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The Degenerate Prabhakar Counting Process and its Renewal Counterpart

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Abstract

The degenerate Prabhakar function of Hachicha and Riahi is the moment generating function of a probability measure μ_λ, which they used to construct the degenerate Prabhakar process as a variance mixture of fractional Brownian motion. We show that this function is a hypergeometric function of type ${}_2F_2$ and that μ_λ is a G distribution in the sense of Dufresne. His results then give, in closed form, the atom of μ_λ at 1, its density on $(0,1)$ and the Lévy measure of its logarithm: μ_λ is the law of e^{-U} with U compound Poisson. As a consequence, the local time of the degenerate Prabhakar process exists almost surely for all parameters, but it is square integrable in (x,ω) only if $\beta > 1/\lambda - 1/2$. We then study the counting analogue obtained by mixing compound Poisson processes with jumps of sizes $1,\dots,k$ against μ_λ. It is a mixed Poisson process whose probability generating function is the degenerate Prabhakar function evaluated at $-t^\rho \sum_j \lambda_j(1-u^j)$; we derive its distribution, an Euler-type equation for the state probabilities, its moments and the power-law tail of the first jump time. Finally, we show that this process is not a renewal process and we introduce its renewal counterpart, whose scaling limit is an inverse stable subordinator.

Keywords Degenerate Prabhakar function Degenerate Prabhakar process G distributions Mixed Poisson process Renewal process Hypergeometric functions Overdispersion

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1 Introduction

Fractional generalizations of the Poisson process have been studied intensively over the last twenty years. In the approach of Laskin [15] and of Beghin and Orsingher [3], the time derivative in the Kolmogorov equations is replaced by a fractional one, and the exponential function in the state probabilities is replaced by a Mittag-Leffler function. Meerschaert, Nane and Vellaisamy [20] later showed that the same process is obtained by running a Poisson process with an independent inverse stable subordinator as clock. Di Crescenzo, Martinucci and Meoli [5] extended this picture to a process with jumps of sizes $1,\dots,k$, the generalized fractional counting process (GFCP), which covers fractional versions of the Poisson process of order k [14, 26]. Mixed and compound representations of related processes, including fractional Pólya–Aeppli and negative binomial processes, were obtained by Beghin and Macci [2].

A different deformation of the exponential comes from the degenerate special functions of Kim and Kim [13]. Starting from the degenerate Gamma function, Hachicha and Riahi [11] defined a degenerate Prabhakar function $\mathcal{E}_\lambda(\beta,\gamma;z)$, showed that $x \mapsto \mathcal{E}_\lambda(\beta,\gamma;x)$ is the moment generating function of a probability measure μ_λ, and used it in place of the exponential in the characteristic functional of Gaussian white noise. The resulting degenerate Prabhakar process (dpp) is self-similar with stationary increments and can be written as $\sqrt{L}\,B^H$, where B^H is a fractional Brownian motion and $L \sim \mu_\lambda$ is independent of it.

The dpp has continuous, symmetric paths, so it cannot describe counts. In this paper we do two things.

First, we determine the structure of the function $\mathcal{E}_\lambda$ and of the measure μ_λ. The function $\mathcal{E}_\lambda(\beta,\gamma;\cdot)$ is a hypergeometric function of type ${}_2F_2$ (Proposition 3.2), and the Mellin transform of μ_λ is a ratio of products of two gamma functions with balanced parameters. Laws with Mellin transforms of this kind are the G distributions studied by Dufresne [7]. For order $(2,2)$ and balanced parameters his results give the support $(0,1]$, the mass of the atom at 1, the density on $(0,1)$ and the Lévy–Khintchine representation of the logarithm, which is always infinitely divisible; a related family, treated by means of a Malmstén-type formula, is studied in [23]. What is new here is the identification of μ_λ as such a G distribution. Theorem 3.4 and Proposition 3.6 then restate Dufresne's results for μ_λ, in the form needed later: μ_λ is the law of e^{-U}, where U has a compound Poisson distribution with an explicit Lévy density. This description has a consequence for [11]: the local time of the dpp exists almost surely for all admissible parameters, but it belongs to $L^2(\mathbb{R}\times\Omega)$ if and only if $\beta > \frac{1}{\lambda} - \frac{1}{2}$ (Proposition 4.8).

Second, we carry the construction over to counting processes. The construction of [11] uses Gaussianity only through the fact that the characteristic functions of $\sqrt{c}\,B^H$ have the form $\exp(-c\Psi)$. Any family of processes with this property can be mixed against μ_λ, and the mixing turns $\exp(-\Psi)$ into $\mathcal{E}_\lambda(\beta,\gamma;-\Psi)$. This is the classical scale-mixture argument and we do not claim it as new; we use it as a dictionary. Applied to the compound Poisson processes $\mathcal{M}(c\,t^\rho)$, where $\mathcal{M}$ is the generalized counting process of [5], it produces the process $\mathcal{M}(L\,t^\rho)$, which we call the degenerate Prabhakar counting process (dPCP).

We stress from the outset what the dPCP is and what it is not. It is a mixed (compound) Poisson process in the sense of Grandell [10], observed along the deterministic clock t^ρ. Overdispersion, non-ergodicity and non-decaying correlations are therefore general features of the class; what is specific to the dPCP is the mixing law μ_λ, which makes the probability mass function, an Euler-type governing equation, the factorial moments and the asymptotic behaviour of the state probabilities and of the first jump time available in closed form. The dPCP is not a fractional process: its governing equation is local, and as $\gamma \downarrow \beta$ it reduces to $\mathcal{M}(t^\rho)$, not to the GFCP. Its relation with the GFCP is that the one-dimensional laws of both processes are of the form $\mathcal{M}(Y\,t^\rho)$, with different mixing variables Y. Nor is the dPCP a renewal process (Proposition 8.1). For this reason we also introduce, in Section 8, the renewal process whose waiting times have the survival function $\mathcal{E}_\lambda(\beta,\gamma;-\Lambda t^\rho)$. This is the analogue of the renewal construction of the fractional Poisson process [17], and when $\rho\beta' < 1$ its scaling limit is an inverse stable subordinator of index $\rho\beta'$.

Several results of [11] are recovered along the way as the Gaussian case of statements proved here: the characteristic function (3.19), the density of Theorem 3.3 and the representation of Theorem 3.5 (Corollary 4.4), and the self-similarity and stationarity of Proposition 3.2 (Proposition 4.7). The existence of μ_λ [11, Thm. 2.3] is reproved with an explicit description of the measure.

The paper is organized as follows. Section 2 collects the necessary background. Section 3 studies the degenerate Prabhakar function and the measure μ_λ. The mixture mechanism, its connection with [11] and the local time of the dpp are presented in Section 4. The dPCP is defined in Section 5, where its distribution and governing equation are derived; moments and dependence are treated in Section 6, and asymptotics in Section 7. Section 8 deals with the renewal structure and the renewal counterpart. Section 9 discusses particular and limiting cases, Section 10 contains worked examples, and Section 11 describes the simulation algorithm. Numerical checks of the formulas are collected in Appendix A, and a simulation study of the estimators in Appendix B.

2 Preliminaries

2.1 Degenerate Gamma and degenerate Prabhakar functions

For $\lambda \in (0,1)$ put $\varepsilon = 1/\lambda$. The degenerate Gamma function [13] is

$$\Gamma_\lambda(s) = \int_0^\infty (1+\lambda t)^{-1/\lambda} t^{s-1}\, dt = \lambda^{-s}\, \frac{\Gamma(s)\Gamma(\varepsilon - s)}{\Gamma(\varepsilon)}, \qquad 0 < \operatorname{Re} s < \varepsilon,$$

extended meromorphically to $\mathbb{C}$ by the right-hand side [11, (2.1)]. Following [11], the degenerate Pochhammer symbol is $(x)_{\lambda,k} = \Gamma_\lambda(x+k)/\Gamma_\lambda(x)$ and the degenerate Prabhakar function (with the first parameter equal to 1, as imposed by [11, Lemma 2.2]) is

$$\mathcal{E}_\lambda(\beta,\gamma;z) = \Gamma_\lambda(\beta) \sum_{k=0}^\infty \frac{(\gamma)_{\lambda,k}}{\Gamma_\lambda(\beta+k)} \frac{z^k}{k!} = \sum_{k=0}^\infty m_k \frac{z^k}{k!}, \qquad m_k := \frac{(\gamma)_{\lambda,k}}{(\beta)_{\lambda,k}}. \tag{2.1}$$

Assumption 2.1 *Throughout,* $0 < \lambda < 1$, $\varepsilon = 1/\lambda$ *and* $\varepsilon - 1 < \beta < \gamma < \varepsilon$. *We set*

$$\beta' := \beta + 1 - \varepsilon, \qquad \gamma' := \gamma + 1 - \varepsilon,$$

so that $0 < \beta' < \gamma' < 1$, $\beta' < \beta$ *and* $\gamma' < \gamma$.

These are exactly the standing assumptions of [11, Sect. 2]; their non-pole conditions [11, (2.8)] hold automatically because $\varepsilon - \beta, \varepsilon - \gamma \in (0,1)$. Under Assumption 2.1, [11, Thm. 2.3] asserts that there is a probability measure μ_λ on $[0,\infty)$ with

$$\int_{[0,\infty)} e^{xs}\, \mu_\lambda(ds) = \mathcal{E}_\lambda(\beta,\gamma;x), \qquad x \in \mathbb{R}. \tag{2.2}$$

Theorem 3.4 below identifies μ_λ explicitly.

2.2 The degenerate Prabhakar process

Let B^H be a fractional Brownian motion with Hurst index $H \in (0,1)$ and covariance $C(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. The dpp P^H of [11, Def. 3.1] satisfies, for $0 \le t_1 < \cdots < t_n$ and $u \in \mathbb{R}^n$ [11, (3.19), Thm. 3.5],

$$\mathbb{E}\big[e^{i\sum_k u_k P^H_{t_k}}\big] = \mathcal{E}_\lambda\big(\beta,\gamma; -\tfrac{1}{2} u^\top \widetilde{\Sigma} u\big), \qquad \widetilde{\Sigma}_{ij} = C(t_i,t_j), \qquad P^H \overset{d}{=} \sqrt{L}\, B^H. \tag{2.3}$$

2.3 Generalized counting processes

Let $k \in \mathbb{N} \cup \{\infty\}$ and let $\lambda_1, \lambda_2, \ldots \ge 0$ (with $\lambda_j = 0$ for $j > k$) satisfy $0 < \Lambda := \sum_j \lambda_j < \infty$. Put

$$\Phi(u) := \sum_{j \ge 1} \lambda_j (1 - u^j), \quad |u| \le 1,$$
$$\Omega(k,n) := \Big\{ x \in \mathbb{N}_0^k : \ \textstyle\sum_j j\, x_j = n \Big\}, \qquad z_x := \textstyle\sum_j x_j.$$

The *generalized counting process* (GCP) $\mathcal{M} = \{\mathcal{M}(\tau)\}_{\tau \ge 0}$ is the compound Poisson process with intensity Λ and jump law $\mathbb{P}(\text{jump} = j) = \lambda_j/\Lambda$. Equivalently, $\mathcal{M}(\tau) = \sum_j j N_j(\tau)$ with independent Poisson processes N_j of rates λ_j. Its generating function and probability mass function are

$$\mathbb{E}\big[u^{\mathcal{M}(\tau)}\big] = e^{-\tau\Phi(u)}, \qquad q_n(\tau) := \mathbb{P}(\mathcal{M}(\tau) = n) = \sum_{x \in \Omega(k,n)} \prod_j \frac{(\lambda_j \tau)^{x_j}}{x_j!}\, e^{-\Lambda\tau}. \tag{2.4}$$

The GFCP M^α of Di Crescenzo, Martinucci and Meoli [5] ($k < \infty$, $0 < \alpha \le 1$) is governed by ${}^C D_t^\alpha p_n = -\Lambda p_n + \sum_{j=1}^{\min(n,k)} \lambda_j p_{n-j}$ with Caputo derivative ${}^C D_t^\alpha$. Its generating function is $E_\alpha(-t^\alpha \Phi(u))$, and

$$p_n^\alpha(t) = \sum_{x \in \Omega(k,n)} \prod_j \frac{\lambda_j^{x_j}}{x_j!}\, z_x!\, t^{\alpha z_x} E^{z_x+1}_{\alpha, \alpha z_x + 1}(-\Lambda t^\alpha), \tag{2.5}$$

where $E^\delta_{\alpha,\beta}$ is the Prabhakar function [24, 9]. For $k = 1$ one recovers the fractional Poisson process [15, 3]. Since the inverse α-stable subordinator Y_α satisfies $Y_\alpha(t) \stackrel{d}{=} t^\alpha Y_\alpha(1)$ and $\mathbb{E} e^{-x Y_\alpha(1)} = E_\alpha(-x)$ [20], the one-dimensional marginals of the GFCP are those of $\mathcal{M}(t^\alpha Y)$ with $Y \stackrel{d}{=} Y_\alpha(1)$ independent of $\mathcal{M}$.

3 The degenerate Prabhakar function and the law μ_λ

Lemma 3.1 *For every $x \in \mathbb{R}$ such that the quantities below are finite and $k \in \mathbb{N}_0$,*

$$\frac{\Gamma_\lambda(x+1)}{\Gamma_\lambda(x)} = \frac{x}{\lambda(\varepsilon - x - 1)}, \qquad (x)_{\lambda,k} = (-\lambda)^{-k} \frac{(x)_k}{(x+1-\varepsilon)_k},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the ordinary Pochhammer symbol.

Proof. The first identity follows from $\Gamma_\lambda(s) = \lambda^{-s}\Gamma(s)\Gamma(\varepsilon - s)/\Gamma(\varepsilon)$ and the functional equation of the Gamma function. Applying it with x replaced by $x+i$, $i = 0, \dots, k-1$, and multiplying, we get $(x)_{\lambda,k} = \lambda^{-k} \prod_{i=0}^{k-1} \frac{x+i}{\varepsilon - x - 1 - i}$, which is the second identity because $\varepsilon - x - 1 - i = -(x + 1 - \varepsilon + i)$. □

Proposition 3.2 *Under Assumption 2.1:*

(a) $m_k = \dfrac{(\gamma)_k (\beta')_k}{(\beta)_k (\gamma')_k}$ *and* $\mathcal{E}_\lambda(\beta, \gamma; z) = {}_2F_2\!\left(\begin{matrix}\gamma, & \beta' \\ \beta, & \gamma'\end{matrix}; z\right)$, *an entire function of z.*

(b) For $r \in \mathbb{N}_0$, $\dfrac{d^r}{dz^r} \mathcal{E}_\lambda(\beta, \gamma; z) = m_r\, {}_2F_2\!\left(\begin{matrix}\gamma + r, & \beta' + r \\ \beta + r, & \gamma' + r\end{matrix}; z\right)$.

(c) With $\theta = z \frac{d}{dz}$, the function $f = \mathcal{E}_\lambda(\beta, \gamma; \cdot)$ solves

$$\theta(\theta + \beta - 1)(\theta + \gamma' - 1) f(z) = z\,(\theta + \gamma)(\theta + \beta') f(z). \tag{3.1}$$

Proof. (a) Apply Lemma 3.1 with $x = \gamma$ and with $x = \beta$. Since $\gamma + 1 - \varepsilon = \gamma'$ and $\beta + 1 - \varepsilon = \beta'$,

$$(\gamma)_{\lambda,k} = (-\lambda)^{-k} \frac{(\gamma)_k}{(\gamma')_k}, \qquad (\beta)_{\lambda,k} = (-\lambda)^{-k} \frac{(\beta)_k}{(\beta')_k},$$

and the factors $(-\lambda)^{-k}$ cancel in $m_k = (\gamma)_{\lambda,k}/(\beta)_{\lambda,k}$. Under Assumption 2.1 the numbers $\beta, \gamma, \beta', \gamma'$ are positive, so no Pochhammer symbol vanishes. Inserting m_k in (2.1) gives exactly the hypergeometric series $\sum_k \frac{(\gamma)_k (\beta')_k}{(\beta)_k (\gamma')_k} \frac{z^k}{k!}$. The ratio of two consecutive coefficients is $\frac{(\gamma+k)(\beta'+k)}{(\beta+k)(\gamma'+k)(k+1)} \to 0$, so the radius of convergence is infinite.

