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Quantum Cointegration: Common Trends and Ordered Correlations

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Abstract

A contrast can remove a common stochastic trend and still leave part of the residual unidentified. For a self-adjoint quantum process, the same cancellation produces a short-memory residual, but the symmetric covariance of that residual does not fix the time-ordered products. In the model $Z_t^{(\sigma)} = Q_t + \sigma P_{t-1}$, both signs have symmetric spectrum $f(\theta) = 1$ and ordered spectrum $S_\sigma(\theta) = 1 - \sigma \sin\theta$. A real linear contrast of one fresh copy has a law independent of σ. The imaginary part of the ordered Weyl witness $\omega(e^{iuZ_t}e^{ivZ_{t-1}})$ changes sign with σ. A finite-dimensional simulation of an idealized differential sensor reproduces the common-trend cancellation, the equality of the real-contrast laws, and the sign of the ordered witness. The simulation is not a hardware measurement, and it does not establish a gain in sensor precision. The statements apply to the processes, initial conditions, and instruments named with them. Full assumptions, proofs, counterexamples, and numerical qualifications are included in the appendices to make the operator and recorded-data claims separately auditable.

Keywords: quantum cointegration; quantum stochastic processes; ordered covariance; common trends; Gaussian measurement; error correction.

1 Introduction

Two time series can share a stochastic trend and still have a stable difference. The question of this paper is what that cancellation identifies when the series are quantum observables, and which time-ordered correlations remain free after the common trend has been removed.

In the classical theory, a linear contrast that annihilates the common loading is stationary, and that stationary residual supports estimation and testing of the cointegrating relation [1–3]. The theory also has a boundary that is easy to miss. A contrast can remove the random walk and still retain a permanent initial offset. That offset is weakly stationary, but its covariance is not absolutely summable, so the residual is not short memory. Classical cointegration therefore identifies a stable relation only after both the trend loading and the initial offset have been controlled.

Noncommuting observables add a second boundary. A short-memory residual has a symmetric covariance, and that covariance determines an even spectrum. The ordered products $\omega(Z_tZ_{t-k})$ also contain an odd spectral part. Gaussian states on the canonical commutation relations are already characterized by admissible covariance kernels [4, 5], and asymmetric quantum noise spectra are a standard part of quantum measurement theory [6]. The present question is narrower. Inside one

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cointegrated construction, which part of that kernel is fixed by the cointegrating relation and by real linear measurements, and which part is not?

The answer has two parts, both restricted to the class defined below. First, a real contrast lies in the quantum cointegration space when it annihilates the long-run loading and the covariance of the initial offset. For a real vector autoregression with one semisimple unit root, this description is equivalent to an initialized error-correction equation. Second, the residual $Z_t^{(\sigma)} = Q_t + \sigma P_{t-1}$ has the same symmetric spectrum for $\sigma = \pm 1$, while its ordered spectrum is $1 - \sigma \sin\theta$. Real linear contrasts of fresh copies do not distinguish the two signs. An ordered Weyl witness does.

The numerical section studies an idealized differential sensor in which one channel carries this residual and both channels carry the same integrated drift. A truncated bosonic simulation shows three facts. The contrast removes the growing channel variance. The real-contrast distributions of the two signs agree exactly on the cutoffs that were diagonalized. The ordered witness changes sign with σ, and the truncation error of that witness is visible at small cutoff. The calculation uses a statevector and a finite-shot sampler. It is not a laboratory record from a differential atom interferometer [7], and it is not the multi-time ion-trap protocol of Wang et al. [8].

2 Quantum cointegration and common trends

The state expectation is written ω. A classical expectation of a commuting record is written $\mathbb{E}$. A centered self-adjoint process is short memory, written QI(0), when it is weakly stationary and its ordered covariance $K(k) = \omega(Z_{t+k} Z_t)$ satisfies $\sum_{k\in\mathbb{Z}} |K(k)| < \infty$. It is integrated of order one, written QI(1), when its centered, mean-zero first difference is QI(0) and has strictly positive long-run variance $\sum_k K_{\Delta X}(k) > 0$, with a finite-variance initial level. Linear variance growth then follows from Theorem F.2. These are second-moment definitions. They do not by themselves classify the operator algebra.

Definition 1 (Quantum cointegration space)**.** Let X_t have QI(1) coordinates. A real vector β is a cointegrating vector when $\beta^\top X_t$ is QI(0). The stable-contrast set is $\mathcal{C}_X = \{0\} \cup \{\beta \neq 0 : \beta^\top X \text{ is QI(0)}\}$. It is a vector space under the regularity conditions below, but need not be closed under addition in general (Proposition F.11). All real combinations used as observables must have the self-adjoint realizations of Definition D.1.

The basic linear class is a long-run loading plus a short-memory remainder,

$$X_t = B \sum_{s=1}^{t} \varepsilon_s + R_t + U.$$

Precisely, impose Definition A.2, including its real weighted-summable increment filter and $\Re G \succ 0$. Theorems E.3 and A.4 give the decomposition and rank formula. Here ε_s is centered white noise with ordered covariance G, R_t is jointly short memory, and U is a centered finite-variance initial offset. For a real contrast, absolute summability forces two separate conditions. The contrast must annihilate B, and it must annihilate the real covariance $\Sigma_U = \Re\omega(UU^\top)$. A nonzero row of B remains integrated. The cointegrating rank in this class is

$$r = n - \operatorname{rank}\begin{bmatrix} B & \Sigma_U^{1/2} \end{bmatrix},$$

when every row of B is nonzero. The first kernel removes the random walk. The second removes a level that does not decay. A classical example already separates them: $(S_t + e_t, S_t)$ has a white-noise contrast, while $(S_t + e_t + V, S_t)$ has contrast $e_t + V$, whose covariance is $\delta_{k0} + 1$.

The same distinction survives in a finite-order real vector autoregression, under hypotheses that should stay visible. The innovations are centered and self-adjoint, the real part of G is positive definite, and the only unit-modulus companion eigenvalue is a semisimple root at 1. All remaining companion eigenvalues lie strictly inside the unit disk; the rank conditions are those of Theorem A.5. The stable block is started in its bilateral stationary solution. The trend block may have any centered finite-variance initial value. Under those conditions the levels have the decomposition above, and

$$\Delta X_t = \alpha\beta^\top X_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t$$

for a full-rank factorization $-\Phi(1) = \alpha\beta^\top$. The columns of β span the kernel of $B^\top$. A contrast in that range is short memory exactly when it also annihilates U. The converse holds after the same root, rank, and initialization conditions are imposed. A Jordan block at 1 is outside the statement, and so is an arbitrary initial condition in the stable block. This is an equivalence inside the stated autoregressive class. It is not a representation theorem for an arbitrary quantum history, and it does not replace the classical Granger representation.

3 Ordered spectra and identification

Once a residual is short memory, its symmetric covariance still leaves an odd spectral component. Write the ordered covariance $K(k) = \omega(Z_t Z_{t-k})$ and the symmetric covariance $C(k) = \Re K(k)$. With the Fourier convention

$$S(\theta) = \sum_k K(k) e^{-ik\theta},$$

the even part f is the transform of C and the odd part a is the transform of the commutator kernel. Spectral positivity gives $|a(\theta)| \le f(\theta)$. The admissible ordered spectra with a prescribed even part are the completions $f + a$ in the Wiener algebra. Theorem B.1, proved in Appendix E, gives a bosonic vacuum realization of every such completion, and the construction uses the characterization of Gaussian CCR processes already given by Lindblad and by Accardi and Hibino. The new content of that statement, in this paper, is the placement of every admissible completion in the same common-trend geometry, together with the measurement consequence below. If the commutator kernel is given as well, then a is fixed and there is nothing left to complete.

The model that makes the gap explicit is

$$Z_t^{(\sigma)} = Q_t + \sigma P_{t-1}, \qquad [Q_t, P_s] = i\delta_{ts} I, \qquad \sigma \in \{-1, +1\}. \tag{1}$$

The vacuum second moments are $1/2$, so $\omega(Z_t^2) = 1$. The ordered kernel is

$$K_\sigma(0) = 1, \qquad K_\sigma(1) = -i\sigma/2, \qquad K_\sigma(-1) = i\sigma/2,$$

and therefore

$$f(\theta) = 1, \qquad S_\sigma(\theta) = 1 - \sigma \sin\theta. \tag{2}$$

The two signs have the same symmetric second-order statistics. Their ordered spectra differ. For the regular Gaussian CCR fields constructed here, f and a determine the ordered Weyl expectations. Distinct choices of a give distinct ordered Weyl laws. The parametrization describes the named time-indexed observables. It does not classify inequivalent representations of an abstract algebra.

Proposition 2 (Real contrasts and an ordered witness)**.** *Within the Gaussian models of Theorem B.1, prepare a fresh independent copy and measure one finite real linear contrast of the levels, or of the residual. The coefficients may depend on earlier outcomes and on model-independent randomness. The outcome law does not depend on the odd completion a. A test between two equal-prior models that differ only in a therefore has success probability $1/2$. The ordered witness*

$$W_\sigma(u,v) = \omega\big(e^{iuZ_t^{(\sigma)}} e^{ivZ_{t-1}^{(\sigma)}}\big) = e^{-(u^2+v^2)/2}\, e^{i\sigma uv/2} \tag{3}$$

does depend on σ. At $(u,v) = (1,1)$,

$$\operatorname{Im} W_\sigma(1,1) = \sigma e^{-1} \sin(1/2). \tag{4}$$

A jointly Gaussian commuting pair with the same symmetric covariance has imaginary part zero.

Proposition B.2 and its proof in Appendix E establish the fresh-copy statement; Theorem A.1 proves the witness formula. The numerical value of (4) is $\sigma \times 0.1764$ in the infinite-dimensional theory. Equation (4) is a property of this model and of this witness. It is not a statement about every quantum measurement. Sequential measurements of Z_t and Z_{t-1} on one copy are a different protocol, because the first measurement disturbs the state used for the second.

4 Measurement and statistical inference

An identity among operators is not yet a classical time series. A legitimate readout has to produce a commuting record. One ideal instrument, in the sense of a joint Gaussian pointer [9], replaces the conjugate pair (Q, P) by a commuting pair whose marginal variances are larger than the vacuum moments. In the normalization used here, the operator residual has variance 1, while the ideal ancillary readout of the same residual has variance 2. Those are different objects. The variance 2 is an analytical Gaussian calculation. The continuous-variable pointer was not built as a circuit, and a cutoff pointer does not obey the exact canonical relation. On a Fock cutoff of dimension d, the defect $[b, b^\dagger] - I$ has operator norm d. Its vacuum expectation can be small. That expectation is not the operator norm.

After a commuting Gaussian record has been formed, cointegration inference is ordinary inference for that record, with the instrument noise kept in the likelihood. Theorems C.2, C.3, and C.4, proved in Appendix H, give a normalized cointegrating regression, a noise-corrected adjustment estimator, and a finite-sample Monte Carlo test. The coverage statement assumes that the noise ratio is known. When the ratio is large, the numerical coverage can fail: at $T = 128$, noise ratio 8, and residual autocorrelation 0.5, the known-ratio interval for the loading covers in 58.5% of 400 histories. An estimated ratio is outside that theorem. A correctly specified classical state-space likelihood for the same Gaussian residual can test the same autocorrelation. This paper does not claim that a quantum estimator improves on that classical likelihood. Nielsen's errors-in-variables cointegrated VAR [10] and the noisy-observation tests studied by Gianfreda, Maranzano, Parisio, and Pelagatti [11] already treat classical measurement error. The supplement records the conditional and unconditional rejection rates, the empty conservative rejection region, and the offset diagnostic. A weak permanent offset is not reliably flagged by the coarse threshold used there.

5 Numerical illustration

The sensor model is an idealized differential pair with an externally imposed integrated drift,

$$X_{1,t}^{(\sigma)} = S_t I + Z_t^{(\sigma)}, \qquad X_{2,t}^{(\sigma)} = S_t I, \qquad S_t = S_{t-1} + \varepsilon_t. \tag{5}$$

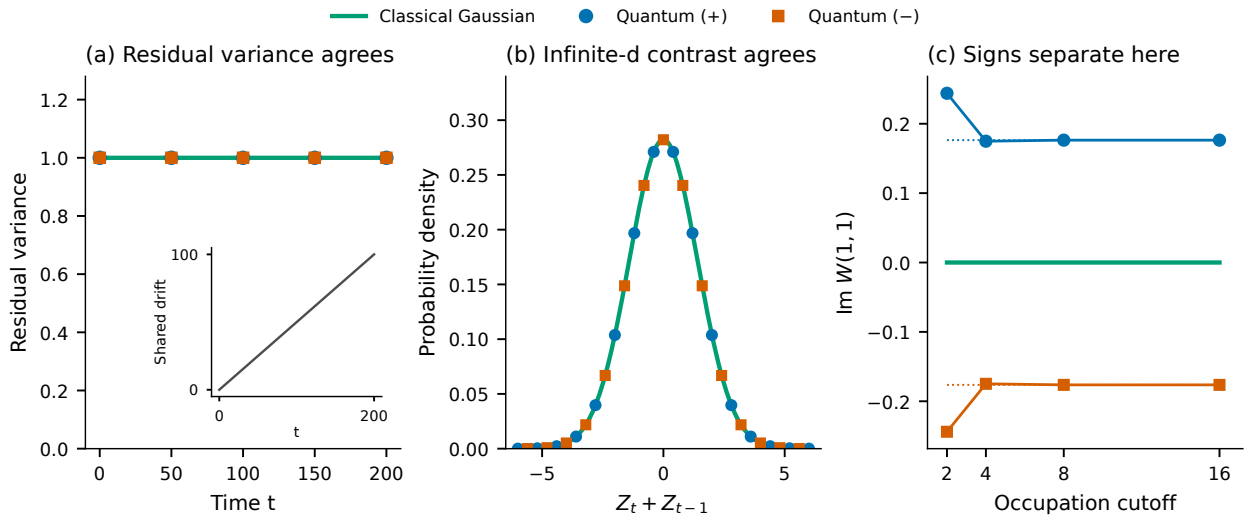


Figure 1: Classical Gaussian, Quantum (+), and Quantum (−), for the sensor model (5). (a) Residual variance after $\beta = (1, -1)^\top$. The three values are exactly 1. The inset is the shared drift variance qt, with $q = 0.5$. (b) Infinite-dimensional law of the real contrast $Z_t + Z_{t-1}$. It is $N(0, 2)$ for all three columns, because the symmetric covariance fixes the variance and the lag-one symmetric covariance is zero. (c) Imaginary part of $W(1, 1)$. The classical value is 0. Dotted lines are the infinite-dimensional quantum values $\pm e^{-1} \sin(1/2)$. Markers are exact cutoff expectations. The marker at cutoff 16 is a factored exponential, not a finite-shot circuit. A Fock histogram is not panel (b): the two quantum signs match each other at every stored cutoff, and their binned distance from $N(0, 2)$ remains 0.40 at cutoff 16.

The drift increment has variance $q = 0.5$, and the cointegrating vector is $\beta = (1, -1)^\top$. The operator contrast is $Z_t^{(\sigma)}$. Its vacuum variance is 1 at every time. The channel variances grow as qt and $qt + 1$. An ideal commuting readout of the contrast has variance 2. The Monte Carlo paths that estimate this readout are classical Gaussian draws. They are not Qiskit output, and they are not an implemented pointer circuit.

The real-contrast and witness calculations use a Fock cutoff of dimension $d \in \{2, 4, 8, 16\}$ and three modes, so that Z_t and Z_{t-1} act on different factors. A real contrast is measured on a fresh copy. It is not a sequential measurement of the two times on one copy. The ordered witness is the vacuum expectation of $e^{iuZ_t} e^{ivZ_{t-1}}$. For $d \leq 8$ that expectation is also the exact output of a Hadamard-test circuit. At $d = 16$ the controlled circuit was not executed. The plotted $d = 16$ value is a factored one-mode exponential, checked against a system-unitary vacuum amplitude. Finite-shot points use a statevector sampler at $d = 2$, $d = 4$, and, for the witness, at $d = 8$ for the pair $(1, 1)$.

Table 1: Where the three models agree, and the one row on which they do not. Every entry is an infinite-dimensional identity. The real-contrast row is the law of one fresh copy. A Fock cutoff keeps the two quantum signs identical on that row and does not reproduce the Gaussian density; see the caption of Figure 1.

	Classical Gaussian	Quantum (+)	Quantum (−)
Common trend	removed by $\beta = (1, -1)^\top$		
Cointegrating relation	residual is short memory, variance 1		
Symmetric spectrum $f(\theta)$	1		
Real contrast, fresh copy	Gaussian law fixed by f		
$\operatorname{Im} W(1,1)$	0	$+0.1764$	-0.1764

Table 1 is the comparison a reader can use before the proofs. The first four rows are the same in all three columns. Cointegration and the symmetric spectrum therefore do not decide between the columns. The last row is the additional structure, and it is visible only to a protocol that keeps the operator order. A forecast, a trading rule, or a classical cointegration test on commuting data is not that protocol. The classical column is the correct benchmark for the shared rows. On the witness row its value is zero because the variables commute. Nothing in the table says that a quantum estimator predicts better, or that a classical test is invalid for classical series.

The shared drift in the inset of Figure 1 was checked on 2000 Gaussian histories, seed 20261150. At $T = 200$ the reference-channel sample variance is 98.742 against $qt = 100$, with standard error 3.123. The operator residual stays exactly 1. An ideal commuting readout of that residual has variance 2, which is a different calculation: the $T = 200$ endpoint of those readout histories is 1.923, and the endpoints at $T = 50, 100, 200$ all lie within four standard errors of 2. That readout is not drawn in Figure 1.

