

Supplementary Information: How Does Misperception Impact Zero-Determinant Strategies in Iterated Prisoner's Dilemma?

Zhaoyang Cheng^{1,2}, Guanpu Chen^{1,2}, and Yiguang Hong^{3,1,*}

¹Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Beijing, 100190, China

²School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100190, China

³Department of Control Science and Engineering, Tongji University, Shanghai, 201804, China

*yghong@iss.ac.cn

1 Proof of Theorem 1

Suppose that any ZD strategy $\mathbf{p}$ of player X in G_1 is also a ZD strategy in G_2 . Then for any α, β, γ , and $\mathbf{p} = \alpha \mathbf{S}_X^{\omega_1} + \beta \mathbf{S}_Y^{\omega_1} + \gamma \mathbf{1} + \mathbf{p}_0$, there exists α', β', γ' such that $\mathbf{p} = \alpha' \mathbf{S}_X^{\omega_2} + \beta' \mathbf{S}_Y^{\omega_2} + \gamma' \mathbf{1} + \mathbf{p}_0$. Then the following equations has at least one solution:

$$\begin{aligned}\alpha' R_2 + \beta' R_2 + \gamma' &= \alpha R_1 + \beta R_1 + \gamma, \\ \alpha' S_2 + \beta' T_2 + \gamma' &= \alpha S_1 + \beta T_1 + \gamma, \\ \alpha' T_2 + \beta' S_2 + \gamma' &= \alpha T_1 + \beta S_1 + \gamma, \\ \alpha' P_2 + \beta' P_2 + \gamma' &= \alpha P_1 + \beta P_1 + \gamma.\end{aligned}\tag{1}$$

Notice that the first three equations of (1) have a solution:

$$\begin{aligned}\alpha' &= \frac{1}{2} \left(\frac{(\alpha + \beta)(2R_1 - T_1 - S_1)}{2R_2 - T_2 - S_2} + \frac{(\alpha - \beta)(S_1 - T_1)}{S_2 - T_2} \right), \\ \beta' &= \frac{1}{2} \left(\frac{(\alpha + \beta)(2R_1 - T_1 - S_1)}{2R_2 - T_2 - S_2} - \frac{(\alpha - \beta)(S_1 - T_1)}{S_2 - T_2} \right), \\ \gamma' &= \alpha R_1 + \beta R_1 + \gamma - R_2 \frac{(\alpha + \beta)(2R_1 - T_1 - S_1)}{2R_2 - T_2 - S_2}.\end{aligned}\tag{2}$$

Since $\det \begin{bmatrix} R_2 & R_2 & 1 \\ S_2 & T_2 & 1 \\ T_2 & S_2 & 1 \end{bmatrix} = (T_2 - S_2)(2R_2 - S_2 - T_2) \neq 0$, (2) is the unique solution of the first three equations of (1). Then

$$\begin{aligned}0 &= \alpha' P_2 + \beta' P_2 + \gamma' - \alpha P_1 + \beta P_1 + \gamma, \\ &= \frac{(\alpha + \beta)(2R_1 - T_1 - S_1)}{2R_2 - T_2 - S_2} P_2 + \alpha R_1 + \beta R_1 + \gamma - R_2 \frac{(\alpha + \beta)(2R_1 - T_1 - S_1)}{2R_2 - T_2 - S_2} - \alpha P_1 + \beta P_1 + \gamma \\ &= \alpha \left(\frac{2R_1 - T_1 - S_1}{2R_2 - T_2 - S_2} (P_2 - R_2) + (R_1 - P_1) \right) + \beta \alpha \left(\frac{2R_1 - T_1 - S_1}{2R_2 - T_2 - S_2} (P_2 - R_2) + (R_1 - P_1) \right).\end{aligned}$$

By the arbitrariness of α, β, γ ,

$$\frac{2R_1 - T_1 - S_1}{2R_2 - T_2 - S_2} (P_2 - R_2) + (R_1 - P_1) = 0.\tag{3}$$

Then (3) is equivalent to

$$\frac{R_1 - P_1}{2R_1 - S_1 - T_1} = \frac{R_2 - P_2}{2R_2 - S_2 - T_2}.$$

On the contrary, suppose that $\frac{R_1 - P_1}{2R_1 - S_1 - T_1} \neq \frac{R_2 - P_2}{2R_2 - S_2 - T_2}$. For any α, β, γ , and $\mathbf{p} = \alpha \mathbf{S}_X^{\omega_1} + \beta \mathbf{S}_Y^{\omega_1} + \gamma \mathbf{1} + \mathbf{p}_0$, take α', β', γ' as (2). Then (1) holds if $\frac{R_1 - P_1}{2R_1 - S_1 - T_1} = \frac{R_2 - P_2}{2R_2 - S_2 - T_2}$. Thus, any ZD strategy $\mathbf{p}$ of player X in G_1 is also a ZD strategy in G_2 .

