Online Appendix for
Growth Promotion Policies when Taxes Cannot be Raised

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This Version: November 8, 2026

A Analysis of the Baseline Model

A.1 Derivation of the Consumption of Newborns

In this section, we derive the consumption of a newly born individual (7). As explained in the main text, each individual maximizes the expected utility (3) subject to the budget constraint (4), initial condition $k_{t,t} = 0$, and the non-Ponzi game condition:

$$\lim_{T\to\infty} e^{-\int_t^T (r_{t''}+\mu)dt''} k_{s,T} \geq 0. \tag{A.1}$$

At the optimum, the non-Ponzi game condition (A.1) holds with equality, because otherwise the individual could consume more without violating it. From (4), (A.1) with equality, and the Euler equation for individuals, $\dot{c}_{s,t} = (r_t - \rho)c_{s,t}$, we obtain

$$c_{s,t} = (\rho + \mu)(k_{s,t} + h_{s,t}), \tag{A.2}$$

where $h_{s,t}$ is the present value of future labor income, or "human wealth," of an individual of generation s evaluated at time t. It is defined by

$$h_{s,t} \equiv \int_t^\infty e^{-\delta(t'-s)} w_{t'} e^{-\int_t^{t'} (r_{t''}+\mu)dt''} dt'. \tag{A.3}$$

By averaging (A.2) for all individuals, we obtain the average consumption as follows:

$$\overline{c}_t = (\rho + \mu)\left(\overline{k}_t + \overline{h}_t\right), \tag{A.4}$$

where $\overline{h}_t$ is the average human wealth, defined by

$$\overline{h}_t \equiv \frac{1}{N_t}\int_{-\infty}^t N_{s,t} h_{s,t} ds. \tag{A.5}$$

Note that, from (A.3), (A.5) and (2), $\overline{h}_t$ and the human wealth of a newborn $h_{t,t}$ are related

as follows:

$$\overline{h}_t = \frac{1}{N_t}\int_{-\infty}^{t} N_{s,t} e^{-\delta(t-s)} \int_{t}^{\infty} e^{-\delta(t'-t)} w_{t'} e^{-\int_t^{t'}(r_{t''}+\mu)dt''} dt' ds = \frac{L_t}{N_t} h_{t,t} = \frac{\beta}{\beta+\delta} h_{t,t}. \quad \text{(A.6)}$$

This relationship means that $h_{t,t} = ((\beta+\delta)/\beta)\overline{h}_t$; i.e., a newborn has $(\beta+\delta)/\beta > 1$ times the human wealth of an average individual. Equation (A.2) with $s = t$ and $k_{t,t} = 0$ implies that the consumption of a newborn is $c_{t,t} = (\rho+\mu)h_{t,t} = (\rho+\mu)(\beta+\delta)\overline{h}_t/\beta$. Eliminating $\overline{h}_t$ from (A.4) yields (7).

A.2 Derivation of the Equilibrium Dynamics

We first derive the interest rate (21). From (14), (20), and (13), the profit of a firm is $\pi_t = (1-\alpha)n_t^{(1-2\alpha)/\alpha} L_t^P$. When R&D is positive, the value of a firm is $v_t = \alpha(1-\theta)Ln_t^{(1-2\alpha)/\alpha}/A$ by (16). Its growth rate is $-(2\alpha-1)AL_t^R/(\alpha L)$ by (15). Substituting these expressions into (19) yields (21).

Next, consider the debt-GDP ratio $D_t = B_t/Y_t$. Since Y_t grows at the rate g_t, the time derivative of D_t is $\dot{D}_t = \dot{B}_t/Y_t - g_t D_t$. The government budget constraint (17) then gives

$$\dot{D}_t = (r_t - g_t)D_t + \frac{\theta\alpha}{\overline{g}(A)} g_t,$$

where we use $w_t = \alpha Y_t/L$ and $L_t^R = Lg_t/\overline{g}(A)$ from (20), (24), and (25). The first term is the change in the ratio when the primary balance is zero. The second is subsidy expenditure relative to full-production GDP. Substituting (27) yields (29).

