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A Comprehensive Review of Tempered Lévy Models for Financial Asset Returns: Theory, Statistical Properties, and Applications

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Abstract

Financial asset returns exhibit heavy tails, skewness, excess kurtosis, volatility clustering, and discontinuities that challenge classical Gaussian models. Tempered Lévy models address these features by modifying the Lévy density to retain stable-like jump behaviour while controlling extreme jumps. This review develops a unified, Lévy-density-centred perspective on the evolution of tempered Lévy models, from early Lévy-based formulations to Variance Gamma, Truncated Lévy Flight, Bilateral Gamma, KoBoL, CGMY, Tempered Stable, and Generalized Tempered Stable models. The evolution is examined through changes in jump activity, tail attenuation, asymmetry, moment properties, estimation, and computational tractability, together with their financial applications in return modelling, volatility, risk measurement, portfolio optimization, option pricing, and high-frequency analysis. The review identifies a key transition in the literature from static historical fitting toward increasingly flexible and data-intensive modelling. It further highlights an emerging research direction in which the Lévy density is treated as an interpretable source of evolving information on financial jump behaviour, providing a basis for dynamic monitoring and potentially real-time risk forecasting. This perspective offers a coherent framework for understanding both the development and future direction of tempered Lévy models in quantitative finance.

Keywords: Lévy process; Lévy density; Tempered Lévy models; Financial return modelling; Option pricing; Heavy-tailed distributions.

JEL Classification: C22 , C58 , G12 , G13

1 Introduction

Accurate modelling of financial asset returns remains a fundamental challenge in quantitative finance because financial markets exhibit stochastic characteristics that are difficult to reconcile with classical Gaussian assumptions. Empirical financial return series commonly display heavy tails, excess kurtosis, skewness, volatility clustering, and abrupt discontinuous movements, indicating that large price changes occur more

frequently than predicted by Gaussian diffusion models [1–3]. In particular, the continuous sample-path structure of the classical Geometric Brownian Motion (GBM) framework does not provide a sufficiently flexible representation of sudden market movements and discontinuous return behaviour. These limitations motivated the development of stochastic models capable of incorporating jumps while retaining a mathematically tractable probabilistic structure.

Lévy processes provide a natural framework for this purpose because they extend Brownian motion by allowing discontinuous jumps while preserving stationary and independent increments. Their mathematical structure is characterized by the Lévy–Khintchine representation, in which the Lévy measure determines the frequency, magnitude, and directional behaviour of jumps [4–6]. Consequently, the Lévy density is not merely a technical component of a stochastic process; it provides a direct description of the underlying jump structure. Modifying the form of the Lévy density therefore provides a systematic mechanism for controlling jump activity, tail behaviour, asymmetry, and the attenuation of extreme jumps. This density-centred perspective provides an important basis for understanding the historical development of Lévy models in financial applications.

Stable Lévy processes initially offered an attractive description of heavy-tailed financial fluctuations, but their power-law tails may lead to infinite moments and excessive probability assigned to very large jumps. The introduction of tempering addressed this limitation by attenuating the tails of the Lévy density while retaining substantial non-Gaussian jump behaviour. The resulting class of tempered Lévy models has developed through a sequence of increasingly flexible formulations, including the Variance Gamma (VG), Truncated Lévy Flight (TLF), Bilateral Gamma (BG), KoBoL, Carr–Geman–Madan–Yor (CGMY), Tempered Stable (TS), and Generalized Tempered Stable (GTS) models [7–11]. Although these models differ in their probabilistic constructions and parameterizations, their development can be interpreted more fundamentally as a progressive modification of the Lévy density to obtain greater control over jump intensity, small-jump activity, tail decay, positive–negative asymmetry, and moment properties.

From this perspective, the evolution of tempered Lévy models is not simply a sequence of alternative probability distributions. It represents a progressive refinement of the mechanism through which discontinuous financial movements are described. Early models introduced finite-activity or tractable jump structures, while subsequent formulations incorporated richer infinite-activity behaviour, asymmetric tail decay, and additional control over small and large jumps. The TLF and related exponentially truncated constructions demonstrated how stable-like behaviour could be retained over a relevant range while suppressing extreme tails [8, 12]. The VG, BG, and related Gamma-based constructions provided analytically convenient alternatives for financial modelling [7]. The KoBoL and CGMY families subsequently provided more general control over activity and tail behaviour, while the TS and GTS formulations further generalized the tempering mechanism and parameter structure [9–11, 13]. Thus, examining the Lévy density across these models provides a unified mathematical perspective on how progressively richer representations of financial jump behaviour have emerged.

The literature also demonstrates that tempered Lévy models have moved well beyond theoretical distributional construction. Empirical studies have applied them to financial returns, high-frequency and low-latency data, volatility modelling, portfolio optimization, derivative pricing, risk measurement, and the simulation of realistic financial trajectories. For example, multivariate tempered stable models have been applied to high-frequency returns and portfolio risk and optimization [14]; tempered stable innovations have been incorporated into fractionally integrated GARCH models for hourly S&P 500 returns [15]; time-changed tempered stable stochastic volatility models have been used for risk measurement and portfolio applications involving the Hang Seng Index [16]; and recent work has examined the fitting of the GTS distribution to Bitcoin, Ethereum, S&P 500, and SPY returns [17]. More recently, the background-driving Lévy density associated with the GTS model has been investigated using daily SPY and Ethereum data, together with stochastic simulation to reproduce realistic cumulative-return trajectories [18]. These developments demonstrate that the role of the Lévy density extends from a mathematical specification of jump behaviour to an empirically relevant component of financial modelling.

However, an important distinction emerges from this literature. Most existing applications estimate the parameters of a selected tempered Lévy model from historical observations and subsequently use the fitted model for a particular task such as pricing, risk measurement, volatility modelling, portfolio construction, or simulation. The estimated Lévy density is therefore predominantly treated as a static model object associated with a historical sample or estimation window. Comparatively less attention has been devoted to considering the evolving Lévy density itself as an interpretable source of financial information that can be repeatedly updated as new market observations become available. This distinction suggests a potentially important transition in the use of tempered Lévy models: from fitting a distribution to historical returns toward monitoring changes in the underlying jump structure.

The remainder of this paper is organized as follows. Section 2 presents the fundamental concepts of financial return modelling, Lévy processes, infinite divisibility, and the Lévy–Khintchine representation that provide the mathematical foundation for tempered Lévy models. Section 3 systematically traces the mathematical evolution of the principal tempered Lévy density models, including the Variance Gamma, Bilateral Gamma, Truncated Lévy Flight, KoBoL, CGMY, Tempered Stable, and Generalized Tempered Stable distributions, with emphasis on their Lévy densities, characteristic functions, theoretical properties, and interrelationships. Section 4 provides a comparative analysis of these models in terms of their Lévy-density structures, jump activity, tail behaviour, asymmetry, moment properties, parameter estimation and computational methodologies, advantages, and limitations, and further synthesizes the resulting methodological and empirical research gaps. Section 5 reviews published applications of tempered Lévy models in financial asset return modelling, option pricing, volatility analysis, portfolio optimization, risk measurement, high-frequency and low-latency financial analysis, and financial simulation, with particular emphasis on the empirical datasets, objectives, and contributions reported in the literature. Finally, Section 6 summarizes the principal findings of the review and discusses future

research directions arising from the density-centred perspective developed throughout the paper.

2 Preliminaries

2.1 Financial Returns and the Need for Jump Models

The mathematical development of financial return models has progressed from discrete-time random walk formulations to continuous-time stochastic processes with increasingly flexible descriptions of uncertainty. Bachelier's random walk model represents price changes as successive random increments,

$$S_t = S_{t-1} + \varepsilon_t,$$

where ε_t denotes a sequence of independent and identically distributed random variables [19]. Its continuous-time analogue is the Wiener process $W_t, t \geq 0$, characterized by

$$W_0 = 0, \qquad W_t - W_s \sim N(0, t-s), \qquad 0 \leq s < t,$$

with independent and stationary increments [20]. Introducing deterministic drift and scale parameters gives the Arithmetic Brownian Motion,

$$dS_t = \mu dt + \sigma dW_t,$$

where μ and σ denote the drift and diffusion coefficients, respectively.

Because the additive structure of the Arithmetic Brownian Motion permits negative values of the modeled asset price, the Geometric Brownian Motion (GBM) provides a multiplicative alternative,

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

with solution

$$S_t = S_0 \exp\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

The GBM framework forms the stochastic foundation of the classical Black–Scholes model [21, 22]. However, the continuous Gaussian structure of GBM implies normally distributed log-returns over fixed time intervals and continuous sample paths. These assumptions are inconsistent with several well-documented empirical characteristics of financial returns, including heavy tails, excess kurtosis, skewness, volatility clustering, and discontinuous price movements [1–3]. This limitation motivates a transition from purely continuous diffusion models to stochastic processes in which discontinuities and jumps can be represented explicitly.

For the purposes of this review, the important mathematical transition is therefore not simply from one financial pricing model to another, but from a Gaussian diffusion structure to a jump-process framework in which the distribution and frequency of discontinuous movements can be controlled through a Lévy measure. This provides the mathematical basis for the subsequent development of tempered Lévy models.

2.2 Lévy Processes and Infinite Divisibility

A stochastic process $X_{t,t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Lévy process* if it satisfies the following properties [4, 5]:

(i) $X_0 = 0$ almost surely;

(ii) **Independent increments:** for every

$$0 \leq t_0 < t_1 < \cdots < t_n,$$

the increments

$$X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1},\ \ldots,\ X_{t_n} - X_{t_{n-1}}$$

are mutually independent;

(iii) **Stationary increments:** for every $0 \leq s < t$,

$$X_t - X_s \overset{d}{=} X_{t-s};$$

(iv) **Stochastic continuity:** for every $\varepsilon > 0$,

$$\lim_{h \to 0} \mathbb{P}\left(|X_{t+h} - X_t| > \varepsilon\right) = 0.$$

The independent and stationary increment properties imply that the distribution of a Lévy-process increment depends only on the length of the observation interval. An important consequence is that the marginal distributions of a Lévy process are infinitely divisible. A random variable X is infinitely divisible if, for every integer $n \geq 1$,

$$X \overset{d}{=} X_1 + \cdots + X_n,$$

where $X_1, \ldots, X_n$ are independent and identically distributed random variables [4]. Infinite divisibility is particularly important for financial modelling because it permits a consistent probabilistic description of returns across different time horizons.

The characteristic function of a Lévy process can be written as

$$\mathbb{E}\left[e^{iuX_t}\right] = \exp t\Psi(u), \tag{1}$$

where $\Psi(u)$ is the characteristic exponent. The Lévy–Khintchine representation gives

$$\Psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|<1}\right) \nu(dx), \tag{2}$$

where $\gamma \in \mathbb{R}$ is the drift coefficient, $\sigma^2 \geq 0$ is the variance coefficient of the Gaussian component, and $\nu(dx)$ is the Lévy measure. The Lévy measure satisfies the integrability condition

$$\int_{\mathbb{R}} \min\left(1, x^2\right) \nu(dx) < \infty.$$

Equation (2) provides the fundamental decomposition of a Lévy process into a deterministic drift, a continuous Gaussian component, and a discontinuous jump component. For the class of models considered in this review, the jump component is

of primary interest because the Lévy measure determines how jumps are distributed across different magnitudes.

3 Evolution of Tempered Lévy Models

3.1 The Classical Lévy Density Framework

The development of Lévy-based models for financial data originates from the recognition that asset returns exhibit statistical characteristics that are difficult to reconcile with the Gaussian assumption. Empirical studies have repeatedly documented heavy tails, excess kurtosis, skewness and departures from normality in stock-index and asset returns, motivating the use of stable and more general Lévy distributions as alternatives to Brownian-motion-based models [1, 2, 23, 24]. From the perspective of the present review, this development is important because the Lévy density provides a direct description of how jumps of different magnitudes contribute to financial return dynamics, rather than describing only the resulting probability density of returns.

For a symmetric α-stable Lévy process, the Lévy density can be written as

$$k(x) = \frac{C}{|x|^{1+\alpha}}, \qquad 0 < \alpha < 2, \tag{3}$$

where $C > 0$ controls the scale of jump activity and α determines the power-law behaviour of the jump measure. More generally, positive and negative jumps may be assigned different intensities through

$$k(x) = \frac{C_+}{x^{1+\alpha_+}}\mathbf{1}_{(0,\infty)}(x) + \frac{C_-}{|x|^{1+\alpha_-}}\mathbf{1}_{(-\infty,0)}(x).$$

The density in (3) is singular at the origin and decays according to a power law as $|x|$ increases. Thus, the same functional structure simultaneously describes a large accumulation of small jumps and a substantial probability of extreme jumps. This feature made stable Lévy models attractive for financial returns, particularly in the early empirical literature on stock-market fluctuations [1, 2, 23].

The singularity near the origin has a direct interpretation in terms of jump activity. Since

$$k(x) \propto |x|^{-1-\alpha}, \qquad x \to 0,$$

the Lévy measure has infinite mass in every neighbourhood of zero for $\alpha > 0$. Consequently, the corresponding process has infinitely many small jumps over every finite time interval. At the same time, the power-law tail remains sufficiently persistent that stable distributions with $0 < \alpha < 2$ do not possess a finite second moment. This property is fundamental to the stable construction, but it also represents a major limitation when the model is used for financial data, since empirical asset returns generally require finite variance for conventional risk measurement, covariance analysis and portfolio applications [3, 6].

The empirical motivation for the stable framework is nevertheless substantial. Mandelbrot [1] proposed stable Lévy behaviour as a model for cotton-price fluctuations,

while Fama [2] provided empirical evidence that stable distributions could provide a better description of stock-market returns than the Gaussian distribution. Subsequent investigations compared stable specifications with financial return data and demonstrated that heavy-tailed models can capture the central and intermediate parts of empirical return distributions substantially better than the normal model [23]. These studies established an important principle for the subsequent development of Lévy models: the distributional behaviour of financial returns is closely connected to the jump structure imposed at the Lévy-measure level.

However, the unrestricted power-law Lévy density introduces an important conflict between empirical tail behaviour and mathematical tractability. A pure power-law tail implies substantially more extreme observations than are typically supported by sufficiently long financial datasets and leads to the absence of finite moments of orders greater than or equal to α. Empirical investigations subsequently suggested that although financial returns possess heavy tails, their distant tails may decay more rapidly than those of an unrestricted stable distribution [12, 25].

An important observation from this evolution is that increasingly sophisticated financial Lévy models can be understood as modifications of the same basic density structure. The power-law component determines how densely jumps accumulate near zero, while the tempering component controls the contribution of large jumps. Subsequent models differ primarily in which aspects of these two mechanisms are fixed, generalized or allowed to vary independently. This provides a more coherent interpretation of the historical development than treating Variance Gamma, Bilateral Gamma, Truncated Lévy Flight, CGMY, KoBoL and Tempered Stable models as unrelated distributions.

The classical Lévy density therefore establishes the benchmark against which the subsequent tempered models can be interpreted. Its principal strengths are the natural representation of heavy-tailed jump behaviour, infinite divisibility and a direct connection between the power-law exponent and small-jump activity. Its principal limitations for financial applications are the unrestricted power-law tails and the resulting absence of finite variance for the stable range $0 < \alpha < 2$. These limitations motivated the introduction of truncation and exponential tempering, which progressively transformed the original stable Lévy density into forms capable of retaining non-Gaussian jump behaviour while controlling extreme financial movements.