2 Proof of Corollary 1

Suppose that player X 's any equalizer strategy $\mathbf{p}$ in G_1 is also an equalizer strategy in G_2 . Then for any β, γ , and $\mathbf{p} = \beta \mathbf{S}_Y^{\omega_1} + \gamma \mathbf{1} + \mathbf{p}_0$, there exists β', γ' such that $\mathbf{p} = \beta' \mathbf{S}_Y^{\omega_2} + \gamma' \mathbf{1} + \mathbf{p}_0$. Then the following equations has at least one solution:

$$\begin{aligned}\beta' R_2 + \gamma' &= \beta R_1 + \gamma, \\ \beta' T_2 + \gamma' &= \beta T_1 + \gamma, \\ \beta' S_2 + \gamma' &= \beta S_1 + \gamma, \\ \beta' P_2 + \gamma' &= \beta P_1 + \gamma.\end{aligned}\tag{4}$$

The first two equations of (4) have a solution:

$$\begin{aligned}\beta' &= \beta \frac{R_1 - T_1}{R_2 - T_2}, \\ \gamma' &= \beta R_1 + \gamma - \beta R_2 \frac{R_1 - T_1}{R_2 - T_2}.\end{aligned}\tag{5}$$

Since $\det \begin{bmatrix} R_2 & 1 \\ T_2 & 1 \end{bmatrix} = R_2 - T_2 \neq 0$, (5) is the unique solution of the first two equations of (4). Then the last two equations of (4) are converted to:

$$\begin{aligned}\beta \frac{R_1 - T_1}{R_2 - T_2} &= \beta \frac{R_1 - S_1}{R_2 - S_2}, \\ \beta \frac{R_1 - T_1}{R_2 - T_2} &= \beta \frac{R_1 - P_1}{R_2 - P_2}.\end{aligned}\tag{6}$$

By the arbitrariness of β, γ , we have

$$\frac{R_1 - P_1}{R_2 - P_2} = \frac{R_1 - T_1}{R_2 - T_2} = \frac{R_1 - S_1}{R_2 - S_2}.$$

On the contrary, suppose that $\frac{R_1 - P_1}{R_2 - P_2} = \frac{R_1 - T_1}{R_2 - T_2} = \frac{R_1 - S_1}{R_2 - S_2}$. For any β, γ , and $\mathbf{p} = \beta \mathbf{S}_Y^{\omega_1} + \gamma \mathbf{1} + \mathbf{p}_0$, take β', γ' as (5). (4) holds if $\frac{R_1 - P_1}{R_2 - P_2} = \frac{R_1 - T_1}{R_2 - T_2} = \frac{R_1 - S_1}{R_2 - S_2}$. Thus, player X 's any equalizer strategy $\mathbf{p}$ in G_1 is also an equalizer strategy in G_2 .

3 Proof of Corollary 2

For any $\phi, \chi > 1$, and $\mathbf{p} = \phi[(\mathbf{S}_X^{\omega_1} - P_1 \mathbf{1}) - \chi(\mathbf{S}_Y^{\omega_1} - P_1 \mathbf{1})] + \mathbf{p}_0$, we want to find if there exists ϕ', χ' such that

$$\mathbf{p} = \phi'[(\mathbf{S}_X^{\omega_2} - P_2 \mathbf{1}) - \chi'(\mathbf{S}_Y^{\omega_2} - P_2 \mathbf{1})] + \mathbf{p}_0, \chi' \geq 1.\tag{7}$$

(7) is equivalent to the following equations:

$$\begin{aligned}\phi'[(R_2 - P_2) - \chi'(R_2 - P_2)] &= \phi[(R_1 - P_1) - \chi(R_1 - P_1)], \\ \phi'[(S_2 - P_2) - \chi'(S_2 - P_2)] &= \phi[(S_1 - P_1) - \chi(S_1 - P_1)], \\ \phi'[(T_2 - P_2) - \chi'(T_2 - P_2)] &= \phi[(T_1 - P_1) - \chi(T_1 - P_1)].\end{aligned}\tag{8}$$

If $\phi \neq 0$, we have $\phi' \neq 0$. The first two equations of (8) have a solution:

$$\begin{aligned}\phi' &= \frac{\phi[(S_1 - P_1) - \chi(T_1 - P_1)](R_2 - P_2) - \phi[(R_1 - P_1) - \chi(R_1 - P_1)](T_2 - P_2)}{(R_2 - P_2)(S_2 - T_2)}, \\ \chi' &= 1 - \frac{\phi[(R_1 - P_1) - \chi(R_1 - P_1)]}{\phi'(R_2 - P_2)}.\end{aligned}\tag{9}$$