Finally, we derive the dynamics of g_t. The goods market clears when $C_t = X_t$. Hence, (14), (24), and the labor-market condition imply $C_t/Y_t = L_t^P/L = 1 - g_t/\overline{g}(A)$. Differentiating gives $\dot{g}_t = -(\overline{g}(A) - g_t)(\dot{C}_t/C_t - g_t)$. If consumption grows more slowly than full-production GDP, fewer workers are needed in production, and the labor released into research raises the growth rate. The assets of individuals consist of government bonds and shares of firms. Using (18) and the value of a firm obtained above, we have

$$\frac{K_t}{Y_t} = D_t + \frac{\alpha(1-\theta)}{A}, \qquad \frac{C_t}{Y_t} = \frac{\overline{g}(A) - g_t}{\overline{g}(A)}. \quad \text{(A.7)}$$

Substituting these ratios and the aggregate Euler equation (8) into the expression for $\dot{g}_t$ yields (28). A higher interest rate raises consumption growth and reduces the labor available for research. A larger amount of aggregate assets has the opposite effect through the replacement of existing individuals by newborns without financial wealth, as explained in Section 3.1.

A.3 The Conventional Debt-GDP Ratio

The equilibrium dynamics of the model involve three variables, government debt B_t, the number of goods n_t, and R&D labor L_t^R, of which B_t and n_t are predetermined. In the main text, we solve them in two steps. First, we construct $D_0 = B_0/Y_0$ from n_0 and B_0, and, given D_0, we solve the dynamics of g_t and D_t in (28) and (29). Second, given n_0, we obtain the path of n_t from that of g_t, which gives the paths of $B_t = D_t Y_t$ and $L_t^R = Lg_t/\overline{g}(A)$. In this subsection, we show that the qualitative results are the same if we use the conventional debt-GDP ratio, $D_t^{\text{conv}} \equiv B_t/\text{GDP}_t$, instead of D_t in the first step.

By (22), (24), and (25), actual GDP is $\text{GDP}_t = \big(1 - (1-\alpha)g_t/\overline{g}(A)\big)Y_t$, so the two ratios are related by

$$D_t^{\text{conv}} = \frac{\overline{g}(A)D_t}{\overline{g}(A) - (1-\alpha)g_t}, \tag{A.8}$$

and they correspond one to one for a given g_t. Differentiating (A.8) with respect to time gives

$$\dot{D}_t^{\text{conv}} = \frac{\overline{g}(A)\dot{D}_t + (1-\alpha)D_t^{\text{conv}}\dot{g}_t}{\overline{g}(A) - (1-\alpha)g_t}. \tag{A.9}$$

We substitute (29) into (A.9). Eliminating D_t from the result and from (28) by (A.8) and using (26), we obtain

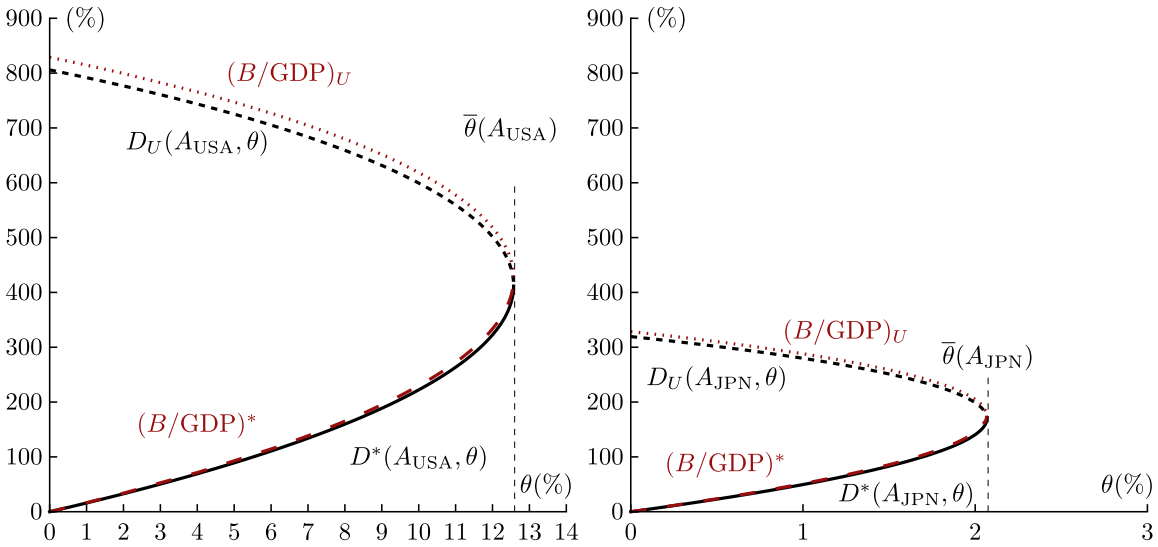
$$\begin{aligned}\dot{g}_t = {} & \big(\overline{g}(A) - g_t\big)\big(s(\theta)(g_t - \widetilde{g}(A,\theta)) - \delta + \rho\big) \\ & + (\rho+\mu)(\delta+\mu)\Big(\big(\overline{g}(A) - (1-\alpha)g_t\big)D_t^{\text{conv}} + (1-\alpha)(1-\theta)\Big),\end{aligned} \tag{A.10}$$