From the perspective of Lévy-density evolution, the resulting progression can be expressed schematically as

$$\begin{aligned}&\text{Stable Lévy density} \longrightarrow \text{Truncated/Tempered Lévy density} \longrightarrow \text{VG/BG}\\ &\longrightarrow \text{CGMY/KoBoL} \longrightarrow \text{Tempered Stable} \longrightarrow \text{Generalized Tempered Stable.}\end{aligned}$$

Each stage represents a modification of the jump-density mechanism to address a specific empirical or mathematical limitation of its predecessor. The importance of this progression for financial data is that it moves the modelling emphasis from simply reproducing heavy-tailed return distributions toward identifying and separately controlling the mechanisms responsible for small-jump activity, directional asymmetry and extreme-jump attenuation.

Accordingly, the classical Lévy density should be viewed as the structural starting point of the tempered Lévy framework rather than merely as an early heavy-tailed distribution. It establishes the power-law mechanism from which the subsequent models inherit their small-jump behaviour, while the empirical inadequacy of an unrestricted power-law tail provides the motivation for introducing tempering. This density-based interpretation creates a consistent foundation for the subsequent sections of this review, where Variance Gamma, Bilateral Gamma, Truncated Lévy Flight, CGMY, KoBoL, Tempered Stable and Generalized Tempered Stable models are examined according to how their Lévy densities modify the balance between small-jump activity, positive–negative asymmetry and large-jump attenuation in financial data.

3.2 The Variance-Gamma Lévy Density

The Variance-Gamma (VG) process introduced by Madan and Seneta [7] constitutes an important step in the evolution of Lévy models for financial returns. Its relevance to financial modelling arises from its ability to replace the continuous Gaussian diffusion mechanism with an infinite-activity pure-jump structure while retaining finite variation and finite moments. This construction was motivated by the empirical departure of financial returns from Gaussian behaviour, particularly the presence of excess kurtosis and asymmetry. The VG process is obtained by evaluating a Brownian motion with drift at a random time governed by an independent Gamma process, thereby generating a flexible non-Gaussian return model with an analytically tractable Lévy structure. Madan, Carr and Chang [26] subsequently demonstrated its usefulness for modelling log-stock prices and option prices, showing that the additional parameters of the VG process provide control over skewness and kurtosis and improve the representation of empirical and risk-neutral return distributions.

For the present review, the principal importance of the VG model lies not only in its stochastic representation but in the explicit form of its Lévy density. In the bilateral Gamma representation, the VG Lévy density can be written as

$$k_{VG}(x) = \begin{cases} \dfrac{\mu_n^2}{\nu_n} \dfrac{\exp\left(-\dfrac{\mu_n}{\nu_n}|x|\right)}{|x|}, & x < 0, \\ \dfrac{\mu_p^2}{\nu_p} \dfrac{\exp\left(-\dfrac{\mu_p}{\nu_p}x\right)}{x}, & x > 0. \end{cases} \tag{4}$$

where $\mu_p, \mu_n > 0$ and $\nu_p, \nu_n > 0$ characterize the positive- and negative-jump components. This representation is particularly informative for financial applications because it separates the jump behaviour on the two sides of the origin. The parameters associated with the positive and negative components can therefore accommodate asymmetry in the jump structure, while the exponential terms determine the rate at which the frequency of larger jumps decreases.

The central feature of (4) is its behaviour close to the origin. As $x \to 0$,

$$k_{VG}(x) \propto \frac{1}{|x|},$$

which implies an infinite number of small jumps over every finite time interval. Nevertheless,

$$\int_{\mathbb{R}} \min(1, |x|) k_{VG}(x) dx < \infty,$$

so that the process has finite variation. Thus, the VG model occupies a distinctive position between finite-activity jump models and more active stable-type processes: it allows infinitely many small jumps, but their aggregate variation remains finite. At the opposite end of the jump-size range, the exponential factors in(4) provide strong attenuation of large jumps. This combination of a singularity at the origin and exponential tail decay is the essential Lévy-density mechanism through which the VG model generates non-Gaussian financial returns.

The financial relevance of the VG Lévy density was established through a sequence of empirical and pricing studies. Madan and Seneta [7] showed that the VG framework provides an effective representation of share-market returns, establishing the empirical motivation for replacing the Gaussian return mechanism with an infinite-activity jump process. Madan, Carr and Chang [26] subsequently showed that the model could reproduce important features of both statistical and risk-neutral distributions of S&P 500 returns and option prices. In particular, their analysis found that the statistical distribution displayed excess kurtosis, while the risk-neutral distribution exhibited negative skewness and greater kurtosis. These findings demonstrate that the jump structure represented by the VG Lévy density is relevant not only for describing historical return behaviour but also for explaining information embedded in option prices.

The applicability of the VG specification was subsequently examined in other financial markets. Daal and Madan [27] investigated foreign-currency options and reported superior out-of-sample pricing performance of the VG model relative to modified Black–Scholes and Merton jump-diffusion alternatives. This provides evidence that the infinite-activity structure of the VG model is not restricted to equity returns but can also be useful for financial data exhibiting asymmetric and non-Gaussian behaviour across different markets. Similarly, Semeraro [28] extended the VG construction to a multivariate setting, demonstrating that its Lévy structure can be incorporated into models of dependence among several financial assets.

An important development in the VG literature concerns the estimation of its parameters from financial data. Seneta [29] investigated moment-based fitting of the VG distribution while allowing for possible dependence in log-return increments. Finlay and Seneta [30] subsequently compared several estimation procedures and emphasized the sensitivity of parameter estimation to the dependence structure of financial observations. These contributions are particularly relevant from a Lévy-density perspective because estimation of the parameters governing the positive- and negative-jump components directly determines the estimated intensity and decay of

the empirical jump measure. Consequently, the usefulness of the VG density in financial applications depends not only on its theoretical flexibility but also on the reliability of the procedure used to estimate its parameters.

Further empirical evidence suggests that the VG Lévy structure may vary with market conditions. Rathgeber, Stadler and Stöckl [31] examined a broad daily data set of individual stocks and employed regime-switching techniques to distinguish normal and turbulent market periods. Their results indicated that VG parameters vary over time and that the model's fitting performance changes across market regimes. This finding is particularly important for the evolution of Lévy-density-based financial modelling. If the parameters governing jump intensity and tail attenuation change between calm and turbulent periods, then a single fixed Lévy-density shape may not fully represent the evolving jump characteristics of financial returns.

The mathematical development of the VG distribution has also strengthened the theoretical foundation of this model. Fischer, Gaunt and Sarantsev [32] provide a comprehensive modern treatment of the VG family, including its representations, moments, cumulants, parameter estimation and financial applications. Their review confirms the broad role of VG distributions within the class of infinitely divisible distributions and highlights their continuing relevance in financial modelling. The broader literature therefore establishes the VG framework as both theoretically tractable and empirically relevant, while simultaneously indicating that its fixed Lévy-density structure imposes restrictions on the representation of small-jump activity.

The principal Lévy-density characteristics of the VG model can consequently be summarized as

(i) infinite jump activity resulting from the $1/|x|$ singularity at the origin;
(ii) finite variation due to the integrability of $|x|$ with respect to the Lévy measure near zero;
(iii) exponentially tempered large jumps, resulting in finite moments of all orders;
(iv) separate positive- and negative-jump components, allowing asymmetric jump behaviour;
(v) an analytically tractable density that can be estimated and applied to financial return and option data.

From the evolutionary perspective of Lévy-density modelling, however, the $1/|x|$ behaviour represents both the strength and the limitation of the VG model. It provides infinite activity while preserving finite variation, but the local power governing the abundance of small jumps is fixed. Consequently, the VG model cannot independently adjust the degree of small-jump activity to accommodate different empirical regimes or asset classes. This limitation motivated the development of more general tempered Lévy densities in which the power-law behaviour near the origin is allowed to vary.

A major subsequent development was the CGMY model of Carr, Geman, Madan and Yor [9], which generalized the VG Lévy-density structure by introducing an additional activity parameter. In the CGMY framework, the corresponding Lévy density has the form

$$k_{CGMY}(x) = C\left[\frac{\exp(-Mx)}{x^{1+Y}}\mathbf{1}_{\{x>0\}} + \frac{\exp(-G|x|)}{|x|^{1+Y}}\mathbf{1}_{\{x<0\}}\right].$$

The VG structure is recovered when the activity parameter takes the boundary value $Y = 0$. Hence, the transition from VG to CGMY is not merely the introduction of another financial model; it represents a direct generalization of the Lévy-density shape. The exponential tempering inherited from VG is retained, while the additional power parameter allows the intensity of small jumps to vary. This development marks a fundamental transition from a fixed tempered Lévy density toward a flexible tempered-stable framework.

This evolution is particularly important for financial data because the empirical jump structure need not remain constant across assets or market states. The empirical findings of Rathgeber et al. [31] concerning parameter variation across normal and turbulent regimes reinforce this point. The VG model established that infinite-activity exponentially tempered jumps can provide an effective representation of financial returns; subsequent tempered-stable models extended this idea by allowing greater control over the local behaviour of the Lévy density. The progression from VG to CGMY and subsequently to generalized tempered-stable specifications therefore represents an increasing emphasis on modelling the Lévy density itself rather than imposing a fixed jump-activity structure.

Accordingly, the VG model represents a foundational stage in the evolution of Lévy-density-based financial modelling. Its contribution is not limited to providing an alternative return distribution to the Gaussian model. More importantly, it established a financially meaningful Lévy density characterized by infinite small-jump activity, finite variation, asymmetric jump components and exponential attenuation of large jumps. The subsequent empirical evidence, together with the development of more flexible tempered-stable models, indicates that these properties form a useful baseline while also motivating additional flexibility in the power-law behaviour near the origin. The VG Lévy density thus provides the natural starting point for the subsequent evolution toward CGMY, KoBoL and generalized tempered-stable Lévy densities considered in the following sections.

3.3 The Bilateral Gamma Lévy Density

The Bilateral Gamma (BG) process represents an important refinement in the evolution of exponentially tempered Lévy models for financial data. Emerging from the Gamma and Variance-Gamma frameworks, the BG construction was developed to allow the positive and negative jump components of a financial return process to be parameterized separately. Küchler and Tappe [33] established the theoretical foundations of the bilateral Gamma process and demonstrated its applicability to financial market data, including German DAX returns. The central contribution of the BG specification from a Lévy-density perspective is therefore not merely the introduction of another non-Gaussian return distribution, but the explicit separation of the upward- and downward-jump mechanisms within a single infinitely divisible model.

Let $Y^+ \sim \Gamma(\alpha^+, \lambda^+)$ and $Y^- \sim \Gamma(\alpha^-, \lambda^-)$, where $\alpha^+, \alpha^- > 0$ are shape parameters and $\lambda^+, \lambda^- > 0$ are rate parameters. The bilateral Gamma random variable is constructed as the difference of two independent Gamma variables, $X = Y^+ - Y^-$.

The corresponding characteristic function is

$$\phi_X(u) = \left(\frac{\lambda^+}{\lambda^+ - iu}\right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + iu}\right)^{\alpha^-}.$$

Although the characteristic function provides an analytically convenient representation, the Lévy-density formulation is more informative for the present review because it directly describes the intensity and attenuation of jumps of different magnitudes.

For the bilateral Gamma process, the Lévy density can be expressed as

$$k_{BG}(x) = \begin{cases} \dfrac{\alpha^+}{x} \exp(-\lambda^+ x), & x > 0, \\ \dfrac{\alpha^-}{|x|} \exp(-\lambda^- |x|), & x < 0. \end{cases} \tag{5}$$

and hence the Lévy measure is

$$\nu_{BG}(dx) = k_{BG}(x)\, dx.$$

This representation makes the financial interpretation of the model particularly transparent. The positive and negative sides of the Lévy density are governed by separate parameter pairs, (α^+, λ^+) and (α^-, λ^-), where α^+ and α^- determine the respective jump intensities, while λ^+ and λ^- control the exponential attenuation of positive and negative jumps. The model can therefore distinguish asymmetry arising from differences in jump frequency from asymmetry arising from differences in the decay of jump magnitudes. Such separation is particularly relevant for financial returns, for which upward and downward movements need not exhibit identical statistical behaviour.

The small-jump behaviour of the BG Lévy density is inherited directly from the Gamma-process structure. As

$$k_{BG}(x) \sim \begin{cases} \dfrac{\alpha^+}{x}, & x \to 0^+, \\ \dfrac{\alpha^-}{|x|}, & x \to 0^-. \end{cases}$$

Thus, the density possesses a $1/|x|$-type singularity at the origin on both sides. This singularity implies infinite jump activity, since the Lévy measure assigns infinite mass to every neighbourhood of zero. Nevertheless,

$$\int_{\mathbb{R}} min(1, |x|) k_{BG}(x)\, dx < \infty,$$

so that the associated process has finite variation. At the opposite end of the jump-size spectrum, the exponential factors $\exp(-\lambda^+ x)$ and $\exp(-\lambda^- |x|)$ rapidly attenuate

large positive and negative jumps. The BG process therefore combines infinite small-jump activity, finite variation and exponentially controlled tails, providing a useful structure for modelling non-Gaussian financial returns.

The four parameters of the Lévy density also provide a direct interpretation of financial asymmetry. Differences between α^+ and α^- modify the relative frequency of upward and downward jumps, whereas differences between λ^+ and λ^- alter their respective tail decay. Consequently, the BG model can distinguish between a market in which negative jumps occur more frequently and one in which negative jumps are less frequent but decay more slowly. This distinction is important because skewness in financial returns is not necessarily generated by a single mechanism. It can arise from differences in jump occurrence, jump magnitude, or both. The BG specification consequently provides a more informative density-based description of asymmetric financial fluctuations than a model imposing identical positive- and negative-jump structures.

The relationship between the BG and VG models is particularly important in the evolution of tempered Lévy densities. Both models retain the characteristic $1/|x|$ behaviour near the origin and exponential attenuation of large jumps. However, the BG representation makes the two sides of the Lévy density independently identifiable through (α^+, λ^+) and (α^-, λ^-). In this sense, the transition from VG to BG can be interpreted as an evolution from a coupled jump representation toward an explicitly two-sided jump mechanism. Küchler and Tappe [33] demonstrated the theoretical and empirical usefulness of this structure for financial market modelling, while subsequent studies investigated its implications for asset pricing and option valuation.

The empirical literature subsequently extended the BG framework from distributional modelling to financial risk and derivative pricing. Madan, Schoutens and Wang [34] examined bilateral Gamma returns and their associated risks and rewards, emphasizing the role of distinct upward and downward jump components in explaining financial risk premia. Their results support the interpretation that positive and negative movements can carry different economic information. This is important for Lévy-density modelling because the separate sides of $k_{BG}(x)$ are not merely mathematical devices for producing skewness; they can represent economically different sources of financial risk.

The flexibility of the BG density has also motivated extensions beyond stationary Lévy processes. Madan and Wang [35] developed additive processes with bilateral Gamma marginals, thereby allowing the distributional characteristics of financial returns to evolve over time. Their empirical calibration using SPY option data demonstrates the potential of time-inhomogeneous bilateral Gamma specifications for financial applications. From the perspective of Lévy-density evolution, this represents an important conceptual step: the model moves from a fixed jump-density structure toward a framework in which jump characteristics may change with time.