The real-contrast equality was also checked on the prespecified contrasts $(1, 0)$, $(1, 1)$, $(1, -1)$, and $(0.5, 1)$, at cutoffs $2, 4, 8, 16$. The exact binned total variation between the two quantum signs is zero in every stored cell. At cutoff 2, eight shot replications of $Z_t + Z_{t-1}$ have mean total variation 0.0375 at 1024 shots and 0.0099 at 16384 shots. Eigenbasis shots at cutoff 16 were not run.

Panel (c) separates truncation from the classical benchmark. At cutoff 2 the exact imaginary part is ± 0.24392, while the complex absolute error of the witness is 0.0685. At cutoff 4 that complex error is 1.55×10^{-3}, at cutoff 8 it is 4.59×10^{-8}, and the factored value at cutoff 16 differs by about 5.6×10^{-17}. Six shot replications at cutoff 2 and $\sigma = +1$ give means 0.2311 at 1024 shots and 0.2445 at 16384 shots. The second mean is approximately 0.0006 from the cutoff value 0.24392, and it remains far from 0.1764. A commuting sample of 20000 draws, seed 20261153, has imaginary part -0.00520 at $(1, 1)$, which is 1.04 standard errors from the classical value 0.

A cutoff commutator was computed for the same code. Its norm equals the cutoff dimension, from 2 through 16, while the vacuum expectation of the ordered kernel keeps the sign of $-\sigma/2$. Reporting only the vacuum expectation would hide the defect.

6 Discussion

Cointegration, in the sense of this paper, identifies a weakly stationary real contrast whose ordered covariance is absolutely summable. It does not identify the odd spectrum of that contrast. The two tasks use different observations. A real contrast, measured on a fresh copy, is insensitive to the sign in (1). The ordered witness (3) is sensitive to it. The second protocol has to be designed and

reported separately. Nothing in that split says that the classical cointegration theorem is false. The classical theorem concerns commuting series. The extra unidentified object is the odd part of a quantum kernel.

The limitation is also narrower than a general no-go theorem. Proposition 2 covers finite real linear contrasts of fresh copies, followed by the ordered Weyl witness written in (3). Other instruments can extract ordered information. The ion-trap protocol of Wang et al. reconstructs multi-time quasi-probabilities from ancilla-assisted measurements [8]. Their later correction [12] was not used to adjust any number in this paper, because this paper does not reuse their experimental values. Baynham et al. demonstrate common-mode laser-phase rejection in a strontium clock gradiometer [7]. Their interferometer is a laboratory instrument with a periodic fringe. Equation (5) is a Gaussian quadrature model with an imposed random-walk drift. The shared idea is differential cancellation. The implementations are different, and this paper does not improve the sensitivity of their instrument.

The Qiskit calculation is a finite-dimensional statevector simulation. Exact expectations, finite-shot frequencies, and the $d = 16$ factored amplitude are labeled separately in Section 5. The continuous pointer that would turn the operator residual into a commuting record of variance 2 was not compiled. A four-outcome joint Pauli measurement used in an earlier diagnostic is not that pointer. Cutoff canonical commutation is defective at every d that was checked. These facts limit the claim to a numerical illustration of the identification boundary.

The statistical supplement has its own boundaries. A rejection rate computed from one shared calibration bank is conditional on that bank. An interval for the unconditional size needs independent banks. One of those unconditional intervals, at $T = 512$ and zero measurement noise, lies entirely below 5%. The known-ratio coverage of 58.5% at high noise is a failure of the asymptotic picture in that cell, and it is reported as such. The conservative grid test has an empty 5% rejection region on the stored grid. A small permanent offset is difficult to detect from the coarse diagnostic. None of these cells was removed after the result was seen.

7 Conclusion

The paper gives an explicit description of the real contrasts that remain short memory after a common quantum trend is removed, including the initial offset that a trend cancellation does not delete. In that residual, symmetric covariance and real linear measurements leave an ordered spectrum unidentified, and the two signs in (2) separate the ordered witness from the real-contrast law. The finite-dimensional simulation makes the same separation visible, with its truncation, its unexecuted circuits, and its unimplemented pointer kept in the record.

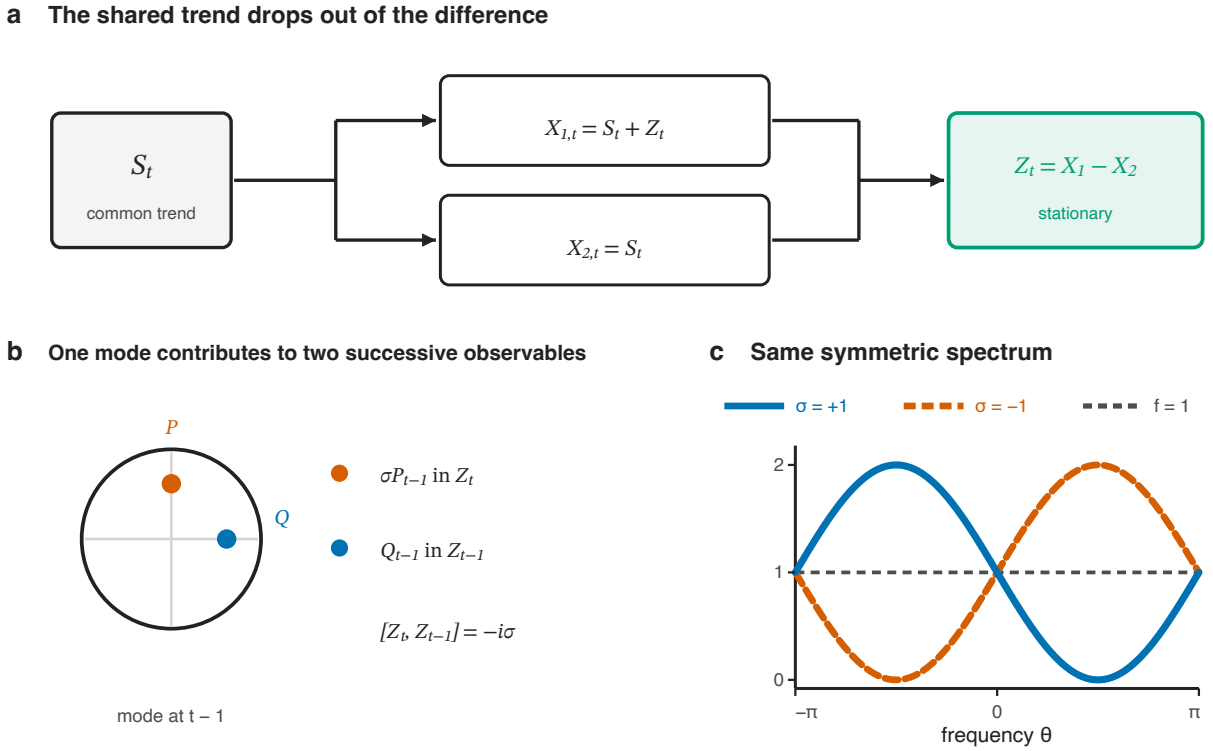


Figure 2: Operator structure of the minimal model, not a laboratory protocol. (a) The common trend S_t enters both observables and cancels in the difference. (b) One bosonic mode at date $t-1$ contributes Q_{t-1} to Z_{t-1} and σP_{t-1} to Z_t, so $[Z_t, Z_{t-1}] = -i\sigma I$. (c) The symmetric spectrum is the constant $f = 1$. The ordered spectra are $1 - \sigma \sin\theta$: the solid curve is $\sigma = +1$ and the dashed curve is $\sigma = -1$. The curves are closed forms, not estimated periodograms.

A When a contrast is short-memory

A.1 One mode in two successive residuals

The series in (5) can be realized with two independent bosonic chains in the product vacuum, in units $\hbar = 1$. Let Q_t and P_t be the position and momentum quadratures of the first chain, so $[Q_t, P_s] = i\delta_{ts} I$ and each vacuum second moment equals $1/2$. Let W_t be the position quadrature of the second chain, and let $S_t = \sum_{j=1}^{t} W_j$ for $t \geq 1$, with $S_0 = 0$. For a sign $\sigma \in \{-1, +1\}$, set

$$Z_t^{(\sigma)} = Q_t + \sigma P_{t-1}, \qquad X_{1,t}^{(\sigma)} = S_t + Z_t^{(\sigma)}, \qquad X_{2,t}^{(\sigma)} = S_t. \tag{A.1}$$

The sign decides which quadrature of the mode at date $t-1$ enters the later residual. It is a contribution to an operator expression, not a second undisturbed reading of that mode. Residuals two or more lags apart are built from different modes. Figure 2 records the cancellation, the shared mode, and the two ordered spectra.

Write $K_Z(k) = \omega(Z_{t+k} Z_t)$, $C_Z(k) = \Re K_Z(k)$, and $J_Z(k) = \Im K_Z(k)$. The ordered spectrum is $S_Z(\theta) = \sum_k K_Z(k) e^{-ik\theta}$. A centered self-adjoint scalar process is short-memory, written QI(0), when it is weakly stationary and $\sum_k |K(k)| < \infty$. It is QI(1) when its centered increments are QI(0), have mean zero, and have positive long-run variance. These are second-order definitions. They do not identify an operator process with a sequence of detector outcomes.

Theorem A.1 (Symmetrically white residuals)**.** *For each sign $\sigma \in \{-1, +1\}$, the processes in* (A.1) *satisfy the following.*

(i) Each coordinate $X_{j,t}^{(\sigma)}$ is QI(1)*. The contrast $\beta_0 = (1, -1)^\top$ has residual $Z_t^{(\sigma)}$, which is moment-stationary and* QI(0)*.*

(ii) The two bivariate systems have the same real symmetrized covariance at every pair of times. For any finitely supported real coefficients, the single observable $\sum_{j,t} a_{j,t} X_{j,t}^{(\sigma)}$ has the same centered Gaussian law for both signs.

(iii) The residual is symmetrically white, $C_Z(k) = \delta_{k0}$, while

$$K_{Z,\sigma}(k) = \delta_{k0} - \frac{i\sigma}{2}\delta_{k,1} + \frac{i\sigma}{2}\delta_{k,-1}, \tag{A.2}$$

$$[Z_t^{(\sigma)}, Z_{t-1}^{(\sigma)}] = -i\sigma I, \qquad S_{Z,\sigma}(\theta) = 1 - \sigma \sin\theta. \tag{A.3}$$

In particular $S_+(\theta) = 1 - \sin\theta$ and $S_-(\theta) = 1 + \sin\theta$.

(iv) For real u, v,

$$\omega\big(e^{iuZ_t^{(\sigma)}} e^{ivZ_{t-1}^{(\sigma)}}\big) = \exp[-(u^2 + v^2)/2] \exp[i\sigma uv/2], \tag{A.4}$$

whereas $\omega\big(e^{i\sum_t h_t Z_t^{(\sigma)}}\big) = \exp[-\frac{1}{2}\sum_t h_t^2]$ for every finitely supported real sequence h.

The proof is in Appendix E. The vacuum calculation uses one same-mode product, $\omega(P_t Q_t) = -i/2$. That product is purely imaginary, so it changes the ordered covariance and not the symmetric covariance. The domain on which the field combinations are essentially self-adjoint is part of the proof. Part (ii) concerns one self-adjoint observable at a time.

A.2 Trend cancellation and the initial level

The example is a rank-one case of a regular linear process. Throughout, X_t is a column of self-adjoint observables. The stable-contrast set is

$$\mathcal{C}_X = \{0\} \cup \{\beta \in \mathbb{R}^n : \beta^\top X \text{ is QI(0)}\}.$$

Higher integration orders, and the distinction between weak and moment stationarity, are in Appendix D.

Definition A.2 (Regular linear class)**.** Let ε_t, $t \in \mathbb{Z}$, be a centered self-adjoint m-vector with $\omega(\varepsilon_t \varepsilon_s^\top) = 1_{\{t=s\}} G$, where $G = G^\dagger \succeq 0$ and $G_R = \Re G \succ 0$. Let real matrices $\Psi_j \in \mathbb{R}^{n\times m}$ satisfy $\sum_{j\geq 0}(j+1)\|\Psi_j\| < \infty$. Set $\Psi(z) = \sum_{j\geq 0} \Psi_j z^j$, $B = \Psi(1)$, and $\Psi^*(z) = (\Psi(z) - B)/(1 - z)$. The remainder $R_t = \Psi^*(L)\varepsilon_t$ is the $L^2(\omega)$ filter, and the levels are

$$X_t = X_0 + \sum_{s=1}^{t} \Psi(L)\varepsilon_s, \qquad t \geq 0. \tag{A.5}$$

The compatible initialization is $X_0 = R_0$. Outputs that are read as observables are assumed to have the self-adjoint realizations required by Definition D.1. Quadratic-mean convergence alone is not that assumption.

Under the compatible initialization, Theorem E.3 gives $X_t = B\sum_{s=1}^{t}\varepsilon_s + R_t$ and $\mathcal{C}_X = \ker_{\mathbb{R}}(B^\top)$. The long-run increment covariance is $D = BGB^\top$. The decomposition has classical and Hilbert-space precedents [13, 14]. The present statement adds the ordered innovation covariance and the initialization under which the identity is exact. A QI(1) process outside the weighted-summable class is constructed in Proposition F.15.

Cancelling the common trend is necessary. The initial level can still prevent a short-memory relation.

Example A.3 (A constant offset). Let e_t and W_t be independent centered commuting white noises of variance one, and let V be a centered variance-one random variable independent of both. For $S_t = \sum_{j=1}^{t} W_j$, compare $(S_t + e_t, S_t)^\top$ with $(S_t + e_t + V, S_t)^\top$. The increments agree. The first contrast is e_t, which is QI(0). The second is $e_t + V$. It is weakly stationary, and its covariance equals $\delta_{k0} + 1$, which is not absolutely summable. The example is classical.

Theorem A.4 (Initial offset and the stable-contrast space)**.** *Retain Definition A.2, and let X_0 be any centered self-adjoint finite-variance initial vector. Set $U = X_0 - R_0$ and $\Sigma_U = \Re\omega(UU^\top)$. No independence of U and the innovations is assumed. If the filtered levels do not meet Definition D.1, read the identities in quadratic mean. Then*

$$X_t = B\sum_{s=1}^{t}\varepsilon_s + R_t + U, \qquad \mathcal{C}_X = \ker_{\mathbb{R}}(B^\top) \cap \ker_{\mathbb{R}}(\Sigma_U). \tag{A.6}$$

Every nonzero row of B still yields a QI(1) *coordinate, and $D = BGB^\top$ does not depend on X_0. If every row of B is nonzero, the cointegrating rank is*

$$r = n - \operatorname{rank}[\, B \;\; \Sigma_U^{1/2} \,]. \tag{A.7}$$

The real zero-frequency increment kernel equals $\mathcal{C}_X$ exactly when $\ker_{\mathbb{R}}(B^\top) \subseteq \ker_{\mathbb{R}}(\Sigma_U)$.

The proof is in Appendix E. If $\beta^\top B \neq 0$, the trend variance grows linearly and a finite-variance offset cannot cancel it. If $\beta^\top B = 0$, the contrast equals a short-memory series plus the constant $\beta^\top U$. Absolute summability forces that constant to vanish in quadratic mean, which for real β is $\Sigma_U\beta = 0$. The two conditions $\beta^\top B = 0$ and $\beta^\top U = 0$ are therefore different: the first removes the random walk, and the second removes a level that does not decay. In Example A.3, $\ker(B^\top)$ is nontrivial and the intersection with $\ker(\Sigma_U)$ is not.

A.3 A conditional error-correction representation

An algebraic rearrangement of a lag polynomial is not, by itself, a representation theorem. The next statement stays inside a finite-dimensional real VAR whose companion has a single unit-modulus eigenvalue, and it uses the initialization of Theorem A.4.

Theorem A.5 (VAR, common trends, and an initialized error correction)**.** *Let $\Phi(L)X_t = \varepsilon_t$ with $\Phi(z) = I_n - \sum_{j=1}^{p} A_j z^j$, real coefficient matrices, and centered self-adjoint white innovations whose ordered covariance G satisfies $\Re G \succ 0$. Suppose every companion eigenvalue lies strictly inside the unit disk except possibly a semisimple root at* 1*, of algebraic multiplicity c. Let the long-run loading B extracted from the spectral projection of that root satisfy* $\operatorname{rank} B = c$ *and* $\operatorname{rank}\Phi(1) = n - c$*. Initialize the stable block in its bilateral stationary solution, and allow the trend block any centered finite-variance initial value. Then*

$$X_t = B\sum_{s=1}^{t}\varepsilon_s + R_t + U, \tag{A.8}$$

with R_t jointly short-memory, and for any full-column-rank factorization $-\Phi(1) = \alpha\beta^\top$,

$$\Delta X_t = \alpha\beta^\top X_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t, \qquad \Gamma_i = -\sum_{j=i+1}^{p} A_j. \tag{A.9}$$

The columns of β span $\ker(B^\top)$. The short-memory space is $\mathcal{C}_X = \mathrm{range}(\beta) \cap \ker(\Sigma_U)$, so a contrast in that range is short-memory exactly when it annihilates U in quadratic mean. Conversely, start from (A.9)*, rebuild the VAR polynomial, and impose the same root, rank, and stable-initialization conditions. Then* (A.8) *and the same description of $\mathcal{C}_X$ follow.*

The proof is in Appendix H. The companion recursion splits into a semisimple unit root, whose projection telescopes, and a stable block, whose powers are absolutely summable once it is started in stationarity. The passage between $\Phi(z)$ and (A.9) reverses as polynomial algebra. The root and initialization conditions do not. A Jordan block at 1 produces an integrated square and is outside the claim. An arbitrary initial condition in the stable block decays geometrically and is not, in general, an exactly stationary process. The equivalence is inside this VAR class. It is not a Granger representation for an arbitrary quantum history. The purely algebraic error-correction identity, without roots or initialization, is Theorem G.1. A noncommuting bosonic model with an exact error correction is Theorem G.5.