Then the last equation of (8) is converted to:

$$\begin{aligned}& \phi'[(T_2 - P_2) - \chi'(S_2 - P_2)] - \phi[(T_1 - P_1) - \chi(S_1 - P_1)] \\ &= \phi' \left[(T_2 - P_2) - \left(1 - \frac{\phi[(R_1 - P_1) - \chi(R_1 - P_1)]}{\phi'(R_2 - P_2)} \right) (S_2 - P_2) \right] - \phi[(T_1 - P_1) - \chi(S_1 - P_1)] \\ &= \phi'(T_2 - S_2) + \frac{1}{R_2 - P_2} \phi[(R_1 - P_1)(S_2 - P_2) - \chi(R_1 - P_1)(S_2 - P_2) - (T_1 - P_1)(R_2 - P_2) + \chi(S_1 - P_1)(R_2 - P_2)] \\ &= \frac{\phi}{R_2 - P_2} [-(S_1 + T_1 - 2P_1)(R_2 - P_2) + (R_1 - P_1)(T_2 + S_2 - 2P_2) + \chi(T_1 + S_1 - 2P_1)(R_2 - P_2) - \chi(R_1 - P_1)(T_2 + S_2 - P_2)] \\ &= \frac{\phi}{R_2 - P_2} [(1 - \chi)((R_1 - P_1)(T_2 + S_2 - 2P_2) - (S_1 + T_1 - 2P_1)(R_2 - P_2))]\end{aligned}$$

Thus, $\phi'[(T_2 - P_2) - \chi'(S_2 - P_2)] - \phi[(T_1 - P_1) - \chi(S_1 - P_1)] = 0$ if $\frac{R_1 - P_1}{2R_1 - S_1 - T_1} = \frac{R_2 - P_2}{2R_2 - S_2 - T_2}$. Besides,

$$\begin{aligned}\chi' &= 1 - \frac{[(R_1 - P_1) - \chi(R_1 - P_1)](S_2 - T_2)}{[(S_1 - P_1) - \chi(T_1 - P_1)](R_2 - P_2) - [(R_1 - P_1) - \chi(R_1 - P_1)](T_2 - P_2)} \\ &= 1 - \frac{(1 - \chi)(R_1 - P_1)(S_2 - T_2)}{(S_1 - P_1)(R_2 - P_2) - (R_1 - P_1)(T_2 - P_2) - \chi((T_1 - P_1)(R_2 - P_2) - (R_1 - P_1)(T_2 - P_2))}\end{aligned}$$

Since $(S_1 - P_1)(R_2 - P_2) - (R_1 - P_1)(T_2 - P_2) - \chi((T_1 - P_1)(R_2 - P_2) - (R_1 - P_1)(T_2 - P_2)) < 0$, we have $\chi' > 1$. Actually, if $\phi = 0$, we have $\phi' = 0$ and $\mathbf{p} = \mathbf{p}_0 = [1, 1, 0, 0]^T$. Then $\mathbf{p}$ is also an extortion strategy in G_2 .

4 Proof of Corollary 3

For any $\phi, \chi > 1$, and $\mathbf{p} = \phi[(\mathbf{S}_X^{\omega_1} - R_1 \mathbf{1}) - \chi(\mathbf{S}_Y^{\omega_1} - R_1 \mathbf{1})]$, we want to find if there exists ϕ', χ' such that

$$\phi'[(\mathbf{S}_X^{\omega_2} - R_2 \mathbf{1}) - \chi'(\mathbf{S}_Y^{\omega_2} - R_2 \mathbf{1})] = \mathbf{p}. \quad (10)$$

(10) is equivalent to the following equations:

$$\begin{aligned}\phi'[(P_2 - R_2) - \chi'(P_2 - R_2)] &= \phi[(P_1 - R_1) - \chi(P_1 - R_1)], \\ \phi'[(S_2 - R_2) - \chi'(T_2 - R_2)] &= \phi[(S_1 - R_1) - \chi(T_1 - R_1)], \\ \phi'[(T_2 - R_2) - \chi'(S_2 - R_2)] &= \phi[(T_1 - R_1) - \chi(S_1 - R_1)].\end{aligned} \quad (11)$$

Similar to the proof of Corollary 2, if $\phi \neq 0$, the first two equations of (12) have a solution:

$$\begin{aligned}\phi' &= \frac{\phi[(S_1 - R_1) - \chi(T_1 - R_1)](R_2 - P_2) - \phi[(R_1 - P_1) - \chi(R_1 - P_1)](T_2 - R_2)}{(R_2 - P_2)(S_2 - T_2)}, \\ \chi' &= 1 - \frac{\phi[(R_1 - P_1) - \chi(R_1 - P_1)]}{\phi'(R_2 - P_2)}.\end{aligned} \quad (12)$$

Also, $\phi'[(T_2 - R_2) - \chi'(S_2 - R_2)] = \phi[(T_1 - R_1) - \chi(S_1 - R_1)]$ if $\frac{R_1 - P_1}{2R_1 - S_1 - T_1} = \frac{R_2 - P_2}{2R_2 - S_2 - T_2}$. Besides, we also have

$$\chi' = 1 - \frac{(1 - \chi)(R_1 - P_1)(S_2 - T_2)}{(S_1 - R_1)(R_2 - P_2) - (R_1 - P_1)(T_2 - R_2) - \chi((T_1 - R_1)(R_2 - P_2) - (R_1 - P_1)(T_2 - R_2))}$$

If $(S_1 - R_1)(R_2 - P_2) - (R_1 - P_1)(T_2 - R_2) - \chi((T_1 - R_1)(R_2 - P_2) - (R_1 - P_1)(T_2 - R_2)) < 0$, we have $\chi' > 1$. Thus, $\mathbf{p}$ is an extortion strategy in G_2 when $\phi \neq 0$. Actually, if $\phi = 0$, we have $\phi' = 0$ and $\mathbf{p} = \mathbf{p}_0 = [1, 1, 0, 0]^T$. Then $\mathbf{p}$ is also a generous strategy in G_2 .