A complementary extension is obtained by introducing dependence across financial assets. Madan and Wang [36] developed a multivariate bilateral Gamma framework and investigated its ability to capture dependence, co-skewness and co-kurtosis. This development demonstrates that the two-sided Lévy-density structure can be retained while extending the model to multivariate financial systems. Such extensions are

particularly relevant for portfolio applications, where modelling marginal return distributions alone is insufficient and the joint behaviour of extreme positive and negative movements must also be considered.

Further developments have addressed the practical calibration and option-pricing performance of bilateral Gamma models. Aguilar and Kirkby [37] developed robust and nearly exact option-pricing methods for bilateral Gamma processes, demonstrating that the analytical structure of the model can be exploited for efficient numerical valuation. Similarly, Kirkby, Aglieri Rinella and Aguilar [38] introduced the bilateral Gamma motion by augmenting the pure-jump bilateral Gamma process with a diffusion component. Their results indicate that the diffusion-augmented specification can provide improved flexibility for reproducing observed option-implied volatility surfaces. This development is particularly informative for the evolution of Lévy models because it identifies a limitation of relying exclusively on the jump component: although the BG Lévy density provides detailed control over asymmetric jumps, some financial option data may contain variation that cannot be adequately represented by the pure-jump component alone.

The BG framework has also been examined from broader risk-management and parameter-selection perspectives. Xie, Wang and Lu [39] investigated the behavioural implications of the bilateral Gamma process, highlighting its relevance for asymmetric financial behaviour. Shirai [40] further considered acceptable bilateral Gamma parameters in financial applications, emphasizing the role of parameter restrictions in determining economically meaningful model configurations. These contributions reinforce an important point for empirical Lévy-density modelling: the parameters governing the positive and negative sides of the density are not interchangeable and should be interpreted in relation to the financial characteristics represented by the model.

The main mathematical and financial characteristics of the bilateral Gamma Lévy density can therefore be summarized as follows:

(i) it defines an infinitely divisible pure-jump process with infinite jump activity;
(ii) the associated process has finite variation because the Lévy density is integrable against $|x|$ in a neighbourhood of the origin;
(iii) the $1/|x|$ singularity determines the local small-jump behaviour;
(iv) α^+ and α^- separately control the intensities of positive and negative jumps;
(v) λ^+ and λ^- independently determine the exponential attenuation of positive and negative jumps;
(vi) the model provides an analytically tractable framework for financial return and option modelling;
(vii) the structure naturally admits additive, multivariate and diffusion-augmented extensions.

From the evolutionary perspective adopted in this review, the principal contribution of the BG model is therefore the introduction of "two-sided parameter flexibility without abandoning the analytically tractable $1/|x|$ exponentially tempered Lévy structure". The model improves upon the more restrictive VG representation by allowing the positive and negative jump mechanisms to possess different intensities and

decay rates. This is particularly useful for financial data because asymmetry is thereby represented directly at the level of the Lévy density rather than being introduced only through the resulting probability distribution.

However, the BG model also reveals a fundamental limitation that becomes important for the subsequent evolution of tempered Lévy models. Although the parameters α^+, α^-, λ^+ and λ^- provide independent control of jump intensity and exponential tail decay, the power governing the singularity at the origin remains fixed:

$$k_{BG}(x) \propto |x|^{-1}, \qquad x \to 0.$$

Thus, the BG model can distinguish the intensity and attenuation of positive and negative jumps, but it cannot independently modify the degree of small-jump activity. In empirical financial data, however, the concentration of small movements and the frequency of increasingly small jumps may vary substantially across assets and market conditions. This limitation provides a natural motivation for the next stage of the evolution, in which the fixed $|x|^{-1}$ component is replaced by a flexible power-law term.

The transition can therefore be expressed schematically as

$$\underbrace{|x|^{-1}e^{-\lambda|x|}}_{\text{VG}} \quad \longrightarrow \quad \underbrace{\begin{cases} \alpha^+ x^{-1} e^{-\lambda^+ x}, & x > 0, \\ \alpha^- |x|^{-1} e^{-\lambda^- |x|}, & x < 0, \end{cases}}_{\text{Bilateral Gamma}} \quad \longrightarrow \quad \underbrace{|x|^{-(1+Y)} e^{-\lambda|x|}}_{\text{CGMY}}$$

where the first transition introduces independent positive- and negative-jump mechanisms, whereas the second introduces an explicit parameter governing the power-law activity of the Lévy density. The bilateral Gamma model therefore occupies a crucial intermediate position between the classical Gamma/VG family and the more flexible tempered-stable models.

This perspective is particularly relevant to the subsequent development of generalized tempered-stable models considered in this review. The BG model demonstrates that financial asymmetry can be incorporated by allowing the two sides of the Lévy density to evolve differently, while the later CGMY and generalized tempered-stable frameworks demonstrate that the "local activity itself can also be made flexible". The evolution is consequently not simply a sequence of increasingly complicated distributions; it represents a progressive enrichment of the Lévy density along three distinct dimensions: "directional asymmetry, large-jump attenuation, and small-jump activity". In this progression, the bilateral Gamma model provides the essential intermediate structure that separates the positive and negative jump mechanisms before the power-law activity parameter is introduced in subsequent tempered-stable specifications.

3.4 The Carr–Geman–Madan–Yor (CGMY) Lévy Density

The Carr–Geman–Madan–Yor (CGMY) model represents a decisive stage in the evolution of Lévy-density-based models for financial data. Earlier Variance-Gamma (VG) and Bilateral Gamma specifications introduced infinite-activity jump structures with

exponentially tempered tails, but their Lévy densities retain a fixed inverse-power behaviour of order $|x|^{-1}$ near the origin. The principal contribution of the CGMY framework is to replace this fixed local structure by a flexible power-law component governed by an explicit activity parameter Y [9]. Consequently, the model separates the local behaviour of small jumps from the attenuation of large jumps and provides a direct mechanism for adapting the Lévy density to the fine structure observed in financial returns.

The CGMY Lévy density is given by

$$k_{CGMY}(x) = \begin{cases} \dfrac{C\exp(-Mx)}{x^{1+Y}}, & x > 0, \\ \dfrac{C\exp(-G|x|)}{|x|^{1+Y}}, & x < 0, \end{cases} \tag{6}$$

where

$$C > 0, \qquad G > 0, \qquad M > 0, \qquad Y < 2.$$

The four parameters have distinct roles in the Lévy-density structure. The parameter C determines the overall scale of jump activity, M controls the exponential attenuation of positive jumps, G controls the exponential attenuation of negative jumps, and Y determines the power-law singularity governing the concentration of small jumps near the origin [9, 41]. This parameter separation is particularly important for financial-data modelling because it distinguishes the frequency of small jumps from the behaviour of large positive and negative movements.

The defining feature of the CGMY density is its local behaviour around the origin. Since

$$\exp(-Mx) \to 1 \qquad \text{and} \qquad \exp(-G|x|) \to 1 \quad \text{as } x \to 0,$$

the Lévy density satisfies

$$k_{CGMY}(x) \sim C|x|^{-(1+Y)}, \qquad x \to 0.$$

Thus, Y directly controls the degree of singularity of the Lévy density. This is a fundamental change from the VG and Bilateral Gamma models, for which the corresponding power is fixed at $|x|^{-1}$. The parameter Y should therefore be interpreted not simply as an additional shape parameter, but as a parameter governing the *fine structure* of the jump process.

This local structure has direct consequences for jump activity and path variation. The total jump activity is determined by

$$\int_{\mathbb{R}} k_{CGMY}(x)\,dx.$$

Near the origin, this integral behaves as

$$\int_0^{\varepsilon} x^{-(1+Y)}\,dx.$$

Consequently, the CGMY process has finite activity for $Y < 0$ and infinite activity for $Y \geq 0$. Furthermore, the finite-variation condition involves

$$\int_{|x|<1} |x| k_{CGMY}(x)\, dx < \infty,$$

which holds for $Y < 1$. Hence, the parameter space naturally separates the process into the regimes

$$\begin{aligned} &Y < 0, && \text{finite activity,} \\ &0 \leq Y < 1, && \text{infinite activity and finite variation,} \\ &1 \leq Y < 2, && \text{infinite activity and infinite variation.} \end{aligned}$$

The same parameter therefore controls a qualitative change in the path structure of the process. This provides a much more informative interpretation of Y than treating it merely as a distributional shape parameter.

The behaviour of large jumps is controlled independently by the exponential tempering terms. For positive and negative jumps,

$$k_{CGMY}(x) \propto \begin{cases} x^{-(1+Y)} e^{-Mx}, & x > 0, \\ |x|^{-(1+Y)} e^{-G|x|}, & x < 0. \end{cases}$$

Thus, the Lévy density contains two distinct scales of behaviour. The power-law component determines the concentration of small jumps, while the exponential factors determine the decay of large jumps. In addition, M and G allow the two sides of the Lévy density to decay at different rates. The CGMY specification consequently provides a density-level decomposition of financial jump behaviour into overall activity, local activity, and directional tail attenuation.

The characteristic function of the CGMY process can be written in closed form as

$$\phi_{CGMY}(u,t) = \exp\left\{ tC\Gamma(-Y)\left[(M - iu)^Y - M^Y + (G + iu)^Y - G^Y \right] \right\},$$

under the corresponding standard parameterization. Although the characteristic-function representation is important for analytical and computational purposes, the Lévy-density representation in (6) is more informative for the evolutionary perspective adopted in this review because it makes the mechanisms governing small and large jumps explicit.

The empirical relevance of this flexibility was established by Carr, Geman, Madan and Yor [9] through their investigation of the fine structure of asset returns. Their analysis demonstrated that financial returns can exhibit substantially different jump structures and that the empirical behaviour of returns cannot be adequately described solely through a fixed Gaussian or fixed-activity jump specification. The CGMY framework provided a means of estimating the parameters governing the Lévy density directly from financial observations, thereby connecting the mathematical structure

of the jump process with measurable characteristics of asset returns. In particular, the estimation of Y provides information about the local concentration of small jumps, while G and M provide information about the asymmetric attenuation of large negative and positive jumps.

The evolution from VG to CGMY becomes particularly transparent by considering the special case $Y = 0$. Substitution into the Lévy density gives

$$k_{CGMY}(x) = \begin{cases} \dfrac{Ce^{-Mx}}{x}, & x > 0, \\ \dfrac{Ce^{-G|x|}}{|x|}, & x < 0, \end{cases}$$

which has the characteristic $1/|x|$ exponentially tempered form of the Variance-Gamma class. Hence, the transition from VG to CGMY can be expressed directly at the Lévy-density level as

$$|x|^{-1} \quad \longrightarrow \quad |x|^{-(1+Y)}.$$

This relationship is important for understanding the historical evolution of tempered Lévy models: VG introduced exponential tempering of an infinite-activity jump density, whereas CGMY released the fixed small-jump exponent and made the activity structure empirically adjustable.

The probabilistic interpretation of this additional flexibility was further developed by Madan and Yor [42], who represented CGMY and Meixner processes as time-changed Brownian motions. Their construction connects the CGMY model with stable subordinators and provides an alternative interpretation of the activity parameter. From the density perspective, this result reinforces the interpretation of Y as a parameter describing the underlying stable-type small-jump structure rather than simply a parameter controlling the overall shape of the return distribution.

The financial relevance of pure-jump Lévy models was further emphasized by Geman [41], who considered Lévy-based asset-price models in which stochastic activity and discontinuous movements play a central role. Within this framework, CGMY provides a particularly flexible representation because the Lévy density can simultaneously accommodate asymmetric jumps, heavy local activity and exponentially tempered extreme movements. The model therefore provides a natural bridge between empirical return modelling and risk-neutral applications such as option pricing.

The flexibility of the CGMY Lévy density has also generated a substantial literature on numerical derivative pricing. Almendral and Oosterlee [43] developed numerical methods for European and American options under the CGMY process and explicitly addressed the numerical consequences of the singular Lévy measure. Wang, Wan and Forsyth [44] similarly considered robust numerical valuation under the CGMY process. These studies are relevant from the density-based perspective because the singularity $x^{-(1+Y)}$ near zero must be handled carefully in numerical integration and discretization. Thus, the activity parameter Y influences not only the statistical behaviour of returns but also the computational properties of models used for derivative valuation.

The relationship between the Lévy-density parameters and higher-order distributional characteristics is another important aspect of the CGMY framework. The role of skewness and kurtosis in tempered-stable models has been examined by Asmussen [45], highlighting how the parameters governing the Lévy density affect observable non-Gaussian characteristics of financial returns. In particular, the combination of Y, G and M determines how probability mass is distributed across small and large jumps and across the positive and negative sides of the return distribution. Consequently, skewness and kurtosis can be viewed as aggregate manifestations of an underlying asymmetric and activity-dependent Lévy density.

The CGMY model has also been extended from historical return modelling to risk-neutral analysis. Kim and Lee [46] investigated relative entropy for CGMY processes and its application to finance, demonstrating that the Lévy-density structure remains important when constructing alternative probability measures for derivative pricing. This illustrates that the four parameters of the CGMY density influence both the statistical representation of observed returns and the transformation required for risk-neutral valuation.

Further developments have extended the CGMY structure to multivariate financial applications. Wu et al. [47] considered multivariate CGMY processes for the valuation adjustment of multi-asset derivatives. Such extensions demonstrate that the fundamental density structure in (6) can be incorporated into models involving dependence between multiple financial assets. The resulting framework retains the ability to represent non-Gaussian and asymmetric jump behaviour while extending the application of CGMY beyond univariate return series.

Recent analytical work has continued to demonstrate the importance of the local power-law component of the CGMY density. Hoffmeyer and Houdré [48] investigated higher-order short-time at-the-money asymptotics for the exponential CGMY model in the infinite-variation regime $Y \in (1, 2)$. Their results illustrate how the singular behaviour of the Lévy density near the origin propagates into short-maturity option prices. This provides an important connection between an apparently local mathematical property of the Lévy density and observable financial quantities.

From the viewpoint of Lévy-density evolution, the CGMY model can therefore be interpreted through three complementary components:

(i) C determines the overall scale of jump activity;
(ii) Y determines the local power-law activity and therefore the fine structure of small jumps;
(iii) G and M determine the directional exponential attenuation of negative and positive jumps.

This decomposition is particularly useful for financial data because it separates characteristics that are often conflated in conventional distributional descriptions. The model does not merely produce heavy tails; it specifies how the jump intensity behaves close to zero and how that intensity is subsequently tempered in the tails.

A further important consequence of the CGMY formulation is that the same activity parameter Y governs both sides of the Lévy density. Although G and M permit

asymmetric tail decay, the local singularity remains symmetric in its power-law order:

$$k_{CGMY}(x) \sim C|x|^{-(1+Y)} \qquad \text{as } x \to 0^{\pm}.$$

Therefore, CGMY provides directional asymmetry in the exponential tempering but not independent positive- and negative-side activity indices. This distinction becomes important when modelling financial returns for which upward and downward movements may exhibit different small-jump structures. From the perspective of the subsequent evolution of tempered Lévy models, this common-Y restriction represents a natural point at which further generalization becomes necessary.

