Supplemental Material

Optimal entanglement swapping with one auxiliary photon: exact fidelity limit and purity requirements

Annan Chen

The main manuscript contains all core theorem proofs in Appendices A–F. This supplement supplies the fixed-instrument acceptance policies, physical comparisons, calibration and mixed-data calculations, and nonzero detection tables for the representative source $c = 0.9$. All probabilities condition on supplying the required photons. The relevant data are available on Zenodo [1].

Contents

S1 Complete fixed-instrument acceptance frontiers . . . 1
S2 Physical source and filtering comparisons . . . 2
S2.1 A centered Gaussian auxiliary with optimized width . . . 2
S2.2 Independent auxiliary count and perfect conditional fidelity . . . 3
S2.3 Target-optimized independent-center Gaussian filters . . . 3
S3 A packet distiller composed with encoded swapping . . . 4
S4 Calibration and mixed-data checks . . . 5
S4.1 The small-angle calibration floor . . . 5
S4.2 Symmetric depolarization of the two data photons . . . 6
S5 Nonzero detection tables at $c = 0.9$ . . . 6
S5.1 One auxiliary: nonzero patterns . . . 6
S5.2 Two auxiliaries: nonzero patterns . . . 7

S1 Complete fixed-instrument acceptance frontiers

Let $p_y = \operatorname{Tr} J_y$ and $w_y = \max_{\Phi \text{ ME}} \langle\Phi|J_y|\Phi\rangle$ for every raw spatial outcome y. Both are zero for dark outcomes. Define the fixed-control LU frontier by

$$P^*(F_{\text{tar}}; U, H) = \max_{0 \le \omega_y \le 1} \left\{ \sum_y \omega_y p_y : \sum_y \omega_y (w_y - F_{\text{tar}} p_y) \ge 0 \right\}. \tag{S1.1}$$

The network, auxiliary and data wave packets remain fixed during this optimization. The magic-basis matrix with columns $(\Phi^+, i\Phi^-, i\Psi^+, \Psi^-)$ is denoted U_{mag}. Every maximally entangled vector has real coefficients in that basis up to one phase. Hence

$$w_y = \lambda_{\max}\big(\operatorname{Re}(U_{\text{mag}}^\dagger J_y U_{\text{mag}})\big). \tag{S1.2}$$

This representation follows from the concurrence characterization of two-qubit maximally entangled vectors [2] and is used for all outcomes, without assuming their matrices have a particular sparsity.

Sort nonzero outcomes by $f_y = w_y/p_y$. Accept in descending order until the fidelity surplus in Eq. (S1.1) is exhausted; only the boundary class need be fractional. To prove optimality, write $b_y = \omega_y p_y$. Moving accepted probability from a lower-f outcome to an unfilled higher-f outcome leaves P unchanged and increases the surplus. Continue these exchanges until all earlier classes are filled. Any remaining surplus buys additional probability. Equal-fidelity ties can be split arbitrarily, and if the full instrument meets the target, the optimum is one. This proves the complete policy solution.

For $U_{\text{opt}}, H = |h_+\rangle\langle h_+|, c = 1/2$, the highest outcome $(1, 1, 0, 0, 1)$ has $p = 1/36$ and $f = 7/8$. The next outcome $(0, 0, 1, 0, 2)$ has $p = 7/144$ and $f = 19/28$. Every other raw outcome has lower recovered fidelity. At target $4/5$ the first outcome supplies surplus $1/480$ and the second consumes $17/2880$ per unit acceptance. Thus its acceptance weight is $6/17$, and

$$P^*(4/5; U_{\text{opt}}, h_+) = \frac{1}{36} + \frac{6}{17}\frac{7}{144} = \frac{55}{1224}. \tag{S1.3}$$

The endpoint itself accepts only the highest outcome. This example shows why the endpoint's one-outcome probability differs from an acceptance frontier at a lower target.