5 Proof of Theorem 2

$S_X^{\omega_i}$, $S_Y^{\omega_i}$, and $\mathbf{1}$ are orthogonal since $\det \begin{bmatrix} R_i & R_i & 1 \\ S_i & T_i & 1 \\ T_i & S_i & 1 \end{bmatrix} = (T_i - S_i)(2R_i - S_i - T_i) \neq 0$. Denote Φ_i as the space spanned by $S_X^{\omega_i}$, $S_Y^{\omega_i}$, and $\mathbf{1}$. Consider $L_i = [2P_i - S_i - T_i, R_i - P_i, R_i - P_i, T_i + S_i - 2R_i]^T$. Then we have

$$\begin{aligned}L_i^T S_X^{\omega_i} &= (2P_i - S_i - T_i)R_i + (R_i - P_i)S_i + (R_i - P_i)T_i + (T_i + S_i - 2R_i)P_i = 0, \\ L_i^T S_Y^{\omega_i} &= (2P_i - S_i - T_i)R_i + (R_i - P_i)T_i + (R_i - P_i)S_i + (T_i + S_i - 2R_i)P_i = 0, \\ L_i^T \mathbf{1} &= (2P_i - S_i - T_i) + (R_i - P_i) + (R_i - P_i) + (T_i + S_i - 2R_i) = 0.\end{aligned}$$

Also, $\|L_i\|_2^2 = (2P_i - S_i - T_i)^2 + 2(R_i - P_i)^2 + (T_i + S_i - 2R_i)^2 > 0$. Thus, $S_X^{\omega_i}$, $S_Y^{\omega_i}$, $\mathbf{1}$, and L_2 are orthogonal in $\mathbb{R}^4$, and L_i is orthogonal to Φ_i . According to Press and Dyson¹, the nonzero canonical angles between ϕ_1 and ϕ_2 is

$$\theta = \arccos \frac{L_1^T L_2}{\|L_1\|_2 \|L_2\|_2}.$$

For any α, β, γ , there exists α', β', γ' such that $\alpha S_X^{\omega_1} + \beta S_Y^{\omega_1} + \gamma \mathbf{1} = \alpha' S_X^{\omega_2} + \beta' S_Y^{\omega_2} + \gamma' \mathbf{1} + \xi \frac{L_2}{\|L_2\|_2}$, where $\xi = \|\mathbf{p}\|_2 \sin \theta$ is the norm of the projection from $\alpha S_X^{\omega_1} + \beta S_Y^{\omega_1} + \gamma \mathbf{1}$ to $\frac{L_2}{\|L_2\|_2}$.

For any $\mathbf{f} = [f_1, f_2, f_3, f_4]^T \in \mathbb{R}^4$, $\mathbf{p}' = [p'_{cc}, p'_{cd}, p'_{dc}, p'_{dd}]^T$, $\mathbf{q}' = [q'_{cc}, q'_{dc}, q'_{cd}, q'_{dd}]^T$, let

$$D(\mathbf{p}', \mathbf{q}', \mathbf{f}) = \det \begin{bmatrix} p'_{cc}q'_{cc} - 1 & p'_{cc} - 1 & q'_{cc} - 1 & f_1 \\ p'_{cd}q'_{dc} & p'_{cd} - 1 & q'_{dc} & f_2 \\ p'_{dc}q'_{cd} & p'_{dc} & q'_{cd} - 1 & f_3 \\ p'_{dd}q'_{dd} & p'_{dd} & q'_{dd} & f_4 \end{bmatrix}.$$

According to Press and Dyson¹,

$$\alpha' u_X^{\omega_2}(\mathbf{p}, \mathbf{q}) + \beta' u_Y^{\omega_2}(\mathbf{p}, \mathbf{q}) + \gamma' = \frac{D(\mathbf{p}, \mathbf{q}, \alpha' \mathbf{S}_X^{\omega_2} + \beta' \mathbf{S}_Y^{\omega_2} + \gamma' \mathbf{1})}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})},$$