B Ordered information in the residual

Once a residual is short-memory, its symmetric covariance still need not fix the ordered products. Let $Z_t = Z_t^*$ be centered and QI(0). Spectral positivity, Theorem E.1, gives $S_Z(\theta) \geq 0$ and $S_Z(-\theta) \geq 0$. The even part $f(\theta) = \frac{1}{2}[S_Z(\theta) + S_Z(-\theta)]$ is the transform of C_Z. The odd part $a(\theta) = \frac{1}{2}[S_Z(\theta) - S_Z(-\theta)]$ is the transform of iJ_Z, under the convention $S_Z(\theta) = \sum_k K_Z(k)e^{-ik\theta}$. Hence

$$S_Z(\theta) = f(\theta) + a(\theta), \qquad a(-\theta) = -a(\theta), \qquad |a(\theta)| \leq f(\theta). \tag{B.1}$$

Theorem A.1 is the case $f = 1$ and $a(\theta) = -\sigma \sin\theta$.

Write $\mathcal{W}(\mathbb{T})$ for the Wiener algebra of continuous 2π-periodic functions with absolutely summable Fourier coefficients. The inverse convention is $K(k) = \int_{-\pi}^{\pi} e^{ik\theta} S(\theta)\, \mathrm{d}\theta/(2\pi)$.

Theorem B.1 (Spectral completion and cointegrated Gaussian realization)**.** *Fix a real, even, nonnegative $f \in \mathcal{W}(\mathbb{T})$. The ordered spectra of centered self-adjoint* QI(0) *processes with symmetric spectrum f are exactly*

$$\mathcal{F}_f = \{f + a : a \in \mathcal{W}(\mathbb{T}) \text{ is real and odd, and } |a(\theta)| \leq f(\theta) \text{ for every } \theta\}. \tag{B.2}$$

Every member has a bosonic vacuum realization $Z_t^{(a)}$ whose finite real linear combinations are essentially self-adjoint on a common domain, whose commutators are scalar, and which is moment-stationary. For each such realization and any $\nu > 0$, an independent commuting vacuum random walk S_t of increment variance ν gives

$$X_{1,t}^{(a)} = S_t + Z_t^{(a)}, \qquad X_{2,t}^{(a)} = S_t,$$

with both coordinates QI(1)*, stable-contrast space $\mathrm{span}_{\mathbb{R}}\{\beta_0\}$, and level variance rates ν. Every single finite real linear contrast of the levels has a distribution independent of a.*

For the regular Gaussian CCR fields constructed here, f and a determine all ordered Weyl expectations. Distinct choices of a give distinct ordered Weyl laws. The parametrization describes laws of the named time-indexed observables. It does not classify inequivalent Hilbert-space representations of an abstract algebra.

Necessity is (B.1) together with absolute summability. Sufficiency builds the field $Z_t = \mathsf{a}(g_t) + \mathsf{a}^*(g_t)$ on symmetric Fock space, with $g_t(\theta) = e^{-it\theta}\sqrt{f(\theta) + a(\theta)}$. Nelson's analytic-vector criterion gives essential self-adjointness of finite real combinations [15]. The vacuum two-point function is the inverse Fourier transform of $f + a$, and the shift that multiplies by $e^{-i\theta}$ fixes the vacuum. Gaussian processes on the CCR algebra were characterized earlier by admissible covariance kernels [4, 5]. The theorem records which part of such a kernel survives a symmetric residual spectrum, and it places every admissible completion in the same common-trend geometry. If the commutator kernel is given as well, then a is fixed and there is nothing left to complete.

If $f + a$ is a trigonometric polynomial, the residual is a finite causal filter of one vacuum chain and the levels lie in Definition A.2. That case is Corollary E.5; the factorization is the scalar Fejér–Riesz theorem [16].

Proposition B.2 (Fresh copies, and an ordered witness)**.** *Consider any finite protocol that, at each round, prepares a fresh independent copy of one Gaussian bivariate model from Theorem B.1 and measures a single finite real linear contrast. The coefficients may depend measurably on previous outcomes and on model-independent randomization. The joint outcome law does not depend on a. A test between two such models with equal prior probabilities therefore has success probability* $1/2$.

For the two signs in Theorem A.1, the bounded self-adjoint contraction

$$\mathcal{O}_{u,v} = \frac{e^{iuZ_t}e^{ivZ_{t-1}} - e^{-ivZ_{t-1}}e^{-iuZ_t}}{2i}, \qquad \omega_\sigma(\mathcal{O}_{u,v}) = e^{-(u^2+v^2)/2}\sin(\sigma uv/2), \tag{B.3}$$

is not a single linear contrast of the levels. Its expectation separates the signs whenever $\sin(uv/2) \neq 0$.

The restriction to fresh copies and to one contrast per copy is part of the statement. A sequential instrument on one retained system is not covered, and neither is a measurement of $\mathcal{O}_{u,v}$. The witness is an observable. No apparatus or sensitivity comparison is claimed. Symmetrized moments alone never certify noncommutativity; Appendix F records that fact and the quasi-free case in which weak stationarity lifts to moment stationarity.

C Inference from a measurement record

C.1 Commuting pointers and a classical record

The observables in (A.9) do not by themselves provide a joint classical series. The following instrument is deliberately narrow. A classical Gaussian control is coupled to independent bosonic vacuum fields. This is classical random modulation together with quantum fluctuations. It is not a general monitored interacting system, and it does not assert that arbitrary noncommuting observables have a disturbance-free joint measurement.

Definition C.1 (Source and fixed observation map)**.** Let V_t be commuting Gaussian control coordinates. Unknown parameters belong to the law of V, not to a data-dependent choice of the measured operator. Independently, take one vacuum mode $(Q_{j,t}, P_{j,t})$ per channel and a mirror vacuum ancilla $(Q'_{j,t}, P'_{j,t})$. Fix a known diagonal matrix $D_0 = \mathrm{diag}(d_j)$ with $d_j \geq 0$ and a sign σ. Define

$$\zeta_{j,t} = Q_{j,t} + \sigma P_{j,t-1}, \qquad X_t = V_t + D_0\zeta_t, \tag{C.1}$$

$$A_{j,t} = Q_{j,t} + Q'_{j,t}, \qquad B_{j,t} = P_{j,t} - P'_{j,t}, \tag{C.2}$$

$$Y_t = V_t + D_0(A_t + \sigma B_{t-1}). \tag{C.3}$$

The prepared past supplies B_{-1} when it is needed. The control has no feedback from the readout.

At each mode, $A_{j,t}$ and $B_{j,t}$ are the two ports of a balanced combination of the system and the ancilla. They are not a sharp joint measurement of $Q_{j,t}$ and $P_{j,t}$. The added variance is the usual cost of a joint quadrature readout [6]. The record uses the pointer kept from the previous date together with the pointer at the current date. Equation (C.3) specifies that lag.

Theorem C.2 (Record law and operator correspondence)**.** *All pointer observables in* (C.2) *commute. Their outcome record has the law*

$$Y_t = V_t + \xi_t, \qquad \xi_t \text{ i.i.d. } N(0, \Sigma_\xi), \qquad \Sigma_\xi = 2D_0^2, \qquad \xi \text{ independent of } V. \tag{C.4}$$

Whenever $d_j > 0$, the source levels fail to commute: $[X_{j,t}, X_{j,t-1}] = -i\sigma d_j^2 I$. For every real vector b, $b^\top X$ is short-memory if and only if $b^\top Y$ is short-memory, and the two have the same integration order 1 whenever the control has positive long-run increment variance in that direction. The stable-contrast spaces agree, including when the control carries a constant random offset.

The commutator $[A_{j,t}, B_{j,t}]$ is $iI - iI$. In vacuum the pair $(A_{j,t}, B_{j,t})$ is standard normal and independent across modes, so $A_{j,t} + \sigma B_{j,t-1}$ is white noise of variance 2, even though the unobserved residuals $\zeta_{j,t}$ do not commute at lag one. Adding or removing a lag-stationary short-memory kernel preserves absolute summability in both directions and does not change a positive long-run increment rate. A nonsummable detector drift, integrated measurement noise that the control does not share, or feedback into V can destroy the correspondence. Those instruments are outside Definition C.1.

The estimation and testing results use a triangular submodel. Take $d_2 = 0$, write m for the scalar readout variance in the first coordinate, and let

$$\begin{aligned} V_{2,t} &= S_t, & S_t &= S_{t-1} + w_t, \quad S_0 = 0, \\ V_{1,t} &= \vartheta S_t + z_t + u, & z_t &= \rho z_{t-1} + e_t. \end{aligned} \tag{C.5}$$

Here $w_t \sim N(0, q_s)$ and $e_t \sim N(0, q_z)$ are independent, with $q_s, q_z > 0$ and $\vartheta \neq 0$, and $u \sim N(0, h)$ is constant within a history. For $|\rho| < 1$, use the bilateral stationary solution for z, independently of the random walk, readout noise, and offset; all innovation chains are independent. The record is $Y_{2,t} = S_t$ and $Y_{1,t} = \vartheta S_t + z_t + u + \xi_t$ with $\xi_t \sim N(0, m)$. The ratio m/q_z is treated as known when a finite-sample size claim is made. For $h = 0$ and $|\rho| < 1$, the normalized cointegrating vector is $(1, -\vartheta)^\top$. For $h > 0$ the same direction cancels the trend and the contrast is not short-memory.

C.2 Estimation and a calibrated test

Throughout this section the observations are the record (C.4) in the triangular model (C.5). The regressor $Y_{2,t}$ is the random walk itself. No generated residual is used as a regressor in the first stage.

Theorem C.3 (Normalized regression and noise-corrected adjustment)**.** *Assume fixed $|\rho| < 1$, $q_z, q_s > 0$, and finite known m. Regress $Y_{1,t}$ on an intercept and $Y_{2,t}$ for $1 \le t \le T$, and write $\hat{\vartheta}$ for the slope, S_t^c for the demeaned random walk, and $D_T = \sum_{t=1}^T (S_t^c)^2$. Let $\Omega_r = q_z/(1-\rho)^2 + m$ be the long-run variance of $z_t + \xi_t$, and let W and B be independent standard Brownian motions, with $W^c(s) = W(s) - \int_0^1 W$. Then*

$$T(\hat{\vartheta} - \vartheta) \Rightarrow \frac{\sqrt{\Omega_r/q_s}\int_0^1 W^c(s)\, dB(s)}{\int_0^1 \big(W^c(s)\big)^2\, ds}. \tag{C.6}$$

In particular $\hat{\vartheta}$ is T-consistent, and $(\hat{\vartheta} - \vartheta)\sqrt{D_T}/\hat{\Omega}_r^{1/2} \Rightarrow N(0,1)$ for a consistent estimator of Ω_r. The slope estimates the normalized trend direction $(1, -\vartheta)^\top$ whether or not $h = 0$. It estimates a short-memory direction only when $h = 0$.

Let $\hat{r}_t$ be the centered residual and let $\hat{\gamma}_j$ be its sample covariance at lag j. Then

$$\hat{\rho} = \frac{\hat{\gamma}_1}{\hat{\gamma}_0 - m} \longrightarrow \rho, \tag{C.7}$$

whereas $\hat{\gamma}_1/\hat{\gamma}_0$ converges to $\rho v_z/(v_z + m)$ with $v_z = q_z/(1 - \rho^2)$. If $\hat{\gamma}_0 - m > 0$ and $|\hat{\rho}| < 1$,

$$\hat{\Omega}_r = (\hat{\gamma}_0 - m)\frac{1 + \hat{\rho}}{1 - \hat{\rho}} + m \tag{C.8}$$

is a consistent plug-in for Ω_r. Under the same conditions, $\sqrt{T}(\hat{\rho} - \rho)$ is asymptotically normal. A nonpositive $\hat{\gamma}_0 - m$, or a value of $\hat{\rho}$ indistinguishable from the unit interval at the sample size in hand, is a weak finite-sample case, not a standard error to be reported as usual.

The intercept removes the realized constant u. The proof of the Brownian limit is in Appendix H. The factor $\sqrt{\Omega_r/q_s}$ is the ratio of the long-run residual scale to the random-walk scale. An earlier writeup of this limit placed q_s in the denominator while calling W standard, which disagrees with the continuous-mapping calculation by a factor $q_s^{-1/2}$. The studentized limit is unaffected, and a direct Gaussian check of that studentization is part of the replication. The T-consistent slope changes the residual sample covariances by $O_p(T^{-1})$ and does not change the $\sqrt{T}$ limit for $\hat{\rho}$. Unknown m must be estimated at a rate faster than $T^{-1/2}$, or its error has to enter the delta method, before these variance formulas remain valid.

The same triangular record separates trend cancellation from short memory. A single demeaned history removes u and cannot reveal h. Independent histories can. Appendix H gives a pilot-and-validation estimator of the intersection $\ker(B^\top) \cap \ker(\Sigma_U)$ and a conservative test of $h = 0$ when the slope is estimated and the stationary parameters lie in a certified compact set. The consistency threshold $T^{-1/2} + M^{-1/4}$ is coarse. In the replication of Section I, with $T = 256$, $M = 128$ validation histories, $\rho = 0.6$, and 500 runs, every realization with $h = 0.2$ remained below the threshold, and every realization with $h = 1$ lay above it.

Theorem C.4 (Monte Carlo residual test)**.** *In the model* (C.5)*, take the null $\rho = 1$, $z_0 = 0$, so that there are two independent trends. Regress $Y_{1,t}$ on $(1, Y_{2,t})$ and let $\hat{r}_t$ be the residual. For $t = 2, \ldots, T$ regress $\Delta\hat{r}_t$ on $\hat{r}_{t-1}$ with no intercept, and let $\mathcal{T}$ be the ordinary t-statistic of that slope, signed so that large negative values favor a stationary residual. Draw B_0 independent null histories at the same T and the same ratio m/q_z, refitting both stages on every history, and set*

$$p_{\mathrm{MC}} = \frac{1 + \sum_{b=1}^{B_0} \mathbf{1}\{\mathcal{T}^{(b)} \le \mathcal{T}_{\mathrm{obs}}\}}{B_0 + 1}. \tag{C.9}$$

If the ratio m/q_z used in the reference histories is the ratio of the instrument, then for every $a \in (0, 1)$,

$$\Pr(p_{\mathrm{MC}} \le a) \le a, \tag{C.10}$$

with equality at the attainable levels of a continuous statistic. Under each fixed alternative $|\rho| < 1$ with stationary initialization and $h = 0$, the rejection probability tends to one as $T \to \infty$, provided $1/(B_0 + 1) \le a$.

Exchangeability is the whole finite-sample argument: the observed history and the reference histories are drawn from one null law, and both regression stages are recomputed on each of them. The null law, after the intercept and the cointegrating slope have been projected out, depends on the sample size and on m/q_z, not on ϑ, q_s, or the constant u. The bound is the classical rank inequality [17]. It is not a new Dickey–Fuller distribution, and it is lost if the reference histories are generated at the wrong noise ratio. A plug-in estimate of m/q_z does not inherit (C.10). A conservative extension, when an independent Gaussian calibration experiment supplies a confidence interval for that ratio, is Proposition H.1. Local alternatives $\rho_T = 1 - c/T$ are outside the fixed-alternative consistency statement.

The intercept makes $\mathcal{T}$ invariant to u. With $|\rho| < 1$ and $h > 0$, the test can reject the two-trend null even though the operator contrast is not short-memory. Rejection then detects trend cancellation, not a short-memory relation. A combined declaration asks for residual rejection and for an offset diagnostic below its vanishing threshold. Under the two-trend null its false-declaration probability remains at most the level of the residual test. There is no uniform distinction between $h = 0$ and arbitrarily small h.

D Mathematical framework and assumptions

This appendix restates the conventions used by every later proof. A quadratic-mean calculation is not a substitute for a self-adjoint realization.

Definition D.1 (Mathematical universe)**.** Let $(\mathcal{A}, \omega)$ be a unital complex $*$-algebra with a normalized positive linear functional ω, so that $\omega(A^*A) \geq 0$ and $\omega(I) = 1$. At each integer time, the centered observables $X_{j,t} = X_{j,t}^*$ belong to a jointly defined polynomial $*$-algebra, with finite moments of every order used below. For unbounded physical representatives, the displayed products are assumed defined on a common invariant dense domain. For spectral-measure or measurement readings, the relevant real combinations are assumed to have self-adjoint closures. The algebra is the algebra of the entire system–environment history.

The GNS seminorm $\|A\|_{2,\omega}^2 = \omega(A^*A)$ controls second-order limits after null vectors are quotiented out and the quotient is completed. The completion is a Hilbert space of equivalence classes. It is not an algebra of physical observables: an $L^2(\omega)$ limit need not have a self-adjoint representative. Let $\tilde{X}_t = X_t - \omega(X_t)I$. For a vector process define

$$K_{ij}(t,s) = \omega(\tilde{X}_{i,t}\tilde{X}_{j,s}), \qquad K(k) = K(t+k,t) \text{ if jointly stationary}, \qquad K(-k) = K(k)^\dagger. \quad \text{(D.1)}$$

For a scalar, centered, weakly stationary self-adjoint signal Z_t,

$$\begin{aligned} K_Z(k) &= \omega(Z_{t+k}Z_t), \\ C_Z(k) &= \tfrac{1}{2}\omega(\{Z_{t+k}, Z_t\}) = \Re K_Z(k), \\ J_Z(k) &= \tfrac{1}{2i}\omega([Z_{t+k}, Z_t]) = \Im K_Z(k), \end{aligned} \quad \text{(D.2)}$$

so $K_Z = C_Z + iJ_Z$. Then $C_Z(-k) = C_Z(k)$ and $J_Z(-k) = -J_Z(k)$. The ordered spectrum is $S_Z(\theta) = \sum_{k\in\mathbb{Z}} K_Z(k)e^{-ik\theta}$. The function C_Z is the symmetrized autocovariance. The function J_Z is the antisymmetric covariance. It is not a quadrature operator; the position quadrature of a bosonic mode is written Q_t.