The evolution from VG through Bilateral Gamma to CGMY can consequently be represented directly at the level of the Lévy density:

$$\underbrace{|x|^{-1}e^{-\lambda|x|}}_{\text{VG}} \quad \longrightarrow \quad \underbrace{\begin{cases} \alpha^{+}x^{-1}e^{-\lambda^{+}x}, & x>0, \\ \alpha^{-}|x|^{-1}e^{-\lambda^{-}|x|}, & x<0, \end{cases}}_{\text{Bilateral Gamma}} \quad \longrightarrow \quad \underbrace{\begin{cases} Cx^{-(1+Y)}e^{-Mx}, & x>0, \\ C|x|^{-(1+Y)}e^{-G|x|}, & x<0, \end{cases}}_{\text{CGMY}}.$$

The first transition introduces independent positive- and negative-side intensity and tempering parameters, whereas the second introduces a continuously adjustable power-law activity index. Thus, the development from VG through Bilateral Gamma to CGMY represents a progressive enrichment of the Lévy density along two fundamental dimensions: *directional asymmetry* and *small-jump activity*.

The principal limitation that remains in the CGMY specification is therefore not its ability to represent heavy tails or asymmetry, but the use of a common activity index Y for both positive and negative jumps. This observation provides a natural motivation for subsequent generalized tempered-stable models in which the power-law behaviour can be formulated more flexibly. In particular, allowing the positive and negative sides to possess distinct activity structures would permit the Lévy density to respond more directly to possible asymmetries in the fine structure of financial returns.

Accordingly, the importance of CGMY in the evolution of financial Lévy models lies in its change from a fixed local jump structure to an empirically adjustable one. The replacement

$$|x|^{-1} \to |x|^{-(1+Y)}$$

makes the small-jump activity an explicit modelling object and connects the mathematical singularity of the Lévy density with observable features of financial returns, path variation, skewness, kurtosis and short-maturity derivative prices. CGMY therefore constitutes a central intermediate stage between the classical exponentially tempered Gamma-type Lévy densities and the more general tempered-stable specifications developed subsequently.

3.5 The KoBoL Lévy Density

The Koponen–Boyarchenko–Levendorskii (KoBoL) class constitutes an important development in the construction of tempered Lévy models for financial data. Its underlying idea is to retain the power-law behaviour of a stable Lévy measure near the origin

while introducing exponential tempering in the tails. This construction is particularly relevant for financial returns because it separates the behaviour of frequent small jumps from that of relatively rare large jumps. The KoBoL framework was developed from the tempered stable perspective of Koponen [8] and subsequently formulated and extended by Boyarchenko and Levendorskii [13, 49].

From the perspective of the present review, the Lévy density is the central object because it provides a direct description of the intensity and magnitude of jumps underlying financial returns. In the parameterization considered here, the KoBoL Lévy density is given by

$$\nu(dx) = \left[Dp \, \frac{\exp(-\lambda_+ x)}{x^{1+\alpha}} \mathbf{1}_{(0,\infty)}(x) + Dq \, \frac{\exp(-\lambda_- |x|)}{|x|^{1+\alpha}} \mathbf{1}_{(-\infty,0)}(x) \right] dx, \tag{7}$$

where

$$D > 0, \qquad 0 < p, q < 1, \qquad p + q = 1, \qquad \lambda_+, \lambda_- > 0, \qquad 0 < \alpha < 2.$$

Here, D determines the overall scale of jump activity, p and q determine the relative contributions of positive and negative jumps, λ_+ and λ_- control the exponential attenuation of positive and negative jumps, respectively, and α determines the local power-law behaviour of the Lévy density. This parameterization therefore provides separate controls for jump intensity, directional asymmetry, small-jump activity and large-jump attenuation.

The most informative feature of (7) is its two-scale structure. As $x \to 0^+$ and $x \to 0^-$,

$$\nu(x) \sim \begin{cases} Dp \, x^{-(1+\alpha)}, & \text{as } x \to 0^+, \\ Dq \, |x|^{-(1+\alpha)}, & \text{as } x \to 0^-. \end{cases}$$

The exponential tempering terms have negligible effect near the origin, so α determines the small-jump activity. As $x \to 0^+$,

$$\nu(x) \sim C_+ x^{-(1+\alpha)},$$

and, for $\alpha > 0$,

$$\int_0^\epsilon x^{-(1+\alpha)}, dx = \infty,$$

implying infinite jump activity. The corresponding variation is determined by

$$\int_0^\epsilon x^{-\alpha} \, dx \begin{cases} < \infty, & 0 < \alpha < 1, \\ = \infty, & 1 \le \alpha < 2. \end{cases}$$

so the process has finite variation for $0 < \alpha < 1$ and infinite variation for $1 \le \alpha < 2$. In contrast, the large-jump behaviour is governed by λ_+ and λ_-:

$$\nu(x) \propto \begin{cases} x^{-(1+\alpha)} e^{-\lambda_+ x}, & x \to +\infty, \\ |x|^{-(1+\alpha)} e^{-\lambda_- |x|}, & x \to -\infty. \end{cases}$$

Thus, α controls small-jump activity and variation, while λ_+ and λ_- control the decay of large positive and negative jumps, respectively.

Hence, the KoBoL density combines stable-like behaviour at small jump sizes with exponentially tempered tails at large jump sizes.

This distinction between local and tail behaviour is particularly important for financial return data. High-frequency or daily financial returns typically contain a large number of relatively small movements together with occasional extreme observations. In the KoBoL Lévy density, the former are represented through the power-law singularity governed by α, whereas the latter are controlled by λ_+ and λ_-. The model therefore provides a direct density-based mechanism for distinguishing the frequency of small fluctuations from the attenuation of extreme positive and negative movements.

The parameter α also determines the path-variation characteristics of the process. For $0 < \alpha < 1$, the process has infinite jump activity but finite variation, whereas $1 \leq \alpha < 2$ corresponds to infinite jump activity and infinite variation. Consequently, α has a direct probabilistic interpretation beyond its role as a shape parameter: it determines the degree of small-jump activity and the resulting fine structure of the sample paths. This feature is particularly useful when Lévy models are employed to represent financial returns in which the distinction between numerous small movements and less frequent large movements is important.

The asymmetric structure of the Lévy density provides a further advantage for financial applications. The parameters p and q allow the relative intensity of positive and negative jumps to differ, while λ_+ and λ_- allow their tail attenuation rates to differ. Thus, two conceptually distinct forms of asymmetry can be represented:

$$\begin{aligned} p \neq q \quad &\Longrightarrow \quad \text{asymmetry in relative jump intensity,} \\ \lambda_+ \neq \lambda_- \quad &\Longrightarrow \quad \text{asymmetry in tail attenuation.} \end{aligned}$$

This distinction is important because skewness in financial returns need not be generated exclusively through unequal jump frequencies. Positive and negative jumps may also exhibit different degrees of persistence in the tails. The KoBoL Lévy density accommodates both effects while retaining a common local activity parameter α.

The characteristic exponent corresponding to the Lévy density in (7) can be written in Lévy–Khintchine form as

$$\Psi(u) = -i\mu u + D\Gamma(-\alpha)\left[p\left\{(\lambda_+ - iu)^\alpha - \lambda_+^\alpha\right\} + q\left\{(\lambda_- + iu)^\alpha - \lambda_-^\alpha\right\}\right], \qquad (8)$$

with characteristic function

$$\phi(u,t) = \exp\{t\Psi(u)\}.$$

The characteristic representation is valuable for analytical derivations and derivative pricing; however, the Lévy-density representation is more informative for the density-based perspective of this review because it makes the mechanisms governing financial jumps explicit.

An important structural connection exists between the KoBoL and CGMY formulations. The CGMY Lévy density can be written as

$$\nu_{CGMY}(dx) = C\left[\frac{\exp(-Mx)}{x^{1+Y}}\mathbf{1}_{(0,\infty)}(x) + \frac{\exp(-G|x|)}{|x|^{1+Y}}\mathbf{1}_{(-\infty,0)}(x)\right]dx. \tag{9}$$

Comparison of (7) and (9) shows that the two representations share the same fundamental tempered power-law structure. Under the parameter correspondence

$$C = D, \qquad Y = \alpha, \qquad M = \lambda_+, \qquad G = \lambda_-,$$

the principal components of the two Lévy densities coincide, with differences arising from parameterization and normalization conventions. Thus, KoBoL and CGMY are best viewed as closely related formulations of the same broader exponentially tempered stable-type Lévy framework rather than as completely separate distributions [9, 13, 49].

This relationship is particularly useful when considering the evolution of Lévy-density-based financial models. The essential structural principle is the combination

Power-Law Small-Jump Behaviour + Exponential Large-Jump Tempering.

The first component determines the local activity of the jump process, while the second limits the contribution of extreme jumps. The KoBoL density therefore provides a direct representation of the two principal scales of financial jump behaviour without requiring the effects of small and large jumps to be described by the same mechanism.

The financial relevance of the KoBoL framework is further demonstrated by its extensive use in Lévy-based derivative pricing. Boyarchenko and Levendorskii [49] developed analytical methods for option-pricing problems under Lévy processes of exponential type, establishing a framework in which the properties of the Lévy measure play a central role. Subsequent studies considered American options and other early-exercise problems under Lévy processes [50, 51]. These contributions are relevant to the present density-based perspective because changes in the jump measure alter the probability of large movements and first-passage events, which in turn affect option values and exercise characteristics.

The influence of the Lévy density is particularly evident for barrier and first-touch derivatives. Boyarchenko, de Innocentis and Levendorskii [52] investigated barrier and first-touch digital options for Lévy-driven models with exponentially decaying jump densities, including KoBoL- and related specifications. Such results demonstrate that the small-jump structure of the Lévy density is not merely a theoretical feature: it can influence the behaviour of financial quantities close to important price boundaries. Consequently, the parameters governing the Lévy density may have implications beyond unconditional return distributions and can affect derivative prices and risk measures.

The connection between KoBoL-type Lévy densities and nonlocal pricing equations provides another important financial interpretation. Fractional-diffusion formulations have been used to represent financial models driven by jump processes, including

CGMY and KoBoL-type specifications [53]. Such formulations emphasize the nonlocal nature of the dynamics generated by the Lévy measure. More specifically, option-pricing equations under the KoBoL model have also been studied using fractional partial differential equation approaches [54]. These developments reinforce the interpretation of the Lévy density as the fundamental mechanism generating nonlocal financial dynamics.

The role of tempering is also relevant when financial dynamics are considered across different time scales. In Lévy-type models, the power-law component can generate pronounced non-Gaussian behaviour at small scales, while exponential tempering suppresses the probability of extremely large jumps. Studies of crossover behaviour between Lévy and Gaussian regimes therefore provide additional motivation for tempered specifications in financial modelling [55]. From this perspective, KoBoL offers a mechanism through which stable-like behaviour can coexist with a controlled contribution from extreme jumps.

For financial return modelling, the parameters of the KoBoL Lévy density can therefore be interpreted directly in terms of observable aspects of the data:

$$\begin{gathered} D \longrightarrow \text{Overall Jump Intensity}, \\ (p, q) \longrightarrow \text{Relative Intensity of Positive and Negative Jumps}, \\ \alpha \longrightarrow \text{Small-Jump Activity and Path Variation}, \\ (\lambda_+, \lambda_-) \longrightarrow \text{Positive and Negative Tail Attenuation}. \end{gathered}$$

This parameter-to-mechanism correspondence is one of the main reasons why the Lévy-density representation is particularly useful for empirical financial analysis. Instead of interpreting skewness, kurtosis and tail behaviour only through the final return distribution, the KoBoL framework allows these features to be connected to specific components of the underlying jump mechanism.

Nevertheless, the KoBoL specification considered here imposes a common activity exponent on the two sides of the Lévy density. In particular,$\alpha_+ = \alpha_- = \alpha$ implicitly assumes that positive and negative jumps share the same local power-law activity, even though their relative intensities and tail attenuation rates may differ. Thus, the model can accommodate $p \neq q$ and $\lambda_+ \neq \lambda_-$, but the local concentration of positive and negative small jumps remains governed by the same exponent. This is an important consideration when applying KoBoL-type models to financial data in which the fine-scale behaviour of upward and downward movements may differ.

Overall, the KoBoL Lévy density represents an important development in the density-based modelling of financial returns. Its principal contribution is the explicit combination of stable-like small-jump activity and exponentially controlled large-jump behaviour within a single infinitely divisible framework. The parameter α determines the local jump activity, D controls the overall scale, (p, q) describe directional jump intensity, and (λ_+, λ_-) determine the attenuation of positive and negative extreme jumps. This separation provides a transparent interpretation of the mechanisms underlying non-Gaussian financial returns and explains why KoBoL-type processes have been widely employed in option pricing, barrier problems and other financial applications. From the perspective of Lévy-density evolution, KoBoL is therefore important not simply as another return-distribution model, but as a framework that makes the

distinct roles of small-jump activity, directional asymmetry and tail tempering explicit in the modelling of financial data.

3.6 Truncated Lévy Flight (TLF)

The Truncated Lévy Flight (TLF) represents an important early stage in the development of non-Gaussian models for financial returns. It emerged from the empirical observation that asset-price fluctuations exhibit stable-like scaling and heavy tails over short horizons, while the unrestricted power-law tails of a stable Lévy model are too extreme to remain appropriate over longer horizons. Mantegna and Stanley [12, 56] introduced the TLF framework by suppressing arbitrarily large Lévy jumps. Their central result was the existence of a slow crossover from a Lévy-like regime at short aggregation scales to a Gaussian regime at sufficiently large scales. This crossover provided an important interpretation of why financial returns may display strongly non-Gaussian behaviour at short horizons while exhibiting more Gaussian characteristics after aggregation.

From the Lévy-density perspective, the TLF construction can be represented in a symmetric exponentially tempered form as

$$\nu(dx) = \frac{C\exp(-\lambda|x|)}{|x|^{1+\alpha}}\,dx, \qquad 0<\alpha<2, \quad \lambda>0, \tag{10}$$

where C controls the overall jump intensity, α determines the power-law behaviour near the origin, and λ controls the suppression of large jumps. The density therefore combines two mechanisms that are particularly relevant for financial data: a stable-type concentration of small jumps and an exponential attenuation of extreme jumps.

The local behaviour of (10) is governed by

$$\nu(x) \sim \begin{cases} Cx^{-(1+\alpha)}, & x\to 0^+, \\ C|x|^{-(1+\alpha)}, & x\to 0^-, \end{cases} \qquad \text{since} \quad e^{-\lambda|x|}\to 1 \quad \text{as } x\to 0.$$

Thus, truncation does not alter the fundamental power-law structure of the small jumps. Instead, its principal effect occurs in the tails:

$$\nu(x) \sim C|x|^{-(1+\alpha)}\exp(-\lambda|x|), \qquad |x|\to\infty.$$

The Lévy density consequently provides a direct separation between local jump activity and extreme-jump attenuation. This distinction is central to the subsequent evolution of tempered Lévy models.

The importance of this mechanism for financial data was empirically demonstrated by Mantegna and Stanley [56], who compared the TLF with ARCH and GARCH models using financial-market data. They found that the TLF reproduced important scaling properties and their breakdown across time horizons, although it did not adequately reproduce the volatility fluctuations observed empirically. This result is particularly informative from the Lévy-density viewpoint: the TLF captures the cross-sectional shape and scale dependence generated by heavy-tailed jump behaviour, but

a static jump density alone cannot reproduce the full temporal dependence associated with volatility clustering [56].

The empirical evidence was subsequently extended to different financial markets. Gillemot, Töyli, Kertész and Kaski [57] examined normal, stable, truncated Lévy, generalized stable, mixed diffusion-jump and compound-normal specifications using S&P 500 and Helsinki Stock Exchange data. Their comparison showed that truncated Lévy models provided a reasonable description of financial returns and, more importantly, highlighted the empirical distinction between stable-like behaviour over limited horizons and the eventual breakdown of exact stability. This provides direct empirical support for interpreting truncation as a mechanism controlling the range over which stable-like Lévy-density behaviour is observable.