(b) and (c) are the differentiation formula [6, §16.3(i)] and the generalized hypergeometric equation [6, §16.8(ii)] for the parameters found in (a). □

Remark 3.3 *For the parameters $\lambda = 0.23$, $\beta = 3.7$, $\gamma = 4.1$ used in [11, Sect. 3], Proposition 3.2(a) gives $m_1 = \gamma\beta'/(\beta\gamma') = 0.51883$, which is the constant $C_{\lambda,\beta,\gamma} \approx 0.5188$ reported there. Note also that a ${}_2F_2$ function is entire of order 1, like the exponential: in contrast with the Mittag-Leffler and Prabhakar functions, $\mathcal{E}_\lambda(\beta, \gamma; \cdot)$ has no fractional order of growth.*

The moments m_k of Proposition 3.2(a) are values at the integers of a ratio of products of two gamma functions whose parameters are balanced, $\gamma + \beta' = \beta + \gamma'$. In the terminology of Dufresne [7], a law with such a Mellin transform is a G distribution of order $(2,2)$, with $a = (\beta', \gamma)$ and $b = (\beta, \gamma')$ in his notation. The next theorem is therefore not new: the support, the atom at 1 and its mass are given by [7, Thm. 6.1], and the Lévy–Khintchine kernel of the logarithm appears in [7, Thm. 6.4 and its proof]. We state it for μ_λ and include a short self-contained proof for the reader's convenience, because the factorized form (3.2) of the Lévy density and the finiteness of ν are used throughout the paper.

Theorem 3.4 (Structure of μ_λ) *Let Assumption 2.1 hold and define on* $(0,\infty)$

$$\nu(du) = \frac{\big(1 - e^{-(\varepsilon-1)u}\big)\big(e^{-\beta' u} - e^{-\gamma' u}\big)}{u\,(1 - e^{-u})}\,du. \tag{3.2}$$

Then ν is a finite positive measure with total mass $|\nu| = \log \frac{\Gamma(\gamma)\Gamma(\beta')}{\Gamma(\beta)\Gamma(\gamma')}$. Let U be a compound Poisson random variable with Lévy measure ν, i.e. $U = \sum_{i=1}^{N} J_i$ with $N \sim \mathrm{Poisson}(|\nu|)$ and J_i i.i.d. with law $\nu/|\nu|$, independent of N. Then $L := e^{-U}$ has law μ_λ, and for every $s > -\beta'$,

$$\mathbb{E}\big[L^s\big] = \frac{\Gamma(\beta)\Gamma(\gamma')}{\Gamma(\gamma)\Gamma(\beta')} \cdot \frac{\Gamma(\gamma+s)\Gamma(\beta'+s)}{\Gamma(\beta+s)\Gamma(\gamma'+s)}. \tag{3.3}$$

Proof. Step 1. Since $\varepsilon > 1$ and $\beta' < \gamma'$, the density in (3.2) is positive. It tends to $(\varepsilon-1)(\gamma'-\beta')$ as $u \downarrow 0$ and it is $O(e^{-\beta' u}/u)$ as $u \to \infty$. Hence $|\nu| < \infty$ and $\int e^{\sigma u}\nu(du) < \infty$ for every $\sigma < \beta'$.

Step 2. Fix $s > -\beta'$ and put $f(s) := \frac{\Gamma(\gamma+s)\Gamma(\beta'+s)}{\Gamma(\beta+s)\Gamma(\gamma'+s)}$. We apply Malmstén's formula [6, §5.9],

$$\log\Gamma(z) = \int_0^\infty \Big[(z-1)e^{-u} - \frac{e^{-u} - e^{-zu}}{1 - e^{-u}}\Big]\frac{du}{u}, \qquad \operatorname{Re} z > 0,$$

to the four arguments, which are positive because β' is the smallest of $\beta, \gamma, \beta', \gamma'$. The terms $(z-1)e^{-u}$ cancel because $\gamma + \beta' = \beta + \gamma'$, and so do the terms $e^{-u}/(1-e^{-u})$. What remains is

$$\log f(s) = \int_0^\infty \frac{e^{-(\gamma+s)u} + e^{-(\beta'+s)u} - e^{-(\beta+s)u} - e^{-(\gamma'+s)u}}{u(1-e^{-u})}\,du = \int_0^\infty e^{-su}\,\nu(du),$$

since, with $\beta = \beta' + \varepsilon - 1$ and $\gamma = \gamma' + \varepsilon - 1$, the numerator factors as $e^{-su}\big(1 - e^{-(\varepsilon-1)u}\big)\big(e^{-\beta' u} - e^{-\gamma' u}\big)$. In particular $|\nu| = \log f(0)$.

Step 3. For a compound Poisson variable, $\mathbb{E}e^{-sU} = \exp\big(\int_0^\infty (e^{-su} - 1)\,\nu(du)\big)$, which is finite for $s > -\beta'$ by Step 1 and equals $f(s)/f(0)$ by Step 2. This is (3.3). For $s = k \in \mathbb{N}_0$ it gives $\mathbb{E}L^k = m_k$, by Proposition 3.2(a). Since $0 < L \le 1$, we may sum the exponential series, and $\mathbb{E}e^{xL} = \sum_k m_k x^k/k! = \mathcal{E}_\lambda(\beta, \gamma; x)$ for every $x \in \mathbb{R}$. A probability measure is determined by a moment generating function which is finite on $\mathbb{R}$, so the law of L is μ_λ. □

Corollary 3.5 *Under Assumption 2.1:*

- *(i)* *μ_λ is carried by $(0,1]$; it has an atom at 1 of mass $C' := e^{-|\nu|} = \frac{\Gamma(\beta)\Gamma(\gamma')}{\Gamma(\gamma)\Gamma(\beta')}$, its restriction to $(0,1)$ is absolutely continuous, and $\mu_\lambda(\{0\}) = 0$.*
- *(ii)* *$\mathbb{E}[L^{-s}] < \infty$ if and only if $s < \beta'$.*
- *(iii)* *$\log L$ is infinitely divisible.*

Proof. (i) Each J_i is a.s. positive, so $L = 1$ if and only if $N = 0$, an event of probability $e^{-|\nu|}$. By Step 2 of the proof of Theorem 3.4, $e^{-|\nu|} = 1/f(0) = \frac{\Gamma(\beta)\Gamma(\gamma')}{\Gamma(\gamma)\Gamma(\beta')}$. On $\{N = n\}$ with $n \geq 1$, U is a sum of n independent variables with the absolutely continuous law $\nu/|\nu|$, so it has a density. Hence the law of U restricted to $(0, \infty)$ is absolutely continuous, and so is the law of $L = e^{-U}$ restricted to $(0, 1)$, because $u \mapsto e^{-u}$ is a smooth bijection from $(0, \infty)$ onto $(0, 1)$. Finally $U < \infty$ a.s., so $L > 0$ a.s. and $\mu_\lambda(\{0\}) = 0$.

(ii) For $0 \leq s < \beta'$, Theorem 3.4 applied with $-s > -\beta'$ gives $\mathbb{E}L^{-s} = f(-s)/f(0) < \infty$. When $s \uparrow \beta'$, the factor $\Gamma(\beta' - s)$ in $f(-s)$ tends to $+\infty$. The other factors $\Gamma(\gamma - s)$, $\Gamma(\beta - s)$ and $\Gamma(\gamma' - s)$ stay positive and bounded, because $\gamma, \beta, \gamma' > \beta'$. Hence $\mathbb{E}L^{-s} \to \infty$. If $s \geq \beta'$, then $L^{-s} \geq L^{-r}$ for every $r < \beta'$, since $0 < L \leq 1$, and therefore $\mathbb{E}L^{-s} \geq \sup_{r<\beta'} \mathbb{E}L^{-r} = \infty$.

(iii) $-\log L = U$ has a compound Poisson law, which is infinitely divisible. □

Proposition 3.6 (Density of μ_λ) *Let Assumption 2.1 hold. Then $\mu_\lambda(dx) = C'\delta_1(dx) + g(x)\,dx$, where, for $0 < x < 1$,*

$$g(x) = C'\,(\varepsilon - 1)(\gamma - \beta)\; x^{\beta'-1}\, {}_2F_1\big(1 - \gamma + \beta,\, 2 - \varepsilon;\, 2;\, 1 - x\big). \tag{3.4}$$

In particular $g(x) \to C'(\varepsilon - 1)(\gamma - \beta)$ as $x \uparrow 1$, and $g(x) \sim C'A\,x^{\beta'-1}$ as $x \downarrow 0$, where

$$A := \frac{\Gamma(\gamma - \beta')}{\Gamma(\varepsilon - 1)\Gamma(\gamma - \beta)}.$$

If moreover $\gamma - \beta' \notin \mathbb{N}$, then, with $B := \frac{\Gamma(\beta'-\gamma)}{\Gamma(\beta-\gamma)\Gamma(1-\varepsilon)}$ and the convention $1/\Gamma(-m) = 0$ for $m \in \mathbb{N}_0$,

$$\begin{aligned} g(x) = C'\Big[&A\,x^{\beta'-1}\, {}_2F_1\big(2 - \varepsilon,\, 1 - \gamma + \beta;\, 1 - \gamma + \beta';\, x\big) \\ &+ B\,x^{\gamma-1}\, {}_2F_1\big(\varepsilon,\, 1 + \gamma - \beta;\, 1 + \gamma - \beta';\, x\big)\Big], \qquad 0 < x < 1. \end{aligned} \tag{3.5}$$

Proof. By [7, Thm. 6.2], applied with $a_1 = \gamma$, $a_2 = \beta'$, $b_1 = \beta$, $b_2 = \gamma'$ (so that $a_2 \leq a_1$, $b_1 - a_2 = \varepsilon - 1$ and $b_2 - a_2 = \gamma - \beta$), we have $g(x) = C'(\varepsilon - 1)(\gamma - \beta)\, x^{\gamma-1}\, {}_2F_1(\varepsilon, 1 + \gamma - \beta; 2; 1 - x)$. Euler's transformation ${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b}\, {}_2F_1(c - a, c - b; c; z)$ [6, §15.8], with $z = 1 - x$ and $c - a - b = \beta' - \gamma$, turns this into (3.4). The limit at 1 is immediate. Since $2 - (1 - \gamma + \beta) - (2 - \varepsilon) = \gamma - \beta' > 0$, Gauss's summation theorem gives ${}_2F_1(1 - \gamma + \beta, 2 - \varepsilon; 2; 1) = \frac{\Gamma(\gamma-\beta')}{\Gamma(1+\gamma-\beta)\Gamma(\varepsilon)}$, and $(\varepsilon - 1)(\gamma - \beta)\big/\big(\Gamma(1 + \gamma - \beta)\Gamma(\varepsilon)\big) = 1\big/\big(\Gamma(\gamma - \beta)\Gamma(\varepsilon - 1)\big)$; this gives the behaviour at 0. Finally, let $\gamma - \beta' \notin \mathbb{N}$. The connection formula [6, §15.8]

$$\begin{aligned} {}_2F_1(a, b; c; 1 - x) = &\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\, {}_2F_1(a, b; a + b - c + 1; x) \\ &+ x^{c-a-b}\, \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}\, {}_2F_1(c - a, c - b; c - a - b + 1; x), \end{aligned}$$

valid for $c - a - b \notin \mathbb{Z}$, is applied with $a = 1 - \gamma + \beta$, $b = 2 - \varepsilon$, $c = 2$, so that $c - a - b = \gamma - \beta'$. It turns (3.4) into (3.5): the first coefficient is $C'A$ as above, and the second one is

$$C'(\varepsilon - 1)(\gamma - \beta)\, \frac{\Gamma(\beta' - \gamma)}{\Gamma(1 - \gamma + \beta)\Gamma(2 - \varepsilon)} = C'B,$$

because $\Gamma(1 - \gamma + \beta) = (\beta - \gamma)\Gamma(\beta - \gamma)$ and $\Gamma(2 - \varepsilon) = (1 - \varepsilon)\Gamma(1 - \varepsilon)$. □

Remark 3.7 *(i) On $(0, 1)$, g is the Meijer G-function $C'\,G^{2,0}_{2,2}\big(x \,\big|\, {}^{\beta-1,\,\gamma'-1}_{\gamma-1,\,\beta'-1}\big)$ [6, §16.17]. Formula (3.5) is its expansion at $x = 0$, as a sum of residues, and (3.4) is its expansion at $x = 1$.*

Because the parameters are balanced, the Mellin transform $\mathbb{E}L^p = C' f(p)$ of μ_λ, with f as in the proof of Theorem 3.4, tends to C', and not to 0, as $p \to +\infty$; this is the analytic counterpart of the atom at 1, while the Mellin transform $C'(f(p)-1)$ of g alone tends to 0.

(ii) For numerical purposes (3.4) *should be preferred: it is regular at $x = 1$ and holds for all parameters. In* (3.5)*, each of the two ${}_2F_1$ functions has $c-a-b = -1$ and grows like a multiple of $(1-x)^{-1}$ as $x \uparrow 1$; these singular parts cancel, which causes a loss of accuracy close to 1.*

(iii) If $\gamma - \beta' \in \mathbb{N}$, the poles $-\beta' - n$, $n \geq \gamma - \beta'$, coincide with the poles $-\gamma - m$, $m \geq 0$, and the expansion of g at 0 contains terms of the form $x^{\gamma+m-1} \log x$. Formula (3.4) *is not affected.*

Remark 3.8 (Parametrization) *Since $\gamma' - \beta' = \gamma - \beta$, the Lévy measure* (3.2) *can be written as*

$$\nu(du) = \frac{e^{-\beta' u}\big(1 - e^{-(\varepsilon-1)u}\big)\big(1 - e^{-(\gamma-\beta)u}\big)}{u\,(1 - e^{-u})}\,du.$$

Hence μ_λ depends on (λ, β, γ) only through β' and the unordered pair $\{\varepsilon - 1, \gamma - \beta\}$. In particular, if $\beta < 1$ (which requires $\lambda > \frac{1}{2}$), the triple $(\tilde\lambda, \tilde\beta, \tilde\gamma) = \big(\frac{1}{1+\gamma-\beta}, \gamma', \gamma\big)$ also satisfies Assumption 2.1 and defines the same measure. This is the symmetry of ${}_2F_2(\gamma, \beta'; \beta, \gamma'; z)$ in its two lower parameters. Consequently one may always assume $\varepsilon - 1 \geq \gamma - \beta$ without changing μ_λ. We do not impose this normalization, in order to keep the parametrization of [11], except for estimation (Remark 6.4).

Remark 3.9 (On [11]) *Theorem 3.4 gives a direct proof of the existence result [11, Thm. 2.3], together with the explicit form of μ_λ. It also bears on the local-time analysis of [11, Sect. 4]. By* (2.3) *and Fubini, $\int_{\mathbb{R}} \mathcal{E}_\lambda(\beta, \gamma; -\frac{u^2}{2}|t-s|^{2H})\,du = \sqrt{2\pi}\,|t-s|^{-H}\,\mathbb{E}[L^{-1/2}]$. By Corollary 3.5(ii) this is finite iff $\beta' > \frac{1}{2}$, i.e. $\beta > \frac{1}{\lambda} - \frac{1}{2}$. In that case* (3.3) *with $s = -\frac{1}{2}$ gives $\mathbb{E}L^{-1/2} = C' \frac{\Gamma(\gamma-\frac{1}{2})\Gamma(\beta'-\frac{1}{2})}{\Gamma(\beta-\frac{1}{2})\Gamma(\gamma'-\frac{1}{2})}$. Rewriting C', $\Gamma(\beta'-\frac{1}{2})$ and $\Gamma(\gamma'-\frac{1}{2})$ with the reflection formula $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$ (note $1-\beta' = \frac{1}{\lambda} - \beta$ and $1-\gamma' = \frac{1}{\lambda} - \gamma$) gives*

$$\mathbb{E}[L^{-1/2}] = \frac{\Gamma(\beta)\Gamma(\frac{1}{\lambda} - \beta)}{\Gamma(\gamma)\Gamma(\frac{1}{\lambda} - \gamma)} \cdot \frac{\Gamma(\gamma - \frac{1}{2})\Gamma(\frac{1}{\lambda} - \gamma + \frac{1}{2})}{\Gamma(\beta - \frac{1}{2})\Gamma(\frac{1}{\lambda} - \beta + \frac{1}{2})} \cdot \frac{\tan\pi(\frac{1}{\lambda} - \beta)}{\tan\pi(\frac{1}{\lambda} - \gamma)}.$$

Thus the $L^2(\mathbb{R} \times \Omega)$ criterion of [11, Lemma 4.2, Thm. 4.3] is satisfied exactly when $\beta > \frac{1}{\lambda} - \frac{1}{2}$, with the constant above. We return to this point in Proposition 4.8, where the almost-sure existence of the local time is shown to hold without this restriction.