The accompanying data enumerate all $\binom{7}{3} = 35$ outcomes for each five-mode three-photon circuit and all $\binom{9}{4} = 126$ outcomes for the six-mode four-photon comparator of Sec. S2.2. They give p_y, w_y, f_y and the full numerical policy frontier at $c = 1/2, 0.9, 0.99$. The generation uses normalized creation operators: adding a photon to an occupied spatial–internal mode multiplies its amplitude by $\sqrt{n+1}$. Amplitudes are combined only when the complete internal Fock occupation agrees, and are then grouped by spatial PNR occupation. This retains bosonic factorials, all internal labels, and every dark outcome. The matrices obey $\sum_y J_y = I_4/4$. An independent six-assignment tensor contraction checks the selected three-photon outcomes; the eight-mode tilted witness is also tested against its full 120-outcome instrument. These computations verify the displayed realizations and normalization, while the converse arguments above provide the analytic all-network results.

S2 Physical source and filtering comparisons

S2.1 A centered Gaussian auxiliary with optimized width

Define

$$g_\sigma(\tau) = (2\pi\sigma^2)^{-1/4} e^{-\tau^2/(4\sigma^2)}, \quad a(\tau) = g_\sigma(\tau - \Delta/2), \quad b(\tau) = g_\sigma(\tau + \Delta/2), \quad h_u(\tau) = g_{u\sigma}(\tau).$$

Here σ is the intensity standard deviation in the specified coordinate. Width optimization is performed in the chosen time or frequency coordinate; Fourier-conjugate widths transform inversely. Put $\kappa_G = \Delta^2/(8\sigma^2) = -\log c$. Direct Gaussian integration gives

$$v = |\langle a|h_u\rangle|^2 = |\langle b|h_u\rangle|^2 = \frac{2u}{1+u^2} e^{-\kappa_G/(1+u^2)} \le \frac{1+c}{2}. \tag{S2.1}$$

The source has weight $2v/(1+c)$ along h_+ and the remaining weight in $\mathcal{S}^\perp$. It has no h_- component. For $\Delta > 0$ a Gaussian cannot equal the midpoint, whose envelope contains a nonconstant $\cosh(\tau\Delta/(4\sigma^2))$ factor. The inequality is therefore strict. An exact finite-dimensional representation is

$$\begin{aligned} a &= (\sqrt{(1+c)/2}, \sqrt{(1-c)/2}, 0), \\ b &= (\sqrt{(1+c)/2}, -\sqrt{(1-c)/2}, 0), \\ h &= (\sqrt{2v/(1+c)}, 0, \sqrt{1-2v/(1+c)}). \end{aligned}$$

This pure Gaussian source generally has support beyond the two-dimensional space used in Sec. 4.

Use the four-coupler network U_{opt} in Eq. (14). The complete occupation $(1,1,0,0,1)$ has Bell weights, ordered as $\Phi^+, \Phi^-, \Psi^+, \Psi^-$,

$$n_0 = [1 + c^2 + 2(1+c)v]/144, \tag{S2.2}$$

$$n_1 = n_2 = [1 + c^2 - (1+c)v]/72, \tag{S2.3}$$

$$n_3 = (1-c)(1+c-2v)/48. \tag{S2.4}$$

One obtains these weights by summing the six assignments of the three photons to the occupied ports with their internal Gram overlaps; the input Bell amplitudes contribute $1/4$ to the density. In particular,

$$p_G = \frac{4 + c^2 - 2(2-c)v}{72}, \qquad f_{\Phi^+} = \frac{1 + c^2 + 2(1+c)v}{2[4 + c^2 - 2(2-c)v]}. \tag{S2.5}$$

The target is LU-optimal precisely if $4(1+c)v \ge 1 + c^2$ and $2(2-c)v \ge 1 - 2c^2$; otherwise another Bell weight may be larger. This distinction is material at small c.

On the physical domain let $D_G = 4 + c^2 - 2(2-c)v > 0$. Then $df_{\Phi^+}/dv = 3(c^2 + c + 2)/D_G^2 > 0$ and

$$\frac{d\log v}{du} = \frac{1 - u^4 + 2\kappa_G u^2}{u(1+u^2)^2}.$$

There is exactly one positive stationary value of u^2, and v vanishes at both width endpoints. The unique global fixed-target optimum is

$$u_*^2 = \kappa_G + \sqrt{1 + \kappa_G^2}, \qquad v_* = e^{-\kappa_G/2} e^{(\sqrt{1+\kappa_G^2} - 1)/2} \sqrt{\frac{2}{1 + \sqrt{1+\kappa_G^2}}}. \tag{S2.6}$$

For $c \geq 1/4$, $v_* \geq \sqrt{c} \geq 1/2$ gives an optimized fixed-target fidelity above $1/2$. The other normalized Bell weights decrease with v and have maxima below $1/2$ at $v = 0$, so the same width also globally optimizes LU over this width family on that sufficient domain.