Then

$$\begin{aligned} D(\mathbf{p}, \mathbf{q}, \alpha' \mathbf{S}_X^{\omega_2} + \beta' \mathbf{S}_Y^{\omega_2} + \gamma' \mathbf{1}) &= \det \begin{bmatrix} p_{cc}q_{cc} - 1 & \alpha R_1 + \beta R_1 + \gamma & q_{cc} - 1 & \alpha' R_2 + \beta' R_2 + \gamma' \\ p_{cd}q_{dc} & \alpha S_1 + \beta T_1 + \gamma & q_{dc} & \alpha' S_2 + \beta' T_2 + \gamma' \\ p_{dc}q_{cd} & \alpha T_1 + \beta S_1 + \gamma & q_{cd} - 1 & \alpha' T_2 + \beta' S_2 + \gamma' \\ p_{dd}q_{dd} & \alpha P_1 + \beta P_1 + \gamma & q_{dd} & \alpha' P_2 + \beta' P_2 + \gamma' \end{bmatrix} \\ &= \det \begin{bmatrix} p_{cc}q_{cc} - 1 & \alpha R_1 + \beta R_1 + \gamma & q_{cc} - 1 & \xi \frac{1}{\|L_2\|_2} (2P_2 - S_2 - T_2) \\ p_{cd}q_{dc} & \alpha S_1 + \beta T_1 + \gamma & q_{dc} & \xi \frac{1}{\|L_2\|_2} (R_2 - P_2) \\ p_{dc}q_{cd} & \alpha T_1 + \beta S_1 + \gamma & q_{cd} - 1 & \xi \frac{1}{\|L_2\|_2} (R_2 - P_2) \\ p_{dd}q_{dd} & \alpha P_1 + \beta P_1 + \gamma & q_{dd} & \xi \frac{1}{\|L_2\|_2} (T_2 + S_2 - 2R_2) \end{bmatrix} \\ &= \xi \frac{1}{\|L_2\|_2} \det \begin{bmatrix} p_{cc}q_{cc} - 1 & \alpha R_1 + \beta R_1 + \gamma & q_{cc} - 1 & 2P_2 - S_2 - T_2 \\ p_{cd}q_{dc} & \alpha S_1 + \beta T_1 + \gamma & q_{dc} & R_2 - P_2 \\ p_{dc}q_{cd} & \alpha T_1 + \beta S_1 + \gamma & q_{cd} - 1 & R_2 - P_2 \\ p_{dd}q_{dd} & \alpha P_1 + \beta P_1 + \gamma & q_{dd} & T_2 + S_2 - 2R_2 \end{bmatrix} \\ &= \|\mathbf{p}\|_2 \sin \theta \frac{1}{\|L_2\|_2} D(\mathbf{p}, \mathbf{q}, L_2). \end{aligned}$$

Thus,

$$\alpha' u_X^{\omega_2}(\mathbf{p}, \mathbf{q}) + \beta' u_Y^{\omega_2}(\mathbf{p}, \mathbf{q}) + \gamma' = \|\mathbf{p}\|_2 \sin \theta \frac{1}{\|L_2\|_2} \frac{D(\mathbf{p}, \mathbf{q}, L_2)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}.$$

Actually,

$$\frac{D(\mathbf{p}, \mathbf{q}, L_2)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})} = v_{cc}(2P_2 - S_2 - T_2) + v_{cd}(R_2 - P_2) + v_{dc}(R_2 - P_2) + v_{dd}(T_2 + S_2 - 2R_2),$$

where

$$\begin{aligned} v_{cc} &= -\frac{1}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})} \det \begin{bmatrix} p_{cd}q_{dc} & p_{cd} - 1 & q_{dc} \\ p_{dc}q_{cd} & p_{dc} & q_{cd} - 1 \\ p_{dd}q_{dd} & p_{dd} & q_{dd} \end{bmatrix}, \quad v_{cd} = \frac{1}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})} \det \begin{bmatrix} p_{cc}q_{cc} - 1 & p_{cc} - 1 & q_{cc} - 1 \\ p_{dc}q_{cd} & p_{dc} & q_{cd} - 1 \\ p_{dd}q_{dd} & p_{dd} & q_{dd} \end{bmatrix}, \\ v_{dc} &= -\frac{1}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})} \det \begin{bmatrix} p_{cc}q_{cc} - 1 & p_{cc} - 1 & q_{cc} - 1 \\ p_{cd}q_{dc} & p_{cd} - 1 & q_{dc} \\ p_{dd}q_{dd} & p_{dd} & q_{dd} \end{bmatrix}, \quad v_{dd} = \frac{1}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})} \det \begin{bmatrix} p_{cc}q_{cc} - 1 & p_{cc} - 1 & q_{cc} - 1 \\ p_{cd}q_{dc} & p_{cd} - 1 & q_{dc} \\ p_{dc}q_{cd} & p_{dc} & q_{cd} - 1 \end{bmatrix}. \end{aligned}$$

According to Press and Dyson¹, $v_{xy} \in [0, 1]$, $xy \in \{cc, dc, cd, dd\}$, and $v_{cc} + v_{cd} + v_{dc} + v_{dd} = 1$. Then

$$\left| \frac{D(\mathbf{p}, \mathbf{q}, L_2)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})} \right| \leq \max\{|2P_2 - S_2 - T_2|, |R_2 - P_2|, |R_2 - P_2|, |T_2 + S_2 - 2R_2|\} = \|L_2\|_\infty.$$

Therefore, $|\alpha' u_X^{\omega_2}(\mathbf{p}, \mathbf{q}) + \beta' u_Y^{\omega_2}(\mathbf{p}, \mathbf{q}) + \gamma'| \leq \|\mathbf{p}\|_2 \frac{\|L_2\|_\infty}{\|L_2\|_2} \sin \theta, \forall \mathbf{q}$.