4 Degenerate Prabhakar mixtures

We now describe the mechanism behind [11] in a form that does not refer to Gaussian processes. The observation is elementary: it is the classical scale-mixture argument, familiar from variance mixtures of Gaussian laws and from mixed Poisson processes [10]. We state it once, in a form that covers both [11] and the counting process of Section 5, and use it as a dictionary between the two.

Definition 4.1 *Let T be an arbitrary index set (e.g. $[0, \infty)$, or a space of test functions) and let $S \subset \mathbb{R}^d$ be the state space (for instance $\mathbb{R}^d$ itself, or $\mathbb{N}_0 \subset \mathbb{R}$). A family $(X_c)_{c>0}$ of S-valued random fields indexed by T is called a* linear-exponent family *if, for all n, $t = (t_1, \ldots, t_n) \in T^n$ and $u = (u_1, \ldots, u_n) \in (\mathbb{R}^d)^n$,*

$$\mathbb{E}\exp\Big(i \sum_{k=1}^{n} \langle u_k, X_c(t_k)\rangle\Big) = \exp\big(-c\,\Psi_t(u)\big), \qquad c > 0,$$

for some function $\Psi_t(u)$ not depending on c (necessarily $\operatorname{Re}\Psi_t(u) \geq 0$).

Definition 4.2 *Let $(X_c)_{c>0}$ be a linear-exponent family and let μ_λ be as in* (2.2)*. The* degenerate Prabhakar mixture *$X^{[\lambda]}$ is the process whose finite-dimensional distributions are*

$$\mathbb{P}\big((X^{[\lambda]}(t_1),\dots,X^{[\lambda]}(t_n))\in A\big)=\int_{(0,1]}\mathbb{P}\big((X_s(t_1),\dots,X_s(t_n))\in A\big)\,\mu_\lambda(ds).$$

Equivalently, $X^{[\lambda]}\overset{d}{=}X_L$ with $L\sim\mu_\lambda$ independent of the family; this realizes the mixture whenever $(c,\omega)\mapsto X_c(\cdot,\omega)$ is jointly measurable, as in all the examples below.

Proposition 4.3 *For a linear-exponent family (X_c),*

$$\mathbb{E}\,e^{\,i\sum_{k=1}^n\langle u_k,X^{[\lambda]}(t_k)\rangle}=\mathcal{E}_\lambda\big(\beta,\gamma;-\Psi_t(u)\big).$$

That is, $X^{[\lambda]}$ is obtained from X_1 by replacing exp *with $\mathcal{E}_\lambda(\beta,\gamma;\cdot)$ in all finite-dimensional characteristic functions.*

Proof. Fix n, $t\in T^n$ and u. We first note that $\operatorname{Re}\Psi_t(u)\ge 0$: a characteristic function has modulus at most 1, so $e^{-c\operatorname{Re}\Psi_t(u)}\le 1$ for all $c>0$. By Definition 4.2, the characteristic function of $(X^{[\lambda]}(t_1),\dots,X^{[\lambda]}(t_n))$ is the μ_λ-average of the characteristic functions of $(X_s(t_1),\dots,X_s(t_n))$. The integrand is bounded by 1, so Fubini's theorem gives

$$\mathbb{E}\,e^{\,i\sum_k\langle u_k,X^{[\lambda]}(t_k)\rangle}=\int_{(0,1]}e^{-s\Psi_t(u)}\,\mu_\lambda(ds).$$

Put $F(w):=\int e^{ws}\mu_\lambda(ds)$ for $w\in\mathbb{C}$. Since μ_λ is carried by $(0,1]$ (Corollary 3.5), $|e^{ws}|\le e^{|w|}$, and expanding e^{ws} in power series and integrating term by term (legitimate because $\sum_k|w|^ks^k/k!=e^{|w|s}$ is integrable) gives $F(w)=\sum_k\frac{w^k}{k!}\int s^k\mu_\lambda(ds)=\sum_k m_k\frac{w^k}{k!}=\mathcal{E}_\lambda(\beta,\gamma;w)$ for all $w\in\mathbb{C}$. Taking $w=-\Psi_t(u)$ completes the proof. □

Corollary 4.4 (The Gaussian member is the dpp of [11]) *Let $X_c=\sqrt{c}\,B^H$. Then $\Psi_t(u)=\frac12 u^\top\widetilde{\Sigma}u$, and $X^{[\lambda]}$ is the degenerate Prabhakar process P^H of [11]: Proposition 4.3 is [11, (3.19)], Definition 4.2 is [11, Thm. 3.5], and the mixture of the Gaussian densities of $N(0,s\widetilde{\Sigma})$ is the density [11, (3.14)] (well defined since $\mu_\lambda(\{0\})=0$ by Corollary 3.5). In particular $\mathbb{E}(P_t^H)^2=m_1t^{2H}$ and $\mathbb{E}(P_t^H)^4=3m_2t^{4H}$.*

Proof. For $t_1,\dots,t_n$, the vector $(\sqrt{c}\,B^H_{t_k})_k$ is centred Gaussian with covariance $c\widetilde{\Sigma}$, so its characteristic function is $\exp(-\frac{c}{2}u^\top\widetilde{\Sigma}u)$ and $\Psi_t(u)=\frac12u^\top\widetilde{\Sigma}u$. Proposition 4.3 gives $\mathcal{E}_\lambda(\beta,\gamma;-\frac12u^\top\widetilde{\Sigma}u)$, which is (2.3). Since finite-dimensional distributions are determined by their characteristic functions, $X^{[\lambda]}$ and P^H have the same finite-dimensional distributions, and $X^{[\lambda]}\overset{d}{=}X_L=\sqrt{L}\,B^H$. Conditionally on $L=s$, P_t^H is centred Gaussian with variance st^{2H}, which gives the two moments, with $\mathbb{E}L=m_1$ and $\mathbb{E}L^2=m_2$.

For the density, let $0<t_1<\cdots<t_n$; then $\widetilde{\Sigma}$ is positive definite. Conditionally on $L=s$, the vector has the density

$$\varphi_s(x)=(2\pi)^{-n/2}s^{-n/2}|\widetilde{\Sigma}|^{-1/2}\exp\Big(-\frac{x^\top\widetilde{\Sigma}^{-1}x}{2s}\Big).$$

Since $\mu_\lambda(\{0\})=0$, Tonelli's theorem shows that $\int\varphi_s(x)\,\mu_\lambda(ds)$ is a probability density, and it is the density of $X^{[\lambda]}$. With $K=m_1$ and $\Sigma=K\widetilde{\Sigma}$ one has $|\Sigma|^{-1/2}K^{n/2}=|\widetilde{\Sigma}|^{-1/2}$ and $K\Sigma^{-1}=\widetilde{\Sigma}^{-1}$, so this expression is [11, (3.14)]. □

Remark 4.5 *The same argument applies to $X_c = \sqrt{c}\,Z$ with $Z \sim N(0, I_d)$, and to $X_c(\xi) = \sqrt{c}\,\langle w, \xi\rangle$ with w a Gaussian white noise on the nuclear space of [11]. It identifies the finite-dimensional degenerate Prabhakar measure [11, Def. 2.5] and the degenerate Prabhakar white noise measure [11, Def. 2.6] as μ_λ-mixtures of centred Gaussian measures, and it gives the moment formula [11, Prop. 2.7] with the constant $m_m = \int s^m \mu_\lambda(ds)$. We omit the details, which are not needed for counting processes.*

Corollary 4.6 (The counting member) *Let $\mathcal{M}$ be a GCP and $\rho > 0$. Then $X_c(t) := \mathcal{M}(c\,t^\rho)$ is a linear-exponent family with $\mathbb{N}_0$-valued paths, and its degenerate Prabhakar mixture is the dPCP of Definition 5.1.*

Proof. Order the times as $0 \le t_1 < \cdots < t_n$ and put $t_0 = 0$, so that $X_c(t_0) = \mathcal{M}(0) = 0$. Summation by parts gives

$$\sum_{k=1}^{n} u_k X_c(t_k) = \sum_{k=1}^{n} U_k\big(X_c(t_k) - X_c(t_{k-1})\big), \qquad U_k := u_k + \cdots + u_n.$$

The map $t \mapsto ct^\rho$ is increasing, so the increments $X_c(t_k) - X_c(t_{k-1}) = \mathcal{M}(ct_k^\rho) - \mathcal{M}(ct_{k-1}^\rho)$ are increments of $\mathcal{M}$ over disjoint intervals. They are therefore independent. Since $\mathcal{M}$ is a compound Poisson process with stationary increments, their characteristic functions are $\exp\big(-c(t_k^\rho - t_{k-1}^\rho)\phi(v)\big)$, with $\phi(v) = \sum_j \lambda_j(1 - e^{ijv})$. Hence

$$\mathbb{E}e^{i\sum_k u_k X_c(t_k)} = \exp\big(-c\,\Psi_t(u)\big), \qquad \Psi_t(u) = \sum_{k=1}^{n}(t_k^\rho - t_{k-1}^\rho)\,\phi(U_k),$$

and $\Psi_t(u)$ does not depend on c. The paths of X_c take values in $\mathbb{N}_0$. Finally $X_L(t) = \mathcal{M}(Lt^\rho)$, which is the process of Definition 5.1. □

Proposition 4.7 (Inherited scaling and stationarity) *Let (X_c) be a linear-exponent family.*

(a) If, for each $a > 0$, there is a measurable map $g_a : S \to S$ such that $X_c(a\,\cdot) \overset{d}{=} g_a\big(X_c(\cdot)\big)$ for all $c > 0$ (equality of finite-dimensional distributions, g_a acting pointwise), then $X^{[\lambda]}(a\,\cdot) \overset{d}{=} g_a\big(X^{[\lambda]}(\cdot)\big)$.

(b) If every X_c has stationary increments, so does $X^{[\lambda]}$.

In particular, for $X_c = \sqrt{c}\,B^H$ one obtains the H-self-similarity and the stationarity of increments of the dpp [11, Prop. 3.2]. For the counting family of Corollary 4.6, $X_c(a\,\cdot) = X_{a^\rho c}(\cdot)$, so the dPCP at time at has the law of the dPCP at time t with all rates λ_j multiplied by a^ρ; for $\rho = 1$ its increments are stationary.

Proof. (a) Fix $a > 0$, times $t_1, \dots, t_n$ and a Borel set $A \subset S^n$. By Definition 4.2 and the hypothesis applied to each X_s,

$$\mathbb{P}\big((X^{[\lambda]}(at_k))_k \in A\big) = \int \mathbb{P}\big((X_s(at_k))_k \in A\big)\mu_\lambda(ds) = \int \mathbb{P}\big((g_a(X_s(t_k)))_k \in A\big)\mu_\lambda(ds).$$

The last integral equals $\mathbb{P}\big((g_a(X^{[\lambda]}(t_k)))_k \in A\big)$, again by Definition 4.2, applied to the set $\{x : (g_a(x_k))_k \in A\}$.

(b) In the same way, for $h > 0$ the finite-dimensional distributions of $\big(X^{[\lambda]}(t+h) - X^{[\lambda]}(h)\big)_t$ are the μ_λ-mixtures of those of $\big(X_s(t+h) - X_s(h)\big)_t$. By hypothesis these coincide with those

of $\big(X_s(t) - X_s(0)\big)_t$, and mixing again gives the finite-dimensional distributions of $\big(X^{[\lambda]}(t) - X^{[\lambda]}(0)\big)_t$.

For $X_c = \sqrt{c}\, B^H$, the self-similarity of fractional Brownian motion gives $\sqrt{c}\, B^H(a\,\cdot) \overset{d}{=} a^H \sqrt{c}\, B^H(\cdot)$, so (a) holds with $g_a(x) = a^H x$, and B^H has stationary increments. For the counting family, $\mathcal{M}(c(at)^\rho) = \mathcal{M}((a^\rho c)\, t^\rho)$ identically. Moreover, a compound Poisson process with rates λ_j run up to time $a^\rho c\, t^\rho$ has the same finite-dimensional distributions as one with rates $a^\rho \lambda_j$ run up to time $c\, t^\rho$, since both have independent increments with the same characteristic functions. Mixing over c gives the claim. For $\rho = 1$, $X_c = \mathcal{M}(c\,\cdot)$ is a Lévy process, so its increments are stationary and (b) applies. □

Proposition 4.8 (Inherited path properties and local times) *Assume that $(c, \omega) \mapsto X_c(\cdot, \omega)$ is jointly measurable and that $L \sim \mu_\lambda$ is independent of the family. If a measurable path property holds almost surely for X_c for every $c \in (0, 1]$, it holds almost surely for X_L. In particular:*

(a) for every admissible (λ, β, γ) and $H \in (0, 1)$, the dpp $P^H = \sqrt{L}\, B^H$ admits almost surely a local time $\ell(\cdot, [0, T]) \in L^2(\mathbb{R}, dx)$;

(b) this local time belongs to $L^2(\mathbb{R} \times \Omega)$ if and only if $\beta' > \frac{1}{2}$, i.e. $\beta > \frac{1}{\lambda} - \frac{1}{2}$;

(c) the paths of the dPCP are a.s. non-decreasing, right-continuous, $\mathbb{N}_0$-valued, with finitely many jumps on compact intervals.

Proof. By joint measurability, $\omega \mapsto X_{L(\omega)}(\cdot, \omega)$ is a random path. Let A be a measurable set of paths such that $\mathbb{P}(X_c \in A) = 1$ for every $c \in (0, 1]$. By independence of L and the family, and Fubini's theorem on the product space,

$$\mathbb{P}(X_L \in A) = \int_{(0,1]} \mathbb{P}(X_s \in A)\, \mu_\lambda(ds) = 1.$$

(a) For a continuous path x on $[0, T]$, the existence of an occupation density in $L^2(\mathbb{R}, dx)$ is equivalent, by Plancherel's theorem, to $\int_{\mathbb{R}} \big| \int_0^T e^{iux(t)} dt \big|^2 du < \infty$. This is a measurable condition on x. Fractional Brownian motion with $H \in (0, 1)$ has a.s. a square-integrable occupation density ℓ_B on $[0, T]$ [1, 8]. For $s > 0$, the occupation measure of $\sqrt{s}\, B^H$ is $A \mapsto \int_0^T \mathbf{1}_A(\sqrt{s}\, B_t^H) dt = \int_{A/\sqrt{s}} \ell_B(y)\, dy = \int_A s^{-1/2} \ell_B(x/\sqrt{s})\, dx$. So $\sqrt{s}\, B^H$ has the occupation density $\ell_s(x) = s^{-1/2} \ell_B(x/\sqrt{s})$, which is square integrable. Since $L > 0$ a.s. by Corollary 3.5(i), the first part of the proof applies to $X_c = \sqrt{c}\, B^H$.

(b) By the change of variables $x = \sqrt{s}\, y$, $\int \ell_s(x)^2 dx = s^{-1} \int \ell_B(x/\sqrt{s})^2 dx = s^{-1/2} \int \ell_B(y)^2 dy$. Conditioning on L, we obtain $\mathbb{E} \int \ell(x)^2 dx = \mathbb{E}[L^{-1/2}] \cdot \mathbb{E} \int \ell_B(y)^2 dy$. By Plancherel's theorem and the Gaussian characteristic function,

$$\mathbb{E} \int \ell_B(y)^2 dy = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^T \int_0^T e^{-u^2 |t-r|^{2H}/2}\, dr\, dt\, du = \frac{1}{\sqrt{2\pi}} \int_0^T \int_0^T |t - r|^{-H} dr\, dt,$$

which is positive and finite since $H < 1$. Hence $\mathbb{E} \int \ell^2 < \infty$ if and only if $\mathbb{E} L^{-1/2} < \infty$, i.e. if and only if $\beta' > \frac{1}{2}$ by Corollary 3.5(ii).