The exact identity

$$F_{\text{opt}} - f_{\Phi^+} = \frac{3(c^2 + c + 2)[(1 + c)/2 - v]}{(2 - c + 2c^2)D_G}$$

and the expansion $(1 + c)/2 - v_* = \kappa_G^4/96 - \kappa_G^5/192 + O(\kappa_G^6)$ imply, after substituting $\kappa_G = -\log(1 - t)$,

$$F_{\text{opt}} - f_{\Phi^+}(v_*) = t^4/72 + O(t^5), \qquad 1 - f_{\Phi^+}(v_*) = t^2/3 + O(t^3).$$

Equal widths give instead $1 - f_{\Phi^+} = t^2/2 + O(t^3)$. The probability also changes: $p_G(v) - P_\star = (2 - c)[(1 + c)/2 - v]/36$.

S2.2 Independent auxiliary count and perfect conditional fidelity

The zero-auxiliary ceiling was proved in Sec. A.2, and the universal one-auxiliary ceiling is attained at finite probability. The following two-auxiliary construction attains perfect conditional fidelity at positive probability. Prepare two *independent* auxiliary states, each in the calibrated pure state $h_- = (a - b)/\sqrt{2(1 - c)}$, in input modes $4, 5$. Use

$$U_{\text{pair}} = B_{15}(\pi/4)B_{04}(\pi/4)B_{13}(\pi/4)B_{02}(\pi/4) \tag{S2.7}$$

and accept the complete four-photon outcome $(1, 1, 0, 0, 1, 1)$. There must be two photons in each component $\{0, 2, 4\}$ and $\{1, 3, 5\}$, so logical inputs $00, 11$ cannot contribute. For logical 01 or 10 the first layer has data transmission amplitude $1/2$. A balanced-coupler coincidence sends data internal state x and auxiliary h to

$$K(x, h) = (h \otimes x - x \otimes h)/2 \tag{S2.8}$$

up to a common sign. For $h = h_-$, $K(a, h) = K(b, h)$ and their squared norm is $(1 + c)/4$. Thus both surviving logical amplitudes have the same internal tensor. Including the $1/2$ Bell input amplitude, each odd diagonal and off-diagonal equals $(1 + c)^2/256$, giving

$$J_{\boldsymbol{n}} = \frac{(1 + c)^2}{128}|\Psi^+\rangle\langle\Psi^+|. \tag{S2.9}$$

Consequently, in the specified independently controllable pure-source resource model,

$$F_{\max}^{(0)} = (1 + c^2)/2, \qquad F_{\max}^{(1)} = F_{\text{opt}}(c), \qquad F_{\max}^{(2)} = 1. \tag{S2.10}$$

One auxiliary is the minimum nontrivial resource in this ideal-pure-auxiliary hierarchy. Thus perfect conditional fidelity at $0 < c < 1$ requires at least two independent auxiliary states and is possible with two; the sources are calibrated to a, b.

S2.3 Target-optimized independent-center Gaussian filters

Take normalized incident amplitudes $g_i(\omega) = (2\pi\sigma^2)^{-1/4} \exp[-(\omega - \mu_i)^2/(4\sigma^2)]$ with $\mu_A = -\mu_B$. The two amplitude filters $K_i(\omega) = \exp[-(\omega - \nu_i)^2/(4\kappa^2)]$ have equal width and independent centers. They act equally on both logical rails of each data photon. Put $r_G = \kappa^2/(\sigma^2 + \kappa^2)$ and $m_i = r_G\mu_i + (1 - r_G)\nu_i$. Completing the square gives

$$\frac{(\omega - \mu_i)^2}{4\sigma^2} + \frac{(\omega - \nu_i)^2}{4\kappa^2} = \frac{(\omega - m_i)^2}{4r_G\sigma^2} + \frac{(\mu_i - \nu_i)^2}{4(\sigma^2 + \kappa^2)}. \tag{S2.11}$$