$$\dot{D}_t^{\text{conv}} = -s(\theta)\big(g_t - \widetilde{g}(A,\theta)\big)D_t^{\text{conv}} + \frac{\theta\alpha g_t + (1-\alpha)D_t^{\text{conv}}\,\dot{g}_t}{\overline{g}(A) - (1-\alpha)g_t}. \tag{A.11}$$

Substituting (A.10) for $\dot{g}_t$ in (A.11), we obtain a system of two first-order differential equations in g_t and D_t^{conv}.

Setting $\dot{D}_t = \dot{g}_t = 0$ in (A.9) gives $\dot{D}_t^{\text{conv}} = 0$. That is, a steady state of the system in D_t and g_t is also a steady state of the system in D_t^{conv} and g_t. The converse holds in the same way. Since the relation (A.8) maps each path of one system to a path of the other, Proposition 1 holds in either system, with the long-term BGP and the unsustainable threshold expressed in the corresponding ratio, once the initial condition is also converted, as we explain below. The thresholds of A in Propositions 2 and 3, $\overline{A}$ and $\widehat{A}$, and the subsidy rates $\overline{\theta}(A)$ and $\widehat{\theta}(A)$ are defined independently of government debt. Since the two systems describe the same economy with different variables, these propositions do not depend on the choice between D_t and D_t^{conv} either.

We use D_t in the main text for two reasons. First, the dynamics of D_t in (29) do



(a) When $A > \widehat{A}$ ($A_{\text{USA}} = 0.399$) (b) When $A < \widehat{A}$ ($A_{\text{JPN}} = 0.120$)

Figure A.1: The long-term conventional debt-GDP ratio $(B/\text{GDP})^*$ and the unsustainable threshold $(B/\text{GDP})_U$, compared with $D^*(A,\theta)$ and $D_U(A,\theta)$.

not involve $\dot{g}_t$ on the RHS, which makes the analysis much simpler. Second, the initial condition is the vertical line $D_t = D_0$ in the phase diagram. In contrast, n_0 and B_0 do not determine D_0^{conv}, because GDP_0 depends on the jump variable g_0. When the system is described in terms of g_t and D_t^{conv}, the initial condition is defined by a curve in the phase diagram. From (22), (23), (25), and (26), we can write the initial condition as

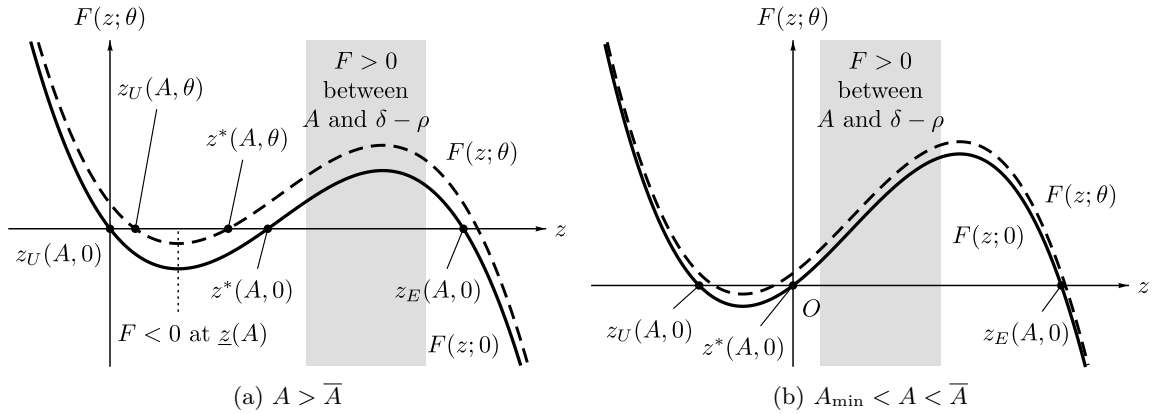
$$g_0 = \frac{A}{\alpha}\left(1 - \frac{B_0}{n_0^{(1-\alpha)/\alpha} L\, D_0^{\text{conv}}}\right), \quad g_0 \in [0, \overline{g}(A)].$$

When B_0 is zero, D_0^{conv} must also be zero from the definition $D_0^{\text{conv}} = B_0/\text{GDP}_0$. Then, the RHS becomes indeterminate (involving 0/0), which means g_0 can take any value in $[0, \overline{g}(A)]$.

