\right|^2 \left(u_1 u_2^*\right) e^{i\frac{\omega_D}{2}t}\right)
\end{aligned}
\tag{S1-34}
$$

$$
\begin{aligned}
x_1^2 x_2^4 &= \frac{1}{64}\begin{pmatrix} u_2^4 e^{2i\omega_D t} + 4\left|u_2\right|^2 u_2^2 e^{i\omega_D t} + 6\left|u_2\right|^4 \\ +4\left|u_2\right|^2 u_2^{*2} e^{-i\omega_D t} + u_1^{*4} e^{-2i\omega_D t} \end{pmatrix} \\
&\quad \left(u_1^2 e^{2i\omega_D t} + u_1^{*2} e^{-2i\omega_D t} + 2\left|u_1\right|^2\right) \\
&\sim \frac{1}{64}\left(4\left|u_2\right|^2 u_2^{*2} u_1^2 + 8\left|u_1\right|^2\left|u_2\right|^2 u_2^2 e^{i\omega_D t}\right)
\end{aligned}
\tag{S1-35}
$$

$$
\begin{aligned}
x_1 x_2^5 &= \frac{1}{64}\begin{pmatrix} u_2^5 e^{\frac{5i\omega_D}{2}t} + 5u_2^4 u_2^* e^{\frac{3i\omega_D}{2}t} \\ +10u_2^3 u_2^{*2} e^{\frac{i\omega_D}{2}t} \\ +10u_2^2 u_2^{*3} e^{-\frac{i\omega_D}{2}t} + u_2^{*5} e^{-\frac{5\omega_D}{2}t} \end{pmatrix} \\
&\quad \left(u_1 e^{i\omega_D t} + u_1^* e^{-i\omega_D t}\right) \\
&\sim \frac{5}{32}\left|u_2\right|^4 \left(u_1 u_2^*\right) e^{\frac{i\omega_D}{2}t}
\end{aligned}
\tag{S1-36}
$$

$$
\begin{aligned}
x_2^6 &= \frac{1}{64}\begin{pmatrix} u_2^6 e^{3i\omega_D t} + 6u_2^5 u_2^* e^{2i\omega_D t} \\ +15u_2^4 u_2^{*2} e^{i\omega_D t} + 20u_2^3 u_2^{*3} \\ +15u_2^2 u_2^{*4} e^{-i\omega_D t} + 6u_2 u_2^{*5} e^{-2i\omega_D t} \\ + u_2^{*6} e^{-3\omega_D t} \end{pmatrix} \\
&\sim \frac{15}{64}\left|u_2\right|^4 u_2^2 e^{i\omega_D t}
\end{aligned}
\tag{S1-37}
$$

Supplementary Section S2: Frequency Parameters of the Existence Boundary of Phononic Frequency Combs

S2-1. Center frequency of existence band of phononic frequency combs

The frequency matching condition for phononic frequency combs is governed by eq. (S2.1).

$$
\left(1 + \Delta_1^2 + 2\gamma + 2\Delta_1\Delta_2\right) \leq 0 \tag{S2.1}
$$

Since $\Delta_2 = \frac{\Delta_1}{2} + \overline{\kappa}$, we have the following expression.

$$\left(1+\Delta_1^2+2\gamma+2\Delta_1\left(\frac{\overline{\Delta}_1}{2}+\overline{\kappa}\right)\right)\leq 0 \tag{S2.2}$$

$$\left(2\Delta_1^2+2\Delta_1\overline{\kappa}+(2\gamma+1)\right)\leq 0 \tag{S2.3}$$

$$\frac{-\overline{\kappa}-\sqrt{\overline{\kappa}^2-2(2\gamma+1)}}{2}\leq\Delta_1\leq\frac{-\overline{\kappa}+\sqrt{\overline{\kappa}^2-2(2\gamma+1)}}{2} \tag{S2.4}$$

The center frequency of existence band of phononic frequency combs is calculated as follows.

$$\Delta_{1,center}=-\frac{\overline{\kappa}}{2} \tag{S2.5}$$

$$\Rightarrow\frac{\omega_{D,center}-\omega_1}{\gamma_1}=-\frac{\omega_1-2\omega_2}{4\gamma_1}$$

$$\Rightarrow\omega_{D,center}=\frac{3\omega_1+2\omega_2}{4}$$

S2-2. Range of existence band of phononic frequency combs

The range of existence band of phononic frequency combs is calculated as follows.

