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Sharp Regularity Threshold for Green Kernels of Fractional Difference Equations Involving h-Fourier–Sine Convolution

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Abstract

This paper establishes the sharp regularity threshold for the Green kernel of a multi-term fractional difference equation involving generalized $\boldsymbol{h}$-Fourier–sine convolution on a bilateral lattice. Let $\boldsymbol{\alpha_*}$ denote the smallest noninteger order with a nonzero coefficient. If $\boldsymbol{P_h}$ is nonvanishing and $|\boldsymbol{k(nh)}| \leq \boldsymbol{M(1+|n|)^{-1-\beta}}$ with $\boldsymbol{\beta > \alpha_*}$, we prove that $\boldsymbol{G_h \in \mathcal{W}_s}$ for $\frac{1}{2} \leq s < \alpha_* + \frac{1}{2}$, whereas $\boldsymbol{G_h \notin \mathcal{W}_{\alpha_*+1/2}}$. Thus, $\boldsymbol{s_0 = \alpha_* + \frac{1}{2}}$ is the sharp regularity threshold with respect to the dyadic-block norm. In the single-term case without an interaction kernel, a compensated combination of Green kernels raises the regularity threshold from $\boldsymbol{\alpha + \frac{1}{2}}$ to $\boldsymbol{2\alpha + \frac{1}{2}}$. The results are illustrated through finite-support kernel approximation and ECG signal denoising.

Keywords: $\boldsymbol{h}$-Fourier–sine convolution; fractional difference equations; Green kernel; sharp regularity threshold; spectral invertibility condition

1 Introduction

In fractional calculus, Lubich studied convolution quadrature [9]. On time scales, the corresponding Fourier theory and convolution operations were developed in [10, 11]. For discrete lattices, convolutions associated with Fourier sine and cosine transforms were investigated in [12]. Generalized h-Fourier–sine convolution was introduced in [13, Definition 4.4], while the algebraic properties of h-Hartley convolution and the

solvability of related problems were studied in [14]. Young and Hausdorff–Young inequalities for weighted convolutions were established in [15, 16].

For $h > 0$, let $\mathbb{T}_h = h\mathbb{Z}$, $I_h = [-\pi/h, \pi/h]$. The h-Fourier and h-sine transforms are defined by

$$\begin{aligned}(\mathcal{F}u)(y) &= h\sum_{n\in\mathbb{Z}} u(nh)e^{-inhy},\\ (T_s u)(y) &= h\sum_{n\in\mathbb{Z}} u(nh)\sin(nhy), \qquad y \in I_h.\end{aligned} \tag{1.1}$$

Multi-term fractional equations in spaces of mean-square growth were studied in [2–4]. On the bilateral lattice, we investigate the Green kernel through the symbol identity $P_h = P_h^0 - iT_s k$ and Wiener's inversion theorem [18]. The principal result establishes the sharp regularity threshold $s_0 = \alpha_* + \frac{1}{2}$, including the endpoint nonmembership $G_h \notin \mathcal{W}_{s_0}$. The single-term case is further examined through compensated combinations of Green kernels, finite-support kernel approximation, and an application to ECG signal denoising.

2 Preliminaries

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For $1 \le p < \infty$, consider the following sequence spaces and their associated norms:

$$\begin{aligned}\ell_p(\mathbb{T}_h) &= \{u : \mathbb{T}_h \to \mathbb{C} : \|u\|_p < \infty\}, \qquad \|u\|_p = \Big(h\sum_{n\in\mathbb{Z}} |u(nh)|^p\Big)^{1/p},\\ \ell_\infty(\mathbb{T}_h) &= \{u : \mathbb{T}_h \to \mathbb{C} : \|u\|_\infty < \infty\}, \quad \|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(nh)|.\end{aligned}$$

For a kernel c, let $|c|_1 = \sum_{n\in\mathbb{Z}} |c(nh)|$, so that $\|c\|_1 = h|c|_1$. Consider the discrete convolution

$$(c \star u)(nh) = \sum_{m\in\mathbb{Z}} c(mh)\, u((n-m)h) \tag{2.1}$$

whenever the series converges absolutely. Let $\delta_j(nh) = \mathbf{1}_{\{j\}}(n)$, $\delta = \delta_0$, $\tau_k u(nh) = u((n-k)h)$, $Ru(nh) = u(-nh)$, and $u_o = (u - Ru)/2$. For $|c|_1 < \infty$, define

$$\sigma[c](y) = \sum_{n\in\mathbb{Z}} c(nh)\, e^{-inhy}, \qquad y \in I_h. \tag{2.2}$$

By (2.2), $\sigma[c \star d] = \sigma[c]\,\sigma[d]$ and $\sigma[Rc](y) = \sigma[c](-y)$. For $\eta \in I_h$, let $e_\eta(nh) = e^{inh\eta}$. It follows from (2.1) that

$$(c \star e_\eta)(nh) = \sum_{m\in\mathbb{Z}} c(mh)e^{i(n-m)h\eta} = \sigma[c](\eta)\, e_\eta(nh). \tag{2.3}$$

For $u \in \ell_1(\mathbb{T}_h)$, equation (1.1) gives $\mathcal{F}u = h\,\sigma[u]$. For $J \subset I_h$, we use the measure $dy/(2\pi)$; for $1 \le t < \infty$, define

$$L_t(J) = \{H : J \to \mathbb{C} : \|H\|_{L_t(J)} < \infty\},$$
$$\|H\|_{L_t(J)} = \left(\frac{1}{2\pi}\int_J |H(y)|^t\,dy\right)^{1/t}.$$

For $t = \infty$, the norm is $\|H\|_{L_\infty(J)} = \operatorname{ess\,sup}_{y\in J}|H(y)|$.

The Fourier inversion formula and Plancherel identity corresponding to (1.1) are

$$u(nh) = \frac{1}{2\pi}\int_{I_h} (\mathcal{F}u)(y)e^{inhy}\,dy,$$
$$\|\mathcal{F}u\|^2_{L_2(I_h)} = \|u\|_2^2.$$

In terms of (2.2), the Plancherel identity takes the form

$$\|c\|_2^2 = \frac{h^2}{2\pi}\int_{I_h} |\sigma[c](y)|^2\,dy.$$

Definition 2.1. The generalized h-Fourier–sine convolution introduced in [13, Definition 4.4] extends from $f, g \in \ell_1(\mathbb{T}_h)$ to $f \in \ell_\infty(\mathbb{T}_h)$, $g \in \ell_1(\mathbb{T}_h)$ through the absolutely convergent series

$$\Big(f \underset{\mathcal{F}T_s}{*} g\Big)(nh) = \frac{h}{2}\sum_{m\in\mathbb{Z}} f(mh)\big[g((n-m)h) - g((m-n)h)\big], \qquad n \in \mathbb{Z},\ h > 0.$$

Lemma 2.2. *Let $f \in \ell_\infty(\mathbb{T}_h)$ and $g \in \ell_1(\mathbb{T}_h)$. Then:*

(a) $f \underset{\mathcal{F}T_s}{*} g = f \star (hg_o)$ *and* $\sigma[hg_o](y) = -i(T_s g)(y)$, $\forall y \in I_h$;

(b) *If $f \in \ell_1(\mathbb{T}_h)$, then $\|f \underset{\mathcal{F}T_s}{*} g\|_1 \le \|f\|_1\|g_o\|_1 \le \|f\|_1\|g\|_1$, and*

$$\mathcal{F}\Big(f \underset{\mathcal{F}T_s}{*} g\Big)(y) = -i\,(\mathcal{F}f)(y)\,(T_s g)(y), \qquad y \in I_h. \tag{2.4}$$

If $g : \mathbb{T}_h \to \mathbb{R}$, then $T_s g$ is real-valued and odd, and $\|T_s g\|_\infty \le \|g_o\|_1$.

Proof The proof is analogous to that of [13, Lemma 4.5]. □

Definition 2.3. For $\alpha > 0$, the backward Grünwald–Letnikov–Weyl fractional difference of order α on $\mathbb{T}_h$ is defined by

$$(\nabla_h^\alpha u)(nh) = h^{-\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}u((n-k)h) = (b^{(\alpha)} \star u)(nh),$$

where $b^{(\alpha)}(kh) = h^{-\alpha}(-1)^k\binom{\alpha}{k}$ if $k \geq 0$ and $b^{(\alpha)}(kh) = 0$ if $k < 0$.

For $\alpha = m \in \mathbb{N}$, ∇_h^m is the usual backward difference. Using the principal branch of the complex power, let $\omega_{\alpha,h}(y) = \big((1 - e^{-ihy})/h\big)^{\alpha}$, $y \in I_h$.

Lemma 2.4. *Let $\alpha > 0$.*

(a) *If $\alpha \in \mathbb{N}$, then $b^{(\alpha)}$ is finitely supported. For $\alpha \notin \mathbb{N}$ and $k \geq 0$,*

$$\begin{aligned} b^{(\alpha)}(kh) &= \frac{h^{-\alpha}\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}, \\ b^{(\alpha)}(kh) &= \frac{h^{-\alpha}}{\Gamma(-\alpha)} k^{-1-\alpha}\big(1 + O(k^{-1})\big), \quad k \to \infty. \end{aligned} \tag{2.5}$$

Moreover, $\operatorname{sgn} b^{(\alpha)}(kh) = \operatorname{sgn}\Gamma(-\alpha)$ *for $k > \alpha$.*

(b) *We have $\sigma[b^{(\alpha)}] = \omega_{\alpha,h}$ on I_h, $\omega_{\alpha,h}(0) = 0$, $\omega_{\alpha,h}(-y) = \overline{\omega_{\alpha,h}(y)}$; for $0 < hy \leq \pi$,*

$$\omega_{\alpha,h}(y) = \left(\frac{2}{h}\right)^{\alpha} \sin^{\alpha}\left(\frac{hy}{2}\right) e^{i\alpha(\pi - hy)/2}. \tag{2.6}$$

Proof Part (a) follows from the binomial–Gamma identity and the asymptotic formula for Gamma-function ratios [20, 24]. Part (b) follows from the binomial expansion and (2.6). □

Definition 2.5. For a sequence u on $\mathbb{T}_h$, $N \in \mathbb{N}_0$, and $\gamma \geq 0$, define

$$\begin{aligned} A_N(u) &= \left(\frac{1}{2N+1} \sum_{|n| \leq N} |u(nh)|^2\right)^{1/2}, \\ X_\gamma &= \{u : \mathbb{T}_h \to \mathbb{C} : \|u\|_{X_\gamma} < \infty\}, \\ \|u\|_{X_\gamma} &= \sup_{N \geq 0} (N+1)^{-\gamma} A_N(u). \end{aligned}$$

To estimate convolution on X_γ, consider the dyadic decomposition

$$D_0 = \{0\}, \qquad D_j = \{n \in \mathbb{Z} : 2^{j-1} \leq |n| < 2^j\} \quad (j \geq 1), \qquad |D_j| = 2^j,$$

For a kernel c, let $c_j = c\mathbf{1}_{D_j}$ and $\|c_j\|_2^\sharp = (\sum_{n \in D_j} |c(nh)|^2)^{1/2}$.

Definition 2.6. For $s \geq \frac{1}{2}$, the space $\mathcal{W}_s$ on $\mathbb{T}_h$ is defined by

$$\begin{aligned} \mathcal{W}_s &= \{c : \mathbb{T}_h \to \mathbb{C} : \|c\|_{\mathcal{W}_s} < \infty\}, \\ \|c\|_{\mathcal{W}_s} &= \sum_{j=0}^{\infty} 2^{js} \|c_j\|_2^\sharp. \end{aligned}$$

Lemma 2.7. *Let $s \geq \frac{1}{2}$.*

(a) $|c|_1 \leq \|c\|_{\mathcal{W}_{1/2}} \leq \|c\|_{\mathcal{W}_s}$, $\|Rc\|_{\mathcal{W}_s} = \|c\|_{\mathcal{W}_s}$, *and* $\|c_o\|_{\mathcal{W}_s} \leq \|c\|_{\mathcal{W}_s}$.

(b) $\|c\|_{\mathcal{W}_s} \leq 2^s \sum_{n\in\mathbb{Z}} (1+|n|)^s |c(nh)|$; *in particular, every finitely supported kernel belongs to* $\mathcal{W}_s$.

(c) *If* $\beta > s - \frac{1}{2}$, $|c(nh)| \leq M(1+|n|)^{-1-\beta}$ $(\forall n \in \mathbb{Z}) \implies c \in \mathcal{W}_s$. *If* $\beta \leq s - \frac{1}{2}$ *and* $\kappa \neq 0$, $c(nh) = \kappa n^{-1-\beta}(1+o(1))$ $(n \to +\infty) \implies c \notin \mathcal{W}_s$.

(d) $\mathcal{W}_s$ *is a Banach space, and* $\|c \star d\|_{\mathcal{W}_s} \leq K_s \|c\|_{\mathcal{W}_s} \|d\|_{\mathcal{W}_s}$ *with* $K_s = 2^{2s+1}/(2^s - 1)$.