The transmission is $T_i = \sqrt{r_G} \exp[-(\mu_i - \nu_i)^2/(2(\sigma^2 + \kappa^2))]$. For fixed separation of the two output means, their mean position is optimally zero. The optimal filter centers are therefore opposite. Rescale to $\sigma = 1$, write incident means $\pm\mu$, $a_G = \mu^2 = -2\log c$, output means $\pm m$, and $x = m/(r_G\mu)$. The squared transmitted overlap and joint transmission become

$$\begin{aligned} \tau_f &= e^{-a_G r_G x^2}, \\ T_A T_B &= r_G \exp[-a_G(1 - r_G x)^2/(1 - r_G)]. \end{aligned} \tag{S2.12}$$

Proposition S2.1 (Acceptance optimum in the equal-width family)**.** *For $0 < c < 1$ and target $(1+c^2)/2 < F_{\mathrm{tar}} \le 1$, put $t_F = 2F_{\mathrm{tar}} - 1$, $r_{G,F} = -\log t_F/a_G$ and $b_G = \sqrt{r_{G,F}}$. The optimum over both centers, the common width and all bare-analyzer outcomes is*

$$\begin{aligned} P_G(c, F_{\mathrm{tar}}) &= \frac{q_G^2}{2}\exp[-a_G(1-b_G q_G)^2/(1-q_G^2)], \\ (1-q_G^2)^2 &= a_G q_G(q_G - b_G)(1-b_G q_G), \qquad b_G < q_G < 1. \end{aligned} \tag{S2.13}$$

The root is unique and the optimal transmitted overlap satisfies $\tau_f = t_F$.

Proof. Feasibility requires $\tau_f \ge t_F$. Lemma C.4 bounds the accepted fraction after transmission by $\tau_f/(2t_F)$, whether or not that bound exceeds one. Thus

$$P \le \frac{r_G c^2}{2t_F}\exp[-a_G r_G(x-1)^2/(1-r_G)]. \tag{S2.14}$$

For $0 < r_G \le r_{G,F}$ this is at most $r_{G,F}c^2/(2t_F)$, attained at $r_G = r_{G,F}, x = 1$. When $r_{G,F} = 0$, this branch is empty. For $r_G \ge r_{G,F}$ the overlap constraint is $|x| \le \sqrt{r_{G,F}/r_G} \le 1$. The bound is maximal at $x = \sqrt{r_{G,F}/r_G}$, where $\tau_f = t_F$ and the bare accepted fraction is exactly $1/2$. It is therefore attainable, and the full problem reduces to

$$\max_{r_{G,F} \le r_G < 1} \frac{r_G}{2}\exp[-a_G(1-\sqrt{r_G r_{G,F}})^2/(1-r_G)].$$

Differentiation after setting $q_G = \sqrt{r_G}$ gives the equation in (S2.13). Its left side is strictly decreasing on $(b_G, 1)$; the right side is strictly increasing there. Indeed the derivative of $q_G(q_G - b_G)(1 - b_G q_G)$ is $2(1+b_G^2)q_G - b_G - 3b_G q_G^2$, a concave quadratic with positive endpoint values $b_G(1-b_G^2)$ and $2(1-b_G)^2$ for $0 < b_G < 1$. The case $b_G = 0$ is immediate. Opposite signs of the equation residual at b_G and 1 give a unique interior maximum. Its center parameter is $s_G = 1 - b_G/q_G$ and $\nu_i = -s_G r_G \mu_i/(1-r_G)$. □

At $F_{\mathrm{tar}} = 1$ the formulas reduce to

$$\begin{aligned} \nu_i &= -r_G\mu_i/(1-r_G), \\ P_G(c,1) &= \frac{r_G}{2}c^{2/(1-r_G)}, \\ (1-r_G)^2 &= (-2\log c)r_G. \end{aligned} \tag{S2.15}$$

Both transmitted normalized packets are identical. At $c = 0.9$ this gives $r_G = 0.6343808244$ and $P_G = 0.1782455432$ per supplied trial.