The levels of the two ratios are also close. Figure A.1 compares $D^*(A,\theta)$ and $D_U(A,\theta)$ in the lower panels of Figure 8 with the corresponding values of D_t^{conv}, $(B/\text{GDP})^*$ and $(B/\text{GDP})_U$. The latter are higher because actual GDP is smaller than full-production GDP, but only by at most 3.4%. Along the transition paths in the third row of Figure 9, D_t^{conv} is higher than D_t by at most 3.6%.

A.4 Steady States with R&D

We now examine the properties of steady states in the region with R&D. Let z denote the $g - r$ gap at a candidate steady state in this region. By (27), we have $z = s(\theta)\big(g - \widetilde{g}(A,\theta)\big)$. Eliminating D from the two steady-state loci (30) and (31) then shows that the steady-state



(a) $A > \overline{A}$ (b) $A_{\min} < A < \overline{A}$

Figure A.2: The steady-state condition $F(z;\theta) = 0$ for different levels of R&D productivity. The shaded area is the interval between A and $\delta - \rho$, on which $F(z;\theta)$ is positive.

gap must satisfy $F(z;\theta) = 0$, where[86]

$$F(z;\theta) \equiv (\rho+\mu)(\delta+\mu)\frac{\theta\alpha\overline{g}(A)}{1-\theta} - z\Big((A-z)(\delta-\rho-z) - (\rho+\mu)(\delta+\mu)\Big). \tag{A.12}$$

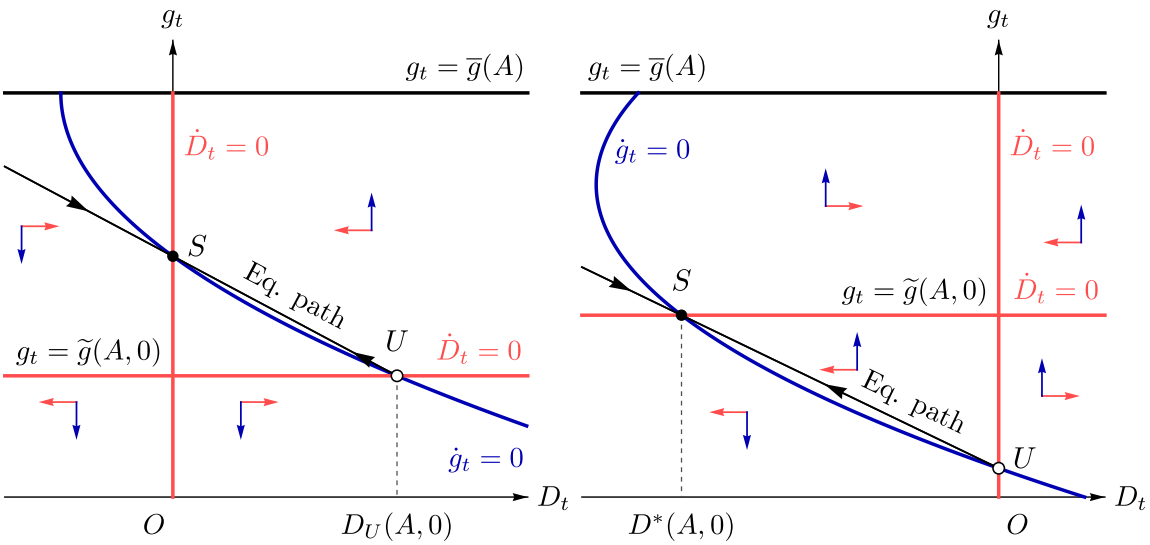
A root of $F(z;\theta) = 0$ corresponds to a steady state with R&D only if the associated growth rate $g = \widetilde{g}(A,\theta) + z/s(\theta)$ satisfies $0 < g \leq \overline{g}(A)$, which implies $z \leq A$. Moreover, the aggregate Euler equation (8) implies that any feasible steady state satisfies $z < \delta - \rho$.[87] The product $(A-z)(\delta-\rho-z)$ in (A.12) is nonpositive between A and $\delta-\rho$, so $F(z;\theta)$ is positive throughout this interval. Since $F(z;\theta)$ is strictly concave for $z > \max\{A, \delta-\rho\}$ and tends to minus infinity as $z \to \infty$, there is exactly one root $z_E > \max\{A, \delta-\rho\}$. It corresponds to $E = (D_E, g_E)$, which is excluded by the labor resource constraint. Thus, at most two roots can correspond to steady states with R&D.