Definition D.2 (Weak and strong stationarity)**.** A self-adjoint scalar process is weakly stationary if its mean is constant and $\omega(\tilde{X}_t\tilde{X}_s)$ depends only on $t - s$. A vector is jointly weakly stationary if

every cross covariance has this property. A process is strongly (moment-)stationary if every finite ordered joint $*$-moment that exists is invariant under a common time shift. These are statements about operator laws.

Weak stationarity fixes expectations of products of at most two observables. Moment stationarity fixes every finite ordered product. If operators at different times fail to commute, a permutation of the factors is a different statistical question.

Definition D.3 (QI(0) and QI(1))**.** A scalar self-adjoint process is QI(0) if it is weakly stationary and $\sum_k |K(k)| < \infty$. This is stronger than weak stationarity and does not ask the spectrum to be positive at frequency zero. Given a finite-variance initial observable X_0, the process X_t is QI(1) if the centered increments $\Delta X_t = X_t - X_{t-1}$ are QI(0), have mean zero, and satisfy $d_1 = \sum_k K_{\Delta X}(k) > 0$. Deterministic polynomial trends are removed before this label is applied.

Definition D.4 (Integer QI(d))**.** For an integer $d \geq 1$, a self-adjoint process with finite second moments for $\Delta^j X_0$, $0 \leq j < d$, is QI(d) when $\Delta^d X_t$ is centered and QI(0) with $d_d = \sum_k K_{\Delta^d X}(k) > 0$. Fractional orders are not included. The cases $d = 0, 1$ are Definition D.3.

Definition D.5 (Stable contrasts)**.** Let every coordinate of the self-adjoint n-vector X_t be QI(d), and assume the relevant real linear combinations are self-adjoint on a common domain. For an integer $1 \leq \ell \leq d$, a nonzero $\beta \in \mathbb{R}^n$ is a QC(d, ℓ) vector if $\beta^\top X_t$ is QI($d - \ell$). The central case is QC(1, 1), where the contrast is QI(0). Set

$$\mathcal{C}_X = \{0\} \cup \{\beta \in \mathbb{R}^n : \beta^\top X \text{ is QI(0)}\}.$$

A real contrast of a coordinate-wise QI(1) vector is strongly cointegrating when its residual is QI(0) and moment-stationary in the sense of Definition D.2. For QC(d, ℓ) with $d - \ell > 0$, a strong-order definition has to name which differenced residual is moment-stationary. Second moments do not decide that question.

Definition D.6 (Residual ordered spectrum)**.** For a centered QI(0) residual Z_t, the functions K_Z, C_Z, J_Z are those of (D.2), and

$$S_Z(\theta) = \sum_{k \in \mathbb{Z}} K_Z(k) e^{-ik\theta} = \sum_k C_Z(k) e^{-ik\theta} + i \sum_k J_Z(k) e^{-ik\theta}.$$

Theorem E.1 makes S_Z real and nonnegative. The transform of C_Z is even. The transform of iJ_Z is odd. The odd function records the order of the operator product.

Regular linear class, as used by the proofs. The numbered statement is Definition A.2. It is repeated here so that this appendix can be checked without reconstructing the hypotheses. Let ε_t, $t \in \mathbb{Z}$, be a centered self-adjoint m-vector with $\omega(\varepsilon_t \varepsilon_s^\top) = 1_{\{t=s\}} G$, where $G = G^\dagger \succeq 0$ and $G_R = \Re G \succ 0$. Let real matrices $\Psi_j \in \mathbb{R}^{n \times m}$ satisfy $\sum_{j \geq 0} (j+1) \|\Psi_j\| < \infty$. Set $\Psi(z) = \sum_{j \geq 0} \Psi_j z^j$, $B = \Psi(1)$, and $\Psi^*(z) = (\Psi(z) - B)/(1 - z)$. The remainder is the $L^2(\omega)$ filter $R_t = \Psi^*(L)\varepsilon_t$, and

$$X_t = X_0 + \sum_{s=1}^{t} \Psi(L)\varepsilon_s, \qquad t \geq 0.$$

The compatible initialization is $X_0 = R_0$. An observable reading of a filter output assumes the self-adjoint realization required by Definition D.1. Convergence in $L^2(\omega)$ is not that assumption.

Logical map.

1. The quadratic-mean completion used for covariance and spectral arguments is distinct from the physical operator domain. The primitive in Theorem F.7 is not automatically a self-adjoint observable.

2. Vanishing zero-frequency variance is necessary for a stationary contrast under Theorem F.13. It is sufficient for QI(0) only with a stronger assumption, such as the regular class of Theorem E.3.

3. Separately stationary contrasts need not be jointly stationary. A cointegrating rank, as the dimension of a linear space, requires closure. Closure holds in the regular class and fails in Proposition F.11.

4. A real coordinate change preserves stable contrasts as sets. It need not preserve the coordinate-wise integration orders used to attach a $\mathrm{QC}(d, \ell)$ label. This is Proposition F.14.

5. Theorem A.1 is a joint state on a bilateral bosonic history. It does not assert a joint readout of its noncommuting observables.

6. An arbitrary initial level can shrink $\mathcal{C}_X$ while leaving the increment kernel unchanged. Theorem A.4 requires the offset to vanish in quadratic mean on each claimed short-memory direction.

7. Theorem B.1 holds the symmetric kernel fixed and lets the residual commutator vary. The full ordered law uses regular Weyl relations. Proposition B.2 adds fresh preparations and single linear contrasts.

E Proofs of the main results

The statements of Theorems A.1, A.4, and B.1, and of Proposition B.2, are in the main text. Their proofs are complete below. The regular representation and the spectral positivity theorem are proved here as well, because the main arguments call them. No step is left to a numerical check.

Theorem E.1 (Ordered spectral positivity)**.** *Let ξ_t be a jointly weakly stationary centered n-vector of self-adjoint observables with $\sum_k \|K_\xi(k)\| < \infty$. Then*

$$D = \sum_{k\in\mathbb{Z}} K_\xi(k), \qquad S_\xi(\theta) = \sum_{k\in\mathbb{Z}} K_\xi(k) e^{-ik\theta} \tag{E.1}$$

are Hermitian positive semidefinite for every θ, and the series for S_ξ converges uniformly. In particular $K_\xi(-k) = K_\xi(k)^\dagger$ and $D = S_\xi(0) \succeq 0$.

Proof. For finitely supported coefficient vectors $c_t \in \mathbb{C}^n$, put $A = \sum_{t,j} c_{t,j}\xi_{j,t}$. Positivity of the state gives

$$0 \le \omega(A^*A) = \sum_{t,s} c_t^\dagger K_\xi(t-s) c_s.$$

The kernel is positive definite in this sense. Taking adjoints inside ω and using stationarity gives $K_\xi(-k) = K_\xi(k)^\dagger$, so $S_\xi(\theta)$ is Hermitian. Fix $v \in \mathbb{C}^n$ and $\theta \in [-\pi, \pi]$, and apply the same identity to

$$A_T = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} e^{it\theta} \sum_{j=1}^{n} v_j \xi_{j,t}.$$

Grouping terms by the lag $k = t - s$ yields

$$\omega(A_T^* A_T) = \sum_{|k|<T} \Big(1 - \frac{|k|}{T}\Big) e^{-ik\theta}\, v^\dagger K_\xi(k) v \geq 0. \tag{E.2}$$

Absolute summability and dominated convergence give $\lim_T \omega(A_T^* A_T) = v^\dagger S_\xi(\theta) v \geq 0$. Since v is arbitrary, $S_\xi(\theta) \succeq 0$. Absolute summability also gives uniform convergence, hence continuity, and $D = S_\xi(0) \succeq 0$.

The argument uses the ordered product $\omega(A_T^* A_T)$. Positivity is a statement in the GNS inner-product space. It does not construct a physical observable from an arbitrary vector in that space. □

Remark E.2. Write $D = F + iJ_D$ with F real symmetric and J_D real skew-symmetric. A real contrast β has long-run variance rate $\beta^\top F \beta = \beta^\top D \beta$. A complex null vector of D need not be a self-adjoint observable contrast. For a classical real signal, $J_Z = 0$ and the spectrum is even. A quantum signal still has a nonnegative ordered spectrum; the odd part comes from the order of the product.

Theorem E.3 (Regular representation and exact rank)**.** *Under Definition A.2 with the compatible initialization $X_0 = R_0$,*

(i) $\Delta X_t = \Psi(L)\varepsilon_t$ for every $t \geq 1$, in $L^2(\omega)$; R_t is jointly QI(0)*; and $X_t = B\sum_{s=1}^t \varepsilon_s + R_t$.*

(ii) For each real β, the contrast $\beta^\top X_t$ is QI(0) *if and only if $\beta^\top B = 0$.*

(iii) If every row of B is nonzero, each coordinate is QI(1)*. Then $\mathcal{C}_X = \ker_{\mathbb{R}}(B^\top)$, the rank is $r = n - \operatorname{rank}(B)$, and the common-trend subspace is $\mathcal{T}_X = \operatorname{range}_{\mathbb{R}}(B)$.*

(iv) The ordered long-run increment covariance is $D = BGB^\top$, and $\Re D = BG_R B^\top$. Inside this class, the real zero-frequency kernel is $\mathcal{C}_X$.

Proof. Write Ψ_j^* for the coefficient of z^j in $\Psi^*(z)$.

(i) Series rearrangement gives $\Psi_j^* = -\sum_{\ell>j} \Psi_\ell$ and $\sum_j \|\Psi_j^*\| \leq \sum_{\ell\geq 1} \ell \|\Psi_\ell\| < \infty$. Thus $R_t = \sum_{j\geq 0} \Psi_j^* \varepsilon_{t-j}$ exists in $L^2(\omega)$, is jointly weakly stationary, and has absolutely summable lag covariances:

$$\sum_{k\in\mathbb{Z}} \|K_R(k)\| \leq \|G\| \Big(\sum_{j\geq 0} \|\Psi_j^*\|\Big)^2 < \infty.$$

The identity $(1-z)\Psi^*(z) = \Psi(z) - B$ implies $\Psi(L)\varepsilon_t = B\varepsilon_t + R_t - R_{t-1}$. Summing the defining increment equation and using $X_0 = R_0$ gives $X_t = B\sum_{s=1}^t \varepsilon_s + R_t$.

(ii) If $\beta^\top B = 0$, then $\beta^\top X_t = \beta^\top R_t$, a real contrast of a jointly short-memory stationary filter, hence QI(0).

(iii) Conversely, let $h = \beta^\top B \neq 0$ and $Y_T = \sum_{s=1}^T \varepsilon_s$. Temporal whiteness gives $\operatorname{Var}(hY_T) = T\, hG_R h^\top > 0$. The remainder has bounded variance, and the cross term is $O(\sqrt{T})$ by the Cauchy–Schwarz inequality in $L^2(\omega)$. Therefore $\operatorname{Var}(\beta^\top X_T)/T \to hG_R h^\top > 0$, which excludes stationarity. The same calculation with a coordinate vector shows that a nonzero row of B makes that coordinate QI(1). Hence $\mathcal{C}_X = \ker_{\mathbb{R}}(B^\top)$.

(iv) Expanding the lag-covariance sum of $\Delta X_t = \Psi(L)\varepsilon_t$ gives $D = \sum_{j,\ell} \Psi_j G \Psi_\ell^\top = BGB^\top$ and $F = \Re D = BG_R B^\top$. For real β, $\beta^\top F \beta = (B^\top\beta)^\top G_R (B^\top \beta)$, which vanishes if and only if $B^\top \beta = 0$ because $G_R \succ 0$. □

Remark E.4. The Beveridge–Nelson decomposition and the rank relation have classical and Hilbertian precedents [13, 14]. The present statement adds an ordered innovation covariance and a named initialization. Proposition F.15 is a QI(1) process outside the weighted filter class. Fractional and nonregular dynamics are not covered.

Proof of Theorem A.4. The identity $\Psi(L)\varepsilon_s = B\varepsilon_s + R_s - R_{s-1}$ telescopes to (A.6). If $\beta^\top B \neq 0$, the trend variance grows at the positive rate $\beta^\top B G_R B^\top \beta$, while $\beta^\top(R_t + U)$ has uniformly bounded variance. Cauchy–Schwarz bounds the cross term by $O(\sqrt{t})$, so the contrast is not stationary. The same estimate gives the coordinate integration orders.

Now suppose $\beta^\top B = 0$, and set $r_t = \beta^\top R_t$ and $u = \beta^\top U$. If $\|u\|_{2,\omega} = 0$, every second moment of $r_t + u$ equals the corresponding moment of r_t, again by Cauchy–Schwarz, so the contrast is QI(0). Conversely, assume $z_t = r_t + u$ is QI(0). For any centered QI(0) process v_t,

$$\Big\| \frac{1}{T} \sum_{t=1}^{T} v_t \Big\|_{2,\omega}^2 \leq \frac{1}{T} \sum_{k \in \mathbb{Z}} |K_v(k)| \longrightarrow 0.$$

Apply this bound to z and to r. The averages differ by the constant u, so the triangle inequality yields $\|u\|_{2,\omega} = 0$. For real β, $\|u\|_{2,\omega}^2 = \beta^\top \Sigma_U \beta$. Positive semidefiniteness of Σ_U converts the vanishing quadratic form into $\Sigma_U \beta = 0$. The resulting kernel is the orthogonal complement of the columns of $[B \;\; \Sigma_U^{1/2}]$, which is (A.7). The increment covariance does not involve X_0. The last sentence of the theorem is $G_R \succ 0$. □

Proof of Theorem A.1. Let b_t be the annihilation operator of the residual chain, so that $Q_t = (b_t + b_t^\dagger)/\sqrt{2}$ and $P_t = (b_t - b_t^\dagger)/(i\sqrt{2})$, and let d_t be the annihilation operator of the independent trend chain, so that $W_t = (d_t + d_t^\dagger)/\sqrt{2}$. Each of Q_t and P_t is a self-adjoint canonical quadrature. Any finite real combination appearing in the theorem is essentially self-adjoint on the standard finite-mode oscillator domain, so its Weyl exponential is unambiguous. The product vacuum is invariant under a shift of the bilateral mode labels, and $Z_t^{(\sigma)}$ has finite temporal memory, so its finite ordered moments are shift-invariant.

The only nonzero second moments on one mode are

$$\omega(Q_t^2) = \omega(P_t^2) = \tfrac{1}{2}, \qquad \omega(Q_t P_t) = \tfrac{i}{2}, \qquad \omega(P_t Q_t) = -\tfrac{i}{2}. \tag{E.3}$$

Distinct modes commute and have vanishing vacuum cross covariance. At lag 0, the summands Q_t and P_{t-1} belong to different modes, so $K_{Z,\sigma}(0) = 1$. At lag 1, the only surviving same-mode factor in $\omega[(Q_{t+1} + \sigma P_t)(Q_t + \sigma P_{t-1})]$ is $\sigma P_t Q_t$, and $K_{Z,\sigma}(1) = -i\sigma/2$. The adjoint relation gives $K_{Z,\sigma}(-1) = i\sigma/2$. Every other lag vanishes. Therefore

$$\begin{aligned} K_{Z,\sigma}(k) &= \delta_{k0} - \tfrac{i\sigma}{2}\delta_{k1} + \tfrac{i\sigma}{2}\delta_{k,-1}, \\ C_{Z,\sigma}(k) &= \delta_{k0}, \\ J_{Z,\sigma}(k) &= \tfrac{\sigma}{2}(\delta_{k,-1} - \delta_{k1}). \end{aligned} \tag{E.4}$$

The shared mode contributes $\sigma[P_{t-1}, Q_{t-1}] = -i\sigma I$ to the commutator of $Z_t^{(\sigma)}$ and $Z_{t-1}^{(\sigma)}$. Fourier summation gives

$$S_{Z,\sigma}(\theta) = 1 - \tfrac{i\sigma}{2} e^{-i\theta} + \tfrac{i\sigma}{2} e^{i\theta} = 1 - \sigma \sin\theta \in [0, 2].$$

The covariance is finitely supported, so $Z^{(\sigma)}$ is QI(0). Moment stationarity was already noted. This is (iii), together with the residual half of (i).

Since $S_t = \sum_{j=1}^{t} W_j$ with independent vacuum quadratures, $\mathrm{Var}(S_T) = T/2$. The increment ΔZ_t is a finite filter of a stationary residual, so its ordered covariances are summable and the spectral factor $1 - e^{-i\theta}$ vanishes at $\theta = 0$. Both $\Delta X_{1,t} = W_t + \Delta Z_t$ and $\Delta X_{2,t} = W_t$ are centered QI(0) processes with long-run variance rate $1/2$. Each level is therefore QI(1), and the difference is $Z_t^{(\sigma)}$. This finishes (i). The W chain is independent of the b chain, so every symmetrized covariance of the bivariate levels is computed from $\omega(S_t S_s)$ and from $C_{Z,\sigma} = \delta_{k0}$. Neither depends on σ.

For (ii), fix finitely many real coefficients $a_{j,t}$. The observable $L^{(\sigma)} = \sum_{j,t} a_{j,t} X_{j,t}^{(\sigma)}$ is a real linear combination of finitely many canonical quadratures in a product vacuum, hence a centered Gaussian. Its variance is the quadratic form of the symmetrized covariance, identical for both signs. Equality of the individual laws is not a joint measurement law for noncommuting observables at different times.