(c) Almost every path of the compound Poisson process $\mathcal{M}$ is non-decreasing, right-continuous, $\mathbb{N}_0$-valued and has finitely many jumps on compact intervals. For $s > 0$, the map $t \mapsto s t^\rho$ is a continuous increasing bijection of $[0, \infty)$, and composing with it preserves these properties. The first part of the proof then gives the result for $X_L = \mathcal{M}^\rho_\lambda$. □

Proposition 4.8(a)–(b) is the general form of the local-time result [11, Thm. 4.3]: almost-sure existence holds for all parameters, while square integrability in (x, ω), which is what Berman's criterion tests, requires $\beta > \frac{1}{\lambda} - \frac{1}{2}$ (compare Remark 3.9).

Table 1 summarizes the correspondence between the Gaussian member of [11] and the counting member studied below.

Table 1: The dpp of [11] and the dPCP as two degenerate Prabhakar mixtures

	dpp P^H [11]	dPCP $\mathcal{M}^\rho_\lambda$ (this paper)
Family X_c	$\sqrt{c}\,B^H$	$\mathcal{M}(c\,t^\rho)$
Representation	$\sqrt{L}\,B^H$	$\mathcal{M}(L\,t^\rho)$
Transform	$\mathbb{E}e^{iuP_t^H} = \mathcal{E}_\lambda(\beta,\gamma;-\frac{u^2}{2}t^{2H})$	$\mathbb{E}u^{\mathcal{M}^\rho_\lambda(t)} = \mathcal{E}_\lambda(\beta,\gamma;-t^\rho\Phi(u))$
Scaling	H-self-similar	time $\times a \Leftrightarrow$ rates $\times a^\rho$
Second moment	$\mathbb{E}(P_t^H)^2 = m_1 t^{2H}$	$\operatorname{Var}\mathcal{M}^\rho_\lambda(t) = m_1 S_2 t^\rho + (m_2 - m_1^2)S_1^2 t^{2\rho}$
Explicit law	density [11, (3.14)]	pmf, Theorem 5.4
Equation	—	Euler-type, Theorem 5.6
Moments	$m_m(2m-1)!!\langle\xi,\xi\rangle^m$ [11, Prop. 2.7]	factorial moments, Prop. 6.1
Path functional	local time, Prop. 4.8 (cf. [11, Thm. 4.3])	first jump time, Theorem 7.1

5 The degenerate Prabhakar counting process

Definition 5.1 *Let Assumption 2.1 hold, let $\mathcal{M}$ be a GCP with rates $(\lambda_j)_{j\geq 1}$ as in Section 2, let $\rho > 0$, and let $L \sim \mu_\lambda$ be independent of $\mathcal{M}$. The* degenerate Prabhakar counting process *(dPCP) is*

$$\mathcal{M}^\rho_\lambda(t) := \mathcal{M}\big(L\,t^\rho\big), \qquad t \geq 0.$$

We write $p_n(t) = \mathbb{P}(\mathcal{M}^\rho_\lambda(t) = n)$, $\tau = t^\rho$, $S_1 = \sum_j j\lambda_j$ and $S_2 = \sum_j j^2\lambda_j$ whenever finite.

Remark 5.2 (Nature of the process) *For $\rho = 1$ and $k = 1$, $\mathcal{M}^\rho_\lambda$ is a mixed Poisson process in the sense of [10], with structure variable $\lambda_1 L$; for general k it is a mixed compound Poisson process. The exponent ρ only changes the clock: pathwise, the dPCP with exponent ρ at time t is the dPCP with exponent 1 at time t^ρ. No fractional operator is involved, and every statement below for general ρ follows from the case $\rho = 1$. We keep ρ because it fixes the growth rate of the mean and allows a comparison with the factor t^α in the one-dimensional laws of the GFCP. Properties such as overdispersion and dependence through L are common to all mixed Poisson processes; what is specific to the dPCP is that μ_λ makes them explicit. The mixing law also has a simple interpretation. Since $L \in (0,1]$ and $\mathbb{P}(L = 1) = C'$, a proportion C' of the paths are ordinary GCP paths on the clock t^ρ, and the other paths run on a clock slowed down by the random factor $L < 1$, with $\mathbb{P}(L \leq x) \sim (C'A/\beta')\,x^{\beta'}$ as $x \downarrow 0$ (Proposition 3.6). The dPCP thus describes a heterogeneous population in which a fraction of the units is unaffected and the others are slowed down; the parameter β' controls how frequent the very slow units are.*

Theorem 5.3 (Generating function) *For $|u| \leq 1$ and $t \geq 0$, $G(u,t) := \mathbb{E}\big[u^{\mathcal{M}^\rho_\lambda(t)}\big] = \mathcal{E}_\lambda\big(\beta,\gamma;-t^\rho\Phi(u)\big)$.*

Proof. By independence of L and $\mathcal{M}$, and by (2.4), $\mathbb{E}\big[u^{\mathcal{M}(L\tau)} \mid L = s\big] = \mathbb{E}u^{\mathcal{M}(s\tau)} = e^{-s\tau\Phi(u)}$. For $|u| \leq 1$ we have $\operatorname{Re}\Phi(u) = \sum_j \lambda_j(1 - \operatorname{Re}u^j) \geq 0$, so this quantity has modulus at most 1. Integrating over s gives $G(u,t) = \int e^{-s\tau\Phi(u)}\mu_\lambda(ds) = F(-\tau\Phi(u))$, where $F(w) = \int e^{ws}\mu_\lambda(ds) = \mathcal{E}_\lambda(\beta,\gamma;w)$ for all $w \in \mathbb{C}$ by the proof of Proposition 4.3. □

Theorem 5.4 (Probability mass function) *For $n \in \mathbb{N}_0$ and $t \geq 0$,*

$$p_n(t) = \sum_{x\in\Omega(k,n)} \prod_j \frac{(\lambda_j t^\rho)^{x_j}}{x_j!}\,\frac{(\gamma)_{z_x}(\beta')_{z_x}}{(\beta)_{z_x}(\gamma')_{z_x}}\,{}_2F_2\!\left(\begin{matrix}\gamma+z_x,\ \beta'+z_x\\ \beta+z_x,\ \gamma'+z_x\end{matrix};-\Lambda t^\rho\right). \tag{5.1}$$

Equivalently, $p_n(t) = \int_{(0,1]} q_n(s\,t^\rho)\,\mu_\lambda(ds) = C'\,q_n(t^\rho) + \int_{(0,1)} q_n(s\,t^\rho)\,\mu_\lambda(ds)$.

Proof. Conditioning on L gives $p_n(t) = \int q_n(s\tau)\,\mu_\lambda(ds)$, which is the mixture formula. By (2.4),

$$q_n(s\tau) = \sum_{x\in\Omega(k,n)} \prod_j \frac{(\lambda_j\tau)^{x_j}}{x_j!}\,s^{z_x}e^{-s\Lambda\tau},$$

because $\prod_j (\lambda_j s\tau)^{x_j} = s^{z_x} \prod_j (\lambda_j \tau)^{x_j}$. The set $\Omega(k,n)$ is finite, since $x_j = 0$ whenever $j > n$, so the sum and the integral can be exchanged:

$$p_n(t) = \sum_{x \in \Omega(k,n)} \prod_j \frac{(\lambda_j \tau)^{x_j}}{x_j!} \int s^{z_x} e^{-s\Lambda\tau}\, \mu_\lambda(ds).$$

Let $F(w) = \int e^{ws} \mu_\lambda(ds)$. Since μ_λ has compact support, we may differentiate under the integral sign, and $F^{(r)}(w) = \int s^r e^{ws} \mu_\lambda(ds)$. Hence $\int s^{z_x} e^{-s\Lambda\tau} \mu_\lambda(ds) = F^{(z_x)}(-\Lambda\tau)$. Since $F = \mathcal{E}_\lambda(\beta, \gamma; \cdot)$, Proposition 3.2(b) gives $F^{(z_x)}(-\Lambda\tau) = m_{z_x}\, {}_2F_2(\gamma + z_x, \beta' + z_x; \beta + z_x, \gamma' + z_x; -\Lambda\tau)$, which is (5.1). The last expression of the theorem follows from the decomposition $\mu_\lambda = C'\delta_1 + \mu_\lambda|_{(0,1)}$ of Corollary 3.5(i). □

Remark 5.5 *It is instructive to compare* (5.1) *with the GFCP formula* (2.5)*. The factor*

$$z_x!\, t^{\alpha z_x} E^{z_x+1}_{\alpha, \alpha z_x + 1}(-\Lambda t^\alpha) \quad \textit{is replaced by} \quad t^{\rho z_x} m_{z_x}\, {}_2F_2(\cdots; -\Lambda t^\rho).$$

Both are integrals of $s^{z_x} e^{-s\Lambda t}$ against a mixing law, namely the Mittag-Leffler law in the first case and μ_λ in the second.

Theorem 5.6 (Governing equation) *Let $\delta_t := \frac{t}{\rho}\frac{\partial}{\partial t}$. The functions p_n are the unique solutions of the system*

$$\begin{aligned} &\delta_t(\delta_t + \beta - 1)(\delta_t + \gamma' - 1)\, p_n(t) \\ &\qquad = -\, t^\rho\, (\delta_t + \gamma)(\delta_t + \beta') \Big[\Lambda\, p_n(t) - \sum_{j=1}^{n} \lambda_j\, p_{n-j}(t) \Big], \qquad n \in \mathbb{N}_0, \end{aligned} \tag{5.2}$$

subject to $p_n(t) - \delta_{n,0} = O(t^\rho)$ as $t \downarrow 0$. Equivalently, G is the unique solution of

$$\delta_t(\delta_t + \beta - 1)(\delta_t + \gamma' - 1) G = -t^\rho \Phi(u)(\delta_t + \gamma)(\delta_t + \beta') G$$

with $G(u,t) - 1 = O(t^\rho)$.

Proof. Step 1: change of variable. Fix u with $|u| < 1$ and $\Phi(u) \neq 0$, and put $z = -t^\rho \Phi(u)$. Then $\frac{dz}{dt} = \rho z/t$, so for any differentiable function f,

$$z \frac{d}{dz} f(z) = z \Big(\frac{dz}{dt} \Big)^{-1} \frac{d}{dt} f(z) = \frac{t}{\rho} \frac{d}{dt} f(z) = \delta_t f(z).$$

By Theorem 5.3, $G(u,t) = f(z)$ with $f = \mathcal{E}_\lambda(\beta, \gamma; \cdot)$. Hence equation (3.1) for f becomes

$$\delta_t(\delta_t + \beta - 1)(\delta_t + \gamma' - 1) G = -t^\rho \Phi(u)(\delta_t + \gamma)(\delta_t + \beta') G.$$

If $\Phi(u) = 0$, then $G \equiv 1$ and both sides vanish.

Step 2: coefficients. Since $|\Phi(u)| \le 2\Lambda$ for $|u| \le 1$ and Φ is analytic in the open unit disc, the function $(u, \tau) \mapsto F(-\tau\Phi(u))$ is analytic on $\{|u| < 1\} \times \mathbb{C}$. Its Taylor coefficients in u are the functions p_n, which are entire functions of $\tau = t^\rho$ by (5.1). By Cauchy's integral formula in u, applying δ_t to G and taking the coefficient of u^n commute. Moreover, $\Phi(u) = \Lambda - \sum_j \lambda_j u^j$, so the coefficient of u^n in $\Phi(u)H(u)$ is $\Lambda H_n - \sum_{j=1}^n \lambda_j H_{n-j}$ for any power series $H = \sum_n H_n u^n$. Identifying the coefficients of u^n in the equation of Step 1 gives (5.2).

Step 3: behaviour at $t = 0$. For $n \ge 1$, every $x \in \Omega(k,n)$ has $z_x \ge 1$, so each term of (5.1) contains the factor τ^{z_x} multiplied by a function bounded near $\tau = 0$; hence $p_n(t) = O(t^\rho)$. For $n = 0$, $p_0(t) = {}_2F_2(\gamma, \beta'; \beta, \gamma'; -\Lambda\tau) = 1 + O(\tau)$.

Step 4: uniqueness. We argue by induction on n. Suppose that $p_0, \ldots, p_{n-1}$ are given and that p_n and $\tilde{p}_n$ both satisfy (5.2) and $p_n - \delta_{n,0} = O(t^\rho)$, $\tilde{p}_n - \delta_{n,0} = O(t^\rho)$. Then $h := p_n - \tilde{p}_n$

satisfies $h = O(t^\rho)$ and the homogeneous equation $\delta_t(\delta_t+\beta-1)(\delta_t+\gamma'-1)h = -\Lambda t^\rho(\delta_t+\gamma)(\delta_t+\beta')h$. In the variable $w = -\Lambda t^\rho$, for which $w\frac{d}{dw} = \delta_t$ by Step 1, this is

$$\theta(\theta+\beta-1)(\theta+\gamma'-1)h = w(\theta+\gamma)(\theta+\beta')h, \qquad \theta = w\frac{d}{dw},$$

the equation (3.1). Since $\theta^3 = w^3D^3 + 3w^2D^2 + wD$ with $D = \frac{d}{dw}$, dividing by w^3 puts it in the form $h''' + a(w)h'' + b(w)h' + c(w)h = 0$, where a, b, c have poles of order at most 1, 2, 3 at $w = 0$. So $w = 0$ is a regular singular point. The indicial polynomial comes from the left-hand side only and equals $r(r+\beta-1)(r+\gamma'-1)$, with roots 0, $1-\gamma'$ and $1-\beta$. Under Assumption 2.1 we have $\gamma' \in (0,1)$ and $\beta > 0$, so all three roots are smaller than 1. They need not be distinct, and they may differ by integers: for instance $\beta = 1$ and $\beta = \gamma'$ are possible when $\lambda > \frac{1}{2}$. In every case, Frobenius' theory provides a basis y_0, y_1, y_2 of solutions on $w < 0$ such that $y_i(w) = |w|^{r_i}(\log|w|)^{m_i}(1+o(1))$ as $w \to 0$, where each r_i is a root of the indicial polynomial, $m_i \in \{0,1,2\}$, and the pairs (r_i, m_i) are pairwise distinct. Consequently, any non-trivial combination $h = a_0y_0 + a_1y_1 + a_2y_2$ is equivalent, as $w \to 0$, to $c\,|w|^{r_i}(\log|w|)^{m_i}$ with $c \neq 0$, for the dominant index i among those with $a_i \neq 0$. Since all roots are smaller than 1, such an h is not $O(w)$. Hence $h \equiv 0$. The case $n = 0$ is identical, with the condition $p_0(t) - 1 = O(t^\rho)$. The same argument gives uniqueness for G, for each fixed real u with $\Phi(u) \neq 0$; if $\Phi(u) = 0$ the equation reduces to $\delta_t(\delta_t+\beta-1)(\delta_t+\gamma'-1)G = 0$, whose solutions are combinations of 1, $t^{\rho(1-\beta)}$, $t^{\rho(1-\gamma')}$ and possibly logarithmic terms, and the condition $G - 1 = O(t^\rho)$ again forces $G \equiv 1$. □

Remark 5.7 *Equation* (5.2) *plays for the dPCP the role of the Caputo system for the GFCP [5], but it is of a different nature. The operator δ_t is local:* (5.2) *is an ordinary differential system of Euler type and involves no fractional derivative. It characterizes the one-dimensional laws only; the dependence between different times comes from the common variable L (Section 6) and is not encoded in the equation.*

6 Moments, overdispersion and dependence

For $i \in \mathbb{N}$ put $S_{(i)} := \sum_j \binom{j}{i}\lambda_j$, so $S_{(1)} = S_1$ and $2S_{(2)} = S_2 - S_1$.