As illustrated in Figure A.2, there can be two roots of $F(z;\theta) = 0$ in the R&D region. When two steady states exist, the one with the larger z has the higher growth rate, since g is increasing in z. It also has the lower debt-GDP ratio. For $\theta > 0$, $F(0;\theta) > 0$ implies that the two roots other than z_E have the same sign. On each side of $z = 0$, the $\dot{D}_t = 0$ locus given by (31) is decreasing in g, as illustrated in Figure 5(a) for positive debt and Figure A.7(c) for negative debt. The case of $\theta = 0$ is examined below. We denote the higher-growth, lower-debt steady state by $S = (D^*, g^*)$ and the other by $U = (D_U, g_U)$, and write z_U for the gap at U. Appendix A.7 shows why S is the long-term BGP and D_U is the unsustainable threshold.

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[86]Strictly speaking, the $\dot{D}_t = 0$ locus (31) is undefined at $z = 0$ when $\theta = 0$, because this case corresponds to $g = \widetilde{g}(A,0)$. However, this case also satisfies the original steady-state conditions $\dot{g}_t = \dot{D}_t = 0$ for an appropriate value of D_t and is included among the roots of $F(z;0) = 0$.

[87]On a BGP, $\dot{C}/C = g$, so (8) gives $z = \delta - \rho - (\rho+\mu)(\delta+\mu)K/C$. When $z > 0$, (29) implies $D \geq 0$, and hence $K > 0$ by (A.7). When $z \leq 0$, the inequality follows from $\delta > \rho$.



(a) $A > \overline{A}$ and $\theta = 0$ (b) $A_{\min} < A < \overline{A}$ and $\theta = 0$

Figure A.3: Stationary loci without subsidies. Panels (a) and (b) compare the two sides of $A = \overline{A}$.

Without subsidies, the $\dot{D}_t = 0$ locus consists of the two lines $D = 0$ and $g = \widetilde{g}(A, 0)$, and $F(z; 0) = 0$ has the root $z = 0$, which corresponds to the second line. The other roots satisfy $(A - z)(\delta - \rho - z) = (\rho + \mu)(\delta + \mu)$. The intersection on the line $g = \widetilde{g}(A, 0)$ has $D = \alpha(1/\overline{A} - 1/A)$ by (30). Thus, S is the intersection on $D = 0$ when $A > \overline{A}$, as in Figure A.3(a), whereas U is the intersection on $D = 0$ when $A < \overline{A}$, as in panel (b).

To express the growth rate at S in terms of the $g - r$ gap, let $z^* \equiv g^* - r^* = s(\theta)\big(g^* - \widetilde{g}(A, \theta)\big)$. Substituting z^* into $F(z^*; \theta) = 0$ and using (27) to eliminate θ, we obtain

$$g^* = \overline{g}(A)\frac{z^*(\delta - \rho - z^*) + (1 - \alpha)(\rho + \mu)(\delta + \mu)}{z^*(\delta - \rho - z^*) + (\rho + \mu)(\delta + \mu)}. \tag{A.13}$$