Proof (a) By the Cauchy–Schwarz inequality and $|D_j| = 2^j$, $|c_j|_1 \leq 2^{j/2} \|c_j\|_2^\sharp$. Summing the estimates and using $RD_j = D_j$ gives the assertion.

(b) For $n \in D_j$, $2^{js} \leq 2^s (1+|n|)^s$ and $\|c_j\|_2^\sharp \leq |c_j|_1$.

(c) The upper and lower tail estimates, respectively, yield

$$2^{js} \|c_j\|_2^\sharp = O\big(2^{j(s-1/2-\beta)}\big), \qquad 2^{js} \|c_j\|_2^\sharp \gtrsim 2^{j(s-1/2-\beta)}.$$

The latter estimate uses the 2^{j-1} positive indices in D_j and the assumption that the leading asymptotic coefficient is nonzero.

(d) Completeness follows from the weighted ℓ_1-sum of the blocks $\ell_2(D_j)$. Let $\|v\|_{2,\sharp} = (\sum_{n\in\mathbb{Z}} |v(nh)|^2)^{1/2}$. For $M = \max\{i,j\}$, the kernel $c_i \star d_j$ is supported in $\{|n| < 2^{M+1}\}$. Thus, Young's inequality gives

$$\|c_i \star d_j\|_{\mathcal{W}_s} \leq \sum_{\ell=0}^{M+1} 2^{\ell s} \|c_i \star d_j\|_{2,\sharp} \leq \frac{2^{(M+2)s}}{2^s - 1} \|c_i \star d_j\|_{2,\sharp},$$

$$\|c_i \star d_j\|_{2,\sharp} \leq \min\big\{|c_i|_1 \|d_j\|_2^\sharp, \|c_i\|_2^\sharp |d_j|_1\big\}.$$

If $i \leq j$, then $|c_i|_1 \leq 2^{i/2} \|c_i\|_2^\sharp$ and $s \geq \frac{1}{2}$ yield

$$2^{js} \|c_i \star d_j\|_{2,\sharp} \leq (2^{is} \|c_i\|_2^\sharp)(2^{js} \|d_j\|_2^\sharp).$$

The case $i > j$ is analogous. Summing over (i,j) yields $\|c \star d\|_{\mathcal{W}_s} \leq K_s \|c\|_{\mathcal{W}_s} \|d\|_{\mathcal{W}_s}$. □

Lemma 2.8. *Let* $\gamma \geq 0$ *and* $s = \gamma + \frac{1}{2}$.

(a) $|u(nh)| \leq \sqrt{2}(1+|n|)^s \|u\|_{X_\gamma}$, $\forall n \in \mathbb{Z}$.

(b) $\ell_p(\mathbb{T}_h) \subset \ell_\infty(\mathbb{T}_h) \subset X_0 \subset X_\gamma$, *with* $\|u\|_\infty \leq h^{-1/p} \|u\|_p$ *and* $\|u\|_{X_\gamma} \leq \|u\|_\infty$; *the reflection operator* R *is an isometry on* X_γ.

(c) *For* $c \in \mathcal{W}_s$ *and* $u \in X_\gamma$, *the convolution* (2.1) *converges absolutely at every* n *and satisfies*

$$\|c \star u\|_{X_\gamma} \leq 2^{1+\gamma} \|c\|_{\mathcal{W}_s} \|u\|_{X_\gamma}. \tag{2.7}$$

(d) *For* $c, d \in \mathcal{W}_s$ *and* $u \in X_\gamma$, *we have* $c \star (d \star u) = (c \star d) \star u$ *and* $c \star d = d \star c$.

Proof (a) By Definition 2.5,

$$|u(nh)|^2 \leq (2|n|+1)(|n|+1)^{2\gamma} \|u\|_{X_\gamma}^2 \leq 2(1+|n|)^{2s} \|u\|_{X_\gamma}^2.$$

(b) The inclusions follow directly from the norm definitions; moreover, $A_N(Ru) = A_N(u)$.

(c) For $N, L \geq 0$, Definition 2.5 gives

$$\sum_{|m|\leq N+L} |u(mh)|^2 \leq (2N+2L+1)(N+L+1)^{2\gamma}\|u\|_{X_\gamma}^2. \tag{2.8}$$

If $2^j \leq N+1$, the triangle inequality and (2.8) yield

$$(N+1)^{-\gamma} A_N(c_j \star u) \leq \sqrt{2}\, 2^{\gamma} 2^{j/2} \|c_j\|_2^\sharp \|u\|_{X_\gamma}.$$

If $2^j > N+1$, the Cauchy–Schwarz inequality and (2.8) yield

$$(N+1)^{-\gamma} A_N(c_j \star u) \leq 2^{1+\gamma} 2^{js} \|c_j\|_2^\sharp \|u\|_{X_\gamma}.$$

On the other hand, for each $n \in \mathbb{Z}$, the Cauchy–Schwarz inequality and the definition of X_γ give

$$\begin{aligned}\sum_{m\in D_j} |c(mh)u((n-m)h)| &\leq \|c_j\|_2^\sharp \left(\sum_{|\ell|\leq |n|+2^j} |u(\ell h)|^2 \right)^{1/2} \\ &\leq \sqrt{2}\, 2^s (1+|n|)^s 2^{js} \|c_j\|_2^\sharp \|u\|_{X_\gamma}.\end{aligned}$$

Summing over j establishes absolute convergence at every n; summing the bounds over j gives (2.7).

(d) By (c), $|c| \star (|d| \star |u|) \in X_\gamma$ and

$$\sum_{j,m\in\mathbb{Z}} |c(jh)d(mh)u((n-j-m)h)| < \infty.$$

Fubini's theorem yields associativity, while commutativity follows from (2.1).

□

Proposition 2.9. *Let $\gamma \geq 0$, $s = \gamma + \frac{1}{2}$, $u \in X_\gamma$, and $k \in \mathcal{W}_s$. The identities*

$$u \underset{\mathcal{F}T_s}{*} k = (hk_o) \star u, \qquad k \underset{\mathcal{F}T_s}{*} u = \tfrac{h}{2}(k \star u - k \star Ru) \tag{2.9}$$

define bounded operators on X_γ and agree with the defining series whenever both sides converge. Moreover,

$$\|u \underset{\mathcal{F}T_s}{*} k\|_{X_\gamma} \leq h 2^{1+\gamma} \|k_o\|_{\mathcal{W}_s} \|u\|_{X_\gamma}, \qquad \|k \underset{\mathcal{F}T_s}{*} u\|_{X_\gamma} \leq h 2^{1+\gamma} \|k\|_{\mathcal{W}_s} \|u\|_{X_\gamma}.$$

Proof By Definition 2.1, we obtain (2.9); the estimates follow from Lemma 2.8(c) and $\|Ru\|_{X_\gamma} = \|u\|_{X_\gamma}$. □

Wiener-type inversion theorems for weighted discrete convolution algebras are given in [17, 18]. We establish the corresponding result for $\mathcal{W}_s$.

Lemma 2.10. *Let $s \geq \frac{1}{2}$, $c \in \mathcal{W}_s$, and suppose that $\sigma[c](y) \neq 0$ for every $y \in I_h$. Then there exists a unique $d \in \mathcal{W}_s$ such that $c \star d = \delta$, $\sigma[d] = 1/\sigma[c]$, and*

$$d(nh) = \frac{h}{2\pi} \int_{I_h} \frac{e^{inhy}}{\sigma[c](y)}\, dy, \qquad n \in \mathbb{Z}. \tag{2.10}$$

Proof By Lemma 2.7(d), the equivalent operator norm

$$|||c||| := \sup_{\|d\|_{\mathcal{W}_s}\le 1} \|c \star d\|_{\mathcal{W}_s}$$

makes $\mathcal{W}_s$ a unital commutative Banach algebra. For each character φ, let $z = \varphi(\delta_1)$. Since $\delta_n = \delta_1^{\star n}$ and $\|\delta_{\pm n}\|_{\mathcal{W}_s} = O((1+n)^s)$,

$$|z|^n \le C(1+n)^s, \qquad |z|^{-n} \le C(1+n)^s,$$

it follows that $|z| = 1$. Density of finitely supported kernels gives $\varphi(c) = \sigma[c](y_0)$ for some $y_0 \in I_h$. Gelfand's invertibility criterion [18, 19] then gives $d \in \mathcal{W}_s$ with $\sigma[d] = 1/\sigma[c]$. The Fourier inversion formula yields (2.10) and uniqueness. □

Lemma 2.11. (a) *For $x > 0$ and $s \ge 0$, we have $\Gamma(x+s)/\Gamma(x) \le (x+s)^s$.*

(b) *For $n \in \mathbb{N}$ and $\beta > 0$, we have $\dfrac{\Gamma(n+\beta)}{\Gamma(n+1)n^{\beta-1}} = 1 + \varepsilon$, where $|\varepsilon| \le \dfrac{B}{n} e^{B/n}$, $B = \max\{\beta^2, 1\}$.*

Proof Part (a) follows from Wendel's inequality and the Gamma-function recurrence [24]. Part (b) follows from a digamma-function estimate [22] and $|e^z - 1| \le |z| e^{|z|}$. □

Let $\mathbb{T}_h^0 = h\mathbb{N}_0$ and use the ℓ_p norms restricted to the half-lattice as in Section 2. For a finitely supported sequence $f : \mathbb{T}_h^0 \to \mathbb{C}$, define

$$(T_c^* f)(y) = h \sum_{n\ge 0} f(nh)\cos(nhy), \tag{2.11}$$

$$(T_s^* f)(y) = h \sum_{n\ge 0} f(nh)\sin(nhy), \tag{2.12}$$

for $0 \le y \le \pi/h$. Orthogonality yields

$$\|T_c^* f\|_{L_2(0,\pi/h)} \le 2^{-1/2}\|f\|_2, \qquad \|T_s^* f\|_{L_2(0,\pi/h)} \le \tfrac{1}{2}\|f\|_2. \tag{2.13}$$

For $1 \le t \le \infty$, let $1/t + 1/t' = 1$, where t' is the conjugate exponent of t and $1/\infty = 0$. Let $F = \mathcal{F}$. The Hausdorff–Young inequalities and their inverse forms [19] read

$$\|Ff\|_{L_{p'}(I_h)} \le \|f\|_p \quad (1 \le p \le 2), \tag{2.14}$$

$$\|F^{-1}H\|_{r'} \le \|H\|_{L_r(I_h)} \quad (1 \le r \le 2). \tag{2.15}$$

Lemma 2.12. *For $1 \le p \le 2$ and $f \in \ell_p(\mathbb{T}_h^0)$, we have*

$$\|T_c^* f\|_{L_{p'}(0,\pi/h)} \le 2^{-1/p'}\|f\|_p, \tag{2.16}$$

$$\|T_s^* f\|_{L_{p'}(0,\pi/h)} \le 2^{-2/p'}\|f\|_p. \tag{2.17}$$

Proof Applying Riesz–Thorin interpolation [19] to the bounds $\ell_1 \to L_\infty$ and the $\ell_2 \to L_2$ bounds in (2.13) yields (2.16)–(2.17). □

For $M \in \mathbb{N}_0$, let

$$d := hk_o, \qquad d^{[M]}(nh) := d(nh)\mathbf{1}_{\{|n|\le M\}},$$
$$\mathcal{K}_d u := d \star u, \qquad \mathcal{K}_M u := d^{[M]} \star u.$$

Let

$$\varepsilon_M = \sum_{|n|>M} |d(nh)|, \qquad M \in \mathbb{N}_0.$$

Since $d \in \ell_1$, we have $\varepsilon_M \to 0$; symmetric truncation preserves the oddness of d.

Theorem 2.13. *For every $1 \le p \le \infty$,*

$$\|\mathcal{K}_d - \mathcal{K}_M\|_{\ell_p \to \ell_p} \le \varepsilon_M. \tag{2.18}$$

For $r > 0$, let $A_r(d) = \sum_{n\in\mathbb{Z}} (1+|n|)^r |d(nh)|$.

(a) *If $A_r(d) < \infty$, then*

$$\varepsilon_M \le (M+2)^{-r} A_r(d). \tag{2.19}$$

The exponent r is uniformly sharp over the class of odd kernels satisfying $A_r(d) \le 1$.

(b) *If $\varepsilon_M \le A(M+1)^{-r}$ for every $M \ge 0$, then $A_q(d) < \infty$ for every $q \in (0, r)$.*

Proof Young's inequality gives (2.18).

(a) The inequality $|n| > M$ implies $1 + |n| \ge M + 2$, giving (2.19). Equality is attained by the odd kernel supported at $\pm(M+1)h$ with values $\pm[2(M+2)^r]^{-1}$.