A particularly relevant application was provided by Miranda and Riera [58], who analysed approximately fifteen years of daily data for the São Paulo Stock Exchange Index (IBOVESPA). They found that the return dynamics were well described by an exponentially truncated Lévy flight with a Lévy exponent approximately in the range $\alpha \simeq 1.6$–1.7 and a non-zero truncation parameter. Their analysis also showed that the model could account for short-term large fluctuations while explaining the gradual convergence toward Gaussian behaviour at longer horizons. The estimated crossover scale was approximately twenty trading days, providing an explicit connection between the truncation mechanism and an empirically relevant financial time scale [58].

This result is particularly important for the interpretation of the Lévy density. The parameter α determines the strength of the stable-like singularity near the origin, whereas λ determines the scale beyond which the probability of increasingly large jumps is suppressed. Hence, the two parameters have fundamentally different financial interpretations:

$$\alpha \longrightarrow \text{Small-Jump Activity and Scaling Behaviour},$$
$$\lambda \longrightarrow \text{Large-Jump Attenuation and Crossover Scale}.$$

The TLF therefore provides an early density-based mechanism for separating the behaviour of ordinary fluctuations from that of extreme financial movements.

The role of truncation is also relevant for risk measurement. Lehnert and Wolff [59] developed a scale-consistent framework based on truncated Lévy innovations for financial returns. Their analysis emphasized that exponential tail decay provides finite moments while retaining fat-tailed behaviour over short horizons. This property is particularly relevant for risk measures because the tail of the return density determines the frequency of extreme losses and therefore directly affects quantities such as Value-at-Risk. In a related contribution, Lehnert and Wolff [60] incorporated truncated Lévy innovations into a scale-consistent Value-at-Risk framework and reported improved back-testing performance relative to a conventional RiskMetrics specification. The results illustrate that the usefulness of a truncated Lévy density is not restricted to descriptive fitting; its tail structure can also influence practical financial risk assessment.

Another important empirical contribution comes from the analysis of different return horizons. Gillemot et al. [57] examined financial returns from the New York and Helsinki stock exchanges over horizons ranging from intraday observations to

approximately thirty days. Their results demonstrated that the statistical characteristics of returns change with the observation horizon and that periods of approximately Gaussian behaviour can coexist with periods in which non-Gaussian dependence and volatility effects become more pronounced. This evidence reinforces the interpretation of truncation as a scale-dependent mechanism rather than simply as a static modification of a stable tail.

An important theoretical development followed from Koponen[8], who provided an analytical treatment of truncated Lévy flights and their convergence toward Gaussian behaviour. The exponential modification of the stable-type jump structure can be expressed directly through a Lévy-density formulation, making it possible to preserve the stable-like local behaviour while ensuring substantially better-controlled tails. This development was conceptually important for the later emergence of exponentially tempered Lévy processes.

The TLF framework therefore reveals an important empirical tension in financial data. At short horizons, returns may exhibit scaling and heavy-tailed behaviour that is consistent with a stable-type power law, whereas at larger horizons the empirical distribution becomes increasingly compatible with Gaussian aggregation. Mantegna and Stanley [12] interpreted this behaviour through the slow Lévy-to-Gaussian crossover generated by truncation. The resulting picture is substantially different from an unrestricted stable model, for which the power-law tail persists at all scales and the variance may be infinite.

From the perspective of the Lévy density, this can be summarized by the two limiting regimes

$$\nu(x) \approx \begin{cases} C|x|^{-(1+\alpha)}, & |x| \ll \lambda^{-1}, \\ C|x|^{-(1+\alpha)} e^{-\lambda|x|}, & |x| \gg \lambda^{-1}. \end{cases}$$

The characteristic length scale λ^{-1} therefore provides a natural measure of the range over which the stable-like behaviour remains important before exponential suppression dominates. This interpretation gives the truncation parameter a direct financial meaning: it determines the scale at which extreme jumps become progressively less influential.

The TLF framework also exposed an important limitation of density-only approaches to financial returns. Although truncation successfully controls the extreme tail and reproduces important scaling properties, Mantegna and Stanley [56] found that the model does not adequately capture the empirically observed fluctuations in volatility. This distinction is important because the Lévy density describes the jump-size mechanism, whereas volatility clustering represents temporal dependence in the sequence of returns. Consequently, a model may reproduce the marginal return density reasonably well while still failing to reproduce the dynamic evolution of financial volatility.

A further limitation concerns the original TLF construction itself. The truncation mechanism was introduced primarily to modify the stable distribution and its scaling properties, rather than as a general four-parameter Lévy-process construction. Koponen [8] subsequently provided an analytical framework for truncated Lévy flights, while later tempered-stable models such as KoBoL and CGMY formulated the

exponential modification directly within infinitely divisible Lévy processes. Thus, TLF should be regarded as an important conceptual precursor to modern tempered Lévy models rather than being identified without qualification with the later Lévy-process formulations.

This distinction is particularly important for the evolution of Lévy-density models used in finance. TLF demonstrated empirically that the unrestricted stable tail could be incompatible with observed financial returns and that an explicit suppression of extreme jumps could reconcile heavy-tailed short-horizon behaviour with finite-moment long-horizon behaviour. The subsequent KoBoL and CGMY developments retained this central idea but embedded it in analytically tractable Lévy-process families with explicit control over small-jump activity and positive and negative tail attenuation.

The financial contribution of TLF can therefore be interpreted through three interconnected density-level mechanisms:

$$\alpha \longrightarrow \text{Stable-Like Local Jump Structure}, \quad \lambda \longrightarrow \text{Large-Jump Truncation},$$
$$\lambda^{-1} \longrightarrow \text{Characteristic Crossover Scale}.$$

These mechanisms explain why the model was able to capture several important empirical features of financial returns while simultaneously revealing its limitations in describing time-varying volatility and dependence.

The principal characteristics of the TLF Lévy-density framework may therefore be summarized as follows:

(i) it retains stable-type power-law behaviour for small jumps;
(ii) the parameter α controls the local jump activity and scaling behaviour;
(iii) exponential truncation suppresses extreme jumps and produces finite moments;
(iv) the parameter λ determines the characteristic scale at which the stable-like regime gives way to exponential attenuation;
(v) empirical studies show that the model can describe short-horizon heavy tails and scaling properties of financial returns;
(vi) the truncation mechanism provides a natural explanation for the observed crossover from Lévy-like to approximately Gaussian behaviour under aggregation;
(vii) the model has been applied to financial return modelling, risk measurement and option pricing;
(viii) its inability to reproduce volatility clustering demonstrates the distinction between the marginal Lévy-density structure and the temporal dependence of financial returns.

From the evolutionary perspective of tempered Lévy models, TLF therefore constitutes a fundamental conceptual step: it demonstrated that the empirically useful stable-like behaviour of financial returns could be retained while the unrealistic contribution of arbitrarily large jumps could be suppressed. Its main limitation was that the truncation idea preceded the more systematic construction of general infinitely divisible tempered Lévy processes. The subsequent development of exponentially tempered Lévy families transformed this empirical insight into a flexible Lévy-density framework

in which small-jump activity and large-jump attenuation could be controlled analytically. TLF thus provides an important empirical and conceptual foundation for the later evolution of tempered Lévy densities used in financial-data modelling.

3.7 The Tempered Stable Lévy Density

The Tempered Stable (TS) class represents an important stage in the evolution of Lévy-density-based models for financial data. Its principal contribution is to retain the power-law structure of stable Lévy measures for small jumps while attenuating the occurrence of very large jumps through exponential tempering. This construction addresses an important limitation of the unrestricted stable framework: although stable models provide a natural representation of heavy-tailed jump behaviour, their tails may be excessively persistent and, depending on the stability index, may not possess the moments required for financial risk analysis. Tempering therefore preserves stable-like local jump behaviour while providing greater control over extreme jumps [10, 11].

From the perspective of the present review, the Lévy density is the central object because it directly describes the intensity and structure of jumps. A general two-sided tempered stable Lévy density can be written as

$$\nu(dx) = \left(\frac{\alpha_+ e^{-\lambda_+ x}}{x^{1+\beta_+}} \mathbf{1}_{(0,\infty)}(x) + \frac{\alpha_- e^{-\lambda_- |x|}}{|x|^{1+\beta_-}} \mathbf{1}_{(-\infty,0)}(x) \right) dx, \tag{11}$$

where

$$\alpha_+, \alpha_- > 0, \qquad \lambda_+, \lambda_- > 0, \qquad 0 \leq \beta_+, \beta_- < 1.$$

Here, α_+ and α_- are scale parameters that determine the overall magnitude of positive and negative jump activity, respectively. The parameters β_+ and β_- govern the local power-law behaviour of the positive and negative components of the Lévy density and therefore characterize the degree of small-jump activity. The parameters λ_+ and λ_- are exponential tempering parameters that determine the rate at which large positive and negative jumps are attenuated. Thus, the Lévy density separates the scale of jump activity, the local small-jump structure, and the attenuation of extreme jumps into distinct parameter components.

The principal structural feature of (11) is the separation between small- and large-jump behaviour. As $x \to 0^+$ and $x \to 0^-$, the exponential terms approach unity, and hence

$$\nu(x) \sim \begin{cases} \alpha_+ x^{-(1+\beta_+)}, & x \to 0^+, \\ \alpha_- |x|^{-(1+\beta_-)}, & x \to 0^-. \end{cases}$$

Consequently, the parameters β_+ and β_- determine the local concentration of the Lévy measure around the origin, whereas exponential tempering has a negligible effect on the asymptotic behaviour of sufficiently small jumps. For large positive and negative

jumps, respectively,

$$\nu(x) \sim \begin{cases} \alpha_+ x^{-(1+\beta_+)} e^{-\lambda_+ x}, & x \to +\infty, \\ \alpha_- |x|^{-(1+\beta_-)} e^{-\lambda_- |x|}, & x \to -\infty. \end{cases}$$

The power-law component therefore governs the local jump structure, while the exponential component controls the attenuation of extreme jumps.

For $0 < \beta_\pm < 1$, the Lévy density is non-integrable in every neighbourhood of the origin, since

$$\int_{|x|<\varepsilon} \nu(dx) = \infty, \qquad \varepsilon > 0.$$

Hence, the corresponding process has infinite jump activity, meaning that infinitely many small jumps occur over every finite time interval. At the same time, for the parameter range $0 < \beta_\pm < 1$,

$$\int_{\mathbb{R}} \min(1, |x|)\nu(dx) < \infty,$$

so the process has finite variation. Thus, in the present parameterization, the activity and variation properties are determined by the power-law indices β_+ and β_- rather than by the tempering parameters λ_+ and λ_-.

The two-sided specification also provides a direct representation of jump asymmetry. Differences between α_+ and α_- correspond to differences in the overall scale of positive and negative jump activity. Differences between β_+ and β_- allow the local small-jump behaviour to differ across the two directions, while differences between λ_+ and λ_- imply different rates of attenuation for large positive and negative jumps. Consequently, asymmetry can arise through distinct mechanisms: different jump intensities, different local activity structures, or different tail-decay rates. This decomposition is particularly relevant for financial returns, where upward and downward movements may exhibit different empirical characteristics.

The TS Lévy density can therefore be interpreted through the decomposition

$$\underbrace{|x|^{-(1+\beta_\pm)}}_{\text{local jump activity}} \times \underbrace{e^{-\lambda_\pm |x|}}_{\text{large-jump attenuation}},$$

with $\alpha_\pm$ controlling the scale of the corresponding jump component. The first factor determines how strongly the Lévy measure accumulates around the origin, whereas the second determines how rapidly the density decreases for large jump magnitudes. This parameter-to-density correspondence provides a direct connection between the mathematical structure of the Lévy measure and financial characteristics such as small-jump activity, directional asymmetry, tail thickness, and extreme-return frequency.

The characteristic function of a tempered stable process follows from the Lévy–Khintchine representation and can be expressed as

$$\phi(u, t) = \exp\{t\Psi(u)\},$$

where $\Psi(u)$ is the characteristic exponent determined by the Lévy density in (11), together with the corresponding drift and compensation convention. Although the characteristic exponent is essential for analytical derivations, option pricing, and simulation, the Lévy density provides the more transparent representation for the evolutionary perspective adopted in this review because it permits the local and tail components of jump behaviour to be examined separately.

The TS framework also establishes important relationships with earlier tempered Lévy models. The Variance Gamma and Bilateral Gamma models possess a fixed $|x|^{-1}$-type local structure, whereas the TS formulation allows a more general power-law behaviour through the parameters β_+ and β_-. The CGMY model can be represented as a restricted two-sided tempered stable specification by imposing

$$\beta_+ = \beta_- = Y, \qquad \alpha_+ = \alpha_- = C, \qquad \lambda_+ = M, \qquad \lambda_- = G,$$

where C is the common scale parameter and Y is the common activity index in the conventional CGMY notation. This relationship demonstrates that the principal tempered Lévy models can be viewed within a common density-based framework, with differences arising from restrictions imposed on the intensity, activity, symmetry, and tempering parameters [9, 11].

The financial relevance of the tempered stable construction has been demonstrated in empirical and theoretical studies. Bianchi, Rachev, Kim and Fabozzi [61] investigated tempered stable distributions in financial applications and showed their ability to represent heavy tails, skewness, and excess kurtosis in financial returns. Fallahgoul and Loeper [62] further examined tempered stable models in financial tail-risk settings, highlighting their usefulness for representing extreme-return behaviour. These studies illustrate the transition from the theoretical modification of stable Lévy densities toward their direct application in financial return and risk modelling.

The same jump structure has also been incorporated into derivative-pricing frameworks. Poirot and Tankov [63] developed efficient Monte Carlo methods for option pricing under tempered stable processes, demonstrating that the tempered Lévy mechanism used to describe financial returns can also be employed in derivative valuation. Thus, the parameters governing the Lévy density have implications beyond the unconditional return distribution, since changes in jump intensity and tail attenuation can influence quantities relevant to option prices and implied-volatility behaviour.

Subsequent research extended the tempered stable framework by allowing its parameters to vary over time. Azzone and Baviera [64] developed additive normal tempered stable processes with time-dependent parameters for equity derivatives, demonstrating the relevance of parameter dynamics for representing changing implied-volatility surfaces. Similarly, Kim, Roh and Douady [65] considered tempered stable processes with time-varying exponential tails and applied the framework to S&P 500 returns and option prices. These developments are particularly important from the Lévy-density perspective because they demonstrate that the parameters governing jump behaviour need not necessarily be treated as fixed quantities. In particular, time variation in λ_+ and λ_- provides a mechanism for representing changing attenuation of extreme positive and negative jumps.

Tempered stable jump mechanisms have also been incorporated into dynamic and mean-reverting processes. Qu, Dassios and Zhao [66] developed exact simulation methods for tempered stable Ornstein–Uhlenbeck processes, while Sabino [67] and Sabino and Cufaro Petroni [68] investigated exact and fast simulation methods for normal tempered stable and tempered stable Ornstein–Uhlenbeck processes. These developments show that tempered jump structures can be embedded within dynamic models capable of representing temporal dependence and mean reversion, extending the role of the Lévy density beyond static return-distribution modelling.