Proposition 6.1 (Factorial moments) *Let $r \in \mathbb{N}$ and assume $\sum_j j^r\lambda_j < \infty$ (automatic if $k < \infty$). Then*

$$\mathbb{E}\big[(\mathcal{M}^\rho_\lambda(t))_{(r)}\big] = \sum_{m=1}^{r} \frac{r!}{m!}\, m_m\, t^{\rho m} \sum_{\substack{i_1+\cdots+i_m=r \\ i_l \geq 1}} \prod_{l=1}^{m} S_{(i_l)},$$

where $(x)_{(r)} = x(x-1)\cdots(x-r+1)$ is the falling factorial.

Proof. Conditionally on $L = s$, for $v \in (-1,0]$ we have $\mathbb{E}[(1+v)^{\mathcal{M}(s\tau)}] = \exp(s\tau\Psi(v))$, where $\Psi(v) = -\Phi(1+v) = \sum_j \lambda_j\big((1+v)^j - 1\big)$. For $v \in (-1,0]$ and $i \leq r$, the i-th derivative of $(1+v)^j$ is bounded by j^i. Under the moment assumption, the series defining Ψ can therefore be differentiated r times, by dominated convergence. This gives $\Psi^{(i)}(0) = \sum_j \lambda_j j(j-1)\cdots(j-i+1) = i!\,S_{(i)}$, and hence $\Psi(v) = \sum_{i=1}^r S_{(i)}v^i + o(v^r)$ as $v \uparrow 0$. Expanding the exponential,

$$\exp(s\tau\Psi(v)) = \sum_{m=0}^{r} \frac{(s\tau)^m\Psi(v)^m}{m!} + o(v^r),$$

uniformly in $s \in (0,1]$. Integrating against $\mu_\lambda(ds)$ and using $\int s^m\mu_\lambda(ds) = m_m$ gives $\mathbb{E}[(1+v)^{\mathcal{M}^\rho_\lambda(t)}] = \sum_{m=0}^r m_m\tau^m\Psi(v)^m/m! + o(v^r)$. The r-th left derivative at $v = 0$ of $\mathbb{E}[(1+v)^X]$ is $\mathbb{E}[(X)_{(r)}]$, and it equals $r!$ times the coefficient of v^r in the polynomial part. The coefficient of v^r in $\Psi(v)^m$ is $\sum_{i_1+\cdots+i_m=r,\, i_l\geq 1}\prod_l S_{(i_l)}$, which gives the formula. □

Corollary 6.2 *Assume $S_2 < \infty$ and put $m_1 = \frac{\gamma\beta'}{\beta\gamma'}$, $m_2 = m_1\frac{(\gamma+1)(\beta'+1)}{(\beta+1)(\gamma'+1)}$. Then $m_2 - m_1^2 = \operatorname{Var} L > 0$ and*

$$\mathbb{E}\,\mathcal{M}_\lambda^\rho(t) = m_1 S_1\, t^\rho, \qquad \operatorname{Var}\mathcal{M}_\lambda^\rho(t) = m_1 S_2\, t^\rho + (m_2 - m_1^2) S_1^2\, t^{2\rho},$$
$$\operatorname{Cov}\left(\mathcal{M}_\lambda^\rho(s), \mathcal{M}_\lambda^\rho(t)\right) = m_1 S_2\, s^\rho + (m_2 - m_1^2) S_1^2\, (st)^\rho, \qquad 0 \le s \le t.$$

In particular the Fano factor $\operatorname{Var}\mathcal{M}_\lambda^\rho(t)/\mathbb{E}\mathcal{M}_\lambda^\rho(t) = S_2/S_1 + (m_2 - m_1^2)S_1 t^\rho/m_1$ is > 1 for $t > 0$ and increases to ∞: the dPCP is overdispersed.

Proof. Conditionally on $L = c$, $\mathcal{M}(c\tau)$ is a compound Poisson variable with mean $cS_1\tau$ and variance $cS_2\tau$, and for $0 \le s \le t$ the independence of the increments of $\mathcal{M}$ gives $\operatorname{Cov}(\mathcal{M}(cs^\rho), \mathcal{M}(ct^\rho)) = \operatorname{Var}\mathcal{M}(cs^\rho) = cS_2 s^\rho$. The laws of total expectation, variance and covariance, with $\mathbb{E}L = m_1$ and $\operatorname{Var} L = m_2 - m_1^2$, give the three formulas; the mean and the variance also follow from Proposition 6.1 with $r = 1, 2$. Moreover $\operatorname{Var} L > 0$, because L is not a.s. constant: by Corollary 3.5(i) it has an atom of mass $C' < 1$ at 1 and the remaining mass on $(0, 1)$. The Fano factor exceeds 1 for $t > 0$ because $S_2 \ge S_1$ and the second term is positive, and it increases to ∞ because that term is proportional to t^ρ. □

Proposition 6.3 (Non-ergodicity and dependence) *Assume $S_1 < \infty$. Then $\mathcal{M}_\lambda^\rho(t)/t^\rho \to S_1 L$ almost surely as $t \to \infty$, and the limit is not degenerate. If moreover $S_2 < \infty$, then for fixed $s > 0$, with $\operatorname{Var}\mathcal{M}_\lambda^\rho(s)$ as in Corollary 6.2,*

$$\lim_{t\to\infty} \operatorname{Corr}\left(\mathcal{M}_\lambda^\rho(s), \mathcal{M}_\lambda^\rho(t)\right) = \frac{\sqrt{m_2 - m_1^2}\, S_1 s^\rho}{\sqrt{\operatorname{Var}\mathcal{M}_\lambda^\rho(s)}} > 0,$$

so the correlation does not decay. This persistence is the random-effect dependence common to all mixed Poisson processes [10], a form of non-ergodicity; it should not be confused with the long-range dependence of fractional processes studied in [18]. For $\rho \ne 1$ the increments are not stationary, since $\mathbb{E}[\mathcal{M}_\lambda^\rho(t+h) - \mathcal{M}_\lambda^\rho(t)] = m_1 S_1((t+h)^\rho - t^\rho)$ depends on t.

Proof. By the strong law of large numbers for compound Poisson processes, $\mathcal{M}(\tau)/\tau \to S_1$ a.s. as $\tau \to \infty$. Since $L > 0$ a.s. and L is independent of $\mathcal{M}$, we have $Lt^\rho \to \infty$ and $\mathcal{M}_\lambda^\rho(t)/t^\rho = L\,\mathcal{M}(Lt^\rho)/(Lt^\rho) \to S_1 L$ a.s. The limit is not degenerate because $\operatorname{Var} L > 0$ (Corollary 6.2). For the correlation, Corollary 6.2 gives $\operatorname{Cov}(\mathcal{M}_\lambda^\rho(s), \mathcal{M}_\lambda^\rho(t)) = m_1 S_2 s^\rho + (m_2 - m_1^2)S_1^2 s^\rho t^\rho$ and $\operatorname{Var}\mathcal{M}_\lambda^\rho(t) = (m_2 - m_1^2)S_1^2 t^{2\rho}(1 + o(1))$; dividing by $\sqrt{\operatorname{Var}\mathcal{M}_\lambda^\rho(s)\operatorname{Var}\mathcal{M}_\lambda^\rho(t)}$ and letting $t \to \infty$ gives the limit. The last claim follows from the formula for the mean. □

Remark 6.4 (Statistical consequences) *By Proposition 6.3, a single path observed over a long horizon determines $S_1 L$, that is, one realization of L, and nothing more about μ_λ. The parameters (λ, β, γ) therefore cannot be estimated consistently from one trajectory, however long. They can be recovered from independent replications: for known rates with $k < \infty$ and known ρ, the factorial moments of Proposition 6.1 at one fixed time determine the sequence $(m_r)_{r\ge1}$, hence μ_λ, since a law on $[0, 1]$ is determined by its moments. In turn μ_λ determines its Lévy measure ν, hence β' and the pair $\{\varepsilon - 1, \gamma - \beta\}$, which is all that can be identified (Remark 3.8). For instance, let $k = 1$ and let $N_1, \dots, N_n$ be the values at a fixed time $t > 0$ of n independent copies of the process. By Proposition 6.1, $\mathbb{E}\big[(N_i)_{(r)}\big] = m_r(\lambda_1 t^\rho)^r$, so*

$$\hat{m}_r := \frac{1}{n}\sum_{i=1}^{n} \frac{(N_i)_{(r)}}{(\lambda_1 t^\rho)^r}$$

is an unbiased estimator of m_r. Since all moments of N_i are finite, it is strongly consistent and asymptotically normal. In principle β', $\varepsilon - 1$ and $\gamma - \beta$ can then be estimated by solving

$m_r = \hat{m}_r$ *for* $r = 1, 2, 3$. *The simulation study of Appendix B shows, however, that while the* $\hat{m}_r$ *are accurate, this inversion is ill-conditioned. Maximum likelihood is preferable. By Theorem 5.4 the likelihood of the sample is* $\prod_i p_{N_i}(t)$, *which is explicit in terms of* ${}_2F_2$ *functions; it is maximized under the constraint* $\varepsilon - 1 \geq \gamma - \beta$, *which removes the ambiguity of Remark 3.8. In the simulations of Appendix B the maximum likelihood estimator recovers* β' *and* $\gamma - \beta$ *accurately, while* $\varepsilon - 1$ *remains the least well determined parameter.*

7 Asymptotics and waiting times

Theorem 7.1 *Let* $D := C' \frac{\Gamma(\gamma-\beta')}{\Gamma(\varepsilon-1)\Gamma(\gamma-\beta)}$. *For every* $n \in \mathbb{N}_0$ *such that* $p_n \not\equiv 0$ *(that is, such that* $\prod_j \lambda_j^{x_j} > 0$ *for some* $x \in \Omega(k,n)$*), as* $t \to \infty$,

$$p_n(t) \sim D\, t^{-\rho\beta'} \sum_{x \in \Omega(k,n)} \prod_j \frac{\lambda_j^{x_j}}{x_j!} \frac{\Gamma(z_x + \beta')}{\Lambda^{z_x+\beta'}}.$$

In particular the first jump time $T_1 = \inf\{t : \mathcal{M}_\lambda^\rho(t) \geq 1\}$ *satisfies*

$$\mathbb{P}(T_1 > t) = {}_2F_2\left(\begin{matrix}\gamma, & \beta' \\ \beta, & \gamma'\end{matrix}; -\Lambda t^\rho\right) \sim \frac{\Gamma(\beta)\Gamma(\gamma')\Gamma(\gamma-\beta')}{\Gamma(\gamma)\Gamma(\varepsilon-1)\Gamma(\gamma-\beta)} (\Lambda t^\rho)^{-\beta'},$$

and $\mathbb{E}\, T_1^q < \infty$ *iff* $q < \rho\beta'$. *Since* $\rho\beta' < \rho$, $\mathbb{E}T_1 = \infty$ *whenever* $\rho \leq 1$.

Proof. Step 1: asymptotics of ${}_2F_2$. Let $a_1 < a_2$ and $b_1, b_2 > 0$. As $x \to +\infty$, the function ${}_2F_2(a_1, a_2; b_1, b_2; -x)$ is the sum of an algebraic part and an exponential part (see [6, §16.11] or [16]). The leading term of the algebraic part is

$$\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(a_2 - a_1)}{\Gamma(a_2)\Gamma(b_1 - a_1)\Gamma(b_2 - a_1)}\, x^{-a_1}.$$

The next terms are of order x^{-a_1-1} or x^{-a_2} (possibly with a factor $\log x$ when $a_2 - a_1 \in \mathbb{N}$). The exponential part is $O(e^{-x} x^{a_1+a_2-b_1-b_2})$. Hence ${}_2F_2(a_1, a_2; b_1, b_2; -x)$ is equivalent to this leading term whenever the leading coefficient is non-zero.

Step 2: one term of (5.1). Fix $x \in \Omega(k,n)$ and write $z = z_x$. Apply Step 1 with $a_1 = \beta' + z < a_2 = \gamma + z$, $b_1 = \beta + z$, $b_2 = \gamma' + z$. Then $b_1 - a_1 = \beta - \beta' = \varepsilon - 1 > 0$ and $b_2 - a_1 = \gamma' - \beta' = \gamma - \beta > 0$, so the leading coefficient is non-zero, and

$${}_2F_2(\gamma + z, \beta' + z; \beta + z, \gamma' + z; -\Lambda\tau) \sim \frac{\Gamma(\beta + z)\Gamma(\gamma' + z)\Gamma(\gamma - \beta')}{\Gamma(\gamma + z)\Gamma(\varepsilon - 1)\Gamma(\gamma - \beta)} (\Lambda\tau)^{-\beta'-z}.$$

By Proposition 3.2(a),

$$m_z = \frac{\Gamma(\beta)\Gamma(\gamma')}{\Gamma(\gamma)\Gamma(\beta')} \cdot \frac{\Gamma(\gamma + z)\Gamma(\beta' + z)}{\Gamma(\beta + z)\Gamma(\gamma' + z)} = C' \frac{\Gamma(\gamma + z)\Gamma(\beta' + z)}{\Gamma(\beta + z)\Gamma(\gamma' + z)}.$$

In the product the factors $\Gamma(\gamma+z)$, $\Gamma(\beta+z)$ and $\Gamma(\gamma'+z)$ cancel. Multiplying by $\prod_j (\lambda_j\tau)^{x_j}/x_j! = \tau^z \prod_j \lambda_j^{x_j}/x_j!$, the term is equivalent to $D\,\Gamma(z + \beta')\Lambda^{-z-\beta'}\tau^{-\beta'} \prod_j \lambda_j^{x_j}/x_j!$.

Step 3: summation and first jump. The sum over $\Omega(k,n)$ is finite, all its terms are non-negative and, by assumption, at least one is positive, so $p_n(t)$ is equivalent to the sum of the equivalents, which is the stated formula. Since the paths are non-decreasing and start at 0, $\{T_1 > t\} = \{\mathcal{M}_\lambda^\rho(t) = 0\}$, and the case $n = 0$ ($\Omega(k,0) = \{0\}$, $z = 0$) gives $\mathbb{P}(T_1 > t) \sim D\,\Gamma(\beta')(\Lambda\tau)^{-\beta'}$, i.e. the displayed constant. Finally $\mathbb{E}T_1^q = \int_0^\infty q t^{q-1} \mathbb{P}(T_1 > t)\, dt$. The integrand is integrable near 0 for $q > 0$, and near ∞ it behaves like $t^{q-1-\rho\beta'}$, which is integrable if and only if $q < \rho\beta'$. Since $\beta' < 1$, we have $\rho\beta' < 1$ whenever $\rho \leq 1$, and then $\mathbb{E}T_1 = \infty$. □

Proposition 7.2 (First jump of a given amplitude) *Let $T^{(j)}$ be the first time the dPCP performs a jump of amplitude j. Then $\mathbb{P}(T^{(j)} > t) = \mathcal{E}_\lambda(\beta, \gamma; -\lambda_j t^\rho)$, which is the waiting time of the first event of the degenerate Prabhakar Poisson process of rate λ_j (Section 9).*

Proof. A compound Poisson process with jump law $(\lambda_j/\Lambda)_j$ can be written as $\mathcal{M} = \sum_j jN_j$, where the N_j are independent Poisson processes with rates λ_j and N_j counts the jumps of size j. Since $t \mapsto Lt^\rho$ is continuous and increasing, the jumps of size j of $\mathcal{M}^\rho_\lambda$ on $[0,t]$ correspond to the jumps of N_j on $[0, Lt^\rho]$. Hence $\{T^{(j)} > t\} = \{N_j(Lt^\rho) = 0\}$. Conditionally on $L = s$, this event has probability $e^{-\lambda_j s t^\rho}$, and integrating over s gives $\int e^{-\lambda_j s t^\rho} \mu_\lambda(ds) = \mathcal{E}_\lambda(\beta, \gamma; -\lambda_j t^\rho)$. The degenerate Prabhakar Poisson process of rate λ_j satisfies $\mathbb{P}(\text{no event in } [0,t]) = G(0,t) = \mathcal{E}_\lambda(\beta, \gamma; -\lambda_j t^\rho)$ by Theorem 5.3, which is the same expression. □

This extends to the degenerate setting the property, proved in [5] for the GFCP, that the first jump of amplitude j is distributed as the first event of a fractional Poisson process of rate λ_j.