(b) Summation by parts gives

$$A_q(d) = |d|_1 + \sum_{M=0}^{\infty} \big((M+2)^q - (M+1)^q\big)\varepsilon_M.$$

The summands are $O((M+1)^{q-r-1})$; since $q < r$, we obtain $A_q(d) < \infty$.

□

3 Main Results

3.1 Spectral Invertibility, Regularity, and Approximation

Problem 3.1. Let $m \ge 1$, $\alpha_1 > \cdots > \alpha_m > 0$, $a_1, \ldots, a_m \in \mathbb{C}$, $a_1 \ne 0$, $\lambda \in \mathbb{C}$, an interaction kernel k on $\mathbb{T}_h$, and a forcing term f be given. Find u on $\mathbb{T}_h$ satisfying

$$\sum_{j=1}^{m} a_j (\nabla_h^{\alpha_j} u)(nh) + \lambda u(nh) + \Big(u \underset{\mathcal{F}\mathcal{T}_s}{*} k\Big)(nh) = f(nh), \qquad n \in \mathbb{Z}. \tag{3.1}$$

By Lemma 2.2(a), define $L_h u := c_h \star u$, where

$$c_h = c_h^0 + h\, k_o, \qquad c_h^0 = \sum_{j=1}^{m} a_j b^{(\alpha_j)} + \lambda\delta. \tag{3.2}$$

Lemmas 2.2(a) and 2.4(b) yield

$$\begin{aligned} P_h(y) &:= \sigma[c_h](y) = P_h^0(y) - i(T_s k)(y), \\ P_h^0(y) &= \sum_{j=1}^{m} a_j \omega_{\alpha_j,h}(y) + \lambda, \qquad y \in I_h. \end{aligned} \tag{3.3}$$

For $u \in \ell_1(\mathbb{T}_h)$, Lemma 2.2(b) gives $\mathcal{F}(L_h u) = P_h \mathcal{F} u$. Set

$$\begin{gathered} \alpha_* := \min\{\alpha_j :\ \alpha_j \notin \mathbb{N},\ a_j \neq 0\}, \qquad \min \varnothing := +\infty, \\ \alpha_* < \infty \implies a_* := a_{j_*} \neq 0, \quad \alpha_{j_*} = \alpha_*. \end{gathered}$$

Theorem 3.2. *Let $0 \le \gamma < \alpha_*$, $s = \gamma + \frac{1}{2}$, and $k \in \mathcal{W}_s$. The following conditions are equivalent:*

(i) *$P_h(y) \neq 0$, $\forall y \in I_h$;*
(ii) *$\forall f \in X_\gamma$, $\exists! u \in X_\gamma$: $L_h u = f$ theo (3.1);*
(iii) *$\ker(L_h|_{\ell_\infty(\mathbb{T}_h)}) = \{0\}$;*
(iv) *$\exists p \in [1,\infty]$: $L_h : \ell_p(\mathbb{T}_h) \to \ell_p(\mathbb{T}_h)$ is bijective;*
(v) *$\forall p \in [1,\infty]$: $L_h : \ell_p(\mathbb{T}_h) \to \ell_p(\mathbb{T}_h)$ is bijective.*

For each f in the corresponding space, the unique solution is $u = G_h \star f$, where

$$G_h(nh) = \frac{h}{2\pi} \int_{I_h} \frac{e^{inhy}}{P_h(y)}\, dy, \qquad n \in \mathbb{Z}. \tag{3.4}$$

Moreover, $G_h \in \mathcal{W}_s$, $c_h \star G_h = \delta$, and

$$\|u\|_{X_\gamma} \le 2^{1+\gamma} \|G_h\|_{\mathcal{W}_s} \|f\|_{X_\gamma}, \qquad \|u\|_p \le |G_h|_1\, \|f\|_p \quad (1 \le p \le \infty). \tag{3.5}$$

Proof By (2.5), Lemma 2.7(b),(c), and $\gamma < \alpha_*$, $c_h \in \mathcal{W}_s \subset \ell_1$. Hence L_h is bounded on X_γ and on every ℓ_p.

(i) Assume (i). Lemmas 2.10 and 2.8 imply that $G_h \in \mathcal{W}_s$ and $G_h \star c_h = \delta$. Thus $L_h^{-1} f = G_h \star f$ on X_γ and every ℓ_p; (3.5) follows from (2.7) and Young's inequality.

(ii) Condition (ii), together with $\ell_\infty(\mathbb{T}_h) \subset X_\gamma$, implies (iii).

(iii) Assume (iii). If $P_h(y_0) = 0$, then $0 \neq e_{y_0} \in \ell_\infty$ and $L_h e_{y_0} = 0$ by (2.3), a contradiction. Thus (i) holds.

(iv) Assume (iv) and $P_h(y_0) = 0$. The exponential solution above rules out $p = \infty$. For $1 \le p < \infty$, set $v_N = e_{y_0}\mathbf{1}_{\{0\le n<N\}}$. Then

$$\frac{\|L_h v_N\|_p}{\|v_N\|_p} \le \sum_{k\in\mathbb{Z}} |c_h(kh)| \left(\frac{2\min\{|k|, N\}}{N}\right)^{1/p} \longrightarrow 0. \tag{3.6}$$

Since $|c_h|_1 < \infty$, the dominated convergence theorem gives a contradiction to bounded invertibility.

(v) Condition (v) implies (iv) by choosing $p = 2$.

□

The restriction $\gamma < \alpha_*$ in Theorem 3.2 is sharp for the convolution to be defined on all of X_γ.

Proposition 3.3. *Assume that $\alpha_* < \infty$, $\gamma \ge \alpha_*$, and there exist $\beta > \alpha_*$ and $M > 0$ such that $|k(nh)| \le M(1 + |n|)^{-1-\beta}$ $(\forall n \in \mathbb{Z})$. Then there exists $u \in X_\gamma$, $\|u\|_{X_\gamma} \le 1$, for which the series $\sum_{j\in\mathbb{Z}} c_h(jh)u(-jh)$ diverges.*

Proof By (2.5), we have

$$a_* b^{(\alpha_*)}(jh) = \rho e^{i\phi} j^{-1-\alpha_*}\big(1 + O(j^{-1})\big),$$
$$\rho = \frac{|a_*|h^{-\alpha_*}}{|\Gamma(-\alpha_*)|} > 0, \qquad \phi = \arg\left(\frac{a_*}{\Gamma(-\alpha_*)}\right).$$

Choose $u(-jh) = e^{-i\phi}(1+j)^\gamma$ for $j \ge 0$ and $u(nh) = 0$ for $n > 0$. Since $A_N(u)^2 \le \frac{N+1}{2N+1}(N+1)^{2\gamma} \le (N+1)^{2\gamma}$, we obtain $\|u\|_{X_\gamma} \le 1$. Consider the partial sum $S_J = \sum_{j=0}^{J} c_h(jh)u(-jh)$. The smallest noninteger order with a nonzero coefficient gives the leading contribution

$$S_J = \rho\Sigma_J + o(\Sigma_J),$$
$$\Sigma_J = \sum_{j=1}^{J} j^{\gamma-1-\alpha_*} \sim \begin{cases} \log J, & \gamma = \alpha_*, \\ \dfrac{J^{\gamma-\alpha_*}}{\gamma-\alpha_*}, & \gamma > \alpha_*. \end{cases}$$

The integer-order terms and $\lambda\delta$ are finitely supported; the higher noninteger orders $\alpha_j > \alpha_*$ contribute $O(\sum_{j\le J} j^{\gamma-1-\alpha_j})$, whereas hk_o contributes $O(\sum_{j\le J} j^{\gamma-1-\beta})$. All remainder terms are $o(\Sigma_J)$, so $|S_J| \to \infty$. □

Corollary 3.4. *Assume $\alpha_* < \infty$, condition (i) of Theorem 3.2, and $|k(nh)| \le M(1 + |n|)^{-1-\beta}$ $(\forall n \in \mathbb{Z})$, $\beta > \alpha_*$. Then*

$$G_h \in \mathcal{W}_s \quad \big(\forall s \in [\tfrac{1}{2}, \alpha_* + \tfrac{1}{2})\big), \qquad G_h \notin \mathcal{W}_{\alpha_*+1/2}.$$

Thus $s_0 = \alpha_* + \frac{1}{2}$ is the sharp regularity threshold with respect to the dyadic-block norm.

Proof For $s < \alpha_* + \frac{1}{2}$, Lemma 2.7(c) and Theorem 3.2 give $G_h \in \mathcal{W}_s$. Set $s_0 = \alpha_* + \frac{1}{2}$. Since $\beta > \alpha_*$ and all other noninteger orders exceed α_*, equation (2.5) yields

$$c_h(nh) = \frac{a_* h^{-\alpha_*}}{\Gamma(-\alpha_*)} n^{-1-\alpha_*}(1 + o(1)) \qquad (n \to +\infty).$$

Consequently, $c_h \notin \mathcal{W}_{s_0}$ by Lemma 2.7(c). If $G_h \in \mathcal{W}_{s_0}$, Lemma 2.10, applied with $\sigma[G_h] = 1/P_h \neq 0$, would give $c_h = G_h^{\star -1} \in \mathcal{W}_{s_0}$, a contradiction. □

For $h = 1$, $\alpha_* = 0.7$, $\lambda = 1$, and $k = 0.15(\delta_1 - \delta_{-1})$, consider the truncated sum

$$S_J(s) = \sum_{j=0}^{J} 2^{js} \|G_h \mathbf{1}_{D_j}\|_2^\sharp.$$

Figure 1 depicts $\log_{10} S_J(s)$ with threshold $s_0 = \alpha_* + \frac{1}{2} = 1.2$.

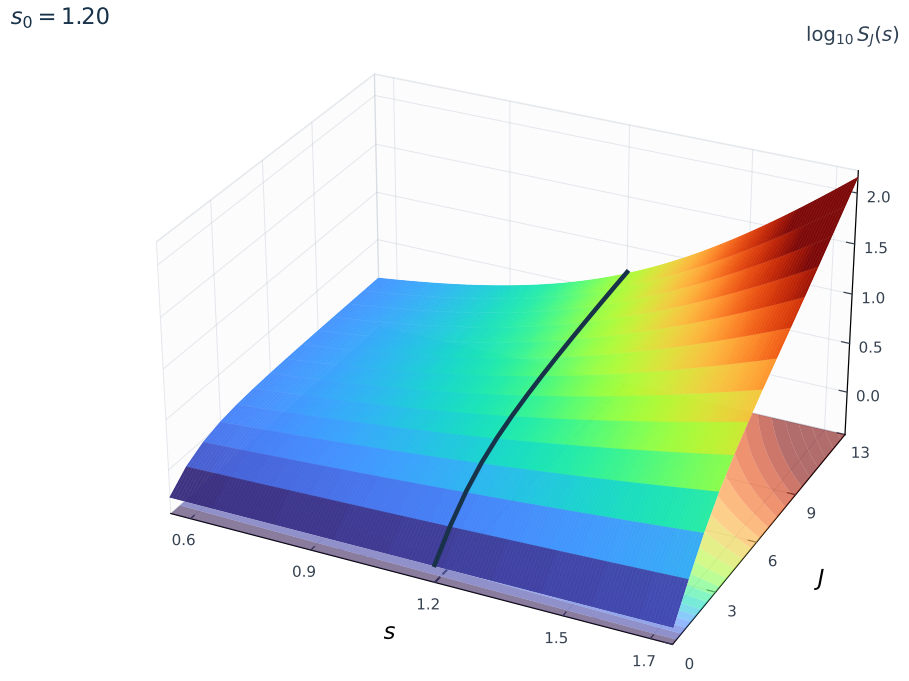

Fig. 1 Values of $\log_{10} S_J(s)$ as functions of s and J with $s_0 = 1.2$.

Theorem 3.5. *Let $s > \frac{1}{2}$ and $d \in \mathcal{W}_s$. Then*

$$\varepsilon_M \leq 2^{s-1/2}(M+1)^{-(s-1/2)} \|d\|_{\mathcal{W}_s}. \tag{3.7}$$

The exponent $s - \frac{1}{2}$ is uniformly sharp on the unit ball of $\mathcal{W}_s$, even when restricted to real-valued odd kernels.

Proof With $J = \lfloor \log_2(M+1) \rfloor$, the Cauchy–Schwarz inequality gives

$$\varepsilon_M \le \sum_{j \ge J} 2^{j/2} \|d_j\|_2^\sharp \le 2^{-J(s-1/2)} \|d\|_{\mathcal{W}_s},$$

which yields (3.7). Sharpness follows by taking $d^{(j)}(nh) = \operatorname{sgn}(n) 2^{-j(s+1/2)} \mathbf{1}_{D_j}(n)$: $\|d^{(j)}\|_{\mathcal{W}_s} = 1$ and, at $M_j = 2^{j-1} - 1$, we have $\varepsilon_{M_j} = 2^{-j(s-1/2)}$. □

Consider $\mathcal{L} = \mathcal{L}_h^0 + \mathcal{K}_d$, $\mathcal{L}_M = \mathcal{L}_h^0 + \mathcal{K}_M$, where $\mathcal{L}_h^0$ corresponds to P_h^0. Then $P_h = P_h^0 + \sigma[d]$, $P_{h,M} = P_h^0 + \sigma[d^{[M]}]$.