An important theoretical property of the TS framework is its ability to retain stable-like behaviour over appropriate scales while modifying the behaviour of large jumps. Rosiński [10] established a general framework for tempered stable processes in which the stable-type structure is preserved locally while the tails are controlled through tempering. Further, Vasudeva and Nagaraja [69] investigated the asymptotic behaviour of extremes of observations from a tempered stable distribution, providing additional insight into its extreme-value and tail properties. Together, these studies provide a theoretical basis for interpreting the TS density as a mechanism that combines rich small-jump behaviour with more rapidly decaying extreme-jump probabilities. For financial data, this structure is useful because the distributional characteristics of returns may depend on the sampling scale, while extreme movements require stronger tail control than an unrestricted stable model provides.

The preceding developments illustrate a broader evolution in the interpretation of the tempered stable Lévy density. Initially, tempering provided a mechanism for reducing the excessive persistence of stable tails. Subsequent studies demonstrated its usefulness for financial return and tail-risk modelling, while later developments incorporated the same jump structure into option pricing, time-varying models, and dynamic mean-reverting processes. The evolution can therefore be summarized as

$$\begin{aligned}\text{Stable Lévy Density} &\longrightarrow \text{Tempered Stable Lévy Density}\\ &\longrightarrow \text{Dynamic Tempered Stable Lévy Density.}\end{aligned}$$

This progression reflects an increasing emphasis on flexibility in representing changing financial jump behaviour while retaining the underlying Lévy-process structure.

From the perspective of the present review, this development also reveals an important research direction. Most applications estimate the parameters of a Lévy density over a historical sample or within a specified estimation window. However, the density itself contains interpretable information about the frequency, magnitude, direction, and attenuation of jumps. Allowing its parameters to evolve therefore provides a natural basis for constructing time-dependent measures of jump behaviour. For example, threshold-based positive and negative jump intensities can be defined as

$$\Lambda_t^+(u) = \int_u^{\infty} \nu_t(dx), \qquad \Lambda_t^-(u) = \int_{-\infty}^{-u} \nu_t(dx), \qquad u > 0,$$

which quantify the contribution of jumps exceeding a specified magnitude threshold in the positive and negative directions. Such quantities illustrate how the Lévy density

can move from being solely a distributional specification toward an interpretable source of information about the evolving jump state of a financial market.

Accordingly, the TS framework represents a pivotal stage in the evolution of Lévy-density-based financial models. Its principal structural contribution is the separation of stable-like small-jump activity from exponential attenuation of large jumps, while its subsequent extensions demonstrate how this structure can be incorporated into financial return modelling, risk measurement, derivative pricing, time-varying models, and dynamic simulation. More broadly, the TS framework provides the mathematical foundation for viewing the Lévy density not only as a mechanism for constructing a return distribution but also as a potentially evolving representation of financial jump behaviour.

3.8 The Generalized Tempered Stable Lévy Density

The Generalized Tempered Stable (GTS) class represents an important development in the evolution of Lévy-density-based models for financial returns. Its principal contribution is the ability to control the positive and negative components of the Lévy density separately. In particular, the GTS framework allows the scale of jump activity, local small-jump behaviour, and attenuation of large jumps to differ between the two directions. This flexibility is particularly relevant for financial returns, where positive and negative movements may exhibit different frequencies, magnitudes, and tail characteristics [11, 70].

From the density-centred perspective adopted in this review, the GTS model can be expressed through the two-sided Lévy density

$$\nu(dx) = \left(\frac{\alpha_+ e^{-\lambda_+ x}}{x^{1+\beta_+}} \mathbf{1}_{(0,\infty)}(x) + \frac{\alpha_- e^{-\lambda_- |x|}}{|x|^{1+\beta_-}} \mathbf{1}_{(-\infty,0)}(x) \right) dx, \tag{12}$$

where

$$\alpha_+, \alpha_- > 0, \qquad \lambda_+, \lambda_- > 0, \qquad 0 \leq \beta_+, \beta_- < 1.$$

Here, α_+ and α_- are scale parameters that determine the overall magnitude of positive and negative jump activity, respectively. The parameters β_+ and β_- govern the local power-law behaviour of the positive and negative components of the Lévy density and therefore characterize their small-jump activity. The parameters λ_+ and λ_- are exponential tempering parameters that determine the rate of attenuation of large positive and negative jumps, respectively. Thus, the GTS Lévy density separates three distinct aspects of the jump mechanism: the scale of jump activity, the local small-jump structure, and the attenuation of extreme jumps.

The flexibility of the GTS specification becomes apparent by considering the positive and negative components separately:

$$\nu(x) = \begin{cases} \alpha_+ x^{-(1+\beta_+)} e^{-\lambda_+ x}, & x > 0, \\ \alpha_- |x|^{-(1+\beta_-)} e^{-\lambda_- |x|}, & x < 0. \end{cases}$$

Unlike more restricted tempered Lévy specifications, the GTS model does not require the two sides of the Lévy density to have identical parameter values. In particular, α_+ and α_- may differ, β_+ and β_- may differ, and λ_+ and λ_- may differ. These differences provide separate mechanisms through which directional asymmetry can be represented in financial returns.

The behaviour of the GTS Lévy density near the origin is determined primarily by the activity parameters. As $x \to 0^+$ and $x \to 0^-$,

$$\nu(x) \sim \begin{cases} \alpha_+ x^{-(1+\beta_+)}, & x \to 0^+, \\ \alpha_- |x|^{-(1+\beta_-)}, & x \to 0^-. \end{cases}$$

Since the exponential tempering terms approach unity near the origin, they do not determine the local activity of the process. Instead, β_+ and β_- determine how strongly the corresponding sides of the Lévy measure accumulate around zero. For $0 < \beta_+ < 1$ and $0 < \beta_- < 1$, the Lévy measure has infinite mass in every neighbourhood of the origin, implying infinite jump activity. Moreover,

$$\int_{|x|<1} min(1, |x|)\nu(dx) < \infty,$$

so the corresponding jump components have finite variation. Thus, within the parameter range considered here, the local activity and variation properties are governed by β_+ and β_- rather than by the tempering parameters.

The large-jump behaviour is controlled by the exponential tempering parameters. As $x \to +\infty$ and $x \to -\infty$,

$$\nu(x) \sim \begin{cases} \alpha_+ x^{-(1+\beta_+)} e^{-\lambda_+ x}, & x \to +\infty, \\ \alpha_- |x|^{-(1+\beta_-)} e^{-\lambda_- |x|}, & x \to -\infty. \end{cases}$$

Hence, a smaller value of λ_+ implies slower attenuation of large positive jumps, whereas a smaller value of λ_- implies slower attenuation of large negative jumps. The parameters λ_+ and λ_- therefore provide a direct mechanism for modelling asymmetric tail behaviour.

The GTS density can consequently be interpreted through the decomposition

$$\underbrace{|x|^{-(1+\beta_\pm)}}_{\text{local jump activity}} \quad \times \quad \underbrace{e^{-\lambda_\pm |x|}}_{\text{large-jump attenuation}},$$

with $\alpha * \pm$ controlling the scale of the corresponding positive and negative jump mechanisms. The first component describes the concentration of small jumps around the origin, while the second determines how rapidly extreme jumps are attenuated. This decomposition provides a direct correspondence between the parameters of the Lévy density and observable characteristics of financial returns, including small-jump activity, directional asymmetry, tail thickness, and extreme-return frequency.

An important advantage of the GTS formulation is therefore its ability to distinguish several sources of asymmetry. Differences between α_+ and α_- represent differences in the overall scale of positive and negative jump activity. Differences between β_+ and β_- represent differences in local small-jump activity, while differences between λ_+ and λ_- represent differences in the attenuation of extreme jumps. Thus, asymmetry in financial returns can be decomposed into differences in jump intensity, local jump structure, and tail attenuation rather than being represented only through an aggregate skewness parameter.

The GTS process may also be represented through its positive and negative jump components as

$$X = X_+ - X_-,$$

where the positive and negative components are associated with tempered stable jump mechanisms having parameter sets $(\alpha_+, \beta_+, \lambda_+)$ and $(\alpha_-, \beta_-, \lambda_-)$, respectively. A location or drift component can subsequently be introduced without changing the underlying Lévy-density specification. This representation separates the directional jump mechanisms and is therefore useful for analysing the contribution of positive and negative jumps to financial returns.

The characteristic exponent of the GTS process follows from the Lévy–Khintchine representation. Under the corresponding parameterization and compensation convention, it can be expressed in the form

$$\Psi(\xi) = i\mu\xi + \alpha_+\Gamma(-\beta_+)\left[(\lambda_+ - i\xi)^{\beta_+} - \lambda_+^{\beta_+}\right] + \alpha_-\Gamma(-\beta_-)\left[(\lambda_- + i\xi)^{\beta_-} - \lambda_-^{\beta_-}\right], \tag{13}$$

where μ denotes the location or drift component. The corresponding characteristic function is

$$\phi(\xi) = \exp\left(\Psi(\xi)\right).$$

The characteristic representation is useful for theoretical analysis, statistical inference, simulation, and financial pricing. Nevertheless, the Lévy density in (12) is more informative for the density-centred perspective of this review because it exposes the distinct mechanisms governing small and large positive and negative jumps.

The GTS formulation can be understood as a natural extension of earlier tempered Lévy models. Variance Gamma and Bilateral Gamma specifications possess a fixed $|x|^{-1}$-type local structure, while the CGMY and related tempered stable classes introduce a common power-law activity parameter. The GTS framework generalizes this construction by permitting the local activity parameters to differ across the two sides of the origin. In addition, the positive and negative sides may have different scale and tempering parameters.

Thus, the principal advance of GTS is not simply the introduction of additional parameters, but the assignment of each parameter to a distinct and interpretable component of the jump mechanism.

The hierarchical structure of the GTS density also permits simpler models to be represented as restricted configurations. Imposing $\alpha_+ = \alpha_-, \quad \beta_+ = \beta_-, \quad \lambda_+ = \lambda_-$ produces a fully symmetric two-sided specification. Alternatively, equality of only one pair removes asymmetry in the corresponding component while allowing the remaining

parameters to differ. This nested structure is useful for empirical model comparison because the contribution of intensity, local activity, and tail asymmetry can be examined separately rather than imposed or interpreted solely through aggregate distributional moments.

The financial relevance of the generalized tempered stable framework has been demonstrated through theoretical, empirical, and computational studies. Küchler and Tappe [11] established a general theoretical framework for tempered stable distributions and processes, providing the mathematical foundation for flexible tempered jump models. Subsequent work by Nzokem and collaborators examined statistical properties, parameter estimation, simulation, and financial applications of generalized tempered stable distributions, including empirical fitting and European option pricing [17, 70]. In particular, Nzokem and Maposa [17] fitted the seven-parameter GTS distribution to Bitcoin, Ethereum, S&P 500, and SPY return data and compared its goodness of fit with alternative tempered Lévy models. These studies demonstrate the progression of the GTS framework from a theoretical construction toward empirical financial modelling and statistical inference.

A further development has extended the analysis from the ordinary GTS Lévy density to the Lévy density of its background-driving Lévy process. Nzokem and Maposa [18] derived the background-driving Lévy density associated with the self-decomposable representation of the GTS distribution. In the notation adopted in this review, the resulting density is

$$N(dx) = \left[\alpha_+ \frac{\beta_+ + \lambda_+ x}{x^{1+\beta_+}} e^{-\lambda_+ x} \mathbf{1}_{(0,\infty)}(x) + \alpha_- \frac{\beta_- + \lambda_- |x|}{|x|^{1+\beta_-}} e^{-\lambda_- |x|} \mathbf{1}_{(-\infty,0)}(x)\right] dx. \tag{14}$$

The density in (14) should be distinguished from the ordinary GTS Lévy density in (12). The latter describes the jump measure of the GTS process itself, whereas $N(dx)$ is the Lévy density associated with the background-driving Lévy process underlying the self-decomposable representation of the GTS distribution. The additional factors $\beta_+ + \lambda_+ x$ and $\beta_- + \lambda_- |x|$ arise from the relationship between the GTS Lévy measure and its background-driving process. Consequently, $N(dx)$ provides a related but distinct description of the underlying jump mechanism.

The background-driving density retains the principal small-jump structure of the GTS density. As $x \to 0^+$ and $x \to 0^-$,

$$N(x) \sim \begin{cases} \alpha_+ \beta_+ x^{-(1+\beta_+)}, & x \to 0^+, \\ \alpha_- \beta_- |x|^{-(1+\beta_-)}, & x \to 0^-. \end{cases}$$

Thus, the power-law order near the origin remains governed by β_+ and β_-, while the multiplicative activity terms modify the local scale. For large positive and negative jumps,

$$N(x) \sim \begin{cases} \alpha_+ \lambda_+ x^{-\beta_+} e^{-\lambda_+ x}, & x \to +\infty, \\ \alpha_- \lambda_- |x|^{-\beta_-} e^{-\lambda_- |x|}, & x \to -\infty. \end{cases}$$

Hence, the background-driving density preserves exponential attenuation in the extreme-jump region while exhibiting a structure distinct from that of the ordinary GTS Lévy density.

This development is particularly relevant to the present review because it demonstrates that the density-centred analysis can be extended beyond the Lévy density of the return process itself. Nzokem and Maposa [18] connected the GTS background-driving Lévy density with Ornstein–Uhlenbeck-type modelling and investigated its financial relevance using daily SPY ETF and Ethereum data over the period 2010–2024. The study demonstrates how a tempered jump mechanism can be incorporated into a dynamic mean-reverting process, thereby linking the Lévy density to both the distributional characteristics and temporal dynamics of financial data.

This progression is important because it demonstrates that Lévy-density analysis need not remain confined to static distributional fitting. Instead, related Lévy densities can be used to characterize the jump mechanism underlying dynamic stochastic processes.

From the perspective of financial-data modelling, the GTS framework therefore represents a transition from a common tempered jump structure toward a more flexible decomposition of positive and negative jump behaviour. Its parameters provide separate information about jump scale, local activity, and extreme-jump attenuation, while its background-driving extension demonstrates how the density structure can be connected with dynamic mean-reverting models. This progression can be viewed more broadly as

$$\begin{aligned}&\text{Tempered Lévy Density} \longrightarrow \text{Generalized and Asymmetric Lévy Density}\\ &\longrightarrow \text{Background-Driving Lévy Density} \longrightarrow \text{Dynamic Financial Jump Modelling.}\end{aligned}$$

The resulting framework strengthens the role of the Lévy density as an interpretable representation of financial jump behaviour, providing a natural foundation for subsequent comparative analysis of model flexibility, estimation, computational requirements, and financial applications.

4 Comparative Analysis of Tempered Lévy Models

The comparison considers four closely related dimensions: the flexibility of the Lévy-density structure, the interpretation and identifiability of the parameters, the evolution of parameter-estimation methods, and the computational requirements associated with estimation. This perspective reveals that the development of tempered Lévy models has involved not only increasing distributional flexibility, but also a progressive shift in the methods required to estimate increasingly complex jump structures.

4.1 Structural Evolution of the Lévy Density

A useful way to understand the evolution of tempered Lévy models is to examine which components of the jump mechanism can be controlled independently. Early models such as the Variance Gamma (VG) and Bilateral Gamma (BG) models provide flexible descriptions of asymmetric and leptokurtic return distributions, but their Lévy

densities have relatively restricted small-jump structures. The VG model, for example, is constructed through a Brownian motion evaluated at a Gamma random time and provides explicit control over skewness and kurtosis through its parameters [26]. Seneta [29] subsequently demonstrated the practical usefulness of moment-based fitting of the VG model, particularly when dependence in financial returns is considered.