Allowing the unfiltered setting completes both Gaussian frontiers below the bare fidelity ceiling. In the notation above,

$$(T_A T_B)\tau_f = r_G c^2 \exp\left[-\frac{a_G r_G(x-1)^2}{1-r_G}\right] \le c^2.$$

For $F_{\mathrm{tar}} > 1/2$, the bare-analyzer acceptance bound and $P \le 1$ therefore give

$$P \le \min\left\{1, \frac{c^2}{2(2F_{\mathrm{tar}} - 1)}\right\}.$$

For $F_{\mathrm{tar}} \le (1+c^2)/2$, this is the bare frontier in Lemma C.4, attained by bypassing the filters and optimizing the unfiltered analyzer's acceptance. For $F_{\mathrm{tar}} \le 1/2$, accepting every unfiltered outcome attains $P = 1$. The common-center family is a subset of the independent-center family and contains the same unfiltered setting, so the two frontiers coincide with the bare frontier throughout this range.

S3 A packet distiller composed with encoded swapping

Consider the two-mode conditional operation used in the framework of Hoch et al. [3]. Write its intensity transmission and reflection as $T = 1/3$, $R = 2/3$, with pure auxiliary $h = h_+$. A coincidence on a packet a produces internal amplitude $T|a\rangle|h\rangle - R|h\rangle|a\rangle$. After tracing the herald photon, its unnormalized density is

$$\mathcal{D}_H(A) = T^2 A + R^2 H - TR(AH + HA). \tag{S3.1}$$

For $A = |a\rangle\langle a|$, $B = |b\rangle\langle b|$ and $H = |h_+\rangle\langle h_+|$, the midpoint overlaps give

$$p_{\rm dist} = \mathrm{Tr}\,\mathcal{D}_H(A) = \frac{3-2c}{9}, \tag{S3.2}$$

$$n_{\rm pkt} = \mathrm{Tr}[B\mathcal{D}_H(A)] = \frac{2-c^2}{9}. \tag{S3.3}$$

Here $n_{\rm pkt}$ is the packet-overlap numerator. Dividing by the herald probability gives the normalized HOM visibility with B:

$$V_{\rm dist} = \frac{2-c^2}{3-2c}, \qquad 1 - V_{\rm dist} = \frac{(1-c)^2}{3-2c}. \tag{S3.4}$$

Thus this internal-state distillation mechanism has quadratic visibility error in the specified pure-source family.

Now apply it only to rail 0 of a dual-rail photon, leaving rail 1 untouched. For a detected internal basis vector $|\mu\rangle$, a Kraus operator on the logical and internal state is, up to a common phase,

$$\begin{aligned} K_\mu = {} & |0\rangle\langle 0| \otimes (Th_\mu I - R|h\rangle\langle\mu|) \\ & + |1\rangle\langle 1| \otimes \sqrt{T}h_\mu I, \qquad h_\mu = \langle\mu|h\rangle. \end{aligned} \tag{S3.5}$$

The logical branches undergo different maps. For the subsequent encoded swapping circuit

$$U_{\rm pre} = B_{13}(\phi)B_{02}(\pi/4)B_{04}(\arccos(1/\sqrt{3})) \tag{S3.6}$$

and detection pattern $(1,1,0,0,1)$, the only nonzero block is

$$J\big|_{\{01,10\}} = \frac{1}{72}\begin{pmatrix} (3-2c)\sin^2\phi & -\sqrt{3}c\sin\phi\cos\phi \\ -\sqrt{3}c\sin\phi\cos\phi & 3\cos^2\phi \end{pmatrix}. \tag{S3.7}$$

The six assignments to the three occupied outputs give this matrix, including the $1/2$ Bell-input amplitude. Balancing its populations sets $\tan^2\phi = 3/(3-2c)$ and yields

$$F_{\rm pre} = \tfrac{1}{2}\left(1 + \frac{c}{\sqrt{3-2c}}\right), \qquad p_{\rm pre} = \frac{3-2c}{24(3-c)}. \tag{S3.8}$$

Writing $t = 1-c$ gives

$$1 - V_{\rm dist} = t^2 - 2t^3 + O(t^4), \tag{S3.9}$$

$$1 - F_{\rm pre} = t - \frac{5}{4}t^2 + O(t^3). \tag{S3.10}$$

The wave-packet visibility refers to the surviving photon, while the Bell fidelity refers to the remote pair produced by the specified one-rail composition. The two logical branches undergo different conditional internal-state transformations, so coherence is reduced even after their populations have been balanced. Appendix A supplies the all-network ceiling for the encoded task.