Remark 7.3 *For the GFCP, the single parameter α fixes both the growth t^α of the mean and the tail $t^{-\alpha}$ of T_1. For the dPCP the mean grows like t^ρ whereas the tail behaves like $t^{-\rho\beta'}$ with $\beta' = \beta + 1 - 1/\lambda$, and β' can be any number in $(0,1)$. The two exponents can therefore be tuned separately (see, however, Remark 6.4 on what can be estimated from data).*

8 Renewal structure and a renewal counterpart

The fractional Poisson process is a renewal process with Mittag-Leffler waiting times [17, 20]. The dPCP does not have this property.

Proposition 8.1 (Waiting times of the dPCP) *Let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ be the jump epochs of $\mathcal{M}^\rho_\lambda$ and put $W_n := \tau^\rho_n - \tau^\rho_{n-1}$, so that for $\rho = 1$ the W_n are the waiting times between jumps. Then, for all $n \geq 1$ and $s_1, \ldots, s_n \geq 0$,*

$$\mathbb{P}(W_1 > s_1, \ldots, W_n > s_n) = \mathcal{E}_\lambda\big(\beta, \gamma; -\Lambda(s_1 + \cdots + s_n)\big).$$

Thus the sequence (W_n) is exchangeable, each W_n has the survival function $\mathcal{E}_\lambda(\beta, \gamma; -\Lambda s)$, and the W_n are not independent. For every $\rho > 0$, the dPCP is not a renewal process. Moreover, the conditional law of L given $(W_1, \ldots, W_n) = (w_1, \ldots, w_n)$ is proportional to $s^n e^{-\Lambda s(w_1 + \cdots + w_n)} \mu_\lambda(ds)$.

Proof. Let $\Gamma_1 < \Gamma_2 < \cdots$ be the jump epochs of $\mathcal{M}$; they form a Poisson process of rate Λ, independent of L. Since $\mathcal{M}^\rho_\lambda(t) = \mathcal{M}(Lt^\rho)$ and $L > 0$ a.s., $\tau^\rho_n = \Gamma_n/L$ and $W_n = (\Gamma_n - \Gamma_{n-1})/L$. Conditionally on $L = s$, the W_n are therefore i.i.d. exponential with parameter Λs, so $\mathbb{P}(W_1 > s_1, \ldots, W_n > s_n \mid L = s) = e^{-\Lambda s(s_1 + \cdots + s_n)}$, and integrating against μ_λ gives the formula, by (2.2). Exchangeability is clear. If W_1 and W_2 were independent, the continuous function $F(s) := \mathcal{E}_\lambda(\beta, \gamma; -\Lambda s)$ would satisfy $F(s_1 + s_2) = F(s_1)F(s_2)$ with $F(0) = 1$, so $F(s) = e^{-as}$ for some $a \geq 0$. Then $\mathbb{E}e^{-xL} = e^{-ax/\Lambda}$ for $x \geq 0$ and L would be a.s. constant, which contradicts $\operatorname{Var} L > 0$ (Corollary 6.2). For $\rho = 1$ this already shows that the dPCP is not a renewal process. For general $\rho > 0$, suppose that the waiting times $\xi_n := \tau_n - \tau_{n-1}$ were i.i.d., and let $J(t) := \max\{n : \tau_n \leq t\}$ be the number of jumps up to time t. Since $J(t)$ is the number of points of the Poisson process (Γ_n) in $[0, Lt^\rho]$, we have $J(t)/t^\rho \to \Lambda L$ a.s. A permutation of $\xi_1, \ldots, \xi_n$ does not change τ_m for $m \geq n$, hence it does not change $J(t)$ for $t \geq \tau_n$. So the limit ΛL is a function of the sequence (ξ_n) which is invariant under finite permutations, and by the Hewitt–Savage zero–one law [12] it is a.s. constant. This contradicts $\operatorname{Var} L > 0$. Finally, given $L = s$ the vector $(W_1, \ldots, W_n)$ has the density $(\Lambda s)^n e^{-\Lambda s(w_1 + \cdots + w_n)}$, and Bayes' formula gives the conditional law of L. □

The proposition shows how the heavy tail of T_1 in Theorem 7.1 arises: a long first waiting time signals a small value of L, and then all later waiting times are long as well. A different object is obtained if the waiting times are drawn independently.

Definition 8.2 *Let $T_1, T_2, \dots$ be i.i.d. with $\mathbb{P}(T_i > t) = \mathcal{E}_\lambda(\beta, \gamma; -\Lambda t^\rho)$, $t \geq 0$, and let $Y_1, Y_2, \dots$ be i.i.d. with $\mathbb{P}(Y_i = j) = \lambda_j/\Lambda$, independent of the T_i. The* renewal counterpart *of the dPCP is*

$$R(t) := \sum_{i=1}^{N(t)} Y_i, \qquad N(t) := \max\{n \geq 0 : \ T_1 + \cdots + T_n \leq t\}.$$

The function $t \mapsto \mathcal{E}_\lambda(\beta, \gamma; -\Lambda t^\rho)$ is indeed a survival function: it is $\mathbb{P}(T_1 > t)$ in Theorem 7.1. Moreover $T_i \stackrel{d}{=} \big(E/(\Lambda L)\big)^{1/\rho}$, where E is a standard exponential variable independent of $L \sim \mu_\lambda$, because $\mathbb{P}\big(E > \Lambda L t^\rho\big) = \mathbb{E}e^{-\Lambda L t^\rho}$. Together with Theorem 3.4 this gives a simulation scheme for R, with a fresh copy of L for every waiting time.

Proposition 8.3 *Assume $S_1 < \infty$.*

(a) R and $\mathcal{M}^\rho_\lambda$ have the same law of the first jump time and of the first jump amplitude, but they are different processes: $\mathcal{M}^\rho_\lambda(t)/t^\rho \to S_1 L > 0$ almost surely, whereas $R(t)/t^\rho \to 0$ in probability.

(b) Let $a := \rho\beta' < 1$ (this holds in particular whenever $\rho \leq 1$) and $K := D\,\Gamma(\beta')\,\Lambda^{-\beta'}$, with D as in Theorem 7.1. Let Y_a be the inverse of the a-stable subordinator with Laplace exponent s^a. Then, as $c \to \infty$, in the sense of finite-dimensional distributions,

$$\Big\{\frac{N(ct)}{c^a}\Big\}_{t\geq 0} \Longrightarrow \Big\{\frac{Y_a(t)}{K\,\Gamma(1-a)}\Big\}_{t\geq 0}, \qquad \Big\{\frac{R(ct)}{c^a}\Big\}_{t\geq 0} \Longrightarrow \Big\{\frac{S_1}{\Lambda}\cdot\frac{Y_a(t)}{K\,\Gamma(1-a)}\Big\}_{t\geq 0}.$$

In particular the one-dimensional limit laws are Mittag-Leffler laws, $\mathbb{E}e^{-xY_a(t)} = E_a(-xt^a)$, as for the fractional Poisson process of order a.

Proof. (b) By Theorem 7.1, $\mathbb{P}(T_1 > t) \sim K\,t^{-a}$ as $t \to \infty$, with $0 < a < 1$. By Karamata's Tauberian theorem [4, Sect. 1.7], $1 - \mathbb{E}e^{-sT_1} = s\int_0^\infty e^{-st}\mathbb{P}(T_1 > t)\,dt \sim K\,\Gamma(1-a)\,s^a$ as $s \downarrow 0$. Put $\kappa := K\Gamma(1-a)$, $S_n := T_1 + \cdots + T_n$ and $S^{(c)}(u) := c^{-1}S_{\lceil c^a u\rceil}$, and let $D(u) := D_a(\kappa u)$, where D_a is the a-stable subordinator with Laplace exponent s^a. For $0 \leq v < u$, the increment $S^{(c)}(u) - S^{(c)}(v)$ is a sum of $\lceil c^a u\rceil - \lceil c^a v\rceil$ independent copies of T_1/c, so its Laplace transform is $\big(1-\kappa(s/c)^a(1+o(1))\big)^{c^a(u-v)+O(1)} \to e^{-(u-v)\kappa s^a}$. Since $S^{(c)}$ and D have independent increments, the finite-dimensional distributions of $S^{(c)}$ converge to those of D. Now $N(t) < n$ if and only if $S_n > t$. Hence, for $t_1, \dots, t_m > 0$ and $u_1, \dots, u_m > 0$,

$$\mathbb{P}\big(c^{-a}N(ct_i) < u_i, \ i \leq m\big) = \mathbb{P}\big(S^{(c)}(u_i) > t_i, \ i \leq m\big) \longrightarrow \mathbb{P}\big(D(u_i) > t_i, \ i \leq m\big),$$

because the law of $(D(u_1), \dots, D(u_m))$ gives no mass to the boundary of the set $\{x_i > t_i, \ i \leq m\}$ (each $D(u_i)$ has a continuous law). Let $Y(t) := \inf\{u : D(u) > t\}$. As D is strictly increasing and right-continuous, $\{Y(t) < u\} \subset \{D(u) > t\} \subset \{Y(t) \leq u\}$, and $\mathbb{P}(Y(t) = u) = 0$, so the limit equals $\mathbb{P}(Y(t_i) < u_i, \ i \leq m)$. Finally $Y(t) = Y_a(t)/\kappa$, which proves the first convergence; functional versions of this limit theorem are given in [19]. For R, note that $N(ct) \to \infty$ a.s. for every $t > 0$ and that $\frac{1}{n}\sum_{i\leq n} Y_i \to S_1/\Lambda$ a.s. by the strong law of large numbers; hence $R(ct)/N(ct) \to S_1/\Lambda$ a.s., and the second convergence follows from the first. The Laplace transform of $Y_a(t)$ is recalled in Section 2.

(a) The first jump time of R is T_1, which has the law of the first jump time of $\mathcal{M}^\rho_\lambda$ by Theorem 7.1, and the first amplitude has the law $(\lambda_j/\Lambda)_j$ in both cases. The limit of $\mathcal{M}^\rho_\lambda(t)/t^\rho$ is given by Proposition 6.3. If $\rho\beta' < 1$, part (b) shows that $R(t)/t^{\rho\beta'}$ converges in distribution,

so $R(t)/t^\rho \to 0$ in probability, because $\beta' < 1$. If $\rho\beta' \geq 1$, then $\rho > 1$, and the renewal theorem gives $N(t)/t \to 1/\mathbb{E}T_1 \in [0,\infty)$ a.s., so that $R(t)/t^\rho \to 0$ a.s. □

For $\rho \neq 1$, $T_1 \stackrel{d}{=} (E/(\Lambda L))^{1/\rho}$ is a mixture of Weibull variables, whose Laplace transform is not available in closed form; Proposition 8.3 and Corollary 8.5 below still apply. For $\rho = 1$ the finite-time laws of both processes can be described through one function. For $y \geq 0$ let

$$F_3(y) := {}_3F_2\big(1, \gamma, \beta'; \beta, \gamma'; -y\big),$$

defined by its series for $y < 1$ and by analytic continuation for $y \geq 1$.

Proposition 8.4 (Laplace transforms for $\rho = 1$) *Let $\rho = 1$. Then $F_3(y) = \mathbb{E}\big[(1+yL)^{-1}\big]$ for all $y \geq 0$, and for $s > 0$ and $u \in [0,1]$:*

(a) $\mathbb{E}e^{-sT_1} = 1 - F_3(\Lambda/s)$;

(b) with $F := F_3(\Lambda/s)$,

$$\int_0^\infty e^{-st}\,\mathbb{E}\big[u^{R(t)}\big]\,dt = \frac{F}{s\,\big[F + \frac{\Phi(u)}{\Lambda}(1-F)\big]}, \qquad \int_0^\infty e^{-st}\,\mathbb{E}N(t)\,dt = \frac{1-F}{s\,F};$$

(c) $\displaystyle\int_0^\infty e^{-st}\,\mathbb{E}\big[u^{\mathcal{M}^\rho_\lambda(t)}\big]\,dt = \frac{1}{s}\,F_3\big(\Phi(u)/s\big).$

Proof. For $0 \leq y < 1$, since $0 < L \leq 1$, $\mathbb{E}[(1+yL)^{-1}] = \sum_k (-y)^k \mathbb{E}L^k = \sum_k m_k(-y)^k$, and by Proposition 3.2(a) this is the series of ${}_3F_2(1,\gamma,\beta';\beta,\gamma';-y)$, because $(1)_k/k! = 1$. Both sides are analytic functions of y in a neighbourhood of $[0,\infty)$, so the identity holds for all $y \geq 0$.

(a) Since $T_1 \stackrel{d}{=} E/(\Lambda L)$ with E standard exponential, $\mathbb{E}e^{-sT_1} = \mathbb{E}\big[\frac{\Lambda L}{s+\Lambda L}\big] = 1 - \mathbb{E}\big[\big(1 + \frac{\Lambda}{s}L\big)^{-1}\big]$.

(b) Put $\varphi := \mathbb{E}e^{-sT_1}$ and $\Pi(u) := \sum_j \lambda_j u^j/\Lambda = 1 - \Phi(u)/\Lambda$. Since $\mathbb{P}(N(t)=n) = \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t)$, we have $\int_0^\infty e^{-st}\mathbb{P}(N(t)=n)\,dt = \varphi^n(1-\varphi)/s$. As $\mathbb{E}[u^{R(t)}] = \sum_n \mathbb{P}(N(t)=n)\Pi(u)^n$, summing the geometric series gives $(1-\varphi)/\big(s(1-\Pi(u)\varphi)\big)$, and $1 - \Pi(u)\varphi = F + \frac{\Phi(u)}{\Lambda}(1-F)$ by (a). Similarly $\sum_n n\varphi^n(1-\varphi)/s = \varphi/\big(s(1-\varphi)\big)$.

(c) By Theorem 5.3, $\mathbb{E}[u^{\mathcal{M}^\rho_\lambda(t)}] = \mathbb{E}e^{-Lt\Phi(u)}$ with $\Phi(u) \geq 0$, and Fubini's theorem gives $\mathbb{E}\big[(s + L\Phi(u))^{-1}\big]$. □

For $u = 0$ both (b) and (c) reduce to $F_3(\Lambda/s)/s$, the Laplace transform of $\mathbb{P}(T_1 > t)$, in accordance with Proposition 8.3(a). For $u > 0$ they differ: in (c) the generating function enters inside F_3, in (b) outside.

For numerical purposes no analytic continuation is needed: for all $y \geq 0$,

$$F_3(y) = \frac{C'}{1+y} + \int_0^1 \frac{g(x)}{1+yx}\,dx,$$

with g given by (3.4). Moreover, since $F_3(\Lambda/s) = 1 - \mathbb{E}e^{-sT_1}$ for $\rho = 1$, the proof of Proposition 8.3(b) shows that $F_3(y) \sim D\,\Gamma(\beta')\Gamma(1-\beta')\,y^{-\beta'}$ as $y \to \infty$.

Corollary 8.5 (Renewal function) *Let $a = \rho\beta' < 1$ and K be as in Proposition 8.3. As $t \to \infty$,*

$$\mathbb{E}N(t) \sim \frac{t^a}{K\,\Gamma(1-a)\Gamma(1+a)}, \qquad \mathbb{E}R(t) = \frac{S_1}{\Lambda}\,\mathbb{E}N(t).$$

Proof. With $\varphi(s) = \mathbb{E}e^{-sT_1}$, the Laplace–Stieltjes transform of the non-decreasing function $t \mapsto \mathbb{E}N(t)$ is $\sum_{n\geq 1} \varphi(s)^n = \varphi(s)/(1-\varphi(s))$. By the proof of Proposition 8.3(b), $1-\varphi(s) \sim K\Gamma(1-a)s^a$ as $s \downarrow 0$, so this transform is equivalent to $s^{-a}/(K\Gamma(1-a))$, and Karamata's Tauberian theorem [4, Sect. 1.7] gives the first claim. The second one is Wald's identity, since the amplitudes Y_i are independent of N. □

The convergence in Corollary 8.5 is slow: for $\rho = 1$, $\Lambda = 1$ and the parameters of [11], the ratio of $\mathbb{E}N(t)$ to its equivalent is 0.86 at $t = 10^2$, 0.97 at $t = 10^4$ and 0.999 at $t = 10^6$ (Table 3).

