Supplementary results for "Limit laws for shallow loops in planar Gaussian kernel density estimates"

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These results use the model and notation of the companion paper. References to Sections 1–8 and Appendices A–B refer to that paper; labels beginning with S refer to this supplement. The main theorem is proved in the companion paper without invoking these additional results.

S.1 Boundary and critical-height localization

Under the model and notation of the companion paper, these estimates give boundary isolation and tightness of the rescaled critical points and their centered heights. Set $S_n = \sqrt{n}\sup_{D_\rho}|\nabla Z_n|$ and $A_s = 2a|s|$, as in Appendix A.5.

Integrating the bound on p'_{ε_n} yields

$$m_{\varepsilon_n}(\rho) - m_{\varepsilon_n}(0) \le A_s n^{-1/3}\rho^2 - \tfrac{1}{2}b\rho^4.$$

Here $m_{\varepsilon_n}(\rho)$ denotes its common value on $|x| = \rho$. The centered difference on that circle has absolute value at most $\rho S_n/\sqrt{n}$. Put $c_\rho = b\rho^4/8$ and define

$$\mathcal{B}_n = \left\{ \sup_{|x|=\rho} [\widehat{f}_n(x) - \widehat{f}_n(0)] \le -c_\rho \right\}. \tag{S.1}$$

For sufficiently large n, $A_s n^{-1/3}\rho^2 \le b\rho^4/8$. On $S_n \le b\rho^3\sqrt{n}/8$, the total boundary difference is at most $-b\rho^4/4$, hence (S.1) holds. Consequently

$$\Pr(\mathcal{B}_n^c) \le \frac{64C}{b^2\rho^6 n}. \tag{S.2}$$

The scaled physical boundary height is at most $-c_\rho n^{2/3}$, which tends to $-\infty$.

For a joint localization statement, put

$$J_n = \sqrt{n}\sup_{D_\rho}\|D^2 Z_n\|_{\mathrm{op}}, \qquad \mathbb{E}J_n^2 \le C.$$

Because $\nabla Z_n(0) = G_n/\sqrt{n}$, Taylor's formula gives

$$x\cdot\nabla Z_n(x) \le n^{-1/2}|G_n|r + n^{-1/2}J_n r^2. \tag{S.3}$$

Fix $M \ge 1$. With $U = as$, take

$$r_0 = \max\{1, \sqrt{8A_s/b}, (8M/b)^{1/3}\}, \qquad C_0 = |U|r_0^2 + br_0^4 + Mr_0 + 1. \tag{S.4}$$

Choose a fixed $R \ge 2r_0$ with

$$R^2 \ge 4|U|/b, \qquad R^3 \ge 4M/b, \qquad R^4 \ge 2(C_0+4)/b. \tag{S.5}$$

For any fixed $\eta > 0$, define

$$\mathcal{J}_n(M,R,\eta) = \{|G_n| \le M\} \cap \{J_n \le n^{1/12}\} \cap \{\Delta_{n,R} \le \min(1,\eta)\} \cap \mathcal{B}_n. \tag{S.6}$$

The last three events have failure probabilities tending to zero by (A.13), (S.2), and $\Pr(J_n > n^{1/12}) \le Cn^{-1/6}$. The union bound and (A.7)–(A.8) give

$$\liminf_n \Pr(\mathcal{J}_n(M,R,\eta)) \ge 1 - e^{-M^2/(2\sigma_g^2)}. \tag{S.7}$$

On this event, for all sufficiently large n, division of (A.19) and (S.3) by r^4, for $r_0 n^{-1/6} \le r \le \rho$, gives

$$\frac{x\cdot\nabla\widehat{f}_n(x)}{r^4} \le \frac{2A_s}{r_0^2} - 2b + \frac{M}{r_0^3} + \frac{n^{-1/12}}{r_0^2} \le -b. \tag{S.8}$$

Thus every interior spatial critical point of $\widehat{f}_n$ in D_ρ lies inside $D_{r_0 n^{-1/6}}$. Since $\sup_{D_{r_0}}|F_n| \le C_0$, its centered scaled height lies in $[-C_0, C_0]$. The explicit radial derivative of $Q_{s,g}$ likewise places all its critical points inside D_{r_0} when $|g| \le M$.

At the selected radius R, (S.5) gives

$$\max_{|u|=R} Q_{s,g}(u) \le -\tfrac{1}{2}bR^4 \le -C_0 - 4, \qquad |g| \le M.$$

Consequently $F_n \le -C_0 - 3$ on that circle. Equation (S.8) extends this bound along every ray over $R \le |u| \le \rho n^{1/6}$. The entire outer annulus is therefore below every central critical value by at least three scaled height units. Equation (S.1) simultaneously gives the fixed physical boundary gap. The persistence pairing is supplied by Theorem B.4.