S4 Calibration and mixed-data checks

S4.1 The small-angle calibration floor

At $c = 1$, put the four angles in propagation order equal to $\pi/4+\delta_{02}$, $\pi/4+\delta_{13}$, $\arccos(1/\sqrt{3})+\delta_{14}$, $\pi/4+\delta_{01}$. The selected reference amplitude is $(-1,0,0,-1)/(4\sqrt{3})$ at the ideal point, with probability $1/24$. The derivatives with respect to the two sensitive angles are $(\sqrt{6},0,0,-\sqrt{6})/12$ and $(0,1,1,0)/6$; the other two derivatives vanish. For a real pure reference amplitude v,

$$1 - F_{\rm LU} = \frac{(v_{00}-v_{11})^2 + (v_{01}+v_{10})^2}{2\sum_{jk}|v_{jk}|^2}$$

near this point. Substitution gives Eq. (17). Two angle offsets of 0.003 rad give 3.0×10^{-5} at quadratic order. At output-end transmissions ζ_j, each path for a fixed pattern is multiplied by $\prod_j \zeta_j^{n_j/2}$; its normalized state is preserved. Losses inserted inside the interferometer require their actual transfer matrix.

S4.2 Symmetric depolarization of the two data photons

In Sec. 4.3, the Bloch vectors are $(r_d\sqrt{1-c^2}, 0, r_d c)$ and $(-r_d\sqrt{1-c^2}, 0, r_d c)$, and $H = |h_+\rangle\langle h_+|$. The invariants in Appendix D become

$$\tau = [1 + r_d^2(2c^2 - 1)]/2, \quad q_A = q_B = (1 + r_d c)/2, \quad \mathrm{Re}\, w = [1 + 2r_d c + r_d^2(2c^2 - 1)]/4.$$

The target Bell weight is $\mathrm{Tr}(\Pi_{\mathrm{sym}}\rho_a \otimes \rho_b \otimes H)/24$, and the pattern probability is

$$p_{\mathrm{sel}}(r_d, c) = \frac{3 - r_d^2 + 2r_d^2c^2 - r_d c}{72}.$$

Taking the target weight divided by this probability gives Eq. (34). It is a fixed-target construction, and coincides with its locally optimized Bell fidelity in a neighborhood of $r_d = c = 1$. Writing $\delta_d = 1 - r_d$, its expansion starts with $[t^2 + \delta_d]/3$; terms of order $t^3, t\delta_d, \delta_d^2$ follow. The complete source-optimized mixed-data network problem is separate from the pure-data theorems proved here.

S5 Nonzero detection tables at $c = 0.9$

The following tables list every nonzero-probability spatial pattern for the fixed four-mixing circuit with h_+ and the two-auxiliary circuit with two copies of h_-, at source overlap $c = 0.9$. Columns give the probability p, locally optimized Bell weight w, and fidelity $F = w/p$. The data archive retains all 35 and 126 patterns, including zero-probability outcomes, at $c = 0.5, 0.9, 0.99$, together with the full acceptance frontiers.