Theorem 3.6. *Assume $d \in \ell_1$, $\delta = \min_{y \in I_h} |P_h(y)| > 0$, and $\varepsilon_M < \delta$. Then $\mathcal{L}^{-1}, \mathcal{L}_M^{-1} \in \mathcal{B}(\ell_2(\mathbb{T}_h))$, and*

$$\|\mathcal{L}^{-1} - \mathcal{L}_M^{-1}\|_{2\to 2} \le \frac{\varepsilon_M}{\delta(\delta - \varepsilon_M)}.$$

If $d \in \mathcal{W}_s$, $s > 1/2$, the right-hand side decays at the rate $(M+1)^{-(s-1/2)}$ for sufficiently large M; if $A_r(d) < \infty$, this improves to $(M+1)^{-r}$.

Proof The resolvent identity, together with (2.18) and Plancherel's identity, gives

$$\|\mathcal{L}^{-1} - \mathcal{L}_M^{-1}\|_{2\to 2} = \left\| \frac{P_{h,M} - P_h}{P_h P_{h,M}} \right\|_\infty \le \frac{\varepsilon_M}{\delta(\delta - \varepsilon_M)}.$$

The convergence rates follow from (2.19) and (3.7). □

Assume that

$$a_j > 0, \quad 0 < \alpha_j \le 2 \ (1 \le j \le m), \quad \lambda > 0, \quad k \in \ell_1(\mathbb{T}_h) \text{ is real-valued.} \tag{O}$$

Define $\kappa(\alpha) = 1$ for $0 < \alpha \le 1$, $\kappa(\alpha) = \sin(\alpha\pi/2)$ for $1 < \alpha < 2$, and $S_h(y) = \sum_{j=1}^m a_j \omega_{\alpha_j,h}(y)$. Then $P_h^0 = \lambda + S_h$.

Theorem 3.7. *Assume* (O).

(a) *If $k = 0$, then $P_h(y) \ne 0$ for all $y \in I_h$; hence L_h is invertible on X_γ when $0 \le \gamma < \alpha_*$ and on ℓ_p for $1 \le p \le \infty$.*

(b) *For $\alpha_1 < 2$, $h > 0$, and $y \in I_h$,*

$$|P_h^0(y)| \ge \lambda \kappa(\alpha_1). \tag{3.8}$$

If $\|T_s k\|_\infty < \lambda\kappa(\alpha_1)$, then $P_h(y) \neq 0$ for all $y \in I_h$; the condition $\|k_o\|_1 < \lambda\kappa(\alpha_1)$ is also sufficient. In this case, for $f \in \ell_2(\mathbb{T}_h)$,

$$\|u\|_2 \le \frac{\|f\|_2}{\lambda\kappa(\alpha_1) - \|T_s k\|_\infty}.$$

Proof By $S_h(-y) = \overline{S_h(y)}$, it suffices to consider $0 \le hy \le \pi$. By (2.6), $\arg \omega_{\alpha_j,h}(y) = \alpha_j(\pi - hy)/2$ khi $y > 0$.

(a) For $0 < hy < \pi$, $\operatorname{Im} P_h^0(y) > 0$. At $y = 0$ and $hy = \pi$, respectively, we have $P_h^0(0) = \lambda > 0$ and $P_h^0(\pi/h) > \lambda$.

(b) The terms in $S_h(y)$ lie in the sector $\{re^{i\varphi} : r \ge 0,\ 0 \le \varphi \le \alpha_1\pi/2\}$. The distance from any point of this sector to $-\lambda$ is at least $\lambda\kappa(\alpha_1)$, yielding (3.8). Moreover,

$$|P_h(y)| \ge \lambda\kappa(\alpha_1) - \|T_s k\|_\infty, \qquad \|T_s k\|_\infty \le \|k_o\|_1.$$

The ℓ_2 estimate follows from Plancherel's identity.

□

Example 3.8. (a) *Sharpness of* (3.8). For $m = 1$, $a_1 = 1$, $\alpha_1 = \alpha < 2$, and $k = 0$, we have $\min_{I_h} |P_h| = \lambda$ for $0 < \alpha \le 1$ and every $h > 0$, whereas

$$\lim_{h\to 0^+} \min_{I_h} |P_h| = \lambda \sin(\alpha\pi/2), \qquad 1 < \alpha < 2.$$

(b) *Resonance.* For $m = 1$, $a_1 = 1$, $\alpha_1 = 2$, $k = 0$, and $\mu(y) = \frac{4}{h^2}\sin^2\frac{hy}{2}$,

$$|P_h(y)|^2 = (\lambda - \mu(y))^2 + \lambda h^2 \mu(y)^2, \qquad \min_{y\in I_h} |P_h(y)| = \frac{\lambda^{3/2} h}{\sqrt{1+\lambda h^2}}, \tag{3.9}$$

and $\|L_h^{-1}\|_{\ell_2\to\ell_2} = \sqrt{1+\lambda h^2}/(\lambda^{3/2}h)$, whereas $\|L_h^{-1}\|_{\ell_p\to\ell_p} \ge \sqrt{1+\lambda h^2}/(\lambda^{3/2}h)$ $(\forall p \in [1,\infty])$.

Proof (a) For $0 < \alpha \le 1$, the minimum is $\lambda = |P_h(0)|$. For $1 < \alpha < 2$, choose $\rho_0 = -\lambda\cos(\alpha\pi/2) > 0$ and y_h such that $|\omega_{\alpha,h}(y_h)| = \rho_0$. Then

$$\sin(hy_h/2) = \tfrac{h}{2}\rho_0^{1/\alpha} \to 0, \qquad P_h(y_h) \to \lambda + \rho_0 e^{i\alpha\pi/2}.$$

Together with (3.8), this gives the required limit.

(b) Set $\mu = 4h^{-2}\sin^2(hy/2)$. By (2.6),

$$|P_h(y)|^2 = (\lambda - \mu)^2 + \lambda h^2 \mu^2, \qquad \mu_* = \frac{\lambda}{1+\lambda h^2} \in [0, 4h^{-2}].$$

Minimizing yields (3.9). Plancherel's identity gives the exact inverse norm on ℓ_2. For $p < \infty$, applying the truncated-exponential argument in (3.6) to G_h at a minimizer gives $\|L_h^{-1}\|_{p\to p} \ge 1/\min|P_h|$; for $p = \infty$, use e_{y_*}.

□

In the purely second-order case, Example 3.8(b) shows that the operator is invertible for each fixed h but not uniformly as $h \to 0$. Adding a positive-coefficient lower-order term yields a spectral lower bound independent of the mesh size. For $0 < \alpha < 2$, set $s(\alpha) = \min\{\sin(\alpha\pi/4), \sin(\alpha\pi/2)\} > 0$. In the continuum case, $P_0(y) = \lambda + \sum_{j=1}^{m} a_j (iy)^{\alpha_j}$, where $(iy)^\alpha = y^\alpha e^{i\alpha\pi/2}$ for $y \geq 0$.

Theorem 3.9. *Assume* (O)*, excluding the case $m = 1$, $\alpha_1 = 2$. Choose j_0 such that $\alpha_{j_0} < 2$ and $\mu_0 \in (0, \infty]$ satisfying $\sum_{j:\alpha_j > 1} a_j \mu_0^{\alpha_j/2} \leq \lambda/2$, with the convention $\mu_0 = \infty$ if no $\alpha_j > 1$ occurs. Then:*

(a) *For $h > 0$, $y \in I_h$,*

$$|P_h^0(y)| \geq c_0 := \min\left\{\lambda/2,\ a_{j_0} s(\alpha_{j_0})\, \mu_0^{\alpha_{j_0}/2}\right\} > 0 \tag{3.10}$$

where $c_0 = \lambda/2$ if $\mu_0 = \infty$. In particular, when $k = 0$, $\|L_h^{-1}\|_{\ell_2 \to \ell_2} \leq 1/c_0$ uniformly in h;

(b) $\lim_{h \to 0^+} \min_{y \in I_h} |P_h^0(y)| = m_0 := \inf_{y \geq 0} |P_0(y)| \in [c_0, \lambda]$*; hence the optimal ℓ_2 stability constant tends to $1/m_0$.*

Proof By conjugate symmetry, consider $0 < \theta = hy \leq \pi$ and $\mu = 4h^{-2}\sin^2(\theta/2)$. Then $|\omega_{\alpha,h}(y)| = \mu^{\alpha/2}$.

(a) If $\theta \geq \pi/2$, then $\operatorname{Re} P_h^0 \geq \lambda$. If $\theta < \pi/2$ and $\mu \leq \mu_0$,

$$\operatorname{Re} P_h^0 \geq \lambda - \sum_{\alpha_j > 1} a_j \mu^{\alpha_j/2} \geq \lambda/2.$$

If $\theta < \pi/2$ and $\mu > \mu_0$,

$$\operatorname{Im} P_h^0 \geq a_{j_0} s(\alpha_{j_0}) \mu^{\alpha_{j_0}/2} \geq a_{j_0} s(\alpha_{j_0}) \mu_0^{\alpha_{j_0}/2}.$$

These lower bounds yield (3.10); the inverse-norm estimate follows from Plancherel's identity.

(b) Uniformly on each interval $[0, Y]$,

$$\frac{1 - e^{-ihy}}{h} \to iy, \qquad P_h^0(y) \to P_0(y), \qquad h \to 0^+.$$

Consequently, $\limsup_{h \to 0} \min_{I_h} |P_h^0| \leq m_0$. For the reverse bound, choose $Y > 0$ such that $a_{j_0} s(\alpha_{j_0})(2Y/\pi)^{\alpha_{j_0}} \geq \lambda$. On $[0, Y]$, uniform convergence gives $|P_h^0| \geq m_0 - \varepsilon$ for sufficiently small h. On $(Y, \pi/h]$, either $\theta \geq \pi/2$ (in which case $|P_h^0| \geq \lambda$) or $\mu^{1/2} \geq 2y/\pi \geq 2Y/\pi$ (in which case (a) yields $|P_h^0| \geq \lambda$). Thus $\liminf_{h \to 0} \min_{I_h} |P_h^0| \geq m_0$.

□

Consider $k = 0$, $\lambda = 1$, and $P_{h,a}(y) = 1 + \omega_{2,h}(y) + a\omega_{1,h}(y)$ for $a \geq 0$. Figure 2 depicts $\log_{10} \|L_{h,a}^{-1}\|_{\ell_2 \to \ell_2}$; at $a = 0$, this norm is given exactly by (3.9).

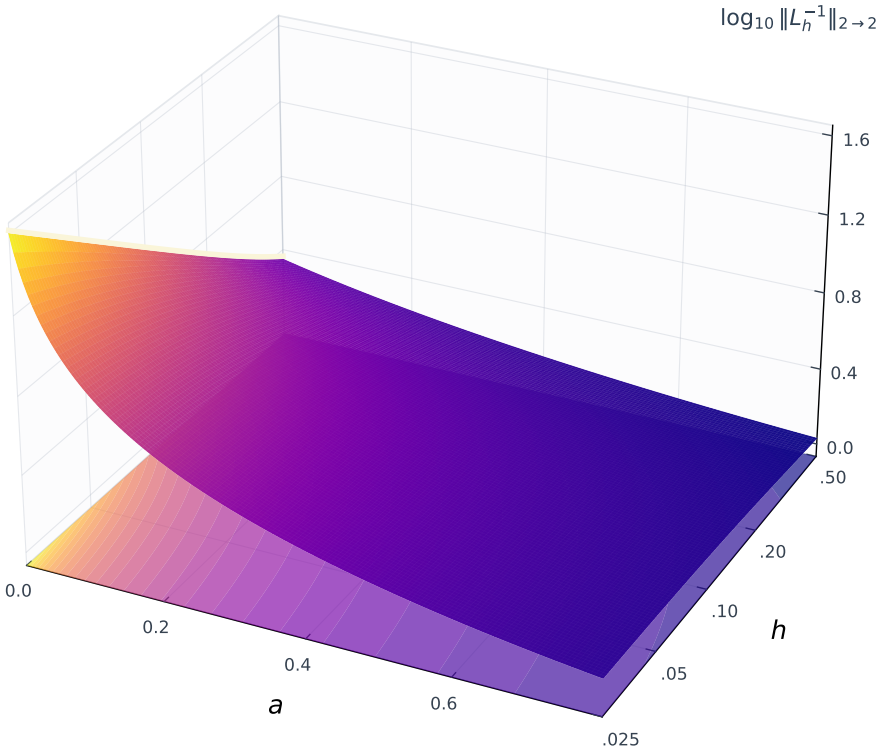

Fig. 2 Values of $\log_{10} \|L_{h,a}^{-1}\|_{\ell_2 \to \ell_2}$ as functions of h and a.