For (iv), set $A = iuZ_t^{(\sigma)}$ and $B_{\mathrm{op}} = ivZ_{t-1}^{(\sigma)}$. The commutator $[A, B_{\mathrm{op}}] = i\sigma uv\, I$ is central, so the Weyl identity gives

$$e^{A} e^{B_{\mathrm{op}}} = e^{i(uZ_t^{(\sigma)} + vZ_{t-1}^{(\sigma)})} e^{i\sigma uv/2}.$$

Part (ii), applied to the single observable $uZ_t^{(\sigma)} + vZ_{t-1}^{(\sigma)}$, uses $C_Z(1) = 0$ and yields the Gaussian factor $e^{-(u^2+v^2)/2}$. The same variance computation gives $\mathrm{Var}(\sum_t h_t Z_t^{(\sigma)}) = \sum_t h_t^2$. □

Proof of Theorem B.1. Spectral positivity gives $S(\theta) \geq 0$ and $S(-\theta) \geq 0$. Their half-sum is the prescribed even function f, and their half-difference is a real odd $a \in \mathcal{W}(\mathbb{T})$. Thus $|a| \leq f$, which is the necessity half of (B.2).

For sufficiency, put $s = f + a \geq 0$ and let $\mathfrak{h} = L^2(\mathbb{T}, \mathrm{d}\theta/(2\pi))$, with the inner product conjugate-linear in its first argument. Set $g_t(\theta) = e^{-it\theta}\sqrt{s(\theta)}$. On the symmetric Fock space over $\mathfrak{h}$, let $\mathsf{a}(g)$ be the annihilation operator, conjugate-linear in g, with $[\mathsf{a}(g), \mathsf{a}^*(h)] = \langle g, h\rangle I$, and let Ω be the vacuum. Define $Z_t = \mathsf{a}(g_t) + \mathsf{a}^*(g_t)$. The algebraic finite-particle domain is common and invariant. The bounds $\|\mathsf{a}(g)\psi_n\| \leq \sqrt{n}\|g\|\|\psi_n\|$ and $\|\mathsf{a}^*(g)\psi_n\| \leq \sqrt{n+1}\|g\|\|\psi_n\|$ show that these vectors are analytic for each finite real field combination: the mth power is at most a constant to the mth power times $\sqrt{(n+m)!/n!}$. Nelson's criterion yields essentially self-adjoint closures [15]. The vacuum calculation gives

$$\omega(Z_t Z_u) = \langle g_t, g_u\rangle = \int e^{i(t-u)\theta} s(\theta) \frac{\mathrm{d}\theta}{2\pi}, \qquad [Z_t, Z_u] = 2i\Im K(t-u)\, I. \tag{E.5}$$

The real part is the inverse transform of f. Absolute summability holds because $s \in \mathcal{W}(\mathbb{T})$. Multiplication by $e^{-i\theta}$ is unitary on $\mathfrak{h}$, and its second quantization shifts Z_t and preserves the vacuum. This is moment stationarity on a common Fock domain.

Let S_t be an independent vacuum random walk with increment variance ν. The spectrum of $\Delta X_{1,t}$ is $\nu + |1 - e^{-i\theta}|^2 s(\theta)$, which lies in $\mathcal{W}(\mathbb{T})$ and equals ν at $\theta = 0$. Both coordinates are QI(1). For real β,

$$\mathrm{Var}(\beta^\top X_t) = \nu t(\beta_1 + \beta_2)^2 + \beta_1^2 K(0).$$

The variance tends to infinity when $\beta_1 + \beta_2 \neq 0$. When the sum vanishes, the contrast is a multiple of $Z_t^{(a)}$, which is QI(0). Thus $\mathcal{C}_X = \mathrm{span}_{\mathbb{R}}\{\beta_0\}$, including the zero contrast. The symmetric level kernel depends on $\nu \min(t, u)$ and on the inverse transform of f, and not on a. Vacuum linear fields are Gaussian, so every single real linear-contrast law is fixed by that kernel.

Write $K = C + iJ$. For any ordered list of times and real coefficients, the Weyl relations and the Gaussian single-exponential characteristic function give

$$\omega\Big(\prod_{j=1}^{m} e^{ih_j Z_{t_j}}\Big) = \exp\Big[-\frac{1}{2}\sum_{j,\ell} h_j h_\ell C(t_j - t_\ell)\Big] \exp\Big[-i\sum_{j<\ell} h_j h_\ell J(t_j - t_\ell)\Big]. \tag{E.6}$$

Distinct functions a have distinct Fourier coefficients, so $J(k)$ differs at some lag. Two factors and sufficiently small nonzero coefficients separate the phases without a 2π ambiguity. Conversely, f and a fix C and J, hence fix (E.6). □

Corollary E.5 (Finite-memory realization in the regular class). If $s = f + a$ in Theorem B.1 is a trigonometric polynomial of degree at most m, the residual is a causal finite real filter of one canonical vacuum chain. The cointegrated levels satisfy Definition A.2 with the compatible initialization.

Proof. Scalar Fejér–Riesz factorization gives $s(\theta) = |p(e^{-i\theta})|^2$ with $p(z) = \sum_{j=0}^{m} p_j z^j$, and strict positivity is not required [16]. For independent vacuum modes b_t,

$$Z_t = \sum_{j=0}^{m} \big(p_j b_{t-j} + \overline{p_j} b^*_{t-j}\big) = \sqrt{2} \sum_{j=0}^{m} \big(\Re p_j \, Q_{t-j} - \Im p_j \, P_{t-j}\big).$$

The kernel is $K(k) = \sum_j p_{j+k}\overline{p_j}$, so the spectrum is s. Let $H(z)$ be the displayed real quadrature row and let $\varepsilon_t = (W_t, Q_t, P_t)^\top$. The increment filter has rows $(1, (1-z)H(z))$ and $(1, 0, 0)$. Here $\mathrm{Var}(W_t) = \nu$, $\Re G = \mathrm{diag}(\nu, 1/2, 1/2) \succ 0$, and the remainder is $(Z_t, 0)^\top$. The filter and initialization hypotheses hold, including when s has zeros or vanishes. □

For symmetric white noise, $f = 1$, the class $\mathcal{F}_f$ contains every real odd Wiener function with $\|a\|_\infty \le 1$. Theorem A.1 is $a = -\sigma \sin\theta$. These scalar Gaussian realizations do not assert that a non-Gaussian process is determined by its second moments, or that a matrix-valued completion obeys the same scalar bound.

Proof of Proposition B.2. Conditional on the transcript and on the protocol's randomization, the next selected contrast has the same Gaussian law for every a. Independence of the fresh preparation makes this the conditional outcome kernel. Induction gives identical transcript laws, and the testing claim follows. For the witness, let $\mathsf{W} = e^{iuZ_t} e^{ivZ_{t-1}}$. Then $\mathcal{O}_{u,v} = (\mathsf{W} - \mathsf{W}^*)/(2i)$ is self-adjoint and has norm at most one. Taking the imaginary part of (A.4) gives the expectation. □

F Auxiliary results and counterexamples

The results in this section are not required to read the main argument in order. They are the growth laws, the closure and stationarity distinctions, and the counterexamples that mark the boundary of the main theorems. Proofs are included.

Proposition F.1 (Implications and counterexamples)**.** *Under finite second moments, strong stationarity implies weak stationarity, and it need not imply summable covariance. Weak stationarity does not imply strong stationarity, even in a commutative algebra.*

Proof. Shift invariance of the first and second moments is weak stationarity. A constant nondegenerate variable $X_t = U$ is strongly stationary and has $K(k) = \mathrm{Var}(U)$ at every lag. For the converse, take independent centered variance-one variables Y_t, standard Gaussian at even times and Rademacher ± 1 at odd times. Then $K(k) = 1_{\{k=0\}}$, while $\mathbb{E}(Y_t^4)$ alternates between 3 and 1. □

Theorem F.2 (Linear variance growth)**.** *Under Definition D.3,* $\lim_{T\to\infty} \mathrm{Var}(X_T)/T = d_1 > 0$.

Proof. Let $Y_T = \sum_{t=1}^{T} \Delta X_t$. Stationarity gives $\mathrm{Var}(Y_T) = \sum_{|k|<T} (T - |k|) K_{\Delta X}(k)$. Divide by T and use absolute summability. The cross term satisfies $|\omega(\tilde{X}_0 Y_T)| \le \sqrt{\mathrm{Var}(X_0)\,\mathrm{Var}(Y_T)} = O(\sqrt{T})$, which is negligible after division by T. □

Theorem F.3 (Growth under repeated integration)**.** *For $d \geq 1$, let $\xi_t = \Delta^d X_t$ be as in Definition D.4, and initialize all lower differences to zero, so that*

$$X_T = \sum_{s=1}^{T} \binom{T-s+d-1}{d-1} \xi_s. \tag{F.1}$$

Then

$$\lim_{T\to\infty} \frac{\mathrm{Var}(X_T)}{T^{2d-1}} = \frac{d_d}{((d-1)!)^2(2d-1)} > 0. \tag{F.2}$$

Polynomial initializations of degree at most $d-1$, with finite second moments, do not change the leading coefficient. A process in this class is not QI(j) *for any $j < d$.*

Proof. The convolution formula follows by induction. Let $w_{T,s} = \binom{T-s+d-1}{d-1}$ for $1 \leq s \leq T$ and $w_{T,s} = 0$ otherwise. Stationarity gives $\mathrm{Var}(X_T) = \sum_k K_\xi(k) \sum_s w_{T,s+k} w_{T,s}$. For each fixed k,

$$T^{-(2d-1)} \sum_s w_{T,s+k} w_{T,s} \longrightarrow \frac{1}{((d-1)!)^2} \int_0^1 (1-u)^{2d-2}\, \mathrm{d}u.$$

Cauchy–Schwarz bounds the absolute normalized inner sum uniformly in T and k. Dominated convergence applies. Initial polynomial terms have variance $O(T^{2d-2})$, and the cross term with the integrated sum is $O(T^{2d-3/2})$. If a lower difference were stationary, one further difference would be a stationary $\Delta^{d-1}X$, contradicting Theorem F.2. □

Example F.4 (Orders $0, 1, 2$). For pairwise commuting vacuum quadratures $u_t = (b_t + b_t^\dagger)/\sqrt{2}$ in independent temporal modes, $\mathrm{Var}(u_t) = 1/2$. The processes u_t, $\sum_{s=1}^t u_s$, and $\sum_{s=1}^t (t-s+1)u_s$ have orders $0, 1, 2$. The last two variance rates are $T/2$ and $T^3/6 + O(T^2)$.

Remark F.5 (A boundedness barrier). A uniformly bounded observable $\|X_t\| \leq M$ has $\mathrm{Var}(X_t) \leq M^2$. It cannot be QI($d$) for any $d \geq 1$ under the growth laws above. The barrier does not exclude integrated outputs on a growing environment, or unbounded bosonic fields.

Proposition F.6 (Zero long-run variance is insufficient)**.** *There exists a centered stationary short-memory classical increment ξ_t with $\sum_k K_\xi(k) = 0$ whose partial sums $X_T = \sum_{t=1}^T \xi_t$ satisfy $\mathrm{Var}(X_T) \asymp T^{1/2}$ and are not stationary.*

Proof. Let ξ_t be a real stationary Gaussian process with spectral density

$$f(\theta) = \big|2\sin(\theta/2)\big|^{1/2},$$

under the convention $K(k) = \int_{-\pi}^{\pi} e^{ik\theta} f(\theta)\, \mathrm{d}\theta/(2\pi)$. Write $f(\theta) = (2 - 2\cos\theta)^\delta$ with $\delta = 1/4$. The generalized binomial identity gives

$$K_\xi(k) = (-1)^k \frac{\Gamma(2\delta+1)}{\Gamma(\delta-k+1)\Gamma(\delta+k+1)},$$

and the reflection formula yields $K_\xi(k) = O(|k|^{-1-2\delta}) = O(|k|^{-3/2})$. The coefficients are absolutely summable, and their sum is $f(0) = 0$. The partial-sum variance is

$$\int_{-\pi}^{\pi} f(\theta) \Big|\sum_{s=1}^{T} e^{is\theta}\Big|^2 \frac{\mathrm{d}\theta}{2\pi}.$$

Integrating over $1/(2T) \leq |\theta| \leq 1/T$, where the integrand is of order $T^{3/2}$, gives a lower bound of order $T^{1/2}$. Splitting the remainder at $1/T$ and using $|\sum_{s=1}^T e^{is\theta}| \leq \min(T, C/|\theta|)$ gives the matching upper bound. The example sits in the commuting sector. □

Theorem F.7 (Spectral stationary-primitive criterion in quadratic mean)**.** *Let $y_t = y_t^*$ be centered and weakly stationary, with positive scalar ordered spectral measure μ_y on $[-\pi, \pi)$. Work in the GNS $L^2(\omega)$ completion of the real linear span of its shifts. The following are equivalent.*

(i) There is a weakly stationary shift-covariant quadratic-mean primitive $z_t = \mathsf{U}^t z_0$ in this completed span with $y_t = z_t - z_{t-1}$ in $L^2(\omega)$.

(ii) The spectral integral is finite:

$$\int_{-\pi}^{\pi} \frac{\mu_y(\mathrm{d}\theta)}{|1 - e^{-i\theta}|^2} < \infty. \tag{F.3}$$

A primitive is unique in this cyclic subspace up to a shift-invariant component. Condition (ii) produces a stationary quadratic-mean primitive. It does not place that primitive in the original operator domain, and it does not make the primitive QI(0)*.*

Proof. Use the complex GNS span of the y_t for the spectral theorem, and retain its closed real span for the primitive. Weak stationarity makes the bilateral shift $\mathsf{U}[y_t] = [y_{t+1}]$ an isometry, which extends to a unitary. Let $v = [y_0]$ and let ν be its spectral measure for U. With the ordered convention, $\langle v, \mathsf{U}^k v\rangle = K_y(-k)$, so ν is the reflection of μ_y. The integrability criterion is unchanged by reflection.

A primitive satisfies $v = (I - \mathsf{U}^{-1})[z_0]$, so (ii) is necessary. Conversely, for $0 < \varrho < 1$ the real causal filter $z_t^{(\varrho)} = \sum_{j\geq 0} \varrho^j y_{t-j}$ converges in quadratic mean. Its multiplier is $h_\varrho(\theta) = (1 - \varrho e^{-i\theta})^{-1}$. The identity $|1-\varrho e^{-i\theta}|^2 = (1-\varrho)^2+\varrho|1-e^{-i\theta}|^2$ bounds $|h_\varrho|^2$ by $2|1-e^{-i\theta}|^{-2}$ for $\varrho \geq 1/2$. Dominated convergence in ν produces an L^2 limit in the closed real span. The limit satisfies $z_t = \mathsf{U}^t z_0$ and $(I - \mathsf{U}^{-1})z_t = y_t$, and it is weakly stationary. The kernel of $I - \mathsf{U}^{-1}$ is the shift-invariant subspace. The argument does not supply a self-adjoint representative. □

Corollary F.8 (A regularity condition for the zero-frequency gap)*.* If y_t has a nonnegative ordered spectral density $f_y \in C^2([-\pi, \pi])$ with $f_y(0) = 0$ and periodic boundary conditions, then a stationary quadratic-mean primitive exists. If in addition $g(\theta) = f_y(\theta)/|1 - e^{-i\theta}|^2$ lies in $\mathcal{W}(\mathbb{T})$, the primitive has summable ordered covariance and is a quadratic-mean QI(0) process. Operator self-adjointness remains a separate requirement.

Proof. Nonnegativity and differentiability give $f_y'(0) = 0$. A bounded second derivative gives $f_y(\theta) = O(\theta^2)$, so (F.3) holds. The primitive has density g, and absolute summability of its Fourier coefficients is the Wiener hypothesis. □

Example F.9 (Two zero-frequency increments, one stationary primitive)*.* If $y_t = u_t - u_{t-1}$ for classical or commuting bosonic white noise of variance one, then $f_y(\theta) = |1 - e^{-i\theta}|^2$ and $z_t = u_t$ is QI(0). The process of Proposition F.6 has $f_y(\theta) \asymp |\theta|^{1/2}$ and fails (F.3), because the integrand diverges as $|\theta|^{-3/2}$. Both spectra vanish at frequency zero. Only the first admits a finite-variance stationary primitive.

Remark F.10*.* Theorem F.7 is the standard spectral criterion for a stationary primitive, applied to the GNS closure. For a specified integrated level X_t, a compatible observable initialization is a further requirement.

Proposition F.11 (Cointegrating sets need not be spaces)**.** *For individually* QI(1) *coordinates, $\mathcal{C}_X$ need not be closed under addition. If a family of candidate contrasts is jointly weakly stationary and all cross-covariance sequences are absolutely summable, its real span consists of* QI(0) *contrasts and is a linear space.*

Proof. Let $\{e_t\}$ and $\{u_t\}$ be independent iid centered unit-variance real white noises. Set $S_t = \sum_{j=1}^{t} u_j$, $U_t = e_t$, $V_t = (-1)^t e_t$, and

$$X_{1,t} = S_t + U_t, \qquad X_{2,t} = S_t, \qquad X_{3,t} = S_t + V_t.$$

Each coordinate is QI(1): $\Delta U_t = e_t - e_{t-1}$ and $\Delta V_t = (-1)^t(e_t + e_{t-1})$ are stationary MA(1) processes with zero long-run variance, and the random-walk coordinate has long-run variance one. The contrasts $\beta_{(1)} = (1, -1, 0)^\top$ and $\beta_{(2)} = (0, -1, 1)^\top$ recover U_t and V_t. Their sum is $[1 + (-1)^t]e_t$, with variance 4 at even times and 0 at odd times, so it is not weakly stationary. The example commutes, and therefore rules out an unconditional claim in the noncommutative setting as well. For the positive half, the covariance of a real linear combination is a finite real bilinear combination of the jointly stationary, absolutely summable cross-covariances. □

Example F.12 (Weak cointegration without strong cointegration). Let $S_t = \sum_{j=1}^{t} u_j$ with iid centered unit-variance real u_j, and let Y_t be independent across time, centered, unit variance, standard Gaussian for even t and Rademacher for odd t, independent of S. Set $X_{1,t} = S_t + Y_t$ and $X_{2,t} = S_t$. Both coordinates are QI(1), because ΔY_t is weakly stationary, supported on lags $0, \pm 1$, and has zero long-run variance. The contrast Y_t is QI(0) and is not strongly stationary: $\omega(Y_{2t}^4) = 3$ and $\omega(Y_{2t+1}^4) = 1$. For centered jointly quasi-free bosonic residuals the implication does hold, by Proposition F.21.