6 Proof of Theorem 3

For any α, β, γ , we have

$$\begin{aligned} &\alpha u_X^{\omega_2}(\mathbf{p}, \mathbf{q}) + \beta u_Y^{\omega_2}(\mathbf{p}, \mathbf{q}) + \gamma \\ &= \frac{D(\mathbf{p}, \mathbf{q}, \alpha \mathbf{S}_X^{\omega_2} + \beta \mathbf{S}_Y^{\omega_2} + \gamma \mathbf{1})}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}, \end{aligned}$$

Acutally,

$$\begin{aligned}
D(\mathbf{p}, \mathbf{q}, \alpha' \mathbf{S}_X^{\omega_2} + \beta' \mathbf{S}_Y^{\omega_2} + \gamma' \mathbf{1}) &= \det \begin{bmatrix} p_{cc}q_{cc} - 1 & \alpha R_1 + \beta R_1 + \gamma & q_{cc} - 1 & \alpha R_2 + \beta R_2 + \gamma \\ p_{cd}q_{dc} & \alpha S_1 + \beta T_1 + \gamma & q_{dc} & \alpha S_2 + \beta T_2 + \gamma \\ p_{dc}q_{cd} & \alpha T_1 + \beta S_1 + \gamma & q_{cd} - 1 & \alpha T_2 + \beta S_2 + \gamma \\ p_{dd}q_{dd} & \alpha P_1 + \beta P_1 + \gamma & q_{dd} & \alpha P_2 + \beta P_2 + \gamma \end{bmatrix} \\
&= \det \begin{bmatrix} p_{cc}q_{cc} - 1 & \alpha R_1 + \beta R_1 + \gamma & q_{cc} - 1 & (\alpha + \beta)(R_2 - R_1) \\ p_{cd}q_{dc} & \alpha S_1 + \beta T_1 + \gamma & q_{dc} & \alpha(S_2 - S_1) + \beta(T_2 - T_1) \\ p_{dc}q_{cd} & \alpha T_1 + \beta S_1 + \gamma & q_{cd} - 1 & \alpha(T_2 - T_1) + \beta(S_2 - S_1) \\ p_{dd}q_{dd} & \alpha P_1 + \beta P_1 + \gamma & q_{dd} & (\alpha + \beta)(P_2 - P_1) \end{bmatrix}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\alpha u_X^{\omega_2}(\mathbf{p}, \mathbf{q}) + \beta u_Y^{\omega_2}(\mathbf{p}, \mathbf{q}) + \gamma \\
&= v_{cc}(\alpha + \beta)(R_2 - R_1) + v_{cd}(\alpha(S_2 - S_1) + \beta(T_2 - T_1)) + v_{dc}(\alpha(T_2 - T_1) + \beta(S_2 - S_1)) + v_{dd}(\alpha + \beta)(P_2 - P_1),
\end{aligned}$$

where $v_{xy}, xy \in \{cc, dc, cd, dd\}$ is the same in the proof of Theorem 2. Since $v_{xy} \in [0, 1], xy \in \{cc, dc, cd, dd\}$, and $v_{cc} + v_{cd} + v_{dc} + v_{dd} = 1$, we have

$$\min(\Gamma) \leq \alpha u_X^{\omega_2}(\mathbf{p}, \mathbf{q}) + \beta u_Y^{\omega_2}(\mathbf{p}, \mathbf{q}) + \gamma \leq \max(\Gamma), \quad (13)$$

where

$$\Gamma = \{(\alpha + \beta)(R_2 - R_1), \alpha(S_2 - S_1) + \beta(T_2 - T_1), \alpha(T_2 - T_1) + \beta(S_2 - S_1), (\alpha + \beta)(P_2 - P_1)\}.$$

7 Proof of Corollary 4

When player Y chooses the always cooperate (ALLC) strategy, $(u_X^{\omega_i}, u_Y^{\omega_i})$ is on $(R_i - S_i)u_X^{\omega_i} + (T_i - R_i)u_Y^{\omega_i} = R_i(T_i - S_i)$ in $G_i, i \in \{1, 2\}$. For any $\beta \neq 0, \gamma$, suppose player X chooses an equalizer strategy $\mathbf{p} = \beta \mathbf{S}_Y^{\omega_1} + \gamma \mathbf{1} + \mathbf{p}_0$ in G_1 . When player Y choose the ALLC strategy, player X get the supremum of its expected utility, $\bar{u}_X^{\omega_1}$, in G_1 . $\bar{u}_X^{\omega_1}$ is the solution of the following equations:

$$\begin{cases} \beta u_Y + \gamma = 0, \\ (R_1 - S_1)u_X + (T_1 - R_1)u_Y - R_1(T_1 - S_1) = 0. \end{cases}$$

$$\text{Then } \bar{u}_X^{\omega_1} = \frac{R_1(T_1 - S_1) + (T_1 - R_1)\frac{\gamma}{\beta}}{(R_1 - S_1)}.$$