Thus the two counting processes built from $\mathcal{E}_\lambda$ behave differently at large times. The dPCP grows like t^ρ with a random slope and keeps the memory of L forever; its renewal counterpart grows like $t^{\rho\beta'}$ and has the same scaling limit as the fractional Poisson process of order $\rho\beta'$. It is the renewal counterpart, not the dPCP, that belongs to the family of fractional-type counting processes.

9 Special cases and limits

Theorem 9.1 (Non-degenerate limit) *Fix λ, β, ρ and the rates. As $\gamma \downarrow \beta$, $\mu_\lambda \Rightarrow \delta_1$ and the finite-dimensional distributions of $\mathcal{M}^\rho_\lambda$ converge to those of $\{\mathcal{M}(t^\rho)\}_{t\geq 0}$. In particular, for $\rho = 1$ one recovers the GCP of [5], for $\rho = 1, k = 1$ the Poisson process, and $\mathcal{E}_\lambda(\beta, \beta; z) = e^z$.*

Proof. Write $\mu^{(\gamma)}$ for the mixing law and $m_k(\gamma)$ for its moments, to stress the dependence on γ. If $\gamma = \beta$ then $\gamma' = \beta'$, so $m_k = 1$ for all k and $\mathcal{E}_\lambda(\beta, \beta; z) = \sum_k z^k/k! = e^z$.

Convergence of the mixing laws. By Proposition 3.2(a), $m_k(\gamma) = \frac{(\gamma)_k(\beta')_k}{(\beta)_k(\gamma')_k}$ with $\gamma' = \gamma + 1 - \varepsilon$. This is a continuous function of γ, since all the Pochhammer symbols in the denominator are positive, and it tends to 1 as $\gamma \downarrow \beta$. So $\int P\, d\mu^{(\gamma)} \to P(1)$ for every polynomial P. All these measures are probability measures on $[0,1]$. By Weierstrass' theorem every $f \in C[0,1]$ is a uniform limit of polynomials, so $\int f\, d\mu^{(\gamma)} \to f(1)$, i.e. $\mu^{(\gamma)} \Rightarrow \delta_1$.

Convergence of the finite-dimensional distributions. Fix $t_1, \dots, t_n$ and a bounded $g : \mathbb{N}_0^n \to \mathbb{R}$, and put $h(s) = \mathbb{E}g\big(\mathcal{M}(st_1^\rho), \dots, \mathcal{M}(st_n^\rho)\big)$ for $s \in [0,1]$. Then

$$\mathbb{E}g\big(\mathcal{M}^\rho_\lambda(t_1), \dots, \mathcal{M}^\rho_\lambda(t_n)\big) = \int h(s)\, \mu^{(\gamma)}(ds).$$

The function h is continuous. Indeed, for $s, s' \in [0,1]$ the two vectors differ only if $\mathcal{M}$ jumps in one of the intervals between st_k^ρ and $s't_k^\rho$, which happens with probability at most $\sum_k \Lambda|s - s'|t_k^\rho$. Hence $|h(s) - h(s')| \leq 2\|g\|_\infty \Lambda|s - s'| \sum_k t_k^\rho$. By weak convergence, $\int h\, d\mu^{(\gamma)} \to h(1) = \mathbb{E}g\big(\mathcal{M}(t_1^\rho), \dots, \mathcal{M}(t_n^\rho)\big)$. For $\rho = 1$ the limit is the GCP, and for $\rho = 1$, $k = 1$ it is the Poisson process of rate λ_1. □

Different choices of the rates (λ_j), that is, of the function Φ, lead to degenerate versions of several known processes. The formulas of Sections 5–7 apply to all of them.

Degenerate Prabhakar Poisson process. Let $k = 1$. Here $\Phi(u) = \lambda_1(1-u)$, and

$$\mathbb{E}u^{\mathcal{M}^\rho_\lambda(t)} = \mathcal{E}_\lambda\big(\beta, \gamma; -\lambda_1 t^\rho(1-u)\big),$$
$$p_n(t) = \frac{(\lambda_1 t^\rho)^n}{n!}\, m_n\, {}_2F_2\left(\begin{matrix}\gamma + n,\ \beta' + n \\ \beta + n,\ \gamma' + n\end{matrix}; -\lambda_1 t^\rho\right).$$

It is a mixed Poisson process with random intensity $\lambda_1 L$. Its generating function is obtained from that of the fractional Poisson process [15, 3, 20], $E_\alpha(-\lambda t^\alpha(1-u))$, by replacing $E_\alpha(-\cdot)$ with $\mathcal{E}_\lambda(\beta, \gamma; -\cdot)$ and t^α with t^ρ. The analogy concerns one-dimensional laws only: unlike the fractional Poisson process, this process is not a renewal process (Proposition 8.1).

Degenerate Poisson process of order k. Let $\lambda_j = \lambda_0$ for $1 \le j \le k$. Then $\Phi(u) = \lambda_0 \sum_{j=1}^{k}(1-u^j)$ and $\Lambda = k\lambda_0$, giving degenerate versions of the Poisson process of order k [14] and of its time-changed variants [26].

Degenerate Pólya–Aeppli process. Let $k = \infty$ and $\lambda_j = \lambda_0(1-\vartheta)\vartheta^{j-1}$ with $0 < \vartheta < 1$. Here $\Lambda = \lambda_0$ and $\Phi(u) = \lambda_0 \frac{1-u}{1-\vartheta u}$, so $\mathbb{E}u^{\mathcal{M}^{\rho}_{\lambda}(t)} = \mathcal{E}_{\lambda}\big(\beta,\gamma;-\lambda_0 t^{\rho}\frac{1-u}{1-\vartheta u}\big)$. This is the degenerate counterpart of the Pólya–Aeppli process [21] and of its fractional version [2]. Truncating the geometric law to $\{1,\dots,k\}$ gives a degenerate Pólya–Aeppli process of order k.

Degenerate negative binomial process. Let $k = \infty$ and $\lambda_j = r\vartheta^j/j$ with $r > 0$, $0 < \vartheta < 1$. Here $\Phi(u) = r\log\frac{1-\vartheta u}{1-\vartheta}$, so $\mathbb{E}u^{\mathcal{M}^{\rho}_{\lambda}(t)} = \mathcal{E}_{\lambda}\big(\beta,\gamma;-rt^{\rho}\log\frac{1-\vartheta u}{1-\vartheta}\big)$. This is the degenerate counterpart of the negative binomial process and of the fractional versions studied in [2, 18].

Relation with the GFCP. Replacing μ_λ by the Mittag-Leffler law of $Y_\alpha(1)$ in Definition 5.1 (with $\rho = \alpha$) produces a process with the same one-dimensional marginals as the GFCP of [5]; see Section 2. Thus the GFCP and the dPCP are two members, with different mixing laws, of the class of counting processes $\mathcal{M}(Yt^{\rho})$. The GFCP is not a special case of the dPCP. Rather, the dPCP is its degenerate Prabhakar analogue, in the same sense in which the dpp of [11] is the degenerate Prabhakar analogue of grey Brownian motion [25, 22]. Note also that the GFCP is defined at the path level by an inverse stable time change, whereas the dPCP is a mixture; the two constructions coincide only for one-dimensional laws.

Coupling with the dpp. Let B^H and $\mathcal{M}$ be independent. Taking the same L in Corollaries 4.4 and 4.6, the processes $\sqrt{L}\,B^H$ and $\mathcal{M}(L\,t^{\rho})$ form a diffusion and a counting process driven by a common random intensity. By Proposition 4.3 applied to the product family $(\sqrt{c}\,B^H, \mathcal{M}(c\,\cdot^{\rho}))$, its joint characteristic function is

$$\mathcal{E}_{\lambda}\big(\beta,\gamma;-\tfrac{1}{2}u^{\top}\widetilde{\Sigma}u - \Psi_t(v)\big).$$

An application: ruin probability. Let $k = 1$, $\rho = 1$, and consider the risk process $X(t) = x + \kappa t - \sum_{i \le \mathcal{M}^{\rho}_{\lambda}(t)} Z_i$, where $x \ge 0$ is the initial capital, $\kappa > 0$ the premium rate, and the claims Z_i are i.i.d. exponential with mean $1/\theta$, independent of $\mathcal{M}^{\rho}_{\lambda}$. Assume $\lambda_1 < \kappa\theta$. Conditionally on $L = s$ this is the classical Cramér–Lundberg model with claim intensity $\lambda_1 s < \kappa\theta$, whose ruin probability is $\frac{\lambda_1 s}{\kappa\theta}\exp\big(-(\theta - \lambda_1 s/\kappa)x\big)$. Integrating against μ_λ, and using $\int s e^{ws}\mu_\lambda(ds) = \frac{d}{dw}\mathcal{E}_{\lambda}(\beta,\gamma;w)$ together with Proposition 3.2(b), we obtain the ruin probability

$$\psi(x) = \mathbb{P}\Big(\inf_{t\ge 0} X(t) < 0\Big) = \frac{\lambda_1}{\kappa\theta}\,e^{-\theta x}\,m_1\,{}_2F_2\!\left(\begin{matrix}\gamma+1,\ \beta'+1\\ \beta+1,\ \gamma'+1\end{matrix};\frac{\lambda_1 x}{\kappa}\right).$$

As $w \to \infty$, $\int s e^{ws}\mu_\lambda(ds) = C'e^{w} + O(e^{w}/w)$, because the absolutely continuous part of μ_λ has a bounded density near 1 (Proposition 3.6). Hence $\psi(x) \sim C'\psi_1(x)$ as $x \to \infty$, where $\psi_1(x) = \frac{\lambda_1}{\kappa\theta}e^{-(\theta-\lambda_1/\kappa)x}$ is the classical ruin probability with intensity λ_1: the adjustment coefficient is unchanged, and the constant is multiplied by the mass of the atom of μ_λ. For the parameters of [11] and $\lambda_1 = \theta = 1$, $\kappa = 1.5$, one finds $\psi(5) = 0.0482$, against $\psi_1(5) = 0.1259$; the ratio $\psi(x)/(C'\psi_1(x))$ equals 1.09 at $x = 20$ and 1.03 at $x = 60$. If $\lambda_1 \ge \kappa\theta$, put $\ell := \kappa\theta/\lambda_1 \in (0,1]$. Given $L = s \ge \ell$ ruin is certain, so that

$$\psi(x) = \mu_\lambda([\ell,1]) + \int_{(0,\ell)} \frac{\lambda_1 s}{\kappa\theta}\,e^{-(\theta-\lambda_1 s/\kappa)x}\,\mu_\lambda(ds) \;\longrightarrow\; \mu_\lambda([\ell,1]) \;\ge C' > 0 \qquad (x\to\infty):$$

in that case no initial capital makes ruin unlikely.

Table 2 summarizes these cases.

Table 2: Special cases of the degenerate Prabhakar mixture and their non-degenerate limits as $\gamma \downarrow \beta$ (Theorem 9.1). The limits are the classical, non-fractional processes; the GFCP and the fractional Poisson process are not limits of the dPCP.

Choice	Degenerate process	Non-degenerate limit as $\gamma \downarrow \beta$ (counting rows: $\rho = 1$)
$X_c = \sqrt{c}\, B^H$ (Gaussian member)	dpp of Hachicha–Riahi [11]	fractional Brownian motion
k general	dPCP	GCP of [5]
$k = 1$	degenerate Prabhakar Poisson	Poisson process
$\lambda_j = \lambda_0$, $j \le k$	degenerate Poisson of order k	Poisson process of order k [14]
$\lambda_j \propto \vartheta^{j-1}$, $k = \infty$	degenerate Pólya–Aeppli	Pólya–Aeppli process [21]
$\lambda_j = r\vartheta^j/j$, $k = \infty$	degenerate negative binomial	negative binomial process

10 Worked examples

All examples use the parameters of [11, Sect. 3], $\lambda = 0.23$, $\beta = 3.7$, $\gamma = 4.1$, and, unless stated otherwise, $\rho = 1$ and $t = 1$. Numerical values are rounded; hypergeometric functions were evaluated by their series in 120-digit arithmetic.

Example 10.1 (The mixing law) *Here $\varepsilon = 4.3478$, $\beta' = 0.3522$, $\gamma' = 0.7522$. By Theorem 3.4 and Corollary 3.5, $L = e^{-U}$ where U is compound Poisson with $|\nu| = 1.21792$; the atom of μ_λ at 1 has mass $C' = 0.29584$, and*

$$m_1 = \mathbb{E}L = 0.51883, \qquad m_2 = \mathbb{E}L^2 = 0.43446, \qquad \operatorname{Var} L = 0.16528.$$

Since $\beta' < \frac{1}{2}$, $\mathbb{E}L^{-1/2} = \infty$ (Corollary 3.5(ii)).

Example 10.2 (Non-Gaussianity of the dpp of [11]) *By Corollary 4.4, $\mathbb{E}(P_t^H)^2 = m_1 t^{2H}$ and $\mathbb{E}(P_t^H)^4 = 3m_2 t^{4H}$. Hence the kurtosis of P_t^H is $3m_2/m_1^2 = 4.842 > 3$ for every $t > 0$ and every H. In general $3m_2/m_1^2 = 3\,\mathbb{E}L^2/(\mathbb{E}L)^2 > 3$ whenever $\gamma > \beta$: the dpp is leptokurtic and never Gaussian, and it becomes Gaussian only in the limit $\gamma \downarrow \beta$ of Theorem 9.1.*

Example 10.3 (Square-integrable local time) *For the parameters of [11], Proposition 4.8 shows that the dpp has a local time almost surely, but $\beta' = 0.352 < \frac{1}{2}$, so this local time is not in $L^2(\mathbb{R}\times\Omega)$. For $\lambda = 0.4$, $\beta = 2.2$, $\gamma = 2.4$ one has $\beta' = 0.7 > \frac{1}{2}$ and, by (3.3), $\mathbb{E}L^{-1/2} = 1.59968$. Berman's integral over $[0,1]^2$ is then $\frac{2\sqrt{2\pi}}{(1-H)(2-H)} \times 1.59968$. A direct numerical integration of $\int_0^\infty x^{-1/2}\mathcal{E}_\lambda(\beta,\gamma;-x)\,dx = \sqrt{\pi}\,\mathbb{E}L^{-1/2}$ gives 2.835368, in agreement with $\sqrt{\pi} \times 1.59968 = 2.835367$.*

Example 10.4 (Degenerate Prabhakar Poisson process) *Take $k = 1$ and $\lambda_1 = 1$. By Theorem 5.4, $p_n(1) = \frac{m_n}{n!}\,{}_2F_2(\gamma+n, \beta'+n; \beta+n, \gamma'+n; -1)$, which gives*

$$(p_0, \dots, p_4)(1) = (0.6453,\ 0.2329,\ 0.0885,\ 0.0259,\ 0.0060),$$

against $(0.3679, 0.3679, 0.1839, 0.0613, 0.0153)$ for the Poisson law of parameter 1. The mean is $m_1 = 0.5188$ and the variance 0.6841, so the Fano factor is 1.319. The first event time satisfies $\mathbb{P}(T_1 > 100) = 0.10443$, while the asymptote of Theorem 7.1 gives 0.10424 (for the Poisson process, $e^{-100} \approx 4 \cdot 10^{-44}$).

Example 10.5 (Degenerate Poisson process of order 2) *Take $k = 2$, $\lambda_1 = \lambda_2 = 1$, so $\Lambda = 2$, $S_1 = 3$, $S_2 = 5$. Then*

$$(p_0, \dots, p_4)(1) = (0.4799,\ 0.1144,\ 0.1521,\ 0.0857,\ 0.0712),$$

with mean 1.5565, variance 4.0816 and Fano factor 2.622. Note that $p_2 > p_1$: jumps of size two make even values relatively more likely, as for the non-degenerate Poisson process of order k [14].