Theorem F.13 (Necessary zero-frequency condition)**.** *Suppose ΔX_t is centered and jointly weakly stationary, with absolutely summable matrix-valued ordered covariance, and X_0 has a finite second moment. If a real contrast $Z_t = \beta^\top X_t$ is* QI(0)*, then $\beta^\top F \beta = 0$ and hence $D\beta = 0$, where $D = \sum_k K_{\Delta X}(k) = F + iJ_D \succeq 0$. The converse is false without a stronger hypothesis.*

Proof. As in Theorem F.2, $T^{-1}\operatorname{Var}(\beta^\top(X_T - X_0)) \to \beta^\top F \beta$. If Z_T is stationary with finite variance, then $\operatorname{Var}(Z_T - Z_0)$ is at most $4\operatorname{Var}(Z_0)$ by Cauchy–Schwarz, so the limit is zero. For real β, $\beta^\top D \beta = \beta^\top F \beta$. A vanishing quadratic form on a positive semidefinite matrix forces $D\beta = 0$. Proposition F.6 is a classical counterexample to the converse. □

Proposition F.14 (Coordinate changes)**.** *For any vector observable process, $\mathcal{C}_X = \{\beta \in \mathbb{R}^n : \beta^\top X$ is* QI(0)*$\}$, including 0. If $Y_t = TX_t$ for $T \in GL_n(\mathbb{R})$, then $\mathcal{C}_Y = T^{-\top}\mathcal{C}_X$, and a linear stable-contrast space has invariant dimension. An invertible T need not preserve coordinate-wise* QI(1)*. The label* QC(1, 1) *is invariant only under those changes that keep every coordinate integrated of order one. A basis of a cointegrating space is unique only up to invertible change of basis.*

Proof. The identity $\beta^\top Y = (T^\top \beta)^\top X$ is the set equality. For the caveat, let $X_1 = S + u$ and $X_2 = S$, with S a QI(1) random walk and u an independent QI(0) noise. For $T = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, the new coordinates are $Y_1 = u$, of order 0, and $Y_2 = S$, of order 1. The stable-contrast set is still mapped by $T^{-\top}$. □

Proposition F.15 (A QI(1) process outside the regular class)**.** *Not every* QI(1) *process belongs to Definition A.2. In the commutative subalgebra, let ξ_t be centered, real, and Gaussian, with spectral density $f_\xi(\theta) = 1 + |2\sin(\theta/2)|^{1/2}$, and set $X_T = \sum_{t=1}^{T} \xi_t$. Then X is* QI(1) *and has no moving-average representation satisfying the weighted summability in Definition A.2 for a temporally white finite-dimensional innovation.*

Proof. Proposition F.6 supplies absolute summability of the increment covariance through the same binomial formula. Since $f_\xi(0) = 1 > 0$, the process is QI(1). A representation $\xi_t = \Psi(L)\varepsilon_t$ of the required type would give $f_\xi(\theta) = \Psi(e^{-i\theta}) G \Psi(e^{-i\theta})^\dagger$. Weighted summability makes $\Psi(e^{-i\theta})$ and its

first derivative continuous, so f_ξ would be continuously differentiable. The function $1+|2\sin(\theta/2)|^{1/2}$ has a cusp at $\theta = 0$. □

Proposition F.16 (Spectral asymmetry and the positivity bound)**.** *For a scalar self-adjoint stationary residual with summable ordered covariance, $S_Z(\theta)$ is real and nonnegative, and not necessarily even. Its even and odd parts satisfy $f \geq 0$ and $|a| \leq f$.*

Proof. Theorem E.1 gives $S_Z(\theta) \geq 0$ and $S_Z(-\theta) \geq 0$. The Fourier coefficients of C_Z are even and real. Those of J_Z are odd and real. □

Proposition F.17 (No independent real-commutator rank correction)**.** *Let $D = F + iJ_D \succeq 0$, with F real symmetric and J_D real skew-symmetric. Then*

$$\{\beta \in \mathbb{R}^n : \beta^\top F\beta = 0\} = \ker_{\mathbb{C}}(D) \cap \mathbb{R}^n = \{\beta \in \mathbb{R}^n : F\beta = J_D\beta = 0\}.$$

A separate commutator-kernel restriction adds no further rank bound once the ordered positive-semidefinite condition is in force.

Proof. For real β, $\beta^\top D\beta = \beta^\top F\beta$. For a Hermitian positive semidefinite matrix, $\beta^\top D\beta = 0$ if and only if $D\beta = 0$. Separating real and imaginary parts gives the claim. The matrix $D = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ -i & 1 \end{smallmatrix}\right)$ has a one-dimensional complex kernel, while $F = I_2/2$ has no nonzero real null direction. □

Theorem F.18 (Equal symmetric laws, distinct phase-rotated residuals)**.** *Fix $0 < \rho < 1$ and $\phi \notin \pi\mathbb{Z}$. Form two stationary vacuum oscillators $q_t^{(+)}$ and $q_t^{(-)}$ with collisions $\rho e^{-i\phi}$ and $\rho e^{i\phi}$. Couple each to an independent copy of the same commuting cumulative noise S_t, and set $X_{1,t}^{(\pm)} = S_t + q_t^{(\pm)}$, $X_{2,t}^{(\pm)} = S_t$. Both systems are* QC(1, 1) *along β_0. All symmetrized two-time covariance matrices agree, while*

$$S_\pm(\theta) = \frac{1-\rho^2}{2[1+\rho^2-2\rho\cos(\theta \pm \phi)]}, \qquad S_-(\theta) = S_+(-\theta), \qquad S_+ \neq S_-. \tag{F.4}$$

Proof. For each model, $K_\pm(k) = \frac{1}{2}\rho^{|k|}e^{\mp ik\phi}$. The real parts agree and the imaginary parts are opposite and not identically zero. Independence of S and $q^{(\pm)}$ gives the same symmetrized covariance for every pair $X_i(t), X_j(s)$. The trend cancels in $X_1 - X_2 = q$. Summation of the Fourier series produces (F.4). The collisions preserve the CCR and the state is vacuum Gaussian, as in Theorem G.5. □

Proposition F.19 (Ordered spectrum of a regular cointegrating contrast)**.** *Under Definition A.2 with compatible initialization, let $\beta^\top B = 0$ and set $H_\beta(z) = \beta^\top\Psi^*(z) = \beta^\top\Psi(z)/(1-z)$, with the singularity at $z = 1$ removed. The residual $Z_t = \beta^\top X_t = H_\beta(L)\varepsilon_t$ has*

$$S_Z(\theta) = H_\beta(e^{-i\theta})\, G\, H_\beta(e^{-i\theta})^\dagger \geq 0. \tag{F.5}$$

The even and odd parts of this spectrum are the symmetric and antisymmetric components. The identity is the linear-filter formula for this residual.

Proof. The condition $\beta^\top B = 0$ gives $\beta^\top X_t = \beta^\top R_t$. Weighted summability produces absolutely summable real coefficients h_j of H_β. Temporal whiteness gives $K_Z(k) = \sum_j h_{j+k}Gh_j^\top$, with out-of-range coefficients read as zero. Absolute convergence permits the Fourier reordering that yields (F.5). Positivity is $G \succeq 0$. □

Proposition F.20 (Classical shadow of the symmetrized kernel)**.** *For any countable family of centered self-adjoint observables $\{X_a : a \in \mathcal{I}\}$ with finite second moments, the real symmetric kernel $\Gamma(a,b) = \frac{1}{2}\omega(\{X_a, X_b\}) = \Re\omega(X_aX_b)$ is the covariance of a centered classical real Gaussian process indexed by $\mathcal{I}$. The full collection of symmetrized two-time covariances never by itself certifies noncommutativity.*

Proof. For finitely supported real coefficients c_a, $\sum_{a,b} c_a\Gamma(a,b)c_b = \omega((\sum_a c_aX_a)^2) \geq 0$, because the sum is self-adjoint. The consistent centered Gaussian laws extend by the Kolmogorov theorem. □

Proposition F.21 (Gaussian lift from second moments to all moments)**.** *For a centered jointly bosonic quasi-free family of linear field observables, joint weak stationarity implies strong moment stationarity. Without the quasi-free property the implication fails.*

Proof. Even ordered moments of a centered quasi-free family are sums over pairings of the ordered two-point function, and odd centered moments vanish. Each term is shift-invariant under joint weak stationarity. Proposition F.1 is a commuting counterexample that is not quasi-free. □

Theorem F.22 (Classical reduction)**.** *If the countable family $\{X_{i,t}\}$ strongly commutes, admits a joint spectral measure, and the given moments are induced by a normal state on the generated abelian von Neumann algebra, then ω induces a classical joint law. Weak stationarity, summable ordered covariance, $\mathrm{QI}(d)$, $\mathrm{QC}(d,\ell)$, the regular representation, and the lag-polynomial error-correction identity then agree with the corresponding short-memory classical formulations under the same initializations. Instrument records are not asserted to coincide with joint spectral observations.*

Proof. The joint spectral theorem identifies every real polynomial in the observables with an ordinary random variable under the measure induced by ω. Moments, differences, linear filters, and covariance limits are preserved, and the algebraic identities reduce term by term. □

G Error correction and physical constructions

Theorem G.1 (Autoregressive form and the error-correction identity)**.** *Let real matrix polynomials $\Phi(z)$ and $\Lambda(z)$ satisfy $\Phi(0) = I$, and suppose $\Phi(L)X_t = \Lambda(L)\varepsilon_t$ holds with a consistent initial history. If $\Phi(1) = -\alpha\beta^\top$ for real matrices $\alpha \in \mathbb{R}^{n\times r}$ and $\beta \in \mathbb{R}^{n\times r}$, the quotient $\Phi_*(z) = [\Phi(z) - z\Phi(1)]/(1-z)$ is a polynomial, and*

$$\Phi_*(L)\Delta X_t = \alpha\beta^\top X_{t-1} + \Lambda(L)\varepsilon_t. \tag{G.1}$$

The identity alone does not give $\mathrm{QC}(1,1)$, a stable stationary solution, or a physical realization.

Proof. The polynomial identity $(1-z)\Phi_*(z) = \Phi(z) - z\Phi(1)$ holds entrywise because the right-hand side vanishes at $z = 1$. Real coefficients preserve operator order and self-adjointness. Substitute $\Phi(1) = -\alpha\beta^\top$. □

Proposition G.2 (A stable error correction with stationary initialization)**.** *Suppose $\Delta X_t = \alpha\beta^\top X_{t-1} + e_t$, with β of full column rank r. Assume e_t is a centered jointly $\mathrm{QI}(0)$ observable vector, $\mathcal{S} = I_r + \beta^\top\alpha$ is Schur stable, and $z_t = \beta^\top X_t$ is initialized at the unique stationary solution of $z_t = \mathcal{S}z_{t-1} + \beta^\top e_t$. Then each coordinate of z_t is $\mathrm{QI}(0)$, or a degenerate zero-variance stationary observable. If the coordinates of X are also $\mathrm{QI}(1)$, every nonzero βc for which the stationary contrast is defined is a cointegrating direction. Without stability of $\mathcal{S}$, stationary forcing, or stationary initialization, the conclusion does not follow.*

Proof. The contrast satisfies $z_t = \mathcal{S} z_{t-1} + \beta^\top e_t$. Under the stated hypotheses, $z_t = \sum_{j\geq 0} \mathcal{S}^j \beta^\top e_{t-j}$ in $L^2(\omega)$. This is an absolutely summable real filter of a jointly QI(0) input, with mean zero and absolutely summable ordered cross covariances. □

Remark G.3. Proposition G.2 does not produce a stable solution from an arbitrary cointegrated model. The recursion $z_t = 2z_{t-1} + e_t$ has the shape of an error correction and is not a stationary relation for a general initial condition.

Example G.4 (The wrong initial condition destroys exact stationarity). For $z_t = \kappa z_{t-1} + e_t$ with $0 < \kappa < 1$ and iid centered forcing of variance σ^2, the choice $z_0 = 0$ gives $\mathrm{Var}(z_t) = \sigma^2(1-\kappa^{2t})/(1-\kappa^2)$, which depends on t. The solution is asymptotically stationary and is not weakly stationary. A stable error correction yields a QI(0) contrast only with an initialization that the definition can see.

Theorem G.5 (A noncommuting QC(1, 1) model with an exact error correction)**.** *Take two independent bosonic input chains in a product vacuum. For $0 < \rho < 1$, $\phi \notin \pi\mathbb{Z}$, and $0 < \kappa < 1$, define the stationary collisions*

$$c_t = \rho e^{-i\phi} c_{t-1} + \sqrt{1-\rho^2}\, b_t, \qquad q_t = (c_t + c_t^\dagger)/\sqrt{2},$$
$$\zeta_t = \kappa\, \zeta_{t-1} + \sqrt{1-\kappa^2}\, \eta_t, \qquad z_t = (\zeta_t + \zeta_t^\dagger)/\sqrt{2}.$$

Use the stationary infinite-past vacuum solutions. Let $S_t = \sum_{j=1}^t q_j$, $h = (1, 2)^\top$, $v = (1/2, -1)^\top$, $\beta = (1, -1/2)^\top$, and $X_t = hS_t + vz_t$. Then every coordinate is QI(1)*, the contrast $\beta^\top X_t = z_t$ is strongly stationary and* QI(0)*, and*

$$\Delta X_t = (\kappa - 1) v\, \beta^\top X_{t-1} + h q_t + v\sqrt{1-\kappa^2}\, (\eta_t + \eta_t^\dagger)/\sqrt{2}. \tag{G.2}$$

Furthermore $\omega([q_t, q_{t-k}]) = -i\rho^{|k|} \sin(k\phi)$ for $k > 0$. The forcing term is stationary and, in general, colored.

Proof. The collisions expand as a real exponentially decaying filter of the vacuum quadratures $(b_t + b_t^\dagger)/\sqrt{2}$, $(b_t - b_t^\dagger)/(i\sqrt{2})$, and $(\eta_t + \eta_t^\dagger)/\sqrt{2}$. The increments therefore have a regular real linear innovation representation. Membership of the levels in Definition A.2 still requires the initial condition named there. The canonical relations and independence of the chains give $[c_t, c_t^\dagger] = 1$, and likewise for ζ_t. In vacuum, $\omega(q_t q_{t-k}) = \rho^k e^{-ik\phi}/2$ for $k \geq 0$, and $\omega(z_t z_{t-k}) = \kappa^{|k|}/2$. Thus z is QI(0) and moment-stationary by the quasi-free vacuum law. The long-run variance rate of q is $(1-\rho^2)/\{2(1 - 2\rho\cos\phi + \rho^2)\} > 0$, so S is QI(1). Since $\beta^\top h = 0$ and $\beta^\top v = 1$, the residual is z. The remaining claims are substitution and the field commutator. □

The constant offset in this construction lies along the loading h and is annihilated by β. Theorem A.4 is the general form of that check. Admissible offsets are selected by the short-memory geometry.

Field coefficients for the minimal example. The calculation below uses the same vacuum history as Theorem A.1. It is an operator identity, checked algebraically in the archive, and it is not a hardware protocol. Let d_t be the annihilation operator of the trend chain, so $W_t = (d_t + d_t^\dagger)/\sqrt{2}$, and let $V_t = (d_t - d_t^\dagger)/(i\sqrt{2})$ be its momentum quadrature. The residual chain is the mode b_t already used for Q_t and P_t. The nonzero canonical commutators are $[Q_t, P_s] = [W_t, V_s] = i\delta_{ts} I$. The chains commute and the vacuum factorizes. On the dense domain of finite-particle vectors with finitely supported mode labels, every signal below is a finite real linear field. The domain is invariant under the displayed polynomials, and these fields are essentially self-adjoint there.

For $t \geq 0$, set $S_0 = 0$ and use (A.1), so $X^{(\sigma)}_{1,0} = Z^{(\sigma)}_0$ and $X^{(\sigma)}_{2,0} = 0$. Vacuum factorization and (E.4) give, for $t, s \geq 0$,

$$\omega(X^{(\sigma)}_{i,t} X^{(\sigma)}_{j,s}) = \frac{\min(t,s)}{2} + \mathbf{1}_{\{i=j=1\}} K_{Z,\sigma}(t-s). \tag{G.3}$$

Equal-time evaluation gives $\mathrm{Var}(X^{(\sigma)}_{1,t}) = t/2+1$, $\mathrm{Var}(X^{(\sigma)}_{2,t}) = t/2$, and $\mathrm{Var}(Z^{(\sigma)}_t) = 1$. For a general real contrast,

$$\mathrm{Var}(\beta^\top X^{(\sigma)}_t) = \tfrac{t}{2}(\beta_1 + \beta_2)^2 + \beta_1^2. \tag{G.4}$$

If $\beta_1 + \beta_2 \neq 0$, the variance diverges. If the sum is zero, the contrast is a multiple of $Z^{(\sigma)}_t$. Hence

$$\mathcal{C}_X = \mathrm{span}_{\mathbb{R}}\{\beta_0\}, \qquad \mathcal{T}_X = \mathrm{span}_{\mathbb{R}}\{(1,1)^\top\}. \tag{G.5}$$

Stationarity uses the shift-invariant vacuum law.