S5.1 One auxiliary: nonzero patterns

Pattern	p	w	F
$(3,0,0,0,0)$	0.05018519	0.02509259	0.50000000
$(2,1,0,0,0)$	0.02715278	0.02510418	0.92455302
$(2,0,1,0,0)$	0.01947917	0.01255208	0.64438503
$(2,0,0,1,0)$	0.04260417	0.03765625	0.88386308
$(2,0,0,0,1)$	0.02715278	0.02510418	0.92455302
$(1,2,0,0,0)$	0.02715278	0.02510418	0.92455302
$(1,1,1,0,0)$	0.03104167	0.02510417	0.80872483
$(1,1,0,1,0)$	0.00604167	0.00197917	0.32758621
$(1,1,0,0,1)$	0.03777778	0.03763889	0.99632353
$(1,0,2,0,0)$	0.03770833	0.01885417	0.50000000
$(1,0,1,1,0)$	0.04166667	0.03770833	0.90500000
$(1,0,1,0,1)$	0.01229167	0.00635417	0.51694915
$(1,0,0,2,0)$	0.03770833	0.01885417	0.50000000
$(1,0,0,1,1)$	0.02479167	0.01885417	0.76050420
$(1,0,0,0,2)$	0.02715278	0.02510418	0.92455302
$(0,3,0,0,0)$	0.05018519	0.02509259	0.50000000
$(0,2,1,0,0)$	0.01947917	0.01255208	0.64438503
$(0,2,0,1,0)$	0.04260417	0.03765625	0.88386308
$(0,2,0,0,1)$	0.02715278	0.02510418	0.92455302
$(0,1,2,0,0)$	0.03770833	0.01885417	0.50000000
$(0,1,1,1,0)$	0.04166667	0.03770833	0.90500000
$(0,1,1,0,1)$	0.01229167	0.00635417	0.51694915
$(0,1,0,2,0)$	0.03770833	0.01885417	0.50000000
$(0,1,0,1,1)$	0.02479167	0.01885417	0.76050420
$(0,1,0,0,2)$	0.02715278	0.02510418	0.92455302
$(0,0,2,0,1)$	0.03770833	0.01885417	0.50000000
$(0,0,1,1,1)$	0.04166667	0.03770833	0.90500000
$(0,0,1,0,2)$	0.05416667	0.05020833	0.92692308
$(0,0,0,2,1)$	0.03770833	0.01885417	0.50000000
$(0,0,0,1,2)$	0.00791667	0.00395833	0.50000000
$(0,0,0,0,3)$	0.05018519	0.02509259	0.50000000