$$\Delta_{1,max}-\Delta_{1,min}=\sqrt{\overline{\kappa}^2-2(2\gamma+1)} \tag{S2.6}$$

$$\Rightarrow\left(\omega_{D,max}-\omega_{D,min}\right)^2=\gamma_1^2\left(\frac{(\omega_1-2\omega_2)^2}{4\gamma_1^2}-\frac{2\left(2\gamma_2+\gamma_1\right)}{\gamma_1}\right)$$

$$\Rightarrow R^2=\left(\frac{(\omega_1-2\omega_2)^2-8\left(2\gamma_1\gamma_2+\gamma_1^2\right)}{4}\right)$$

$$\Rightarrow R^2=\left(\frac{\omega_1^2-4\omega_1\omega_2+4\omega_2^2-8\left(\frac{2\omega_1\omega_2}{Q_1Q_2}+\frac{\omega_1^2}{4Q_1^2}\right)}{4}\right)$$

$$\Rightarrow R^2=\left(\omega_2^2-\left(1+\frac{4}{Q_1Q_2}\right)\omega_1\omega_2+\frac{\omega_1^2}{4}\left(1-\frac{2}{Q_1^2}\right)\right)$$

$$\Rightarrow R^2=\left(\omega_2^2-\omega_1\omega_2+\frac{\omega_1^2}{4}\right)-\left(\frac{4\omega_1\omega_2}{Q_1Q_2}\right)-\left(\frac{\omega_1^2}{2Q_1^2}\right)$$

For $Q_1^2\gg 2$, $\frac{\omega_1^2}{4}\gg\frac{\omega_1^2}{2Q_1^2}$. Hence, $\frac{\omega_1^2}{2Q_1^2}$ can be neglected from eq. (S2.6).

$$R^2=\left(\omega_2^2-\omega_1\omega_2+\frac{\omega_1^2}{4}\right)-\left(\frac{4\omega_1\omega_2}{Q_1Q_2}\right) \tag{S2.7}$$

$$\Rightarrow R^2=\left(\omega_2^2-\omega_1\omega_2\left(1+\frac{4}{Q_1Q_2}\right)+\frac{\omega_1^2}{4}\right)$$

The quadratic on ω_2 in the RHS of eq. (S2.7) can be written as

$$R^2 = \left(\omega_2 - \omega_1 \left[\frac{\left(1 + \frac{4}{Q_1 Q_2}\right) + \sqrt{\left(1 + \frac{4}{Q_1 Q_2}\right)^2 - 1}}{2} \right] \right)^2$$
$$R^2 = \left(\omega_2 - \frac{\omega_1}{2} \left(\left(1 + \frac{4}{Q_1 Q_2}\right) + \sqrt{\frac{8}{Q_1 Q_2} + \frac{16}{Q_1^2 Q_2^2}} \right) \right)^2 \tag{S2.8}$$

For $Q_1 Q_2 \gg 2$, $\frac{8}{Q_1 Q_2} \gg \frac{16}{Q_1^2 Q_2^2}$. Hence, the term $\frac{16}{Q_1^2 Q_2^2}$ can be neglected in eq. (S2.8).

$$R^2 = \left(\omega_2 - \frac{\omega_1}{2} \left(1 + \frac{4}{Q_1 Q_2} + \sqrt{\frac{8}{Q_1 Q_2}} \right) \right)^2 \tag{S2.9}$$

For $Q_1 Q_2 \gg 2$, $\sqrt{\frac{8}{Q_1 Q_2}} \gg \frac{4}{Q_1 Q_2}$. Hence, the term $\frac{4}{Q_1 Q_2}$ can be neglected in eq. (S2.9).

$$R^2 = \left(\omega_2 - \frac{\omega_1}{2} \left(1 + \sqrt{\frac{8}{Q_1 Q_2}} \right) \right)^2 \tag{S2.10}$$

An expression for range can hence be obtained as follows.

$$R = \left| \omega_2 - \frac{\omega_1}{2} - \frac{\sqrt{2}\omega_1}{\sqrt{Q_1 Q_2}} \right| \tag{S2.11}$$

S2-3. Critical detuning for the generation of phononic frequency combs

For $\omega_2 - \frac{\omega_1}{2} < 0$, the right bound of regime of phononic frequency combs is closer to ω_1, i.e., $\omega_{D,center} + \frac{R}{2}$.

$$\begin{aligned} \omega_{D,edge} &= \omega_{D,center} + \frac{R}{2} = \frac{3\omega_1 + 2\omega_2}{4} - \frac{\omega_2}{2} + \frac{\omega_1}{4} + \frac{\omega_1}{\sqrt{2Q_1 Q_2}} \\ \Rightarrow \omega_{D,edge} &= \omega_{D,center} + \frac{R}{2} = \omega_1 + \frac{\omega_1}{\sqrt{2Q_1 Q_2}} \end{aligned} \tag{S2.12}$$