3.2 Interpolation Estimates for Discrete Transforms and Convolution Operators

Combining (2.4) with the Hausdorff–Young inequalities (2.14)–(2.15) gives interpolation estimates for the convolution in Definition 2.1; see also the Fourier-multiplier approach in [15, 16].

Theorem 3.10. *Let $p, q, r \in [1, 2]$ have conjugate exponents p', q', r' satisfying $1/p' + 1/q' = 1/r$. For $f \in \ell_p(\mathbb{T}_h)$ and $g \in \ell_q(\mathbb{T}_h^0)$, where g is extended by 0 for $n < 0$, we have*

$$\|f \underset{\mathcal{F}T_s}{*} g\|_{r'} \le 2^{-1/q'} \|f\|_p \|g\|_q. \tag{3.11}$$

Proof The proof follows as in [16, Theorem 2.2] and [15, Theorem 3.3], using (2.4) and (2.17). □

3.3 Green Kernels and the Continuum Limit

On the principal branch, set $q_h(z) = \sum_{j=1}^{m} a_j h^{-\alpha_j}(1-z)^{\alpha_j} + \lambda$. This function is holomorphic on $\mathbb{C} \setminus [1,\infty)$ and continuous up to the boundary of the unit disk. If $k = 0$, then $P_h(y) = q_h(e^{-ihy})$. Set

$$\vartheta_h = \frac{1}{|a_1|}\Big(|\lambda| h^{\alpha_1} + \sum_{j=2}^{m} |a_j| h^{\alpha_1 - \alpha_j}\Big), \qquad \beta_\ell = \alpha_1(1+\ell_0) + \sum_{j=2}^{m} \ell_j(\alpha_1 - \alpha_j),$$

where $\ell = (\ell_0, \ell_2, \dots, \ell_m) \in \mathbb{N}_0^m$ is a multi-index, $|\ell| = \ell_0 + \ell_2 + \cdots + \ell_m$, and $\binom{|\ell|}{\ell} = |\ell|!/(\ell_0!\ell_2!\cdots\ell_m!)$.

Theorem 3.11. *Assume $k = 0$ and $P_h(y) \neq 0$ for all $y \in I_h$. Consider condition* (H)*:*

$$q_h(z) \neq 0 \quad (\forall |z| \leq 1). \tag{H}$$

The Green kernel G_h^0 in Theorem 3.2 satisfies

$$G_h^0(nh) = 0 \ (\forall n < 0) \quad \Longleftrightarrow \quad \text{(H)}.$$

If (H) *holds, then $G_h^0(nh) = [z^n] q_h(z)^{-1}$ for $n \geq 0$. If $\vartheta_h < 1$ and $n \geq 0$,*

$$G_h^0(nh) = \frac{h^{\alpha_1}}{a_1} \sum_{r=0}^{\infty} (-1)^r \sum_{|\ell|=r} \binom{r}{\ell} \Big(\frac{\lambda h^{\alpha_1}}{a_1}\Big)^{\ell_0} \prod_{j=2}^{m} \Big(\frac{a_j h^{\alpha_1 - \alpha_j}}{a_1}\Big)^{\ell_j} \frac{\Gamma(n+\beta_\ell)}{\Gamma(\beta_\ell)\,\Gamma(n+1)}, \tag{3.12}$$

where the series in r converges absolutely. Under assumption (O)*, condition* (H) *holds for all $h > 0$.*

Proof In (3.4), set $\theta = hy$ and then replace $\theta \mapsto -\theta$ to obtain

$$G_h^0(nh) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-in\theta}}{q_h(e^{i\theta})}\, d\theta.$$

By (H), the boundary Fourier coefficients of q_h^{-1} coincide with its nonnegative-index Taylor coefficients, whereas its negative-index coefficients vanish. Conversely, causality and $G_h^0 \in \ell_1$ imply that $\Phi(z) = \sum_{n \geq 0} G_h^0(nh) z^n$ is continuous on $|z| \leq 1$ and

$$q_h(z)\Phi(z) = \sum_{n \geq 0} (c_h^0 \star G_h^0)(nh) z^n = 1 \qquad (|z| \leq 1),$$

which proves (H).

To prove (3.12), set $A(z) = a_1 h^{-\alpha_1}(1-z)^{\alpha_1}$ and $B(z) = q_h(z) - A(z)$. If $\vartheta_h < 1$, then $\sup_{|z|=\varepsilon} |B(z)/A(z)| < 1$ for sufficiently small $\varepsilon > 0$, and hence

$$q_h^{-1} = \sum_{r \geq 0} (-1)^r B^r A^{-r-1} \quad (|z| = \varepsilon).$$

Cauchy's estimates ensure absolute convergence coefficientwise. The multinomial expansion and the identity $[z^n](1-z)^{-\beta} = \Gamma(n+\beta)/(\Gamma(\beta)\Gamma(n+1))$ yield (3.12).

Under (O), let $w = 1 - z$, $|z| < 1$. If $\arg w \neq 0$, the imaginary parts of all $a_j h^{-\alpha_j} w^{\alpha_j}$ have the same nonzero sign; if $\arg w = 0$, then $q_h(z) > 0$. On $|z| = 1$, nonvanishing follows from Theorem 3.7(a). □

For $\nu \in \mathbb{R}$, $t \in h\mathbb{N}$, define the rising factorial $t_h^{\overline{\nu}} = h^\nu \Gamma(t/h + \nu)/\Gamma(t/h)$ on $h\mathbb{Z}$ [6, 8], and set

$$\mathcal{E}^h_{\alpha,\beta}(\mu; t) = \sum_{r=0}^{\infty} \mu^r \frac{t_h^{\overline{\alpha r+\beta-1}}}{\Gamma(\alpha r + \beta)}, \qquad t \in h\mathbb{N}, \quad |\mu| h^\alpha < 1.$$

This series defines a nabla-type discrete Mittag–Leffler function [5, 7].

Example 3.12. Assume $m = 1$, $a_1 = 1$, $\alpha_1 = \alpha \in (0, 2]$, $\lambda > 0$, $k = 0$, and $\lambda h^\alpha < 1$. Then $G_h^0(nh) = h\,\mathcal{E}^h_{\alpha,\alpha}(-\lambda; (n+1)h)$ for $n \geq 0$, whereas $G_h^0(nh) = 0$ for $n < 0$. The solution of (3.1) is $u(nh) = h \sum_{m \leq n} \mathcal{E}^h_{\alpha,\alpha}\big(-\lambda; (n-m+1)h\big) f(mh)$.

Proof The proof follows the discrete Mittag–Leffler representation in [7], together with (3.12) and Theorem 3.11. □

Theorem 3.13. *Assume $k = 0$, condition* (H)*, and $\alpha_* < \infty$. Then $\lambda \neq 0$ and*

$$G_h^0(nh) = -\frac{a_* h^{-\alpha_*}}{\lambda^2\,\Gamma(-\alpha_*)}\, n^{-1-\alpha_*}\big(1 + O(n^{-\epsilon})\big), \qquad n \to \infty, \tag{3.13}$$

for some $\epsilon > 0$ depending only on the orders α_j. Under assumption (O)*, the tail of G_h^0 is positive if $\alpha_* < 1$ and negative if $1 < \alpha_* < 2$.*

Proof Set $w = 1 - z$. By (H) and $q_h(1) = \lambda \neq 0$, the expansion near $z = 1$ has the following form for some $A > \alpha_*$ and polynomial p:

$$q_h(z)^{-1} = p(w) - \frac{a_* h^{-\alpha_*}}{\lambda^2} w^{\alpha_*} + O(|w|^A).$$

Applying the asymptotic transfer theorem [21, Theorem 1 and Corollary 2], as in the coefficient expansions in Theorem 3.11, yields (3.13). □

For exponentially decaying k, consider the two tails of G_h.

Theorem 3.14. *Assume $\alpha_* < \infty$, P_h is nonvanishing on I_h, and $|k(nh)| \leq M e^{-\varepsilon_0 |n|}$ with $M, \varepsilon_0 > 0$. Then $\lambda \neq 0$ and:*

(a) *(negative-index tail) $\exists r \in (0,1)$, $C > 0$: $|G_h(nh)| \leq C r^{|n|}$ $(\forall n \leq 0)$;*

(b) *(positive-index tail) as $n \to +\infty$,*

$$G_h(nh) = -\frac{a_* h^{-\alpha_*}}{\lambda^2\,\Gamma(-\alpha_*)}\, n^{-1-\alpha_*}\big(1 + O(n^{-\epsilon})\big) \tag{3.14}$$

for some $\epsilon > 0$. In particular, the leading coefficient of the algebraically decaying tail does not depend on k.

Proof Set $\mathcal{K}(z) = \sum_{n\in\mathbb{Z}} hk_o(nh)z^n$ and $\widetilde{Q}(z) = q_h(z) + \mathcal{K}(z)$. The interaction kernel has a Laurent series analytic in an annulus containing the unit circle; $\mathcal{K}(1) = 0$ and $\widetilde{Q}(1) = \lambda \neq 0$.

(a) Since P_h is nonvanishing, choose $r \in (e^{-\varepsilon_0}, 1)$ such that $\widetilde{Q}$ is nonvanishing on $r \leq |z| \leq 1$. The Laurent coefficient formula gives

$$|G_h(nh)| \leq r^{|n|} \max_{|z|=r} |\widetilde{Q}(z)|^{-1}, \qquad n \leq 0.$$

(b) Since $\mathcal{K}$ is analytic in an annulus around the unit circle and $\widetilde{Q}$ is nonvanishing on the circle, $\widetilde{Q}^{-1}$ admits analytic continuation to a Δ-domain at $z = 1$, apart from the branch point $z = 1$; every other point on the circle has a holomorphic neighborhood. Since $\mathcal{K}(1) = 0$ and $\mathcal{K}$ is analytic, the first noninteger term in $\widetilde{Q}^{-1}$ is $-a_* h^{-\alpha_*}(1-z)^{\alpha_*}/\lambda^2$. The negative-power Laurent part is analytic across $z = 1$ and does not contribute to the leading algebraic coefficient on the positive side. The asymptotic transfer theorem, applied as in the proof of Theorem 3.13 [21], gives (3.14).

□

If k has only algebraic decay, a singularity of $\mathcal{K}$ at $z = 1$ may change the leading coefficient in (3.14); dyadic-block regularity is still determined by Corollary 3.4 under its assumptions.

Theorem 3.15. *Let $0 \leq \gamma < \alpha_*$, $s = \gamma + \frac{1}{2}$, and $k \in \mathcal{W}_s$. Assume that P_h^0 is nonvanishing on I_h and has a Green kernel $G_h^0 \in \mathcal{W}_s$. Set $K = hk_o$, $B = G_h^0 \star K$, and*

$$q := \max_{y\in I_h} \frac{|(T_s k)(y)|}{|P_h^0(y)|}.$$

If $q < 1$, then P_h is nonvanishing and the Green kernel of (3.1) *is*

$$G_h = \sum_{r=0}^{\infty} (-1)^r B^{\star r} \star G_h^0.$$

The series converges in $\mathcal{W}_s$, and the unique solution in X_γ is $u = \sum_{r\geq 0} (-1)^r B^{\star r} \star G_h^0 \star f$. If, in addition, $\rho_1 := |B|_1 < 1$, then $\|u\|_p \leq \frac{|G_h^0|_1}{1-\rho_1}\|f\|_p$ for $1 \leq p \leq \infty$.

Proof From $c_h = c_h^0 \star (\delta + B)$ and (3.3), $\sigma[B] = -iT_s k/P_h^0$. Applying the spectral-radius formula in Banach algebras [18], as in the proof of Lemma 2.10, yields $\lim_{r\to\infty} |||B^{\star r}|||^{1/r} = q < 1$. Hence the Neumann series converges in $\mathcal{W}_s$; the remaining conclusions follow from Theorem 3.2 and Young's inequality. □

In this part, assume $k = 0$, $0 < \alpha_m < \cdots < \alpha_1 \le 2$, and $\mu := \lceil \alpha_1 \rceil \in \{1, 2\}$. For $v : h\mathbb{N}_0 \to \mathbb{C}$, let $\bar{v}$ denote the extension of v by zero to $\mathbb{T}_h$, and consider

$$(\nabla^{\alpha}_{h,0} v)(nh) = (b^{(\alpha)} \star \bar{v})(nh) = \sum_{k=0}^{n} b^{(\alpha)}(kh)\, v((n-k)h), \qquad n \ge 0,$$

the Grünwald–Letnikov fractional difference with initial point 0 (the discrete Riemann–Liouville form) [6, 8]. Let the initial data be $u_0 \in \mathbb{C}$ and, if $\mu = 2$, also $u_1 \in \mathbb{C}$. Define $T(nh) = u_0$ if $\mu = 1$ and $T(nh) = u_0 + nhu_1$ if $\mu = 2$. For each j, set $T_j = u_0$ if $\alpha_j \le 1$ and $T_j = T$ if $1 < \alpha_j \le 2$. The Caputo-type difference is defined by ${}^{C}\nabla^{\alpha_j}_{h,0} u = \nabla^{\alpha_j}_{h,0}(u - T_j)$.