The Bilateral Gamma model extends the Gamma-process construction by allowing separate positive and negative jump components. Küchler and Tappe [33] showed that the bilateral Gamma process can represent financial fluctuations through two independent Gamma processes, providing separate control over positive and negative jump mechanisms. However, the corresponding Lévy density retains a singularity $1/|x|$ near the origin, which limits its ability to distinguish different degrees of small-jump activity.

The introduction of stable-type power-law behaviour into the Lévy density provides substantially greater control over the concentration of small jumps. The Tempered Stable (TS) class combines a power-law singularity near the origin with exponential attenuation for large jumps. Consequently, the activity parameters determine the local jump concentration, while the tempering parameters control the attenuation of extreme jumps. Rosiński [10] provided a general theoretical framework for tempered stable processes and established important properties concerning their stable-like short-time behaviour and tempered long-time behaviour. Küchler and Tappe [11] further developed the theory of tempered stable distributions and processes, including their parameterization, density properties, and parameter estimation.

The Generalized Tempered Stable (GTS) model represents a further increase in flexibility because the positive and negative sides can possess different activity, intensity, and tempering parameters. Thus, the model can simultaneously accommodate asymmetry in jump intensity, asymmetry in small-jump activity, and asymmetry in tail attenuation. This distinction is important because these three forms of asymmetry represent different aspects of the underlying jump mechanism and should not be treated as a single source of skewness.

From this perspective, the structural evolution can be summarized as a progression from relatively constrained jump specifications toward Lévy densities in which different features of the jump mechanism can be controlled separately. The important comparative issue is therefore not simply whether one model has more parameters than another, but whether each additional parameter corresponds to a distinct and interpretable feature of the Lévy density.

4.2 Evolution of Parameter-Estimation Methods

The increasing structural flexibility of tempered Lévy models has been accompanied by a corresponding evolution in parameter-estimation methodology. Models with analytically tractable densities can be estimated directly through likelihood-based procedures, whereas models whose densities are unavailable in closed form require indirect or numerically assisted estimation procedures. This relationship between model complexity and estimation methodology is one of the most important outcomes of the comparative analysis.

For the VG model, maximum likelihood estimation is possible because the return density has an explicit representation. Madan, Carr, and Chang [26] estimated the statistical and risk-neutral densities using the VG framework, demonstrating the feasibility of likelihood-based inference. Seneta [29] subsequently considered a method-of-moments approach for fitting the VG model and emphasized its usefulness when dependence in log returns makes conventional independent-and-identically-distributed estimation arguments less appropriate.

For the Bilateral Gamma model, estimation benefits from the relatively tractable Gamma-process construction. Küchler and Tappe [33] developed the bilateral Gamma distribution and associated process, while subsequent work considered its density and parameter properties. Consequently, conventional likelihood and moment-based procedures remain comparatively accessible for this model.

The situation becomes more complicated for the broader tempered stable class. Küchler and Tappe [11] considered parameter estimation for tempered stable distributions, demonstrating that estimation can be undertaken despite the increasing complexity of the density representation. More recently, Figueroa-López, Gong, and Han [71] proposed a semiparametric estimation procedure for tempered stable Lévy models based on an approximate method of moments combined with truncated realized quadratic variations. Their approach is particularly important because it illustrates a shift from purely parametric distribution fitting toward estimation of structural characteristics of the underlying Lévy process, such as volatility and the Blumenthal–Getoor activity index.

The estimation problem becomes particularly challenging for the GTS class because its probability density does not generally possess a convenient closed-form expression. Nzokem et al. [70] addressed this difficulty using the Fractional Fourier Transform (FRFT) to numerically evaluate the density and its derivatives within a maximum-likelihood procedure. Their contribution illustrates an important methodological transition: maximum likelihood remains the statistical estimation principle, but numerical transform methods become necessary to make likelihood evaluation computationally feasible.

Nzokem and Maposa [17] subsequently developed a fitting procedure for the seven-parameter GTS model using the characteristic function and FRFT. Their results demonstrate that numerical transform methods can overcome the absence of a closed-form GTS density and allow statistical inference for a highly flexible asymmetric tempered Lévy model.

The estimation methods identified in the literature can therefore be organized as follows.

Table 1: Evolution of parameter-estimation methods for major tempered Lévy models.

Model	**Estimation approach**	**Representative reference**	**Comparative interpretation**
Variance Gamma (VG)	Maximum Likelihood Estimation	Madan, Carr and Chang (1998) [26]	Direct likelihood-based estimation is feasible because the VG return density has an explicit analytical representation.
Variance Gamma (VG)	Method of Moments	Seneta (2004) [29]	Provides a comparatively simple estimation strategy and can be useful when dependence in financial returns needs to be considered.
Bilateral Gamma (BG)	Maximum Likelihood / parametric fitting	U Küchler, S Tappe (2008) [33]	The Gamma-process construction provides a relatively tractable framework for statistical fitting and interpretation.
Bilateral Gamma (BG)	Moment-based fitting	Küchler and Tappe (2013) [11]	Useful for parameter characterization and preliminary estimation within the broader tempered stable framework.
KoBoL	Likelihood-based and characteristic-function methods	Boyarchenko and Levendorskiĭ (2002) [49]	The increasing complexity of the Lévy density makes numerical evaluation and transform-based methods important for inference.
CGMY	Maximum Likelihood Estimation	Carr et al. (2002) [9]	Provides parametric statistical estimation for a flexible tempered stable specification with additional control over jump activity.
CGMY	Characteristic-function / transform methods	Cont and Tankov (2004) [6]	Transform-based procedures are useful when direct likelihood evaluation becomes numerically demanding.
Tempered Stable (TS)	Parametric estimation	Küchler and Tappe (2013) [11]	Provides estimation within the general tempered stable framework while accounting for the absence of simple density expressions in some parameterizations.
Tempered Stable (TS)	Semiparametric method of moments	Figueroa-López et al. (2022) [71]	Uses high-frequency information and truncated realized quadratic variation to estimate structural characteristics such as volatility and jump activity.
GTS	FRFT-assisted Maximum Likelihood	Nzokem and et al. (2025) [70]	FRFT is used as a numerical device for evaluating the density and its derivatives required in likelihood estimation.
GTS	Characteristic-function and FRFT-based fitting	Nzokem and Maposa (2024) [17]	Addresses the absence of a closed-form GTS density through characteristic-function and FRFT-based numerical evaluation.
GTS	Recurrent fractal interpolation with GTS noise	Kumar et al. (2026) [72]	Represents a newer computational direction in which GTS noise is incorporated into a recurrent interpolation framework rather than relying solely on conventional distributional estimation.

Table 1 reveals an important pattern that is not apparent from a model-by-model theoretical description. The evolution of tempered Lévy models has been accompanied by an evolution from direct statistical estimation toward increasingly numerical and hybrid estimation procedures. For VG and BG models, the availability of relatively tractable density representations permits conventional likelihood or moment-based estimation. For TS, CGMY, KoBoL, and especially GTS models, the increasing complexity of the probability density creates a greater dependence on characteristic functions, numerical integration, FFT-type procedures, or FRFT-based density evaluation.

This distinction is important because a numerical transform should not itself be interpreted as a statistical estimator. For example, FRFT is primarily a computational mechanism for evaluating quantities required by maximum likelihood. Nzokem et al. [70] explicitly use FRFT to evaluate the GTS density and its derivatives within maximum likelihood, while Nzokem and Maposa [17] use characteristic-function and FRFT machinery to overcome the absence of a closed-form density. Thus, the appropriate comparison is between the underlying statistical estimation principles and the numerical machinery required to implement them.

4.3 Statistical Estimation versus Computational Calibration

Another important distinction emerging from the literature is between parameter estimation from observations and numerical calibration using transform-based techniques. Maximum likelihood, method of moments, and semiparametric estimation are statistical inference procedures because they define how parameters are inferred from observed data. Fourier, FFT, and FRFT procedures, in contrast, are generally computational tools used to evaluate densities, characteristic functions, likelihood components, or pricing quantities.

This distinction becomes especially relevant for models such as KoBoL and CGMY, where Fourier methods are frequently associated with option-pricing calculations. Consequently, the presence of an FFT or Fourier method in the literature does not necessarily imply that the underlying model parameters were statistically estimated using an FFT-based estimator. A comparative review should therefore distinguish between *estimation methodology* and *numerical implementation methodology*. This distinction prevents pricing-oriented transform techniques from being incorrectly classified as independent statistical estimators.

The GTS literature provides a clear example of this distinction. Nzokem et.al., [70] retain maximum likelihood as the inferential principle while using FRFT to numerically obtain the density and its derivatives. Similarly, Nzokem and Maposa [17] identify the lack of a closed-form GTS density as the central obstacle to direct MLE and develop a characteristic-function/FRFT procedure to overcome that obstacle.

This leads to a useful comparative classification:

$$\text{Statistical Principle} \longrightarrow \text{Numerical Evaluation Method} \longrightarrow \text{Parameter Inference}.$$

For example, the statistical principle may be maximum likelihood, while the numerical evaluation method may be FRFT, and the final output is the estimated parameter

vector. This three-level distinction is more informative than simply classifying a model as using "FFT estimation" or "FRFT estimation."

4.4 Flexibility, Identifiability and Interpretability

Increasing the number of parameters does not automatically imply a better statistical model. A model may become more flexible while simultaneously becoming more difficult to identify and estimate reliably. This issue is particularly relevant for generalized tempered stable models because separate parameters may control different aspects of the positive and negative jump mechanisms.

The evolution from VG and BG toward TS and GTS illustrates this trade-off. In the simpler models, a smaller number of parameters provides relatively direct interpretation of skewness, scale, and tail behaviour. In the GTS framework, positive and negative jump intensities, activity indices, and tempering parameters can be separated. Such flexibility can improve the representation of asymmetric financial returns, but it also increases the dimensionality of the estimation problem.

4.5 Comparison of the Three Sources of Asymmetry

A further insight from the comparative analysis is that asymmetry should be decomposed into separate mechanisms. In a two-sided tempered Lévy density, differences between positive and negative jumps can arise from at least three sources:

$$\begin{aligned} \alpha_+ \neq \alpha_- \quad &\Longrightarrow \quad \text{asymmetry in jump intensity,} \\ \beta_+ \neq \beta_- \quad &\Longrightarrow \quad \text{asymmetry in small-jump activity,} \\ \lambda_+ \neq \lambda_- \quad &\Longrightarrow \quad \text{asymmetry in tail attenuation.} \end{aligned}$$

This decomposition provides a more informative comparison than describing a model simply as "symmetric" or "asymmetric." Two models may both reproduce negative skewness in financial returns while achieving it through fundamentally different changes in the underlying Lévy density. Consequently, model comparison should examine not only the final fitted distribution but also which structural mechanism is responsible for the observed asymmetry.

4.6 Research Gaps

The comparative evidence reveals several methodological gaps that remain important for the development of tempered Lévy models.

First, direct Lévy-density-based inference remains relatively underdeveloped. Much of the estimation literature focuses on the probability density, characteristic function, option prices, or transform-based numerical representations. This is understandable because the probability density of many tempered Lévy models is not available in a simple closed form. Nevertheless, such approaches can obscure the direct interpretation of the underlying Lévy-density parameters. The semiparametric work of Figueroa-López et al. [71] moves toward estimating structural jump characteristics, but a general framework for direct inference on flexible asymmetric Lévy-density parameters remains an important methodological opportunity.

Second, estimation complexity increases rapidly with structural flexibility. The progression from VG and BG to TS and GTS demonstrates that additional control over jump activity and asymmetry introduces additional computational and statistical difficulties. In the GTS case, the absence of a closed-form density motivates characteristic-function and FRFT-based procedures [17, 70]. A remaining challenge is therefore to develop estimation procedures that retain the structural flexibility of generalized models while reducing dependence on computationally intensive numerical transformations.

Third, parameter identifiability requires greater attention in highly flexible models. When intensity, activity, and tempering parameters are estimated simultaneously, different combinations of parameters may produce similar empirical density shapes over a finite sample. Consequently, goodness-of-fit alone may not establish that all estimated parameters are statistically well identified. Future studies should therefore report parameter uncertainty, sensitivity to initialization, finite-sample performance, and stability across different samples or market periods.

Fourth, the temporal stability of Lévy-density parameters remains insufficiently investigated. Most traditional estimation procedures treat the parameters as fixed over the estimation window. However, financial markets undergo changing volatility, liquidity, and jump conditions. A flexible Lévy-density model raises the possibility that the structural parameters themselves may evolve over time. This motivates research on time-varying estimation of jump intensity, activity, and tempering parameters rather than relying exclusively on static parameter estimates.

Fifth, there is a methodological gap between statistical distribution fitting and predictive modelling. Existing estimation studies primarily focus on fitting the return distribution or estimating structural characteristics of the Lévy process. Comparatively less attention has been given to incorporating estimated Lévy-density parameters into modern predictive architectures. This creates an opportunity to investigate whether structural information contained in the Lévy density can improve forecasting and probabilistic prediction beyond conventional point-forecasting approaches.

Sixth, model comparison should move beyond goodness-of-fit statistics alone. A model may provide a statistically acceptable fit while having parameters that are unstable, difficult to interpret, or computationally expensive to estimate. Therefore, a more comprehensive comparison should jointly consider

$$\text{Goodness of Fit} + \text{Parameter Stability} + \text{Identifiability}$$
$$+\text{Computational Efficiency} + \text{Structural Interpretability}.$$

This broader criterion is particularly relevant when comparing highly parameterized generalized tempered stable models with simpler alternatives.

4.7 Comparative Synthesis

The evolution of tempered Lévy models can therefore be interpreted as a simultaneous evolution in three dimensions:

$$\text{Lévy-Density Flexibility} \longrightarrow \text{Estimation Complexity} \longrightarrow \text{Need for Structural Inference.}$$

The early VG and BG models demonstrate that tractable density structures permit relatively direct statistical estimation. The TS and related models introduce richer stable-like small-jump behaviour, while CGMY, KoBoL, and GTS provide progressively greater control over jump activity, asymmetry, and tail attenuation. As this structural flexibility increases, conventional likelihood evaluation becomes more difficult and transform-based numerical procedures become increasingly important. The estimation literature consequently evolves from direct likelihood and moment-based methods toward semiparametric, characteristic-function, FFT, and FRFT-assisted procedures [11, 17, 29, 70, 71].

The principal contribution of this comparative perspective is therefore not to identify one universally superior tempered Lévy model. Rather, it demonstrates a fundamental trade-off:

$$\text{Structural Flexibility} \longleftrightarrow \text{Parameter Interpretability} \longleftrightarrow \text{Estimation Complexity}$$

This trade-off provides a unified framework for understanding the development of tempered Lévy models. It also identifies the principal methodological direction for future research: developing estimation procedures that can exploit the increasingly rich information contained in the Lévy density while maintaining statistical identifiability, computational stability, and interpretable parameters.

5 Applications of Tempered Lévy Densities to Real Financial Data

The empirical importance of tempered Lévy models is best demonstrated not by their theoretical flexibility alone, but by the financial datasets and decision-making problems to which they have been applied. Published studies have progressively moved from modelling the distribution of asset returns toward high-frequency volatility modelling, portfolio risk measurement, low-latency econometrics, and simulation of realistic financial trajectories. This empirical progression provides an important basis for identifying the next application frontier of Lévy-density-based financial modelling.