These estimates also imply tightness of the supremum of the scaled radii and the supremum of the absolute centered scaled heights of the interior critical-point set, taking either supremum as zero if the set is empty. To justify measurability without assuming a finite critical set, exhaust $\operatorname{int} D_\rho$ by closed disks. On each such compact disk, the event that there exists x with zero gradient and a continuous score $a(x)$ exceeding $t \ge 0$ is

$$\bigcup_{m \ge 1} \left\{ \min_x \left[|\nabla \widehat{f}_n(x)|^2 + \big(t + 1/m - a(x)\big)_+^2 \right] = 0 \right\}.$$

The compact minimum is continuous in the sample coordinates. A countable union over the exhaustion proves Borel measurability, with $a(x) = n^{1/6}|x|$ or $a(x) = |n^{2/3}(\widehat{f}_n(x) - \widehat{f}_n(0))|$. Given a probability tolerance, first choose M large, then r_0, C_0, R, and finally let $n \to \infty$ in (S.7). These choices prove tightness.

S.2 Translation and angular-average centers

Proposition S.1 (Translation equivariance) *For every finite sample X and displacement $c \in \mathbb{R}^2$,*

$$\widehat{f}_{X+c}(x) = \widehat{f}_X(x - c), \qquad L(X + c) = L(X). \tag{S.9}$$

The first identity induces a natural isomorphism of the entire plane persistence modules. If a center rule C satisfies $C(X + c) = C(X) + c$, its continuous angular-average profile

$$\mathcal{R}_C(X)(r) = \frac{1}{2\pi} \int_0^{2\pi} \widehat{f}_X(C(X) + r(\cos\theta, \sin\theta))\, d\theta$$

also satisfies $\mathcal{R}_C(X + c)(r) = \mathcal{R}_C(X)(r)$ for every $r \ge 0$.

Proof Substitute the translated sample into the kernel sum. The map $x \mapsto x + c$ sends each original superlevel homeomorphically to the translated one and commutes with every inclusion. Substitution of $C(X+c) = C(X)+c$ in the integral proves the second identity. □

The identities give samplewise invariance in a pure location family; the local version uses the translated disk $c + D_\rho$. True-center and estimated-center profiles need not agree: for $n = 1$ and true center zero, the sample-mean profile at $r = 0$ is $\phi_H(0)$, whereas the true-center profile is $\phi_H(X_1) < \phi_H(0)$ whenever $X_1 \ne 0$. Translation equivariance alone does not imply equal estimated-center profile laws for two densities that are not translates, even when their true-center radial laws agree.

S.3 A single added kernel

Proposition S.2 (A subcritical radial background plus one positive kernel) *Use the main model with $q \in (0,1)$, $H > 0$, $H^2 < q$, $v_1 = 1+H$, $v_2 = q+H$, $e = v_2^2/(qv_1^2)$, $A_0 = (2\pi v_1)^{-1}$, and $B = q/(2\pi v_2)$. Let*

$$p_\varepsilon(w) = \frac{A_0 e^{-w/(2v_1)} - \varepsilon B e^{-w/(2v_2)}}{1-\varepsilon q}.$$

For $0 < \varepsilon \le e$, $0 < \lambda < 1$, and $d > 0$, define

$$f(x_1,x_2) = (1-\lambda)p_\varepsilon(x_1^2+x_2^2) + \lambda\phi_H((x_1,x_2) - de_1).$$

Its closed-superlevel H_1 module is zero over any fixed field, at every height, including degenerate critical levels.

Proof Put $\Lambda = 1/(2v_2) - 1/(2v_1) > 0$. Direct differentiation gives

$$p'_\varepsilon(w) = -\frac{A_0 e^{-w/(2v_1)}}{2v_1(1-\varepsilon q)}[1 - (\varepsilon/e)e^{-\Lambda w}] \le 0, \qquad w \ge 0.$$

Moreover

$$\partial_2 f(x) = x_2\left[2(1-\lambda)p'_\varepsilon(|x|^2) - \frac{\lambda}{H}\phi_H(x-de_1)\right],$$

and the bracket is strictly negative everywhere, including $\varepsilon = e$ at the origin. Thus f increases as $|x_2|$ decreases with x_1 fixed. The single homotopy $(x_1,x_2) \mapsto (x_1,(1-\alpha)x_2)$ strongly retracts every closed superlevel to its horizontal-axis section and commutes with all filtration maps. For $c > 0$ this section is compact; $f(\cdot,0) - c$ is a nonconstant real analytic function, so it has finitely many zeros on a compact interval containing the section. The section is a finite union of closed intervals and points, or empty. For $c \le 0$ it is the whole axis, since $f > 0$. All these sets have zero H_1. □

For $e < \varepsilon < 1$, the radial background has a nondegenerate local minimum at the origin. Fix λ. As $d \to \infty$, the added kernel and its first two derivatives vanish uniformly on each fixed neighborhood, preserving this minimum. To see a persistent class directly, choose a fixed circle whose weighted background value $(1-\lambda)p_\varepsilon$ exceeds its value at the origin, and a height strictly between those values. For large d, the circle lies in the superlevel while the origin lies outside. Its winding class remains nonzero in the punctured plane throughout an interval of such heights. Thus the condition $\varepsilon \le e$ is essential to the zero-module conclusion.