Problem 3.16. Find $u : h\mathbb{N}_0 \to \mathbb{C}$ satisfying

$$\sum_{j=1}^{m} a_j\, {}^{C}\nabla^{\alpha_j}_{h,0} u(nh) + \lambda u(nh) = f(nh), \quad n \ge \mu,$$
$$u(0) = u_0, \quad u(h) = u_0 + hu_1 \text{ if } \mu = 2.$$

Theorem 3.17. *Assume* $c^0_h(0) = \sum_{j=1}^{m} a_j h^{-\alpha_j} + \lambda \neq 0$.

(a) *For every f and initial data, Problem 3.16 has a unique solution u given by*

$$u = T + w, \qquad \bar{w} = \mathsf{G}^{\mathrm{caus}}_h \star \bar{F},$$

where $\mathsf{G}^{\mathrm{caus}}_h(nh) = [z^n] q_h(z)^{-1}$ *($n \ge 0$),* $\mathsf{G}^{\mathrm{caus}}_h(nh) = 0$ *($n < 0$),* $F(nh) = 0$ *($n < \mu$), and for $n \ge \mu$,*

$$F(nh) = f(nh) - \lambda T(nh) - u_1 \sum_{j:\, \alpha_j \le 1} a_j \frac{h^{1-\alpha_j}\Gamma(n+1-\alpha_j)}{\Gamma(2-\alpha_j)\Gamma(n)},$$

If $\mu = 1$, the sum is empty.

(b) *Let $\mathcal{S}_0 : f \mapsto u$ be the solution operator corresponding to zero initial data. Then*

$$\text{(H)} \iff \mathsf{G}^{\mathrm{caus}}_h \in \ell_1(h\mathbb{N}_0) \iff \mathcal{S}_0 \in \mathcal{B}\big(\ell_\infty(h\mathbb{N}_0)\big).$$

In this case, $\|\mathcal{S}_0\|_{\ell_p \to \ell_p} \le |\mathsf{G}^{\mathrm{caus}}_h|_1$ for every $p \in [1, \infty]$.

(c) *Under* (O), *$q_h(z) \neq 0$ ($\forall h > 0$, $|z| \le 1$); no additional mesh-size restriction of the form $\lambda h^{\alpha} \le C$ is needed.*

Proof (a) Set $w = u - T$ on $h\mathbb{N}_0$; the initial conditions give $w(nh) = 0$ for $n < \mu$. For $\alpha_j > 1$, ${}^{C}\nabla_{h,0}^{\alpha_j} u = \nabla_{h,0}^{\alpha_j} w$. If $\mu = 2$ and $\alpha_j \le 1$, the term $u_1 \nabla_{h,0}^{\alpha_j} \mathrm{id}$ occurs, where $\mathrm{id}(nh) = nh$ and

$$\nabla_{h,0}^{\alpha} \mathrm{id}(nh) = h^{1-\alpha} \frac{\Gamma(n+1-\alpha)}{\Gamma(2-\alpha)\Gamma(n)} \quad (n \ge 1).$$

The convolution equation reduces to the triangular system $c_h^0 \star \bar{w} = \bar{F}$, with $w(nh) = F(nh) = 0$ for $n < \mu$. Since $c_h^0(0) \neq 0$, formal power-series inversion gives the unique causal solution $\bar{w} = \mathsf{G}_h^{\mathrm{caus}} \star \bar{F}$.

(b) For zero initial data, $\mathcal{S}_0$ is causal convolution with $\mathsf{G}_h^{\mathrm{caus}}$. Young's inequality gives $\mathsf{G}_h^{\mathrm{caus}} \in \ell_1 \Rightarrow \mathcal{S}_0 \in \mathcal{B}(\ell_\infty)$. Conversely, choose $f(nh) = \overline{\operatorname{sgn} \mathsf{G}_h^{\mathrm{caus}}((N-n)h)}$ for $\mu \le n \le N$ and evaluate at Nh to obtain

$$\sum_{j=0}^{N-\mu} |\mathsf{G}_h^{\mathrm{caus}}(jh)| \le \|\mathcal{S}_0\|_{\infty\to\infty}.$$

By the one-sided Wiener lemma [18, 19], $\mathsf{G}_h^{\mathrm{caus}} \in \ell_1$ is equivalent to (H). The ℓ_p bounds follow from Young's inequality.

(c) The assertion follows from Theorem 3.11.

□

When $k = 0$, the continuous Green kernel is defined by the multivariate Mittag–Leffler expansion [1, 23]:

$$\mathcal{G}(t) = \frac{1}{a_1} \sum_{r=0}^{\infty} (-1)^r \sum_{|\ell|=r} \binom{r}{\ell} X_\ell \, \frac{t^{\beta_\ell - 1}}{\Gamma(\beta_\ell)}, \qquad X_\ell = \Big(\frac{\lambda}{a_1}\Big)^{\ell_0} \prod_{j=2}^{m} \Big(\frac{a_j}{a_1}\Big)^{\ell_j}, \quad t > 0. \tag{3.15}$$

Set

$$\delta_0 := \begin{cases} \min\{\alpha_1, \alpha_1 - \alpha_2\}, & m \ge 2, \\ \alpha_1, & m = 1, \end{cases} \qquad \Xi := \frac{|\lambda| + \sum_{j=2}^{m} |a_j|}{|a_1|},$$

$$C_\ell := \frac{1}{|a_1|} \binom{|\ell|}{\ell} |X_\ell|, \qquad \beta_\ell \ge \alpha_1 + \delta_0 |\ell|, \qquad \sum_{|\ell|=r} \binom{r}{\ell} |X_\ell| = \Xi^r.$$

Stirling's bound $\Gamma(\beta) \ge \sqrt{2\pi/\beta}(\beta/e)^\beta$ shows that, for $\kappa \ge 0$, $T > 0$, the series

$$\Phi_\kappa(t) = \sum_{\ell \in \mathbb{N}_0^m} C_\ell \, \frac{t^{\beta_\ell - 1} \max\{\beta_\ell, 1\}^\kappa}{\Gamma(\beta_\ell)}$$

converges uniformly on every interval $[t_0, T] \subset (0, \infty)$.

Theorem 3.18. *Assume $k = 0$, $0 < t_0 < T$, and* (H) *holds for all sufficiently small $h > 0$. Then there exist $h_0, C > 0$ such that, for every $h \in (0, h_0]$, $n \in \mathbb{N}$, $nh \in [t_0, T]$,*

$$\Big| \frac{G_h^0(nh)}{h} - \mathcal{G}(nh) \Big| \le C\, h. \tag{3.16}$$

Proof Choose $\rho \in (0,1)$ and $h_0 > 0$ such that $\Xi\rho^{\delta_0/2} \le \frac{1}{2}$, $h_0 \le \rho/(2e)$, and $\vartheta_{h_0} < 1$. By decreasing h_0 if necessary, we may ensure that (H) holds for $h \le h_0$. By (3.12), the term indexed by ℓ in $G_h^0(nh)/h$ is

$$d_\ell = \frac{(-1)^{|\ell|}}{a_1}\binom{|\ell|}{\ell} X_\ell h^{\beta_\ell - 1}\frac{\Gamma(n+\beta_\ell)}{\Gamma(\beta_\ell)\Gamma(n+1)}.$$

Let $g_\ell(t)$ be the corresponding term in (3.15), with $t = nh \in [t_0, T]$, and set $B_\ell = \max\{\beta_\ell^2, 1\}$. Lemma 2.11(b) yields $d_\ell = g_\ell(t)(1+\varepsilon_\ell)$.

If $B_\ell \le n$, then $|\varepsilon_\ell| \le eB_\ell h/t_0$, so

$$\sum_{B_\ell \le n} |d_\ell - g_\ell(t)| \le \frac{eh}{t_0}\Phi_2(t).$$

If $B_\ell > n$, then $\beta_\ell > 1$. Lemma 2.11(a) gives

$$|d_\ell| \le C_\ell \Psi(\beta_\ell), \qquad \Psi(\beta) := \frac{(T + h_0\beta)^{\beta-1}}{\Gamma(\beta)}.$$

By Stirling's estimate, for sufficiently large β,

$$\beta^2\Psi(\beta) \le C\beta^{5/2}\rho^\beta \le C'\rho^{\beta/2}.$$

Since $\beta_\ell \ge \delta_0|\ell|$ and $\sum_{|\ell|=r}\binom{r}{\ell}|X_\ell| = \Xi^r$,

$$\Psi_* := \sum_\ell C_\ell \beta_\ell^2 \Psi(\beta_\ell) < \infty.$$

Moreover, $B_\ell h/t > 1$ on this index set, and hence

$$\sum_{B_\ell > n} |d_\ell - g_\ell(t)| \le \frac{h}{t_0}\big(\Psi_* + \Phi_2(t)\big).$$

Combining the two bounds and the uniform convergence of Φ_2 on $[t_0, T]$ gives (3.16). □

Corollary 3.19. *Assume $m = 1$, $a_1 = 1$, $\alpha \in (0,2]$, $\lambda > 0$, $f = 0$, $u_0 = 1$, and $u_1 = 0$ if $\mu = 2$. The solution of Problem 3.16 is*

$$u_h(nh) = 1 - \lambda\sum_{j=0}^{n-\mu} G_h^0(jh) \quad (n \ge \mu).$$

For $0 < t_0 < T$, there exists $C > 0$ such that $|u_h(nh) - E_\alpha(-\lambda(nh)^\alpha)| \le Ch$ ($nh \in [t_0, T]$, $h > 0$ sufficiently small), where $E_\alpha(z) = \sum_{r\ge 0} z^r/\Gamma(\alpha r + 1)$.

Proof The solution formula follows from Theorem 3.17(a) with $T \equiv 1$ and $F = -\lambda$. Summing (3.12) over the time index and applying the same Gamma-ratio estimate as in the proof of Theorem 3.18 with $\beta_r = \alpha(r+1)+1$ yields an $O(h)$ error on $[t_0, T]$. □

3.4 Reversed-Argument Convolution and Kernel Recovery

Interchanging the two arguments of h-Fourier–sine convolution introduces a reflection operator and a system coupling symmetric frequency pairs.

Consider the equation

$$L'_h u := \sum_{j=1}^{m} a_j \nabla_h^{\alpha_j} u + \lambda u + k \underset{\mathcal{F}T_s}{*} u = f \qquad \text{on } \mathbb{T}_h. \tag{3.17}$$

By Proposition 2.9, the operator with reversed convolution arguments on X_γ takes the form

$$L'_h = C(a') + C(b')R, \qquad a' = c_h^0 + \tfrac{h}{2}k, \quad b' = -\tfrac{h}{2}k,$$

where $C(c)u := c \star u$. Set $\widehat{k} = \mathcal{F}k$ and

$$\mathsf{A}(y) = \sigma[a'](y) = P_h^0(y) + \tfrac{1}{2}\widehat{k}(y), \qquad \mathsf{B}(y) = \sigma[b'](y) = -\tfrac{1}{2}\widehat{k}(y),$$

$$\begin{aligned}\Delta_h(y) &= \mathsf{A}(y)\mathsf{A}(-y) - \mathsf{B}(y)\mathsf{B}(-y)\\ &= P_h^0(y)P_h^0(-y)\\ &\quad + \frac{1}{2}\big(P_h^0(y)\widehat{k}(-y) + P_h^0(-y)\widehat{k}(y)\big).\end{aligned}$$

We have $\Delta_h(-y) = \Delta_h(y)$; unlike (3.1), Δ_h depends on the entire kernel k, not only its odd part. For $\eta \in \{0, \pi/h\}$, $\mathsf{A}(-\eta) = \mathsf{A}(\eta)$, $\mathsf{B}(-\eta) = \mathsf{B}(\eta)$; hence $\Delta_h(\eta) = (\mathsf{A}-\mathsf{B})(\eta)(\mathsf{A}+\mathsf{B})(\eta)$, where $(\mathsf{A}+\mathsf{B})(\eta) = P_h^0(\eta)$.