S5.2 Two auxiliaries: nonzero patterns

Pattern	p	w	F
$(3,1,0,0,0,0)$	0.00710938	0.00355469	0.50000000
$(3,0,0,0,0,1)$	0.00710938	0.00355469	0.50000000
$(2,2,0,0,0,0)$	0.00861328	0.00712891	0.82766440
$(2,1,1,0,0,0)$	0.00445312	0.00222656	0.50000000
$(2,1,0,1,0,0)$	0.01640625	0.01417969	0.86428571
$(2,1,0,0,1,0)$	0.02117187	0.01058594	0.50000000
$(2,1,0,0,0,1)$	0.01558594	0.01410156	0.90476190
$(2,0,1,0,0,1)$	0.00445312	0.00222656	0.50000000
$(2,0,0,1,0,1)$	0.01640625	0.01417969	0.86428571
$(2,0,0,0,1,1)$	0.02117187	0.01058594	0.50000000
$(2,0,0,0,0,2)$	0.00861328	0.00712891	0.82766440
$(1,3,0,0,0,0)$	0.00710938	0.00355469	0.50000000
$(1,2,1,0,0,0)$	0.01640625	0.01417969	0.86428571
$(1,2,0,1,0,0)$	0.00445312	0.00222656	0.50000000
$(1,2,0,0,1,0)$	0.01558594	0.01410156	0.90476190
$(1,2,0,0,0,1)$	0.02117187	0.01058594	0.50000000
$(1,1,2,0,0,0)$	0.02828125	0.01414063	0.50000000
$(1,1,1,1,0,0)$	0.03125000	0.02828125	0.90500000
$(1,1,1,0,1,0)$	0.00296875	0.00148438	0.50000000
$(1,1,1,0,0,1)$	0.02968750	0.02820312	0.95000000
$(1,1,0,2,0,0)$	0.02828125	0.01414063	0.50000000
$(1,1,0,1,1,0)$	0.02968750	0.02820312	0.95000000
$(1,1,0,1,0,1)$	0.00296875	0.00148438	0.50000000
$(1,1,0,0,2,0)$	0.02117187	0.01058594	0.50000000
$(1,1,0,0,1,1)$	0.02820312	0.02820312	1.00000000
$(1,1,0,0,0,2)$	0.02117187	0.01058594	0.50000000
$(1,0,2,0,0,1)$	0.02828125	0.01414063	0.50000000
$(1,0,1,1,0,1)$	0.03125000	0.02828125	0.90500000
$(1,0,1,0,1,1)$	0.00296875	0.00148438	0.50000000
$(1,0,1,0,0,2)$	0.01640625	0.01417969	0.86428571
$(1,0,0,2,0,1)$	0.02828125	0.01414063	0.50000000
$(1,0,0,1,1,1)$	0.02968750	0.02820312	0.95000000
$(1,0,0,1,0,2)$	0.00445312	0.00222656	0.50000000
$(1,0,0,0,2,1)$	0.02117187	0.01058594	0.50000000
$(1,0,0,0,1,2)$	0.01558594	0.01410156	0.90476190
$(1,0,0,0,0,3)$	0.00710938	0.00355469	0.50000000
$(0,3,0,0,1,0)$	0.00710938	0.00355469	0.50000000
$(0,2,1,0,1,0)$	0.01640625	0.01417969	0.86428571
$(0,2,0,1,1,0)$	0.00445312	0.00222656	0.50000000
$(0,2,0,0,2,0)$	0.00861328	0.00712891	0.82766440
$(0,2,0,0,1,1)$	0.02117187	0.01058594	0.50000000
$(0,1,2,0,1,0)$	0.02828125	0.01414063	0.50000000
$(0,1,1,1,1,0)$	0.03125000	0.02828125	0.90500000
$(0,1,1,0,2,0)$	0.00445312	0.00222656	0.50000000
$(0,1,1,0,1,1)$	0.02968750	0.02820312	0.95000000
$(0,1,0,2,1,0)$	0.02828125	0.01414063	0.50000000
$(0,1,0,1,2,0)$	0.01640625	0.01417969	0.86428571
$(0,1,0,1,1,1)$	0.00296875	0.00148438	0.50000000
$(0,1,0,0,3,0)$	0.00710938	0.00355469	0.50000000
$(0,1,0,0,2,1)$	0.01558594	0.01410156	0.90476190
$(0,1,0,0,1,2)$	0.02117187	0.01058594	0.50000000
$(0,0,2,0,1,1)$	0.02828125	0.01414063	0.50000000
$(0,0,1,1,1,1)$	0.03125000	0.02828125	0.90500000
$(0,0,1,0,2,1)$	0.00445312	0.00222656	0.50000000
$(0,0,1,0,1,2)$	0.01640625	0.01417969	0.86428571
$(0,0,0,2,1,1)$	0.02828125	0.01414063	0.50000000
$(0,0,0,1,2,1)$	0.01640625	0.01417969	0.86428571
$(0,0,0,1,1,2)$	0.00445312	0.00222656	0.50000000
$(0,0,0,0,3,1)$	0.00710938	0.00355469	0.50000000
$(0,0,0,0,2,2)$	0.00861328	0.00712891	0.82766440
$(0,0,0,0,1,3)$	0.00710938	0.00355469	0.50000000

References

[1] Annan Chen. Numerical data for "Optimal entanglement swapping with one auxiliary photon: exact fidelity limit and purity requirements". Zenodo, 2026. Data set. https://doi.org/10.5281/zenodo.24141275.

[2] William K. Wootters. Entanglement of formation of an arbitrary state of two qubits. *Physical Review Letters*, 80:2245–2248, 1998. doi: 10.1103/PhysRevLett.80.2245.

[3] Francesco Hoch, Anita Camillini, Giovanni Rodari, Eugenio Caruccio, Gonzalo Carvacho, Taira Giordani, Riccardo Albiero, Niki Di Giano, Giacomo Corrielli, Francesco Ceccarelli, Roberto Osellame, Marco Robbio, Leonardo Novo, Nicolò Spagnolo, Ernesto F. Galvão, and Fabio Sciarrino. Optimal distillation of photonic indistinguishability. *Physical Review Research*, 2026. doi: 10.1103/lyw5-qdfx. Accepted 9 October 2026.