Let $\varepsilon_t = (W_t, Q_t, P_t)^\top$. The ordered white covariance and the increment filter are

$$G = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}, \qquad \Psi_\sigma(z) = \begin{pmatrix} 1 & 1-z & \sigma z(1-z) \\ 1 & 0 & 0 \end{pmatrix}. \tag{G.6}$$

The vacuum block of G has eigenvalues $0, 1$, so $G \succeq 0$ and $\Re G = I_3/2 \succ 0$. Direct differencing gives $\Delta X_t = \Psi_\sigma(L)\varepsilon_t$. Furthermore

$$B = \Psi_\sigma(1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Psi^*_\sigma(z) = \begin{pmatrix} 0 & 1 & \sigma z \\ 0 & 0 & 0 \end{pmatrix}. \tag{G.7}$$

The remainder is $(Z^{(\sigma)}_t, 0)^\top$, including at time 0. Finite filter length and this initialization are Definition A.2, so Theorem E.3 applies to the levels. The long-run matrix is

$$D = BGB^\top = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{G.8}$$

Both rows of B are nonzero. The kernel of $B^\top$ recovers (G.5). The error correction

$$\Delta X^{(\sigma)}_t = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \beta_0^\top X^{(\sigma)}_{t-1} + \begin{pmatrix} W_t + Z^{(\sigma)}_t \\ W_t \end{pmatrix} \tag{G.9}$$

has $1 + \beta_0^\top \alpha = 0$ and $\beta_0^\top e_t = Z^{(\sigma)}_t$. The forcing is jointly stationary, finitely correlated, and not a white innovation: its first coordinate keeps the lag-one ordered correlation of the residual. This is an instance of Proposition G.2.

If the canonical history is written R with commutator matrix Ω, and the signal coefficients are the rows of a real matrix L, then $[\mathsf{LR}, (\mathsf{LR})^\top] = i\mathsf{L}\Omega\mathsf{L}^\top$ on the common domain. In particular

$$[X^{(\sigma)}_{1,t}, X^{(\sigma)}_{1,s}] = i\sigma(\delta_{t,s-1} - \delta_{t-1,s})I, \qquad [X^{(\sigma)}_{1,t}, X^{(\sigma)}_{2,s}] = 0. \tag{G.10}$$

The rectangular map is not claimed to be a canonical transformation. Bilateral mode translation preserves the vacuum and the canonical commutators. The residual filter $H_{\beta_0}(z) = (0, 1, \sigma z)$ and Proposition F.19 recover $S_{Z,\sigma}(\theta) = 1 - \sigma \sin\theta$ and symmetric spectrum 1. These spectra integrate to residual variance one and meet (B.1). Figure 2 plots (G.4) and these spectra. A classical Gaussian sample with the same symmetric covariance does not check the ordered correlations. Reading an ordered spectrum from a laboratory record would require a specified instrument.

H Measurement, estimation, and testing

The statements of Theorems A.5, C.2, C.3, and C.4 are in the main text. This section proves them and records two conservative extensions. The companion matrix of a VAR is written $\mathcal{A}$, so that the letter A_t remains the pointer of Definition C.1. The symbol M is not used for a noise covariance.

Proof of Theorem A.5. Write the companion state as $x_t = \mathcal{A}x_{t-1} + K\varepsilon_t$, where $K = (I_n, 0, \dots, 0)^\top$ and $X_t = Jx_t$ with $J = K^\top$. Let Π_1 be the spectral projection of $\mathcal{A}$ onto its semisimple eigenvalue 1, interpreted as zero when $c = 0$. Then $\Pi_1\mathcal{A} = \mathcal{A}\Pi_1 = \Pi_1$, so the projected recursion telescopes:

$$\Pi_1 x_t = \Pi_1 x_0 + \sum_{s=1}^{t} \Pi_1 K\varepsilon_s.$$

On the complementary invariant subspace the eigenvalues lie inside the open disk, and the bilateral stationary solution of that block is

$$(I - \Pi_1)x_t = \sum_{j\geq 0} (\mathcal{A} - \Pi_1)^j (I - \Pi_1)K\varepsilon_{t-j}.$$

Set $B = J\Pi_1 K$ and $U = J\Pi_1 x_0$, and write R_t for the image under J of the stable series. Exponential decay and temporal whiteness give $\sum_k \|K_R(k)\| < \infty$ by the same comparison used for the weighted filter in Theorem E.3. Thus $X_t = B\sum_{s=1}^{t}\varepsilon_s + R_t + U$. The rank hypotheses are $\operatorname{rank} B = c$ and $\operatorname{rank}\Phi(1) = n - c$.

If v is a companion eigenvector at eigenvalue 1, its n-dimensional blocks are identical and $\Phi(1)Jv = 0$. Hence $\Phi(1)B = 0$. The rank assumptions force $\operatorname{range}(B) = \ker\Phi(1)$. For a full-column-rank factorization $-\Phi(1) = \alpha\beta^\top$, passing to orthogonal complements gives $\operatorname{range}(\beta) = \ker(B^\top)$. Substituting $X_{t-j} = X_{t-1} - \sum_{i=1}^{j-1}\Delta X_{t-i}$ into the VAR produces (A.9), with $\Gamma_i = -\sum_{j=i+1}^{p} A_j$. Every coefficient is real, so no reordering of noncommuting factors is required. The polynomial identity reverses. Applying the decomposition already proved to the reconstructed companion gives the converse.

Theorem A.4 applies to the offset U without an independence assumption, and yields $\mathcal{C}_X = \ker(B^\top) \cap \ker(\Sigma_U)$. An arbitrary initial error in the stable block adds a term $J(\mathcal{A} - \Pi_1)^t d$. It decays geometrically and is not in general a stationary process, so exact QI(0) can fail for a contrast that does not annihilate it. Semisimplicity excludes a Jordan block at 1. □

Proof of Theorem C.2. On one mode, $[Q, P] = iI$ and $[Q', P'] = iI$, while the system chain commutes with the ancilla. Therefore

$$[A_{j,t}, B_{j,t}] = [Q_{j,t}, P_{j,t}] - [Q'_{j,t}, P'_{j,t}] = iI - iI = 0.$$

Distinct modes and distinct times commute by construction of the product vacuum, except for the system operators that share a mode across adjacent dates. Those system operators are not the pointers. The pointers at all times therefore commute, and they have a joint spectral measure.

In vacuum, (Q, P) and (Q', P') are independent standard pairs: each quadrature has variance $1/2$, and the only nonzero second moment between a pair is the commutator contribution. Hence $(A_{j,t}, B_{j,t})$ is centered Gaussian with covariance I_2, and independent across (j, t). The linear combination $A_{j,t} + \sigma B_{j,t-1}$ is consequently i.i.d. $N(0, 2)$. Multiplying by the known d_j gives the covariance $\Sigma_\xi = 2D_0^2$ and the record law (C.4). This is a statement about the joint outcome law, not only about its one-dimensional margins.

The ordered covariance of $b^\top X$ differs from that of $b^\top V$ by a kernel supported on lags $0, \pm 1$. The covariance of $b^\top Y$ differs from that of $b^\top V$ by the constant $b^\top \Sigma_\xi b$ at lag zero. Either modification preserves weak stationarity and absolute summability, in both directions. The long-run variance of the increments of these short-memory noises is zero, so a positive long-run increment rate of the control is unchanged. A constant random offset in V is the offset of both X and Y. Finite real combinations of the field operators have the common-domain realizations already used for Theorem A.1, tensored with multiplication by the control variables.

The commutator of the source is the commutator of $D_0\zeta$. For one channel the only same-mode pair is P_{t-1} with Q_{t-1}, so $[Q_t + \sigma P_{t-1}, Q_{t-1} + \sigma P_{t-2}] = \sigma[P_{t-1}, Q_{t-1}] = \sigma(-iI) = -i\sigma I$. The ancilla is absent from ζ. Scaling by d_j produces $[X_{j,t}, X_{j,t-1}] = -i\sigma d_j^2 I$. □

Proof of Theorem C.3. The intercept removes the realized u. Write $r_t = z_t + \xi_t$, which is independent of S, centered, and geometrically strong mixing, with long-run variance $\Omega_r = \sum_k \omega(r_{t+k} r_t)$. For the stationary AR(1) this sum equals $q_z/(1-\rho)^2$, and the white readout adds m.

Demeaned partial-sum convergence gives $S_{\lfloor Ts \rfloor}/\sqrt{T} \Rightarrow \sqrt{q_s}\, W(s)$ and $T^{-1/2} \sum_{t=1}^{\lfloor Ts \rfloor} r_t \Rightarrow \sqrt{\Omega_r}\, B(s)$, jointly, with W and B independent because w_t is independent of r_t. Summation by parts for a demeaned random walk against a short-memory series yields

$$T^{-1} \sum_{t=1}^{T} S_t^c r_t \Rightarrow \sqrt{q_s \Omega_r} \int_0^1 W^c \, dB,$$
$$T^{-2} \sum_{t=1}^{T} (S_t^c)^2 \Rightarrow q_s \int_0^1 (W^c)^2 \, ds.$$

The ratio of these limits is (C.6). The conditional variance is $\sum_{t,s} S_t^c S_s^c \gamma_r(t-s)$; geometric summability and the weighted limit above make its ratio to $\Omega_r D_T$ tend to one in probability. Therefore $(\hat{\vartheta} - \vartheta)\sqrt{D_T}/\sqrt{\Omega_r} \Rightarrow N(0,1)$. Consistency of $\hat{\Omega}_r$ gives the studentized form. The replication's scale check draws this studentized statistic at $q_s = 1$ and at $q_s = 4$; the sample standard deviations are 0.95 and 0.98.

The population residual covariances are $\gamma_0 = v_z + m$ and $\gamma_1 = \rho v_z$, so $\gamma_1/(\gamma_0 - m) = \rho$ and $\gamma_1/\gamma_0 = \rho v_z/(v_z + m)$. Geometric ergodicity gives $\hat{\gamma}_j \to \gamma_j$. Let $\delta_T = \hat{\vartheta} - \vartheta = O_p(T^{-1})$ and let $\hat{r}_t$ be the demeaned regression residual. Then $\hat{r}_t = r_t^c - \delta_T S_t^c$, and

$$T^{-1} \sum_t \hat{r}_t \hat{r}_{t-j} = T^{-1} \sum_t r_t^c r_{t-j}^c - \delta_T\, T^{-1} \sum_t (S_t^c r_{t-j}^c + S_{t-j}^c r_t^c) + \delta_T^2\, T^{-1} \sum_t S_t^c S_{t-j}^c.$$

The cross sum is $O_p(T)$ because it is the order of the regression numerator, so the second term is $O_p(T^{-1})$. The last term is $O_p(T^{-2}) \cdot O_p(T) = O_p(T^{-1})$. Thus $\hat{\gamma}_j = \gamma_j + O_p(T^{-1/2})$ still holds, and the $O_p(T^{-1})$ discrepancy does not affect a $\sqrt{T}$ central limit theorem. The delta method, applied to $g(\gamma_0, \gamma_1) = \gamma_1/(\gamma_0 - m)$ at a point with $\gamma_0 > m$, gives asymptotic normality. The plug-in (C.8) equals $v_z(1+\rho)/(1-\rho) + m$ at the population values, which is Ω_r.

The products $(r_t^2, r_t r_{t-1})$ have exponentially summable cumulants under the Gaussian linear representation, so a Bartlett estimator with bandwidth $\lfloor T^{1/4} \rfloor$ is consistent for their long-run covariance. That estimator is not required for (C.8), which uses the AR(1)-plus-white structure. □

Proof of Theorem C.4. Under the stated null, $Y_{1,t} = \vartheta S_t + u + \sum_{s=1}^t e_s + \xi_t$ and $Y_{2,t} = S_t$. The column space of $(1, Y_{2,t})$ contains the constants and the random walk, so the residual $\hat{r}$ does not depend on (ϑ, u). Scaling (w_t, e_t, ξ_t) by a common positive constant scales $\hat{r}$ and $\Delta\hat{r}$ equally and cancels in the t-statistic. The remaining parameter is m/q_z. Reference histories drawn at that ratio,

with both stages refitted, are exchangeable with the observed statistic. The lower-tail rank of an exchangeable continuous sample is uniform on its attainable grid, and the plus-one formula (C.9) satisfies (C.10) whether or not there are ties.

Under a fixed alternative $|\rho| < 1$ with $h = 0$, Theorem C.3 makes the residual covariances consistent for those of $r = z + \xi$. The population coefficient in the regression of Δr_t on r_{t-1} equals $(\gamma_1 - \gamma_0)/\gamma_0$. Here $\gamma_0 - \gamma_1 = v_z(1 - \rho) + m > 0$, so the coefficient is strictly negative, and the residual variance about that regression is positive. Hence $\mathcal{T}/\sqrt{T}$ converges in probability to a strictly negative constant. Under the null, $\mathcal{T}$ is tight: after the spurious-regression projection, the leading piece is a functional of Brownian motion, and the stationary readout contributes only short-run terms. A tight null reference sample of fixed size B_0 therefore lies above the alternative statistic with probability tending to one. □

The same rank argument applies to a scalar Dickey–Fuller regression with an intercept when the marginal null is a random walk plus independent white noise of known variance ratio. That calibration is not the residual calibration above, and it is not reused under a cointegrated alternative, where the increments of the first coordinate are colored.

Proposition H.1 (Noise ratio known only through a confidence interval)**.** *Suppose that under the null of Theorem C.4 the ratio $\lambda = m/q_z > 0$ is unknown, and that an independent Gaussian calibration experiment of K_c noise draws and K_c innovation draws produces $\hat{\lambda}$ with $\lambda/\hat{\lambda} \sim F_{K_c,K_c}$. Let C_δ be the resulting two-sided level-$(1-\delta)$ interval. On a finite grid G whose cells cover C_δ, let p_g be the Monte Carlo p-value of Theorem C.4 computed at the fixed ratio g, from a bank independent of the data and of the calibration. Let $k_1, \dots, k_T$ be the eigenvalues of the matrix $(\min(t,s))_{t,s\le T}$, and define*

$$d(\lambda, g) = \left(\frac{1}{4} \sum_{i=1}^{T} \Big[\frac{k_i + \lambda}{k_i + g} - 1 - \log \frac{k_i + \lambda}{k_i + g} \Big] \right)^{1/2}.$$

Set $\epsilon(C_\delta) = \sup_{\lambda \in C_\delta} \min_{g \in G} d(\lambda, g)$, the minimum being the grid point assigned to λ. Then

$$p_{\mathrm{cal}} = \min\big\{1,\ \delta + \epsilon(C_\delta) + \max_{g \in G} p_g\big\}$$

satisfies $\Pr_\lambda(p_{\mathrm{cal}} \le a) \le a$ for every $\lambda > 0$ and every $a \in (0,1)$. A grid without the term $\epsilon(C_\delta)$ is not covered.

Proof. The quantity $d(\lambda, g)$ equals $\sqrt{\mathrm{KL}(N(0, \Sigma_\lambda) \,\|\, N(0, \Sigma_g))/2}$, where $\Sigma_\lambda = K + \lambda I_T$. Pinsker's inequality bounds the total variation of these centered Gaussian laws by $d(\lambda, g)$. The map $\lambda \mapsto d(\lambda, g)^2$ is convex, so the supremum of d on a cell is attained by checking the endpoints.

With probability at least $1 - \delta$, the true λ lies in C_δ. Conditional on that event and on the calibration, the assigned grid point g satisfies $d(\lambda, g) \le \epsilon(C_\delta)$. The law of the observed record together with its Monte Carlo bank is then within total variation $\epsilon(C_\delta)$ of the law at the fixed ratio g. At that fixed ratio, p_g is superuniform. Conditional on coverage, the probability that $p_g \le a - \delta - \epsilon(C_\delta)$ is at most $\max\{0, a - \delta - \epsilon(C_\delta)\} + \epsilon(C_\delta)$. Rejection by p_{cal} implies this event. Adding the probability at most δ of missing the confidence interval gives the bound. If $a \le \delta + \epsilon(C_\delta)$, then $p_{\mathrm{cal}} \le a$ is impossible. Banks at different grid points need not be independent of one another. □

This is a specialization of maximized Monte Carlo testing over a confidence set [17, 18] to the Gaussian covariance Σ_λ. The ability to draw the calibration sample is an extra experimental assumption.

Proposition H.2 (Offset test with an estimated slope)**.** *In the triangular record, let $|\rho| \le r_* < 1$, $q_z \le q_*$, $m \le m_*$, and $q_s \le q_{s,*}$ be known bounds. A pilot of length T_p, independent of the validation sample, estimates ϑ by the regression of Theorem C.3. Conditional on the pilot regressor, the disturbance has covariance operator norm at most $L_* = q_*/(1-r_*)^2 + m_*$. Hence, with probability at least $1-\delta_\vartheta$,*

$$|\hat{\vartheta} - \vartheta| \le b_p, \qquad b_p = z_{1-\delta_\vartheta/2}\sqrt{L_*/D_p},$$

where D_p is the pilot sum of squares of the demeaned random walk. On M new histories of length T, let H_i be the sample mean of $Y_{1,t}^{(i)} - \hat{\vartheta} Y_{2,t}^{(i)}$ and let s_H^2 be the sample variance of the H_i. Define

$$V_* = \frac{q_*}{(1-r_*^2)T^2}\left(T + 2\sum_{k=1}^{T-1}(T-k)r_*^k\right) + \frac{m_*}{T} + b_p^2 q_{s,*}\frac{(T+1)(2T+1)}{6T}.$$

If $0 < \delta_\vartheta < a$ and $h = 0$, the rule that rejects when $(M-1)s_H^2/V_* > \chi^2_{M-1,\,1-(a-\delta_\vartheta)}$ has type I error at most a.