According to Theorem 3, $\min(\Gamma_1) \leq \beta u_Y^{\omega_2}(\mathbf{p}, \mathbf{q}) + \gamma \leq \max(\Gamma_1)$, where

$$\Gamma = \{\beta(R_2 - R_1), \beta(T_2 - T_1), \beta(S_2 - S_1), \beta(P_2 - P_1)\}.$$

Thus, $|\beta u_Y^{\omega_2}(\mathbf{p}, \mathbf{q}) + \gamma| \leq |\beta| \delta$, where $\delta = \max\{|R_2 - R_1|, |S_2 - S_1|, |T_2 - T_1|, |P_2 - P_1|\}$. Then $u_Y^{\omega_2} \leq \delta - \frac{\gamma}{\beta}$. $u_X^1 = \frac{R_2(T_2 - S_2) - (T_2 - R_2)(\delta - \frac{\gamma}{\beta})}{(R_2 - S_2)}$ is a solution of the following equations:

$$\begin{cases} u_Y = \delta - \frac{\gamma}{\beta}, \\ (R_2 - S_2)u_X + (T_2 - R_2)u_Y = R_2(T_2 - S_2). \end{cases}$$

Also, $u_X^2 = \frac{R_2(T_2 - S_2) - (R_2 - S_2)(\delta - \frac{\gamma}{\beta})}{(T_2 - R_2)}$ is the solution of the following equations:

$$\begin{cases} u_Y = \delta - \frac{\gamma}{\beta}, \\ (T_2 - R_2)u_X + (R_2 - S_2)u_Y = R_2(T_2 - S_2). \end{cases}$$

When $a_i^1 \frac{\gamma}{\beta} > b_i^1, i \in \{1, 2\}$, we have

$$[(R_1 - S_1)(T_2 - R_2) - (R_2 - S_2)(T_1 - R_1)] \frac{\gamma}{\beta} > (R_2 - S_2)R_1(T_1 - S_1) - (R_1 - S_1)R_2(T_2 - S_2) + \delta(R_1 - S_1)(T_2 - R_2),$$

$$[(R_1 - S_1)(R_2 - S_2) - (T_2 - R_2)(T_1 - R_1)] \frac{\gamma}{\beta} > (T_2 - R_2)R_1(T_1 - S_1) - (R_1 - S_1)R_2(T_2 - S_2) + \delta(R_1 - S_1)(R_2 - S_2).$$

Thus, $u_X^1 > \bar{u}_X^{\omega_1}$, and $u_X^2 > \bar{u}_X^{\omega_1}$. Since u_X^1 and u_X^2 is lower than player X 's utility when player Y chooses ALLC strategy, the supremum of player X 's expected utility in G_2 is higher than it in G_1 .

8 Proof of Corollary 5

For any $\phi, \chi > 1$, suppose player X chooses extortion strategy $\mathbf{p} = \phi[(\mathbf{S}_X^{\omega_1} - P_1 \mathbf{1}) - \chi(\mathbf{S}_Y^{\omega_1} - P_1 \mathbf{1})] + \mathbf{p}_0$ in G_1 . When player Y choose the ALLC strategy, player X get the supremum of its expected utility, $\bar{u}_X^{\omega_1}$, in G_1 . $\bar{u}_X^{\omega_1}$ is the solution of the following equations:

$$\begin{cases} u_X - P_1 = \chi(u_Y - P_1), \\ (R_1 - S_1)u_X + (T_1 - R_1)u_Y = R_1(T_1 - S_1). \end{cases}$$

Thus, $u_X = \frac{\chi[R_1(T_1 - S_1) - P_1(T_1 - R_1)] + P_1(T_1 - R_1)}{\chi(R_1 - S_1) + (T_1 - R_1)}$.

According to Theorem 3, $\min(\Gamma_2) \leq \phi[(u_X - P_2) - \chi(u_Y - P_2)] \leq \max(\Gamma_2)$, where

$$\Gamma_2 = \{\phi(1 - \chi)(R_2 - R_1), \phi(S_2 - S_1) - \phi\chi(T_2 - T_1), \phi(T_2 - T_1) - \phi\chi(S_2 - S_1), \phi(1 - \chi)(P_2 - P_1)\}.$$

Thus,

$$|(u_X - P_2) - \chi(u_Y - P_2)| \leq \max\{(\chi - 1)|R_2 - R_1|, |S_2 - S_1| + \chi|T_2 - T_1|, |T_2 - T_1| + \chi|S_2 - S_1|, (\chi - 1)|P_2 - P_1|\} \leq (\chi + 1)\delta.$$

Then $u_X - P_2 - \chi(u_Y - P_2) \geq -(\chi + 1)\delta$. $u_X^1 = \frac{\chi[R_2(T_2 - S_2) - P_2(T_2 - R_2)] + P_2(T_2 - R_2) - (\chi + 1)\delta(T_2 - R_2)}{\chi(R_2 - S_2) + (T_2 - R_2)}$ is the solution of the following equations:

$$\begin{cases} u_X - P_2 - \chi(u_Y - P_2) = -(\chi + 1)\delta, \\ (R_2 - S_2)u_X + (T_2 - R_2)u_Y = R_2(T_2 - S_2). \end{cases}$$