For $\omega_2 - \frac{\omega_1}{2} > 0$, the left bound of regime of phononic frequency combs is closer to ω_1, i.e., $\omega_{D,center} - \frac{R}{2}$.

$$\begin{aligned} \omega_{D,edge} &= \omega_{D,center} - \frac{R}{2} = \frac{3\omega_1 + 2\omega_2}{4} - \frac{\omega_2}{2} + \frac{\omega_1}{4} + \frac{\omega_1}{\sqrt{2Q_1 Q_2}} \\ \Rightarrow \omega_{D,edge} &= \omega_{D,center} - \frac{R}{2} = \omega_1 + \frac{\omega_1}{\sqrt{2Q_1 Q_2}} \end{aligned} \tag{S2.13}$$

Hence, the minimum detuning from ω_1 that is required for exciting phononic combs can be obtained as

$$\delta = \left| \omega_{D,edge} - \omega_1 \right| = \frac{\omega_1}{\sqrt{2Q_1 Q_2}} \tag{S2.14}$$

S2-4. Critical drive frequency ω_D^* at which $|\psi_2|$ reaches its minimum value

$$|\psi_2|^2 \geq -\frac{\gamma_{21}\left(1 + \Delta_1^2\right)\left(1 + \Delta_1^2 + 4\gamma_{21}\left(1 + \gamma_{21}\right)\right)}{4\left(1 + \gamma_{21}\right)^2\left(1 + \Delta_1^2 + 2\gamma_{21} + 2\Delta_1\Delta_2\right)} \tag{S2-15}$$

$$|\psi_2|^2 \geq -\frac{\gamma_{21}}{4\left(1 + \gamma_{21}\right)^2} \frac{\left(\Delta_1^4 + \Delta_1^2\left(4\gamma_{21}^2 + 4\gamma_{21} + 2\right) + \left(4\gamma_{21}^2 + 4\gamma_{21} + 1\right)\right)}{\left(\Delta_1^2(2) + \Delta_1(2\kappa) + (2\gamma_{21} + 1)\right)} \tag{S2-16}$$

$$\frac{d\,|\psi_2|^2}{d\Delta_1} = 0$$

$$\left[\Delta_1^2(2) + \Delta_1(2\kappa) + (2\gamma_{21} + 1)\right]\left[\Delta_1^3(4) + \Delta_1\left(8\gamma_{21}^2 + 8\gamma_{21} + 4\right)\right] - \left[\Delta_1^4 + \Delta_1^2\left(4\gamma_{21}^2 + 4\gamma_{21} + 2\right) + \left(4\gamma_{21}^2 + 4\gamma_{21} + 1\right)\right]\left[\Delta_1(4) + 2\kappa\right] = 0 \tag{S2-17}$$

$$\Delta_1^5(4) + \Delta_1^4(6\kappa) + \Delta_1^3\left(8\gamma_{21} + 4\right) + \Delta_1^2\left(8\gamma_{21}^2\kappa + 8\gamma_{21}\kappa + 4\kappa\right) + \Delta_1\left(16\gamma_{21}^3 + 8\gamma_{21}^2\right) - \left(8\gamma_{21}^2\kappa + 8\gamma_{21}\kappa + 2\kappa\right) = 0 \tag{S2-18}$$

For $Q_2 \ll Q_1$, we have $\gamma_{21} \gg \kappa, \Delta_1$. Hence, we have

$$\Delta_1^2(\kappa) + \Delta_1\left(2\gamma_{21}\right) - (\kappa) = 0 \tag{S2-19}$$

Hence, by solving this quadratic equation, we have $\Delta_1 = \frac{1}{2}\frac{\kappa}{\gamma_{21}^2}$ and $\Delta_1 = -2\left(\frac{\gamma_{21}}{\kappa}\right)$. Since $-2\left(\frac{\gamma_{21}}{\kappa}\right)$ is very large and not feasible, we can only consider $\Delta_1 = \frac{1}{2}\frac{\kappa}{\gamma_{21}^2}$. ω_D can thus be found as $\omega_D = \omega_1 + \frac{1}{4}(\omega_1 - 2\omega_2)\left(\frac{\omega_1}{\omega_2}\right)^2\left(\frac{Q_2}{Q_1}\right)^2$. Since $\frac{\omega_1}{\omega_2} \sim 2$, we have

$$\omega_D = \omega_1 + (\omega_1 - 2\omega_2)\left(\frac{Q_2}{Q_1}\right)^2 \tag{S2-20}$$