Proof. The Gaussian regression tail gives the pilot event of probability at least $1-\delta_\vartheta$. On that event every validation mean is centered and Gaussian, conditionally on the pilot, with variance at most V_*: the first two terms bound the variance of a mean of a stationary AR(1) plus white noise, and the third bounds the variance contributed by an error of size b_p times the mean of an independent random walk. Given the pilot, $(M-1)s_H^2$ divided by the true conditional variance is χ^2_{M-1}. Dividing by the larger constant V_* makes the statistic stochastically smaller, so the conditional rejection probability is at most $a - \delta_\vartheta$. Adding the probability of the complementary pilot event proves the claim. The bound does not give uniform power against small h. □

A cheaper diagnostic, still only pointwise consistent, uses a pilot of length T^2 and declares an offset when the average of the squared validation means exceeds $T^{-1/2} + M^{-1/4}$. Section I evaluates that threshold. It is not a level-a test.

Proposition H.3 (White measurement noise is not a VAR innovation)**.** *Suppose $V_t = \mathcal{A}V_{t-1} + e_t$ with e_t white and independent of the past of V, and suppose $Y_t = V_t + \xi_t$ with ξ_t i.i.d., independent of (V, e), and of covariance Σ_ξ. Then*

$$Y_t = \mathcal{A}Y_{t-1} + \nu_t, \qquad \nu_t = e_t + \xi_t - \mathcal{A}\xi_{t-1},$$

and $\mathrm{Cov}(\nu_t, \nu_{t-1}) = -\mathcal{A}\Sigma_\xi = \mathrm{Cov}(\nu_t, Y_{t-1})$. Unless this matrix vanishes, ν_t is not a white innovation independent of Y_{t-1}.

Proof. Substitute $V = Y - \xi$. The cross terms that survive are those in the shared ξ_{t-1}. A white innovation independent of the lagged observation would have both covariances equal to zero. □

The Gaussian state-space model with transition $\mathcal{A}$, state noise covariance Q, observation noise Σ_ξ, and a specified initial covariance P_0 has the Kalman prediction

$$\begin{aligned} \hat{V}_{t|t-1} &= \mathcal{A}\hat{V}_{t-1|t-1}, & P_{t|t-1} &= \mathcal{A}P_{t-1|t-1}\mathcal{A}^\top + Q, & F_t &= P_{t|t-1} + \Sigma_\xi, \\ v_t &= Y_t - \hat{V}_{t|t-1}, & K_t &= P_{t|t-1}F_t^{-1}, & P_{t|t} &= P_{t|t-1} - K_t F_t K_t^\top. \end{aligned}$$

The log likelihood, up to a parameter-free constant, is $-\frac{1}{2}\sum_t\big(\log\det F_t + v_t^\top F_t^{-1} v_t\big)$. Rank-constrained maximization of this likelihood, and the null distribution of the resulting likelihood-ratio statistic on the boundary of the cointegrating rank, are not proved here. In particular the critical values of the Johansen trace test are not claimed [2].

I Numerical supplementary records

I.1 Operator variance and readout variance

Figure 3 separates the two variance calculations that share the sensor contrast. The operator contrast is the vacuum moment 1. The readout contrast is a classical Gaussian residual whose population variance is 2. The Gaussian draws use seed 20261150 and 2000 histories. At $T = 50, 100, 200$ the readout endpoints are 2.022, 2.168, and 1.923. Their standardized distances from 2 are 0.34, 2.45, and -1.27. The $T = 100$ endpoint is the farthest of the three and is still inside the four-standard-error rule fixed before the run was interpreted.

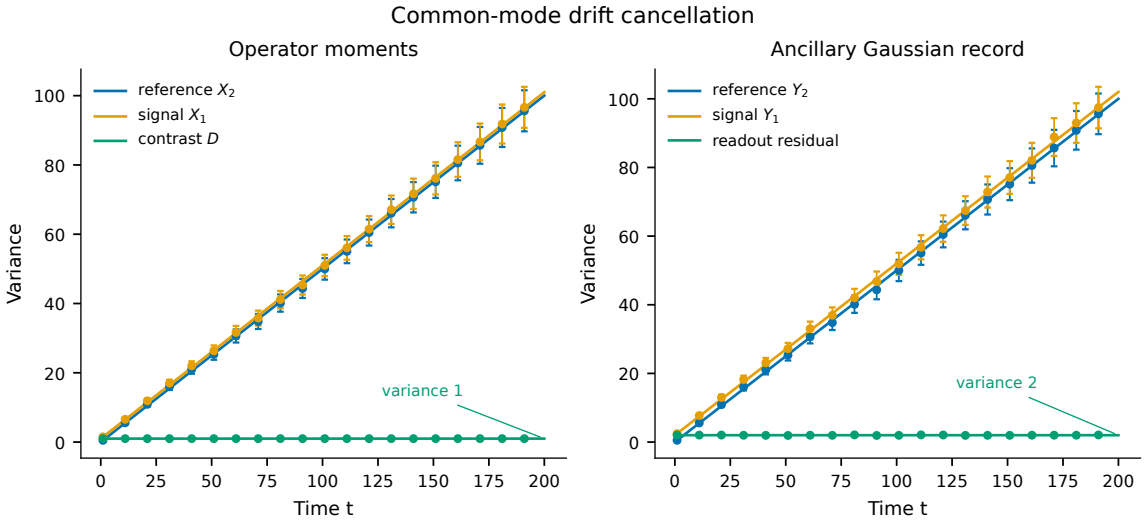


Figure 3: Operator moments and the separate Gaussian readout. The annotation "variance 1" belongs to the operator contrast. The annotation "variance 2" belongs to the analytical readout. The readout panel is not a Qiskit sample.

I.2 Finite-dimensional real contrasts

The exact total variation between $\sigma = +1$ and $\sigma = -1$ is zero for every stored contrast and every stored cutoff. Figure 4 shows the $d = 2$ shot total variation for $Z_t + Z_{t-1}$. Eight replications give mean 0.0375 (standard deviation 0.0143) at 1024 shots and mean 0.0099 (standard deviation 0.0025) at 16384 shots. The finite laws that produce total variation zero are discrete cutoff laws. They are not $N(0, 1)$. Eigenbasis shots at $d = 16$ were not run. The $d = 16$ entry in the exact-total-variation statement is the diagonalized vacuum law only.

I.3 Ordered witness and truncation

Figure 5 collects the cutoff witness and the $d = 2$ shots. The Hadamard circuit applies $e^{ivZ_{t-1}}$ first and e^{iuZ_t} second, on three Fock modes, with the ancilla as the highest qubit. Exact circuit expectations were computed for $d \leq 8$. The controlled circuit at $d = 16$ was not executed. The $d = 16$ witness used in the main text is the factored exponential. Its gap from $e^{-1}\sin(1/2)$ is about 5.6×10^{-17}. At $d = 2$ the exact imaginary part is 0.24392, and 16384 shots do not remove that truncation gap.

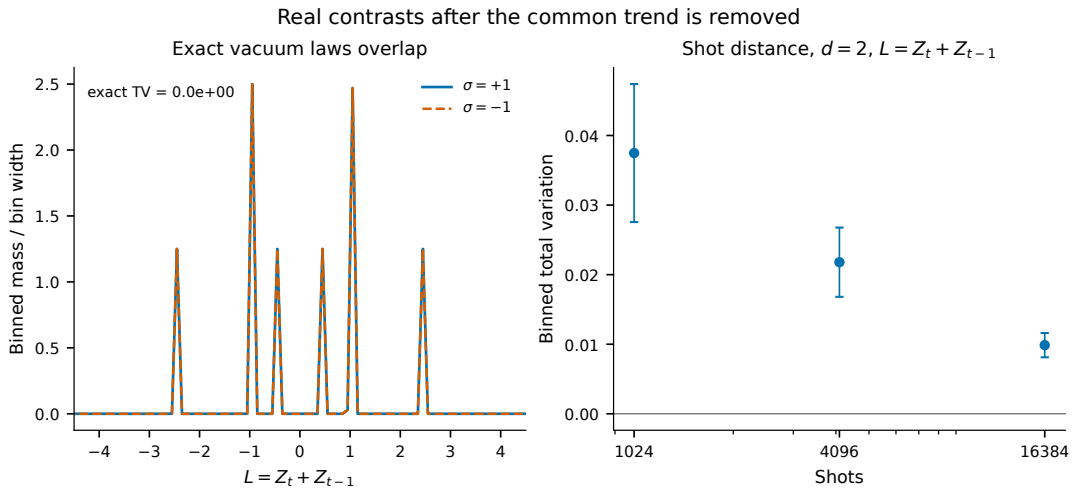


Figure 4: Real-contrast comparison. The left panel overlays exact cutoff laws. The right panel is the finite-shot total variation at $d = 2$, with a 1.96-standard-error interval across replications.

The commutator defect was evaluated on the same cutoffs. Its operator norm equals d at $d = 2, 4, 8, 16$. The vacuum expectation of the lag-one kernel remains $-i\sigma/2$, with an absolute error below 10^{-15} where the kernel was recomputed. A five-mode check with an extra reference quadrature, at cutoff 2 and 4, reproduces the robust variance $1 + \kappa^2/2$ for $\kappa \in \{0, 0.5\}$. The same five-mode check at cutoff 8 was not run.

A commuting classical pair, 20000 draws, seed 20261153, has imaginary part -0.00496 at $(0.5, 0.5)$ and -0.00520 at $(1, 1)$. The ratios of these values to their standard errors are -1.24 and -1.04.

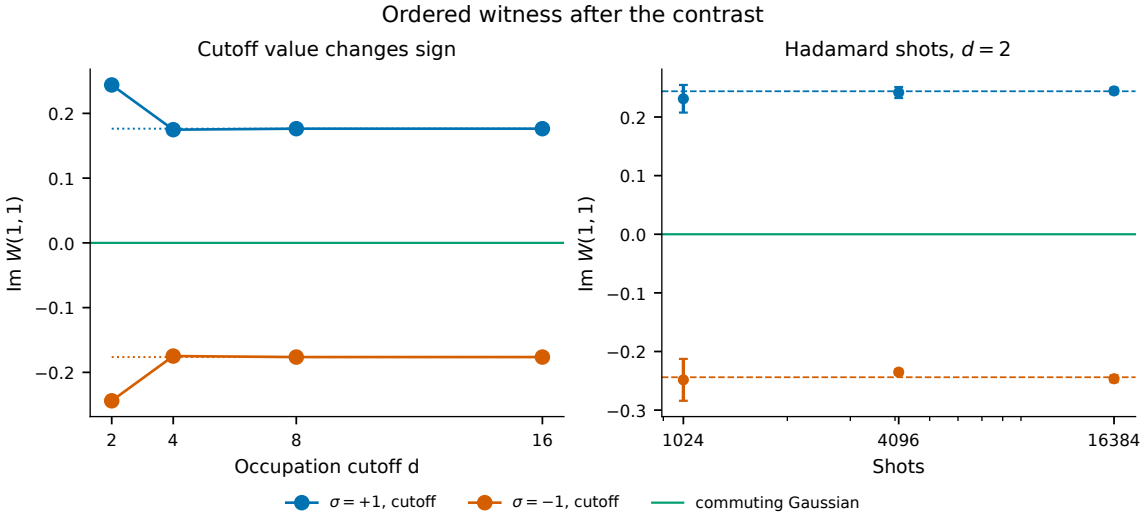


Figure 5: Cutoff witness and $d = 2$ shots. The infinite-dimensional value is the horizontal reference in the main-text panel. Shot intervals are sampling intervals at fixed cutoff. They are not intervals for the truncation gap.

I.4 Conditional size, unconditional size, and calibration noise

The residual Monte Carlo is a Gaussian triangular record, not a qubit simulation and not an equity series. Eight original cells each use one calibration bank of 399 histories and 1000 evaluation histories. The pooled conditional rejection count is $449/8000 = 5.6125\%$. A Wilson interval computed from those 1000 histories is an interval for the rejection rate given that one bank. It is not an interval for the unconditional size. Two of those conditional intervals lie entirely above 5%: 6.4% at $T = 128$, noise ratio 8, and 7.5% at $T = 512$, noise ratio 0.

Table 2 reports a later design with 40 independent banks, 200 evaluation histories per bank, and a Student t interval for the mean of the forty bank-level rates. The master seed is 20261131. Seven intervals contain 5%. The cell $T = 512$, noise ratio 0, has interval $[0.0402, 0.0493]$, which lies entirely below 5%. That cell is retained. The between-bank variance is between 0.95 and 1.99 times the binomial variance of a fixed bank, so a binomial interval would have been too narrow in the larger cells.

Table 2: Unconditional rejection rate of the matched residual test. Each entry is the mean of 40 independent bank-level rates, with a 95% Student t interval.

T	noise ratio	mean	95% interval
128	0	0.0462	[0.0409, 0.0516]
128	0.5	0.0467	[0.0417, 0.0518]
128	2	0.0529	[0.0480, 0.0578]
128	8	0.0535	[0.0473, 0.0597]
512	0	0.0447	[0.0402, 0.0493]
512	0.5	0.0499	[0.0450, 0.0548]
512	2	0.0479	[0.0411, 0.0547]
512	8	0.0511	[0.0448, 0.0574]

I.5 High-noise coverage and the conservative grid

The known-ratio coverage claim does not describe every simulated cell. At $T = 128$, noise ratio 8, and residual autocorrelation 0.5, the loading interval that counts an invalid signal as a miss covers in 0.585 of 400 histories. The corrected residual-autocorrelation root-mean-square error in that cell is 67.67, and the fraction of histories with a valid signal is 0.83. At $T = 512$ in the same noise and autocorrelation cell, the coverage is 0.8025 and the corrected root-mean-square error is 2.57. At $T = 512$, noise ratio 2, and autocorrelation 0.5, the naive autocorrelation bias is -0.305 and the corrected bias is -0.0054.

An independently estimated noise ratio is outside the known-ratio theorem. In the same high-noise cell the estimated-ratio coverage is 0.5625 at $T = 128$ and 0.7275 at $T = 512$.

The conservative test uses a fixed 16-point geometric grid from 0.05 to 40, with $\delta = 0.01$. The grid was not retuned after the results were seen. On all twelve stored cells the conservative 5% rejection count is 0/300. The median grid slack runs from about 0.45 at $T = 128$ to about 1.52 at $T = 512$. An empty rejection region is a property of this conservative calibration, not evidence that the hypothesis is true. The plug-in rate on the same cells ranges from 3.7% to 8.3% and has no exact finite-sample guarantee.

I.6 A classical likelihood for the same readout

A Gaussian state-space likelihood for the ordinary-least-squares residual, with a diffuse initial variance 10^6 on the unit-root state, was evaluated on 60 histories after a calibration bank of 49. Under the null, the rejection counts at noise ratios 0, 2, and 8 are 7/60, 4/60, and 5/60. At residual autocorrelation 0.5 the count is 60/60 in each of those noise cells. At autocorrelation 0.9 the rates are 0.683, 0.533, and 0.583. These counts show that a correctly specified classical likelihood can detect the same Gaussian alternative. The calibration bank is short, and the intervals are wide. The comparison is not a ranking of methods. An earlier run pinned the residual at zero and produced a likelihood ratio with no power. That run was replaced after the initialization was corrected, and the replacement is recorded in the JSON file.

I.7 Permanent offset

The initialization theorem separates a random-walk loading from a constant offset. Demeaning one series does not identify a constant that is common to the contrast. In the coarse diagnostic, with threshold 0.442 at $T = 128$, the fractions of 200 runs falling below the threshold are 1, 1, and 0 at offset scales $h = 0$, 0.2, and 1. The small offset is not declared. The large offset is declared in every stored run. The diagnostic is a coarse screen, and the small-offset result is a limitation of that screen.

I.8 What the equity calculations are not

Three equity pairs were examined in a separate archive and are not used as evidence for any quantum statement. The archive does not estimate an odd spectrum, and it does not represent a price as a noncommuting operator. No result from that archive is cited in the main text.

I.9 Acceptance record of the sensor run

The sensor run that produced the JSON files was scored against thresholds written before the output was interpreted. The stored score is 16 pass, 0 fail, 1 open, and 2 limitations. The open item is the continuous-variable pointer circuit, which was not built. The limitations are the unexecuted $d = 16$ eigenbasis shots and the unexecuted $d = 16$ controlled Hadamard circuit. The packaged smoke test repeats three exact identities at cutoff 2. It does not recompute this score, and a pass in the stored score is a numerical comparison against a stored criterion, not a declaration that every open item has been closed.

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Statements and Declarations

Data availability. The numerical values in the article and the supplement are taken from the JSON records in the ancillary data and replication directories. No laboratory dataset was collected.

Code availability. The operator routines, the witness circuit, the figure data, and a smoke test are in the `anc/replication` directory of this submission. The smoke test checks the kernel sign, one exact Hadamard expectation at cutoff 2, and one exact total variation. It does not repeat the 2000-history Monte Carlo or the $d = 16$ diagonalizations. Those results are the stored records of the completed run.

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Competing interests. The authors declare no competing interests.

Use of artificial intelligence. Cursor, with a language model, was used on 10–11 November 2026 to assemble the replication package, check stored numerical files, and draft prose from the authors' existing proofs and experiment records. The final source assembly and mathematical-consistency checks were assisted by Codex on 11 November 2026. Responsibility for the scientific claims and final approval rests with the authors. The model is not an author.