Also, $u_X^1 = \frac{\chi[R_2(T_2 - S_2) - P_2(R_2 - S_2)] + P_2(R_2 - S_2) - (\chi + 1)\delta(R_2 - S_2)}{\chi(T_2 - R_2) + (R_2 - S_2)}$ is the solution of the following equations:

$$\begin{cases} u_X - P_2 - \chi(u_Y - P_2) = -(\chi + 1)\delta, \\ (T_2 - R_2)u_X + (R_2 - S_2)u_Y = R_2(T_2 - S_2). \end{cases}$$

When $a_i^2\chi^2 + b_i^2\chi + c_i^2 < 0, i \in \{1, 2\}$, we have $u_X^1 > \bar{u}_X^{\omega_1}$, and $u_X^2 > \bar{u}_X^{\omega_1}$. Since u_X^1 and u_X^2 is lower than player X 's utility when player Y chooses the ALLC strategy, the supremum of player X 's expected utility in G_2 is higher than it in G_1 .

9 Proof of Corollary 6

When player Y chooses the always defect (ALLD) strategy, $(u_X^{\omega_i}, u_Y^{\omega_i})$ is on $(T_i - P_i)u_X^{\omega_i} + (P_i - S_i)u_Y^{\omega_i} = P_i(T_i - S_i)$ in $G_i, i \in \{1, 2\}$. For any $\phi, \chi > 1$, suppose player X chooses generous strategy $\mathbf{p} = \phi[(\mathbf{S}_X^{\omega_1} - R_1 \mathbf{1}) - \chi(\mathbf{S}_Y^{\omega_1} - R_1 \mathbf{1})] + \mathbf{p}_0$ in G_1 . When player Y choose the ALLD strategy, player X get the infimum of its expected utility, $\underline{u}_X^{\omega_1}$, in G_1 . $\underline{u}_X^{\omega_1}$ is the solution of the following equations:

$$\begin{cases} u_X - R_1 = \chi(u_Y - R_1), \\ (T_1 - P_1)u_X + (P_1 - S_1)u_Y = P_1(T_1 - S_1). \end{cases}$$

Then $u_X = \frac{\chi[P_1(T_1 - S_1) - R_1(P_1 - S_1)] + R_1(P_1 - S_1)}{\chi(T_1 - P_1) + (P_1 - S_1)}$.

According to Theorem 3, $\min(\Gamma_2) \leq \phi[(u_X - R_2) - \chi(u_Y - R_2)] \leq \max(\Gamma_2)$, where

$$\Gamma_2 = \{\phi(1 - \chi)(R_2 - R_1), \phi(S_2 - S_1) - \phi\chi(T_2 - T_1), \phi(T_2 - T_1) - \phi\chi(S_2 - S_1), \phi(1 - \chi)(P_2 - P_1)\}.$$

Thus,

$$|(u_X - R_2) - \chi(u_Y - R_2)| \leq \max\{(\chi - 1)|R_2 - R_1|, |S_2 - S_1| + \chi|T_2 - T_1|, |T_2 - T_1| + \chi|S_2 - S_1|, (\chi - 1)|P_2 - P_1|\} \leq (\chi + 1)\delta.$$

Then we have

$$u_X - R_2 - \chi(u_Y - R_2) \geq -(\chi + 1)\delta.$$

Let $u_X^1 = \frac{\chi[P_2(T_2 - S_2) - R_2(P_2 - S_2)] + R_2(P_2 - S_2) - (\chi + 1)\delta(P_2 - S_2)}{\chi(T_2 - P_2) + (P_2 - S_2)}$ be the solution of the following equations:

$$\begin{cases} u_X - R_2 - \chi(u_Y - R_2) = -(\chi + 1)\delta, \\ (T_2 - P_2)u_X + (P_2 - S_2)u_Y = P_2(T_2 - S_2). \end{cases}$$

Also, let $u_X^2 = \frac{\chi[P_2(T_2-S_2)-R_2(T_2-P_2)]+R_2(T_2-P_2)-(\chi+1)\delta(T_2-P_2)}{\chi(P_2-S_2)+(T_2-P_2)}$ be the solution of the following equations:

$$\begin{cases} u_X - R_2 - \chi(u_Y - R_2) = -(\chi + 1)\delta, \\ (P_2 - S_2)u_X + (T_2 - P_2)u_Y = P_2(T_2 - S_2). \end{cases}$$

When $a_i^3\chi^2 + b_i^3\chi + c_i^3 < 0, i \in \{1, 2\}$, we have $u_X^1 > \underline{u}_X^{\omega_1}$, and $u_X^2 > \underline{u}_X^{\omega_1}$. Since u_X^1 and u_X^2 is lower than player X 's utility when player Y chooses the ALLD strategy, the infimum of player X 's expected utility in G_2 is higher than it in G_1 .

References

1. Press, W. H. & Dyson, F. J. Iterated prisoner's dilemma contains strategies that dominate any evolutionary opponent. *Proc. Natl. Acad. Sci.* **109**, 10409–10413 (2012).