Example 10.6 (Degenerate Pólya–Aeppli process) *Take* $\lambda_0 = 1$, $\vartheta = \frac{1}{2}$, *so* $\lambda_j = 2^{-j}$, $\Lambda = 1$, $S_1 = 2$, $S_2 = 6$ *and* $\mathbb{E}u^{\mathcal{M}^{\rho}_{\lambda}(t)} = \mathcal{E}_{\lambda}\big(\beta, \gamma; -t^{\rho}\frac{1-u}{1-u/2}\big)$. *Then*

$$(p_0, \dots, p_4)(1) = (0.6453,\ 0.1165,\ 0.0804,\ 0.0545,\ 0.0364),$$

with mean 1.0377, *variance* 3.7741 *and Fano factor* 3.637. *The value of* p_0 *coincides with Example 10.4 because both processes have* $\Lambda = 1$ *(Proposition 7.2).*

In Examples 10.4–10.6 the effect of the mixing is always the same. The mean is multiplied by $m_1 = 0.5188$, the dispersion increases, and $p_0(t)$ decays like a power of t instead of exponentially.

11 Simulation and numerical illustrations

Thanks to Theorem 3.4, paths can be simulated exactly, without inverting $\mathcal{E}_{\lambda}$. The jumps of U are sampled by rejection. Put $c_1 := \varepsilon - 1$ and $c_2 := \gamma - \beta$. By Remark 3.8, the density of ν is $e^{-\beta' u}\, r(u)$ with

$$r(u) := \frac{(1 - e^{-c_1 u})(1 - e^{-c_2 u})}{u\,(1 - e^{-u})} \le c_2 \max(1, c_1) =: \bar{r},$$

because $1 - e^{-c_2 u} \le c_2 u$ and $1 - x^{c_1} \le \max(1, c_1)(1 - x)$ for $x \in [0, 1]$. Hence an exponential proposal of parameter β', accepted with probability $r(u)/\bar{r}$, has the law $\nu/|\nu|$. The acceptance rate is $\beta'|\nu|/\bar{r}$, which equals 0.32 for the parameters used below.

Algorithm 1 Exact simulation of a dPCP path on $[0, T]$

1: Draw $N \sim \text{Poisson}(|\nu|)$ with $|\nu| = \log \frac{\Gamma(\gamma)\Gamma(\beta')}{\Gamma(\beta)\Gamma(\gamma')}$.
2: Draw $J_1, \dots, J_N$ i.i.d. with law $\nu/|\nu|$ by rejection: repeat drawing $V \sim \text{Exp}(\beta')$ and W uniform on $[0, 1]$ until $W\,\bar{r} \le r(V)$, then set $J_i = V$.
3: Set $L = \exp(-(J_1 + \cdots + J_N))$ (so $L = 1$ if $N = 0$).
4: Simulate the GCP on $[0, LT^{\rho}]$: Poisson(Λ) arrival times $\tau_1 < \tau_2 < \dots$ and i.i.d. amplitudes with $\mathbb{P}(j) = \lambda_j/\Lambda$.
5: Return the jump times $t_i = (\tau_i/L)^{1/\rho}$ with the same amplitudes.

All numerical results below use the parameters of [11]: $\lambda = 0.23$, $\beta = 3.7$, $\gamma = 4.1$, so that $\beta' = 0.3522$, $\gamma' = 0.7522$, $|\nu| = 1.21792$, $C' = 0.29584$. We also take $k = 3$ and $(\lambda_1, \lambda_2, \lambda_3) = (1, 0.5, 0.25)$.

A numerical verification of all closed-form results is given in Appendix A.

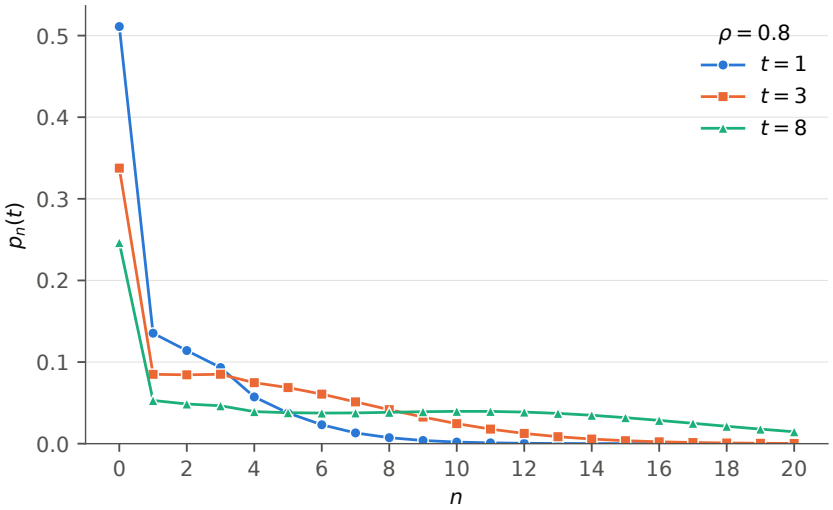

Figure 1: Probability mass function (5.1) at $t = 1, 3, 8$ ($\rho = 0.8$). The mass at $n = 0$ decays slowly, as predicted by Theorem 7.1.

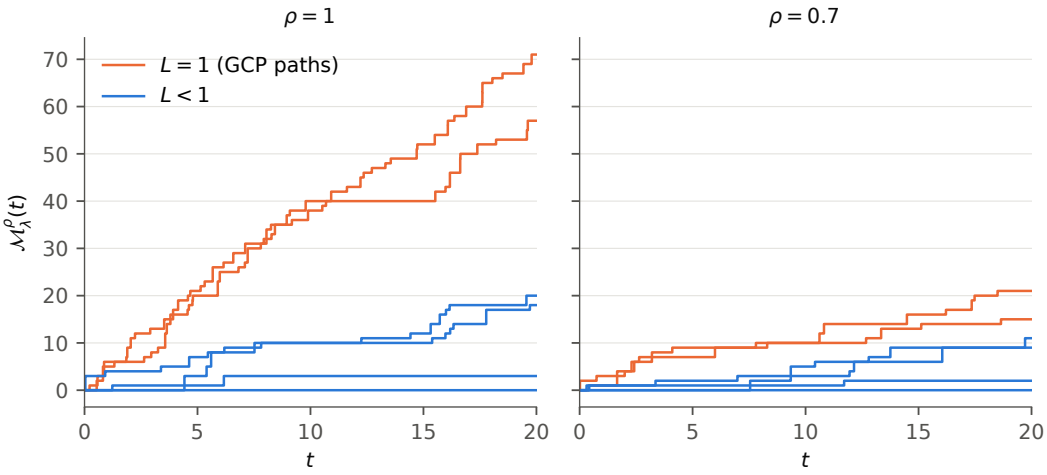

Figure 2: Sample paths from Algorithm 1 for $\rho = 1$ and $\rho = 0.7$. Each path has its own clock speed L; paths with $L = 1$ are GCP paths.

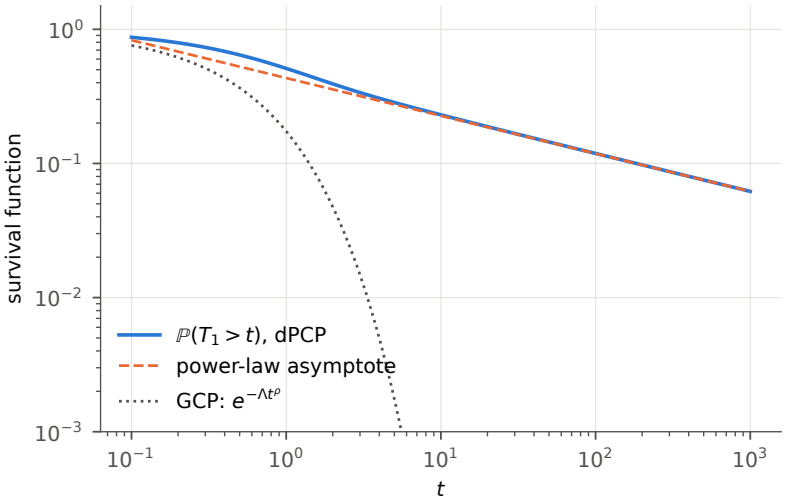

Figure 3: Survival function of the first jump time (log–log scale, $\rho = 0.8$), its power-law asymptote from Theorem 7.1, and the exponential tail of the GCP.

12 Conclusion

The degenerate Prabhakar function is a ${}_2F_2$ function, and its measure μ_λ is a G distribution of order $(2,2)$ with an explicit compound Poisson structure. This makes the degenerate Prabhakar process of [11] and its counting analogue, the dPCP, explicitly computable, and it settles when the local time of the former is square integrable. The dPCP is a mixed compound Poisson process: it is neither fractional nor renewal, and its long-time behaviour is governed by the random rate L. Its renewal counterpart, by contrast, has an inverse stable scaling limit of index $\rho\beta'$. Several questions remain open. A path-level construction of the dPCP marginals through a random time change, in the spirit of [20], would require a subordinator whose inverse has $\mathcal{E}_\lambda(\beta,\gamma;-x)$ as marginal Laplace transform, and we do not know whether such a subordinator exists. The finite-time distribution of the renewal counterpart for $\rho \neq 1$, ruin probabilities for risk models driven by the renewal counterpart, and the estimation of $(\lambda,\beta,\gamma,\rho)$ from replicated count data are other natural directions.

A Numerical verification

Table 3 compares the closed-form results with direct computation and with Monte Carlo simulation ($2\cdot10^6$ draws of L, $4\cdot10^5$ paths, $\rho=0.8$, $t=1.5$). It is a consistency check on the formulas and on their implementation, not a substitute for the proofs. The Monte Carlo standard errors are about $3\cdot10^{-4}$ for $\mathbb{P}(L=1)$ and the moments of L, at most $8\cdot10^{-4}$ for the probabilities p_n, $4\cdot10^{-3}$ for the mean and $2\cdot10^{-2}$ for the variance; all the reported deviations are within two standard errors. Hypergeometric series for large arguments were evaluated with 200-digit arithmetic.

Table 3: Numerical verification

Result	Closed form	Independent computation
Prop. 3.2: m_1, m_2, m_5	0.5188252, 0.4344589, 0.3624370	via Γ_λ in (2.1): identical (10 digits)
Thm. 3.4: $\lvert\nu\rvert$, C'	1.2179226, 0.295844	quadrature 1.2179226; MC $\mathbb{P}(L=1)=0.295891$
Thm. 3.4: $\mathbb{E}L, \mathbb{E}L^2, \mathbb{E}L^3$	0.5188, 0.4345, 0.3974	MC 0.5187, 0.4344, 0.3973
Prop. 3.6: $C'+\int_0^1 x^k g$, $k=0,1,2$	1, 0.5188252, 0.4344589	series (3.5) integrated termwise ($4\cdot10^5$ terms): 0.9999990, 0.5188241, 0.4344579
Prop. 3.6: (3.4) against (3.5) at $x=0.05,0.3,0.6,0.9$	equal	relative difference below $2\cdot10^{-6}$
Thm. 5.4: $\sum_n p_n(t)$	1	1 ± 10^{-16}, all $p_n\geq0$
Thm. 5.4: $p_0,\dots,p_3$	0.4378, 0.1212, 0.1118, 0.1003	MC 0.4370, 0.1216, 0.1114, 0.1007
Cor. 6.2: mean, variance	1.97345, 6.15878	MC 1.97449, 6.14044
Thm. 5.6, $n=0,2,4$ (left side)	0.5109443, -0.1957288, -0.4698536	right side equal to 7 digits
Prop. 6.1: $\mathbb{E}(\mathcal{M}^\rho_\lambda(t))_{(3)}$	40.0877680377	from the pmf: 40.0877680377
Thm. 7.1: p_0 at $\Lambda t^\rho=200$	0.0817385 (two-term asymptote)	series: 0.0817388
Thm. 7.1: p_2 at $t^\rho=60,150$	0.018277, 0.013236	series: 0.018439, 0.013283 (ratios 0.991, 0.996)
Alg. 1: acceptance rate, $\mathbb{P}(L=1)$	0.3203, 0.295844	rejection sampler (10^6 draws): 0.3201, 0.29569
Prop. 8.4(a), (b) at $s=2$ ($k=1$, $\Lambda=1$, $u=\frac{1}{2}$)	0.18458, 0.44916	simulation: 0.18457, 0.44909
Ruin probability $\psi(5)$ (Sect. 9)	0.04821	mixture over 10^6 draws of L: 0.04821
Cor. 8.5: $\mathbb{E}N(t)$ at $t=10^2,10^4,10^6$ ($\rho=1$, $\Lambda=1$)	asymptote 7.75, 39.2, 198.7	simulation ($4\cdot10^4$ paths): 6.64, 38.0, 198.6

B Simulation study of the estimators

We illustrate Remark 6.4 for $k = 1$, $\rho = 1$, $\lambda_1 = 1$, $t = 10$ and the parameters of [11], for which $(m_1, m_2, m_3) = (0.5188, 0.4345, 0.3974)$ and $(\beta', \varepsilon - 1, \gamma - \beta) = (0.352, 3.348, 0.400)$. Samples $N_1, \dots, N_n$ were simulated with L drawn by Algorithm 1.

Moment estimators. Table 4 reports the bias and the root mean square error of $\hat{m}_1, \hat{m}_2, \hat{m}_3$ over 1000 samples. The estimators are unbiased up to simulation error and their error decreases like $n^{-1/2}$; for $\hat{m}_1$ it agrees with the exact standard deviation $\big((m_1\lambda_1 t + \mathrm{Var}(L)\lambda_1^2 t^2)/n\big)^{1/2}/(\lambda_1 t)$, equal to 0.0330, 0.0147 and 0.0066 for the three sample sizes.

Table 4: Moment estimators $\hat{m}_r$: bias and RMSE over 1000 samples

n	bias $(\hat{m}_1, \hat{m}_2, \hat{m}_3)$	RMSE $(\hat{m}_1, \hat{m}_2, \hat{m}_3)$
200	0.0025, 0.0024, 0.0020	0.033, 0.041, 0.053
1000	0.0002, 0.0000, −0.0004	0.015, 0.019, 0.024
5000	−0.0001, −0.0002, −0.0002	0.007, 0.009, 0.011

Parameters. Table 5 compares, over 200 samples, two estimators of $(\beta', \varepsilon - 1, \gamma - \beta)$. The method of moments solves $m_r = \hat{m}_r$, $r = 1, 2, 3$, by least squares in the box $\beta', \gamma - \beta \in (0, 1)$, $\varepsilon - 1 \in (0, 50)$. An exact solution exists only for 22%, 43% and 75% of the samples for $n = 200$, 1000, 5000, and the spread of the estimates is large. The maximum likelihood estimator maximizes $\prod_i p_{N_i}(t)$ numerically (Nelder–Mead, two starting points) under the constraints $\beta' \in (0, 1)$, $0 < \gamma - \beta < 1 - \beta'$ and $\gamma - \beta \le \varepsilon - 1 \le \gamma - \beta + 50$. It is markedly more accurate. The time $t = 10$ was chosen so that the mean count $m_1\lambda_1 t \approx 5.2$ is large enough for the Poisson noise not to hide the variability of L; larger values of $\lambda_1 t$ improve all estimators. With 200 samples and a derivative-free optimizer, the figures for $n = 200$ should be read as orders of magnitude; standard errors based on the Fisher information are not provided. The parameter $\varepsilon - 1$ is the hardest to estimate: for $n = 200$ the likelihood is so flat in this direction that 17.5% of the estimates lie on the upper bound (4.5% for $n = 1000$, none for $n = 5000$).

Table 5: Estimation of $(\beta', \varepsilon - 1, \gamma - \beta) = (0.352, 3.348, 0.400)$: median and, in brackets, interquartile range over 200 samples. The value [49] is limited by the size of the search box

	Method of moments			Maximum likelihood		
n	β'	$\varepsilon - 1$	$\gamma - \beta$	β'	$\varepsilon - 1$	$\gamma - \beta$
200	0.34 [0.12]	5.5 [49]	0.45 [0.38]	0.36 [0.09]	2.9 [8.6]	0.47 [0.26]
1000	0.34 [0.08]	4.6 [28]	0.37 [0.51]	0.35 [0.06]	3.3 [4.1]	0.40 [0.22]
5000	0.35 [0.07]	3.3 [4.4]	0.40 [0.26]	0.36 [0.03]	3.4 [1.8]	0.40 [0.07]

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Declarations

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