Theorem 3.20. *Let $0 \le \gamma < \alpha_*$, $s = \gamma + \frac{1}{2}$, and $k \in \mathcal{W}_s$.*

(a) *If $\Delta_h(y) \neq 0$ for all $y \in [0, \pi/h]$, then for every $f \in X_\gamma$, equation (3.17) has a unique solution $u \in X_\gamma$. The same conclusion holds on every $\ell_p(\mathbb{T}_h)$, $1 \le p \le \infty$, and the solution has the form*

$$u = g_1 \star f + g_2 \star Rf, \qquad g_1, g_2 \in \mathcal{W}_s,$$
$$\sigma[g_1](y) = \frac{\mathsf{A}(-y)}{\Delta_h(y)}, \qquad \sigma[g_2](y) = -\frac{\mathsf{B}(y)}{\Delta_h(y)}.$$

(b) $\forall p \in [1, \infty]$,

$$L'_h|_{\ell_p(\mathbb{T}_h)} \text{ is bijective} \iff \Delta_h(y) \neq 0 \ (\forall y \in [0, \pi/h]).$$

(c) *If $\Delta_h(\eta) = 0$ at some $\eta \in (0, \pi/h)$ or $P_h^0(\eta) = 0$ at $\eta \in \{0, \pi/h\}$, then $\ker(L'_h|_{\ell_\infty(\mathbb{T}_h)}) \neq \{0\}$.*

(d) *If $a_j, \lambda \in \mathbb{R}$ and k is real-valued, then $\Delta_h(y) = |P_h^0(y) + \frac{1}{2}\widehat{k}(y)|^2 - \frac{1}{4}|\widehat{k}(y)|^2$ is real-valued. The condition $|\widehat{k}(y)| < |P_h^0(y)|$ for all y (in particular, $\|k\|_1 < \min_{I_h} |P_h^0|$) implies $\Delta_h > 0$. Under the additional assumptions (O) and $\alpha_1 < 2$, it suffices that $\|k\|_1 < \lambda\kappa(\alpha_1)$.*

Proof Use $RC(c)R = C(Rc)$, $\sigma[Rc](y) = \sigma[c](-y)$, and $\|Ru\|_p = \|u\|_p$ (also on X_γ).

(a) Set $D = a' \star Ra' - b' \star Rb'$, so that $\sigma[D] = \Delta_h$. By Lemma 2.10, $e = D^{\star -1} \in \mathcal{W}_s$. Define

$$g_1 = Ra' \star e, \qquad g_2 = -b' \star e.$$

Since $D = RD$, we have $e = Re$; the convolution–reflection identities give

$$\big(C(g_1) + C(g_2)R\big)L'_h = L'_h\big(C(g_1) + C(g_2)R\big) = I.$$

Lemma 2.8 and Young's inequality ensure boundedness on X_γ and all ℓ_p.

(b) Sufficiency follows from (a). If $\Delta_h(\eta) = 0$ for $0 < \eta < \pi/h$, choose $(x, z) \neq (0, 0)$ in the kernel of the matrix in (c), and set

$$\chi_N(nh) = \mathbf{1}_{\{|n| \le N\}}, \qquad v_N = \chi_N(xe_\eta + ze_{-\eta}).$$

Since $R\chi_N = \chi_N$, the components of $L'_h v_N$ are commutator errors between truncation and convolution with a', b'. For each fixed $j \in \mathbb{Z}$,

$$\frac{\|\tau_j \chi_N - \chi_N\|_p}{\|v_N\|_p} \longrightarrow 0 \quad (1 \le p < \infty), \qquad \|v_N\|_p \asymp N^{1/p}.$$

Since $|a'|_1 + |b'|_1 < \infty$, the dominated convergence theorem gives $\|L'_h v_N\|_p / \|v_N\|_p \to 0$. For $p = \infty$, the exponential solution in (c) lies in the kernel. The case $P_h^0(\eta) = 0$ at an endpoint is handled using e_η.

For the remaining endpoint case $(\mathsf{A} - \mathsf{B})(\eta) = 0$, $P_h^0(\eta) \neq 0$, set

$$\psi_N(nh) = \begin{cases} n/N, & |n| \le N, \\ \operatorname{sgn}(n)(2 - |n|/N), & N < |n| \le 2N, \\ 0, & |n| > 2N, \end{cases} \qquad v_N = e_\eta \psi_N.$$

Then $Rv_N = -v_N$ and $\sigma[a' - b'](\eta) = 0$. For each fixed k,

$$\frac{\|\tau_k \psi_N - \psi_N\|_p}{\|\psi_N\|_p} \longrightarrow 0, \qquad \frac{\|\tau_k \psi_N - \psi_N\|_p}{\|\psi_N\|_p} \le 2 \quad (1 \le p \le \infty).$$

The dominated convergence theorem and $|a' - b'|_1 < \infty$ yield $\|L'_h v_N\|_p / \|v_N\|_p \to 0$. Thus, if Δ_h has a zero, the operator is not bijective.

(c) If $\Delta_h(\eta) = 0$ for $0 < \eta < \pi/h$, then the matrix

$$\begin{pmatrix} \mathsf{A}(\eta) & \mathsf{B}(\eta) \\ \mathsf{B}(-\eta) & \mathsf{A}(-\eta) \end{pmatrix}$$

has a nonzero kernel vector (x, z), hence $L'_h(xe_\eta + ze_{-\eta}) = 0$. If $P_h^0(\eta) = 0$ for $\eta \in \{0, \pi/h\}$, then $L'_h e_\eta = 0$.

(d) For real coefficients,

$$\Delta_h = |P_h^0|^2 + \operatorname{Re}(P_h^0 \overline{\widehat{k}}) \ge |P_h^0|\big(|P_h^0| - |\widehat{k}|\big).$$

Combine $\|\widehat{k}\|_\infty \le \|k\|_1$ with (3.8).

□

When Δ_h vanishes at an endpoint with $\mathsf{A} - \mathsf{B} = 0$ and $P_h^0 \neq 0$, injectivity requires separate consideration. For a bounded sequence v, the Beurling spectrum is defined by

$$\operatorname{sp}(v) = \big\{\xi \in I_h : \ \sigma[c](\xi) = 0 \ \forall c \in \ell_1(\mathbb{T}_h) \text{ such that } c \star v = 0\big\},$$

where the two endpoints $\pm\pi/h$ are identified.

Lemma 3.21. *Let $v \in \ell_\infty(\mathbb{T}_h)$.*

(a) *If $D \in \ell_1(\mathbb{T}_h)$ and $D \star v = 0$, then $\operatorname{sp}(v) \subset \{\xi : \sigma[D](\xi) = 0\}$.*

(b) *If $\operatorname{sp}(v)$ is finite, then $v = \sum_{\xi \in \operatorname{sp}(v)} c_\xi e_\xi$; in particular, if $\operatorname{sp}(v) \subset \{0, \pi/h\}$, then $v(nh) = c_0 + c_1(-1)^n$.*

Proof Part (a) follows from the definition of the spectrum $\operatorname{sp}(v)$. Part (b) follows as in the local Wiener division theorem [18] and the structure theorem for distributions with finite support [25, Theorem 4.19]: from $\mathcal{F}v = \sum_{\xi \in E} \sum_{r=0}^{m_\xi} a_{\xi,r} \delta_\xi^{(r)}$ one obtains $v(nh) = \sum_{\xi \in E} p_\xi(n) e^{inh\xi}$. Since v is bounded, $\deg p_\xi = 0$. □

Theorem 3.22. *For $k \in \ell_1(\mathbb{T}_h)$,*

$$\ker\bigl(L_h'|_{\ell_\infty(\mathbb{T}_h)}\bigr) = \{0\} \quad \Longleftrightarrow \quad \begin{cases} \Delta_h(\eta) \neq 0, & \forall \eta \in (0, \pi/h), \\ P_h^0(\eta) \neq 0, & \forall \eta \in \{0, \pi/h\}. \end{cases} \tag{3.18}$$

If the conditions on the right-hand side of (3.18) *hold but $(\mathsf{A} - \mathsf{B})(\eta) = 0$ for some $\eta \in \{0, \pi/h\}$, then* (3.17) *has at most one bounded solution for each f, although L_h' is not surjective on any $\ell_p(\mathbb{T}_h)$, $1 \le p \le \infty$.*

Proof Necessity follows from Theorem 3.20(c). Conversely, assume the conditions in (3.18) and $L_h' v = 0$ with $v \in \ell_\infty$. The convolution–reflection identities give

$$D \star v = 0, \quad D = a' \star Ra' - b' \star Rb', \quad \sigma[D] = \Delta_h.$$

By periodicity, all zeros of Δ_h then lie in $\{0, \pi/h\}$. By Lemma 3.21,

$$v(nh) = c_0 + c_1(-1)^n, \qquad L_h' v = c_0 P_h^0(0) + c_1 P_h^0(\pi/h)(-1)^n.$$

Hence $c_0 = c_1 = 0$. If $(\mathsf{A} - \mathsf{B})(\eta) = 0$ at an endpoint, Theorem 3.20(b) rules out bijectivity on every ℓ_p; injectivity on $\ell_p \subset \ell_\infty$ then implies that the operator is not surjective. □

Theorem 3.23. *Let $0 \le \gamma < \alpha_*$, $s = \gamma + \frac{1}{2}$, and suppose that a_j, α_j, λ are known.*

(a) *For $k, \tilde{k} \in \mathcal{W}_s$, let $L_h^{(k)}$ and $L_h^{(\tilde{k})}$ denote the operators in* (3.1). *Then*

$$L_h^{(k)}|_{X_\gamma} = L_h^{(\tilde{k})}|_{X_\gamma} \quad \Longleftrightarrow \quad k_o = \tilde{k}_o.$$

The even part of k cannot be identified from this operator.

(b) *Assume $(u, f) \in \ell_1(\mathbb{T}_h) \times X_\gamma$ satisfies* (3.1) *with unknown $k \in \mathcal{W}_s$. Set $g = f - c_h^0 \star u = hk_o \star u \in \ell_1(\mathbb{T}_h)$ and $Z = \{y \in I_h : \sigma[u](y) = 0\}$. Then k_o is uniquely determined by (u, f) in $\mathcal{W}_s$ if and only if $\operatorname{int}(Z \cap (-Z)) = \varnothing$. Under this condition, $\sigma[k_o]$ is the unique continuous extension to I_h of*

$$y \mapsto \frac{\sigma[g](y)}{h\,\sigma[u](y)} \quad (y \notin Z), \qquad y \mapsto -\frac{\sigma[g](-y)}{h\,\sigma[u](-y)} \quad (y \notin -Z), \tag{3.19}$$

and $k_o(nh) = \frac{h}{2\pi}\int_{I_h} \sigma[k_o](y)e^{inhy}\,dy$.

(c) *(Recovery of* λ *from a single observation.) For the causal equation with zero initial data* $c_h^0 \star \bar{u} = \bar{f}$ *on* $h\mathbb{N}_0$ *(*$\forall n \geq 0$*), if* $f(0) \neq 0$*, then* $u(0) = f(0)/c_h^0(0) \neq 0$ *and*

$$\lambda = \frac{f(0)}{u(0)} - \sum_{j=1}^{m} a_j h^{-\alpha_j}.$$

For $|\varepsilon| \leq |u(0)|/2$*, a perturbation* $u(0) \mapsto u(0) + \varepsilon$ *yields* $|\Delta\lambda| \leq 2|f(0)||\varepsilon|/|u(0)|^2$*. For fixed* $f(0) \neq 0$*,* $|u(0)| = |f(0)||a_1|^{-1}h^{\alpha_1}(1 + o(1))$ *as* $h \to 0$*, so the error amplification factor is of order* $h^{-2\alpha_1}$*.*

Proof (a) By (3.2), $L_h^{(k)}$ depends only on k_o. Conversely, applying the difference of the two operators to $\delta \in X_\gamma$ gives $h(k_o - \tilde{k}_o) = 0$.

(b) By (3.2),

$$h\sigma[k_o](y)\sigma[u](y) = \sigma[g](y), \qquad \sigma[k_o](-y) = -\sigma[k_o](y).$$

Thus (3.19) defines an odd function on $I_h \setminus (Z \cap (-Z))$. If $\operatorname{int}(Z \cap (-Z)) = \varnothing$, continuity guarantees uniqueness. Otherwise, choose $J \subset (0, \pi/h)$ and $0 \neq \varphi \in C_c^\infty(J)$ such that $J \cup (-J) \subset Z \cap (-Z)$. The odd function $\Phi(y) = \varphi(y) - \varphi(-y)$ has Fourier coefficients $d \in \mathcal{W}_s$ and satisfies $\sigma[d \star u] = \Phi\sigma[u] = 0$, implying nonuniqueness.

(c) At $n = 0$, $c_h^0(0)u(0) = f(0)$ gives the recovery formula for λ. If $|\varepsilon| \leq |u(0)|/2$,

$$|\Delta\lambda| \leq \frac{2|f(0)||\varepsilon|}{|u(0)|^2}.$$

The asymptotic relation follows from $c_h^0(0) = a_1 h^{-\alpha_1}(1 + o(1))$.