A particularly important development is the application of Lévy models to financial data observed at increasingly short time scales. Barndorff-Nielsen, Pollard, and Shephard [73] investigated integer-valued Lévy processes in the context of low-latency financial econometrics and applied the proposed models to low-latency observations from several types of futures contracts. Their study is important because it demonstrates that Lévy-process modelling is relevant not only for conventional daily financial returns but also for markets in which observations arrive at very short time intervals. The empirical setting therefore establishes an early connection between Lévy processes and the rapidly changing information environment of modern financial markets.

The use of tempered stable models with high-frequency equity data was further developed by Kim [14]. The study considered high-frequency returns for five U.S. stocks and constructed a multivariate tempered stable framework incorporating long-range dependence and time-varying volatility. The model was subsequently used to obtain portfolio Value-at-Risk and average Value-at-Risk measures and to investigate portfolio optimization. This application is particularly relevant because the tempered stable component is not used merely to improve a static return-density fit; it is integrated into a multivariate financial system in which volatility dynamics and cross-asset portfolio risk are simultaneously considered [14].

Another direct high-frequency application was provided by Feng and Shi [15], who incorporated a tempered stable distribution into a fractionally integrated GARCH framework. Their empirical analysis used hourly S&P 500 returns and showed that the resulting FIGARCH specification with tempered stable innovations provided a useful representation of high-frequency financial volatility. The study is important for the present review because it demonstrates that tempered stable behaviour can remain relevant when the objective shifts from modelling unconditional return distributions toward modelling persistent volatility at an intraday frequency [15].

Tempered stable Lévy models have also been applied directly to financial risk measurement. Gong and Zhuang [16] developed time-changed tempered stable stochastic-volatility models and applied them to the Hang Seng Index. Their empirical analysis considered Value-at-Risk, Conditional Value-at-Risk, extreme-event forecasting, and portfolio construction. The resulting framework combined infinite-activity tempered stable jumps with time-varying volatility, thereby allowing the model to represent both discontinuous return movements and volatility clustering. The study illustrates an important practical use of the Lévy structure: the jump component can be transferred directly into quantitative measures of downside risk and portfolio decisions rather than being treated only as a statistical description of returns [16].

The empirical scope of tempered stable modelling has subsequently expanded across conventional financial assets and cryptocurrencies. Nzokem and Maposa [17] applied a seven-parameter Generalized Tempered Stable model to Bitcoin, Ethereum, the S&P 500, and the SPY exchange-traded fund. Bitcoin and Ethereum were selected as heavily tailed financial series, whereas the S&P 500 and SPY represented more peaked return distributions. The authors evaluated the fitted GTS models using Kolmogorov–Smirnov, Anderson–Darling, and Pearson chi-squared tests and compared the results with KoBoL, CGMY, and bilateral Gamma specifications. Their empirical results showed that the GTS framework provided a statistically satisfactory description of the four return datasets and, in several comparisons, improved upon simpler tempered-stable subclasses [17].

This cross-asset evidence is important because it demonstrates that the usefulness of a tempered Lévy structure is not confined to a single type of financial market. Equity-index returns, exchange-traded funds, and cryptocurrency returns exhibit different degrees of tail heaviness and asymmetry, yet the GTS framework can accommodate these differences through its directional and tempering parameters. Thus, the empirical application of the Lévy density has moved from explaining a particular asset class toward providing a common probabilistic structure for heterogeneous financial markets.

A further development is provided by Nzokem and Maposa [18], who investigated the background-driving Lévy density associated with the GTS distribution and applied the resulting framework to real financial data. Their empirical study used daily returns of the SPY ETF and Ethereum over the 2010–2024 period and constructed an Ornstein–Uhlenbeck-type process driven by the GTS background-driving Lévy process. The resulting simulations produced realistic daily cumulative-return trajectories. This application is particularly significant for a Lévy-density-centred perspective because the study moves beyond fitting an observed return distribution and investigates the role of the underlying Lévy structure in generating realistic financial dynamics [18].

Taken together, these empirical studies reveal a clear progression in the practical use of tempered Lévy models:

$$\text{Financial Return Fitting} \longrightarrow \text{High-Frequency Volatility Modelling} \longrightarrow \text{Risk Measurement} \longrightarrow \text{Portfolio Decisions} \longrightarrow \text{Dynamic Return Simulation.}$$

The progression is important because each application uses a different aspect of the Lévy structure. In historical return fitting, the objective is primarily to reproduce heavy tails, asymmetry, and peakedness. In high-frequency volatility modelling, the tempered distribution is combined with temporal dependence. In risk measurement, the resulting jump structure affects VaR and CVaR. In portfolio applications, the estimated risk structure affects asset allocation. In simulation, the Lévy mechanism becomes a generator of realistic return trajectories. Consequently, the financial role of the Lévy density has gradually expanded from a distributional component toward a mechanism for representing financial dynamics.

However, the empirical literature also reveals an important limitation. In the majority of these applications, the Lévy-density parameters are estimated for a historical sample, a specified estimation window, or a model calibrated to a particular dataset. Even when the surrounding model incorporates time-varying volatility or time-changing activity, the individual structural components of the tempered Lévy density are generally not treated as continuously monitored financial quantities. The distinction is important: modelling time-varying volatility is not equivalent to continuously tracking the evolution of the jump-size intensity, local jump activity, directional imbalance, and large-jump attenuation encoded in the Lévy density.

This observation suggests a more focused application frontier for tempered Lévy densities. Instead of asking only whether a tempered stable model provides a better description of a historical return sample, the Lévy density can be investigated as a time-indexed representation of the current jump environment. For newly arriving observations, an estimation window can be updated and the corresponding jump-density structure recalculated. Conceptually,

$$\text{New Financial Observations} \longrightarrow \text{Updated Lévy-Density Estimation} \longrightarrow \text{Jump-Structure Indicators} \longrightarrow \text{Risk and Forecasting Information.}$$

The novelty of this application perspective is therefore not the introduction of another tempered stable distribution. Tempered stable models have already been applied successfully to equity indices, exchange-traded funds, cryptocurrencies, high-frequency stock returns, futures markets, volatility modelling, risk measurement, and portfolio optimization [14–17, 73]. The more open question is whether the Lévy density itself can be transformed into an operational monitoring object for newly arriving financial information.

For this purpose, the positive and negative jump components can be summarized through density-derived quantities rather than relying exclusively on the estimated parameter vector. For a positive jump-size threshold $u > 0$, for example, the positive and negative large-jump intensities may be represented by

$$\Lambda_t^+(u) = \int_u^\infty \nu_t(dx), \qquad \Lambda_t^-(u) = \int_{-\infty}^{-u} \nu_t(dx).$$

These quantities provide a direct interpretation in terms of the expected frequency of jumps exceeding a specified magnitude in the positive and negative directions. Monitoring their evolution would distinguish, for example, an increase in downside jump frequency from a general increase in return volatility. Similarly, changes in the shape of the small-jump region of the Lévy density can provide information about local jump activity that is not directly represented by conventional volatility measures.

Such density-derived indicators create an opportunity to connect model estimation with practical financial monitoring. A continuously updated system could compare the current estimated Lévy density with its historical distribution and identify statistically meaningful changes in directional jump intensity or extreme-jump mass. The output could then be incorporated into risk forecasting, stress testing, portfolio allocation, or probabilistic return prediction. Importantly, this should be treated as an empirical research question rather than an assumption that a particular parameter value automatically corresponds to a market crisis or a calm regime.

The recent GTS background-driving Lévy-density study provides an especially relevant starting point for this research direction. Its application to daily SPY and Ethereum returns demonstrates that the Lévy structure can be connected to dynamic stochastic processes and realistic financial-path simulation [18]. At the same time, the study remains based on daily data and does not establish a continuously updated intraday monitoring system. This leaves open the question of whether the corresponding Lévy-density information can be estimated and exploited at higher frequencies as financial observations arrive.

The high-frequency literature strengthens the importance of this question. Kim [14] demonstrates the feasibility of tempered stable modelling for high-frequency returns of multiple U.S. stocks, while Feng and Shi [15] provide empirical evidence using hourly S&P 500 returns. Barndorff-Nielsen et al. [73] further demonstrate the relevance of Lévy-based modelling in low-latency futures data. These studies establish that sufficiently rapid financial observations are available for studying short-time-scale jump behaviour. What remains comparatively underdeveloped is the integration of these observations with a continuously updated, interpretable representation of the complete tempered Lévy density.

The resulting application frontier can therefore be formulated as

$$\text{Historical Lévy-Density Fitting} \longrightarrow \text{High-Frequency Density Estimation} \longrightarrow \text{Dynamic Jump-State Monitoring} \longrightarrow \text{Real-Time Risk Forecasting}.$$

This direction provides a more concrete form of novelty than simply proposing another application of tempered stable distributions. The existing literature has already established that tempered Lévy models can fit financial returns and support risk measurement, portfolio optimization, volatility modelling, and simulation. The emerging opportunity is to use the estimated Lévy density itself as a continuously updated source of financial information, particularly when combined with high-frequency observations and modern forecasting methods.

Accordingly, the future application of tempered Lévy densities should be viewed not only in terms of improving the fit of a return distribution, but also in terms of extracting interpretable information from the evolving jump mechanism. Such a framework could provide a direct statistical connection between newly arriving market observations, changes in the structure of positive and negative jumps, extreme-jump probabilities, and subsequent financial risk. This would represent a shift from using the Lévy density primarily as a model specification toward using it as an empirically testable representation of evolving financial risk.

6 Conclusion

This review has traced the development of tempered Lévy models from the limitations of Gaussian and stable frameworks to increasingly flexible representations of heavy-tailed, asymmetric, and discontinuous financial returns. The evolution from Variance Gamma and Bilateral Gamma models through truncated and exponentially tempered stable constructions, KoBoL, CGMY, Tempered Stable, and Generalized Tempered Stable models demonstrates a progressive attempt to reconcile two competing requirements in financial modelling: sufficient flexibility to reproduce the empirical characteristics of asset returns and sufficient mathematical structure to support statistical inference and financial applications. Across these developments, the Lévy density has remained the fundamental structural object through which jump intensity, jump-size behaviour, directional asymmetry, and the attenuation of extreme movements are specified.

The review further shows that the role of tempered Lévy models in finance has expanded considerably beyond static return-density fitting. Published empirical studies have applied these models to equity-index returns, exchange-traded funds, cryptocurrencies, high-frequency stock returns, futures-market data, stochastic-volatility modelling, derivative pricing, Value-at-Risk, Conditional Value-at-Risk, portfolio optimization, and realistic financial-path simulation. In particular, applications to high-frequency and low-latency data demonstrate that Lévy-based representations remain relevant when financial information arrives at increasingly short time scales, while recent applications to Bitcoin, Ethereum, the S&P 500, and SPY demonstrate their applicability across heterogeneous financial markets. These empirical developments confirm that tempered Lévy models provide more than a theoretical alternative

to Gaussian modelling; they constitute a practical framework for representing different forms of financial jump behaviour.

A central finding of this review is that the empirical value of tempered Lévy models is closely connected to the information contained directly in their Lévy densities. Conventional financial analysis often reduces the information in a return series to a small number of aggregate measures, such as volatility, skewness, kurtosis, or tail-risk statistics. In contrast, the Lévy density provides a size-dependent representation of jump activity and distinguishes positive and negative jump mechanisms. The density therefore contains several layers of information simultaneously: the intensity of jumps, the behaviour of small jumps, directional asymmetry, and the attenuation of large jumps. This structural representation provides an interpretable link between the observed return process and the underlying jump mechanism.

The review also identifies an important methodological distinction between estimating a tempered Lévy model and using the Lévy density as a source of financial information. Much of the existing literature estimates the parameters of a specified model and subsequently uses the resulting characteristic function, probability density, option-pricing representation, or risk measure. This approach has been highly successful and has produced a substantial body of quantitative-finance applications. Nevertheless, it places the primary emphasis on the final return distribution or financial quantity rather than on the evolving structure of the jump measure itself. The Lévy density consequently remains underexploited as a directly interpretable statistical object.

The empirical evidence reviewed in this study points toward a more dynamic application frontier. High-frequency and low-latency studies demonstrate that financial observations can be analysed at substantially shorter time scales, while applications involving time-varying volatility and time-changed tempered stable processes demonstrate that financial jump behaviour need not be treated as completely static. More recent GTS research has further extended the analysis from return-density fitting to the background-driving Lévy density and its role in generating realistic financial trajectories. Taken together, these developments suggest that the next stage of research should investigate not merely whether a tempered Lévy density fits a historical dataset, but whether changes in its structural components contain useful information about the evolving financial environment.

This observation leads to the principal research gap identified by this review. Although the literature has separately established the usefulness of tempered Lévy models for return modelling, high-frequency data, risk measurement, portfolio analysis, and financial simulation, the integration of these developments into a unified framework for continuously updating and interpreting the underlying Lévy-density structure remains comparatively underdeveloped. In particular, there is scope to investigate whether directional jump intensity, local jump activity, and extreme-jump attenuation can be estimated over successive information windows and evaluated as evolving indicators of financial risk. Such an approach would shift the emphasis from a static distributional description toward an empirically testable representation of changing jump conditions.

The resulting perspective can be summarized by the transition

Static Lévy-Density Modelling ⟶ Data-Adaptive Estimation
⟶ Dynamic Jump-Structure Monitoring ⟶ Risk and Forecasting Applications.

An important advantage of this direction is that it does not require the value of any individual Lévy-density parameter to be interpreted a priori as a particular market regime. Whether changes in jump intensity, asymmetry, local activity, or tempering are associated with calm periods, market stress, abrupt corrections, or other financial states is an empirical question that should be established through out-of-sample evidence. This distinction is essential for developing statistically defensible financial indicators rather than imposing an assumed economic interpretation on model parameters.

The review also highlights the potential for combining the interpretability of Lévy-density models with modern data-driven forecasting methods. Machine-learning and deep-learning approaches can extract nonlinear temporal information from financial observations, whereas tempered Lévy models provide an explicit probabilistic description of discontinuities and heavy-tailed behaviour. Their combination therefore offers a promising direction for probabilistic financial forecasting in which predictive accuracy and structural interpretability are considered simultaneously. Rather than replacing Lévy-based modelling with purely data-driven prediction, such hybrid approaches can use the Lévy density to retain an interpretable representation of jump risk within a flexible forecasting framework.

From an application perspective, the ultimate opportunity is to transform the Lévy density from a mathematical component of a probability model into an operational financial information object. Newly arriving financial observations could be used to update the estimated jump structure, from which density-based measures of directional and extreme-jump activity could be constructed and evaluated for their contribution to risk forecasting, stress testing, portfolio decisions, and probabilistic prediction. The objective is not simply to obtain a better in-sample distributional fit, but to determine whether the structural information contained in the jump density provides incremental predictive or decision-making value.

Overall, the evolution reviewed in this paper indicates that tempered Lévy modelling has reached a stage at which the principal research challenge is no longer only the construction of increasingly flexible distributions. The more consequential challenge is to determine how the information encoded by their Lévy densities can be extracted, updated, interpreted, and validated against real financial outcomes. Future research should therefore place greater emphasis on data-adaptive estimation, high-frequency and sequential observations, density-derived jump-risk indicators, robust statistical inference, out-of-sample validation, and integration with modern forecasting methodologies. Such developments would extend the role of tempered Lévy models from describing the statistical behaviour of financial returns toward providing an interpretable and potentially dynamic framework for understanding and managing evolving financial jump risk.

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