□

For known $u \in \ell_1(\mathbb{T}_h)$, observations at opposite frequencies can be combined to recover the odd part of the kernel. Set $U = \sigma[u]$, $G = \sigma[g]$, and

$$D_u(y) = |U(y)|^2 + |U(-y)|^2, \qquad d_*(u) = \min_{y \in I_h} \sqrt{D_u(y)}.$$

We have $d_*(u) > 0$ if and only if $Z \cap (-Z) = \varnothing$, where $Z = \{y : U(y) = 0\}$. For $g \in \ell_2(\mathbb{T}_h)$, $G = \mathcal{F}g/h$, and the Fourier transform of the recovered kernel is interpreted in the L_2 sense.

Theorem 3.24. *Assume that* $u \in \ell_1(\mathbb{T}_h)$ *is known.*

(a) *Suppose* $d_*(u) > 0$*. Define* $\mathcal{R}_u^{\mathrm{LS}} : \ell_2(\mathbb{T}_h) \to \ell_2(\mathbb{T}_h)$ *by*

$$\sigma[\mathcal{R}_u^{\mathrm{LS}} g](y) = \frac{\overline{U(y)}G(y) - \overline{U(-y)}G(-y)}{hD_u(y)}. \tag{3.20}$$

The recovered kernel is odd, and $\mathcal{R}_u^{\mathrm{LS}}(hk_o \star u) = k_o$ *for every odd kernel* $k_o \in \ell_2(\mathbb{T}_h)$*. Moreover,*

$$\|\mathcal{R}_u^{\mathrm{LS}} g - \mathcal{R}_u^{\mathrm{LS}} \widetilde{g}\|_2 \le \frac{\sqrt{2}}{h\, d_*(u)} \|g - \widetilde{g}\|_2, \qquad \|\mathcal{R}_u^{\mathrm{LS}}\|_{2\to 2} = \frac{\sqrt{2}}{h\, d_*(u)}. \tag{3.21}$$

(b) *If* $d_*(u) > 0$*,* $s \ge \frac{1}{2}$*, and* $u, g \in \mathcal{W}_s$*, then* $\mathcal{R}_u^{\mathrm{LS}} g \in \mathcal{W}_s$*.*

(c) *If* $d_*(u) = 0$*, then the map* $k_o \mapsto hk_o \star u$ *on the class of odd kernels is not bounded below in* ℓ_2*:* $\forall s \ge \frac{1}{2}$*,* $\exists d_\varepsilon \in \mathcal{W}_s \setminus \{0\}$ *odd, with*

$$\frac{\|hd_\varepsilon \star u\|_2}{\|d_\varepsilon\|_2} \to 0, \qquad \frac{\|\sigma[hd_\varepsilon \star u]\|_\infty}{\|\sigma[d_\varepsilon]\|_\infty} \to 0.$$

Thus $Z \cap (-Z) = \varnothing$ *is necessary and sufficient for Lipschitz-stable recovery in* ℓ_2*; uniqueness in* $\mathcal{W}_s$ *requires only* $\operatorname{int}(Z \cap (-Z)) = \varnothing$ *(Theorem 3.23(b)).*

Proof (a) For an odd kernel k_o, the exact data satisfy

$$\begin{pmatrix} G(y) \\ G(-y) \end{pmatrix} = h \begin{pmatrix} U(y) \\ -U(-y) \end{pmatrix} \sigma[k_o](y).$$

Formula (3.20) recovers $\sigma[k_o]$; its numerator is odd, whereas D_u is even. On the frequency pair $\{y, -y\}$, the recovery map has singular value $\sqrt{2}/(h\sqrt{D_u(y)})$. Plancherel's identity yields (3.21); data spectrally concentrated near a minimizer of D_u asymptotically attain the operator norm.

(b) By Lemma 2.7, $\mathcal{W}_s$ is closed under convolution, reflection, and complex conjugation. The numerator and D_u in (3.20) correspond to kernels in $\mathcal{W}_s$. Since $D_u \ge d_*(u)^2 > 0$, Lemma 2.10 shows that D_u^{-1} also belongs to this algebra.

(c) Choose y_0 such that $U(y_0) = U(-y_0) = 0$. Choose a nonzero smooth odd periodic function $\Phi_\varepsilon \not\equiv 0$ supported in shrinking neighborhoods of $\{y_0, -y_0\}$ (using a local odd smooth function at an identified endpoint if necessary), and set $\sigma[d_\varepsilon] = \Phi_\varepsilon$. Then $d_\varepsilon \in \mathcal{W}_s$ for every $s \ge \frac{1}{2}$, and

$$\sigma[hd_\varepsilon \star u] = h\Phi_\varepsilon U, \qquad \frac{\|hd_\varepsilon \star u\|_2}{\|d_\varepsilon\|_2},\ \frac{\|\sigma[hd_\varepsilon \star u]\|_\infty}{\|\sigma[d_\varepsilon]\|_\infty} \le h \sup_{\operatorname{supp}\Phi_\varepsilon} |U| \to 0.$$

□

4 Application to Green-Kernel Approximation and ECG Signal Denoising

Motivated by the Green-kernel representation (3.4) and Corollary 3.4, we consider a compensated combination of Green kernels to improve kernel decay and reduce truncation errors. The application is investigated using ECG signals.

The ECG-ID dataset [26, 27] comprises 310 recordings from 90 subjects, each lasting 20 seconds and sampled at 500 Hz. The filtered channel is used as a surrogate reference signal, while the raw channel is retained for comparison. Three types of synthetic noise are considered: baseline wander (BW), muscle artifacts (MA), and

simulated electrode-motion artifacts (EM-like), at SNR levels of 0, 6, and 12 dB. The calibration set contains 92 recordings from 20 subjects, and the test set contains 218 recordings from the remaining 70 subjects.

For the mean-centered reference signal x, let $y = x + e$, $y_0 = \mathcal{H}_0 y$, where $\mathcal{H}_0$ is a high-pass filter with transfer function

$$H_0(\nu) = \frac{(\nu/0.30)^4}{1 + (\nu/0.30)^4}, \qquad \nu \geq 0.$$

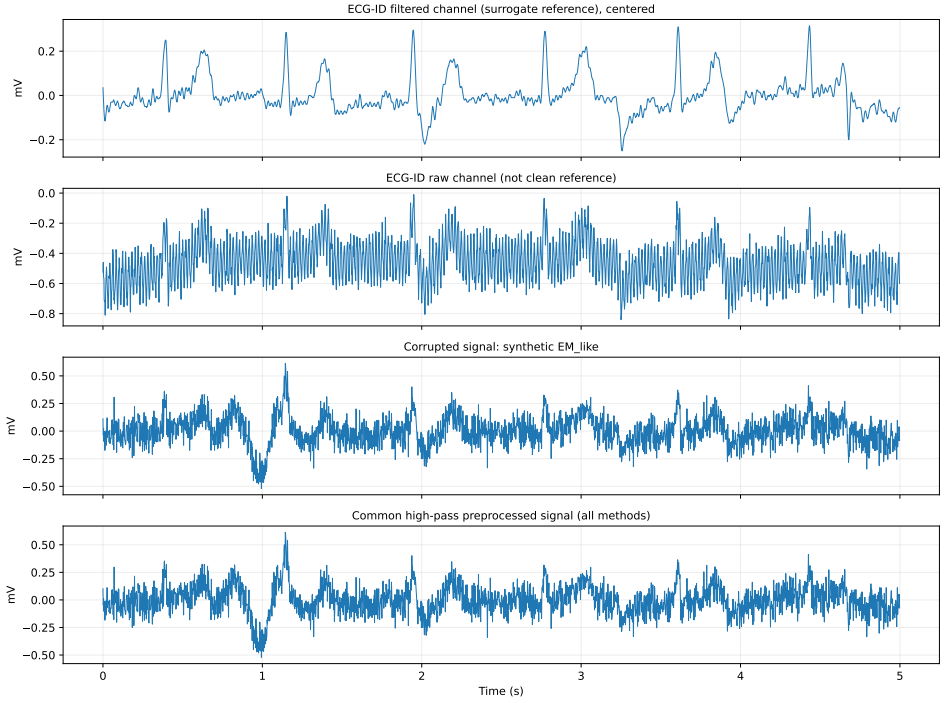

Fig. 3 ECG-ID signals: filtered channel, raw channel, EM-like corrupted signal, and signal after high-pass preprocessing.

For $0 < \alpha < \frac{1}{2}$, let

$$\mathcal{A}_h = a\nabla_h^\alpha + \mathcal{K}_h, \qquad \mathcal{K}_h u = u \underset{\mathcal{F}T_s}{*} k, \qquad \mathcal{R}_t = (I + t\mathcal{A}_h)^{-1}.$$

Let $Q_h = a\omega_{\alpha,h} - iT_s k$ as in (3.3). Assume that $1 + tQ_h$ and $1 + \chi tQ_h$ are nonvanishing on I_h. For $\chi > 1$, by (3.4),

$$\mathcal{C}_{t,\chi} = \frac{\chi\mathcal{R}_t - \mathcal{R}_{\chi t}}{\chi - 1}, \qquad \sigma[c_{t,\chi}](y) = \frac{1 + (\chi + 1)tQ_h(y)}{(1 + tQ_h(y))(1 + \chi tQ_h(y))}. \tag{4.1}$$

When $k = 0$, let $\mu = tah^{-\alpha}$, $q = (1 - z)^{\alpha}$. As $z \to 1$,

$$\frac{1}{1 + \mu q} = 1 - \mu q + O(q^2), \qquad \frac{1 + (\chi + 1)\mu q}{(1 + \mu q)(1 + \chi \mu q)} = 1 - \chi \mu^2 q^2 + O(q^3).$$

For $\mu \neq 0$, the asymptotic transfer theorem [21] and Lemma 2.7(c) yield, as $n \to +\infty$:

$$\begin{aligned} r_t(nh) &\sim -\frac{\mu}{\Gamma(-\alpha)} n^{-1-\alpha}, \qquad s_0(r_t) = \alpha + \tfrac{1}{2}, \\ c_{t,\chi}(nh) &\sim -\frac{\chi \mu^2}{\Gamma(-2\alpha)} n^{-1-2\alpha}, \quad s_0(c_{t,\chi}) = 2\alpha + \tfrac{1}{2}. \end{aligned} \tag{4.2}$$

Consequently, the ℓ_1 truncation errors of the two kernels are of orders $M^{-\alpha}$ and $M^{-2\alpha}$, respectively.

For the ECG filter associated with (4.1), set $h = 1/500$, $\chi = 2$, $\theta = 2\pi\nu/500$, and

$$q_{\alpha,f,\eta}(\theta) = \frac{(1 - e^{-i\theta})^{\alpha}}{[2\sin(\pi f/500)]^{\alpha}} - 2i\eta \sin\theta, \qquad C_f(\theta) = \frac{1 + 3q_{\alpha,f,\eta}(\theta)}{(1 + q_{\alpha,f,\eta}(\theta))(1 + 2q_{\alpha,f,\eta}(\theta))}.$$

The term $-2i\eta \sin\theta$ corresponds to an odd interaction kernel supported at $\pm h$. The filter is defined by

$$B(\theta) = \big(1 - |C_{f_L}(\theta)|^2\big)|C_{f_H}(\theta)|^2, \qquad \widehat{x} = (1 - \rho)y_0 + \rho\, \mathcal{F}^{-1}[B\, \mathcal{F} y_0]\,. \tag{4.3}$$

The parameters are calibrated on the calibration set subject to $|C_f(\theta)| \leq 1$ on the frequency grid; $\rho \in [0, 1]$ is determined from the spectral characteristics of y_0. The comparison methods include db4 wavelet soft-thresholding (DWT) [28, 29] and segment-based non-local means (NLM) [30].

The SNR improvement is computed as

$$\Delta\mathrm{SNR} = 10 \log_{10} \frac{\|y - x\|_2^2}{\|\widehat{x} - x\|_2^2} \quad \text{(dB)}.$$

Table 1 reports the subject-wise mean over 70 subjects across nine noise–SNR combinations, together with 95% confidence intervals computed at the subject level.

Table 1 SNR improvement on the ECG-ID test set (70 subjects; subject-wise mean values).

Method	$\overline{\Delta\mathrm{SNR}}$ (dB)	95% confidence interval
DWT [28]	4.936	[4.872, 5.000]
NLM [30]	4.729	[4.659, 4.799]
Adaptive compensated Green resolvent (CGR)	3.670	[3.592, 3.747]
High-pass preprocessing (HP)	2.376	[2.349, 2.403]
Convex-projected Green combination (PSGF)	1.934	[1.866, 2.002]

CGR achieves a mean SNR improvement 1.294 dB higher than HP, but 1.266 and 1.059 dB lower than DWT and NLM, respectively. The selected CGR parameters are $\alpha = 0.4$, $f_L = 0.2$ Hz, $f_H = 35$ Hz, and $\eta = 0$. For $\alpha = 0.4$ and $k = 0$, the thresholds in (4.2) are 0.9 and 1.3, respectively.

Declarations

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Author contributions. T.K.H. developed the mathematical framework, established the theoretical results, conducted the numerical experiments, and prepared the manuscript. The author read and approved the final manuscript.

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