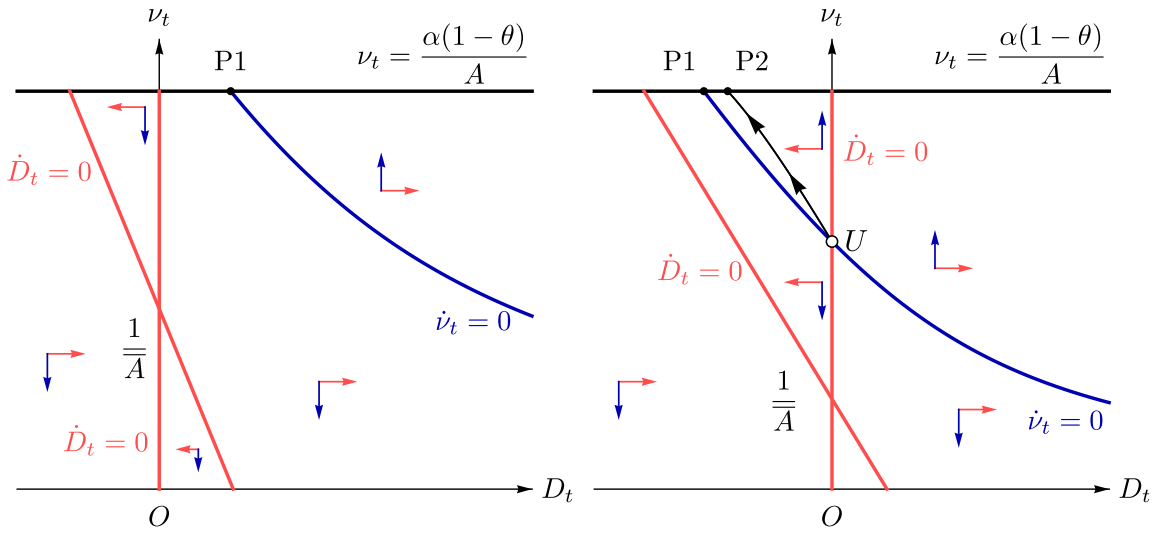
Since (A.13) is increasing in $z^*(\delta - \rho - z^*)$ for feasible steady states, $z^* \leq 0$ implies $g^* \leq (1 - \alpha)\overline{g}(A)$, so at least a fraction α of labor is allocated to production at S.

When research productivity is too low for that intersection to have positive growth, the corresponding steady state is in the region without R&D, examined next.

A.5 Dynamics in the Region without R&D

When research is absent, $g_t = 0$, $L_t^P = L$, and the free-entry condition need not hold with equality. Let $\nu_t = v_t n_t / Y_t$ be the aggregate value of firms relative to full-production GDP. The region is $0 < \nu_t \leq \alpha(1 - \theta)/A$, and free entry holds with equality at its upper boundary. The subsidy affects this boundary but does not affect the dynamics inside, because the government pays no subsidy.

Consumption and full-production GDP are constant in this region. Setting $\dot{C}_t = 0$ in



(a) $A/(1-\theta) > A_{\min}$ (b) $A/(1-\theta) < A_{\min}$

Figure A.4: Phase diagram in D_t and $\nu_t \equiv v_t n_t / Y_t$ in the region without R&D. In panel (a), $A/(1-\theta) > A_{\min}$ and there is no steady state. In panel (b), $A/(1-\theta) < A_{\min}$ and U lies inside the region.

(8) gives

$$r_t = (\rho+\mu)(\delta+\mu)(D_t+\nu_t) - (\delta-\rho). \tag{A.14}$$

The government budget constraint and the no-arbitrage condition then imply

$$\dot{D}_t = \big((\rho+\mu)(\delta+\mu)(D_t+\nu_t) - (\delta-\rho)\big)D_t, \tag{A.15}$$

$$\dot{\nu}_t = \big((\rho+\mu)(\delta+\mu)(D_t+\nu_t) - (\delta-\rho)\big)\nu_t - (1-\alpha). \tag{A.16}$$

The $\dot{D}_t = 0$ locus consists of $D_t = 0$ and $D_t + \nu_t = 1/\overline{A}$. The latter is the line on which $r_t = 0$, where $\dot{\nu}_t = -(1-\alpha) < 0$. Therefore, a steady state has $D_t = 0$. Solving $\dot{\nu}_t = 0$ on this axis gives $U = (0, \nu_U)$, where $\nu_U = \alpha / A_{\min}$ and

$$A_{\min} \equiv \frac{-(\delta-\rho) + \sqrt{(\delta-\rho)^2 + 4(1-\alpha)(\rho+\mu)(\delta+\mu)}}{2(1-\alpha)/\alpha}. \tag{A.17}$$

Thus, U lies inside the region without R&D if $A/(1-\theta) < A_{\min}$, as in Figure A.4(b), whereas no steady state exists in this region if $A/(1-\theta) > A_{\min}$. The interest rate at U is positive by $r_U \nu_U = 1 - \alpha$.

The two regions meet at $g_t = 0$ and $\nu_t = \alpha(1-\theta)/A$. The point P1, where the $\dot{g}_t = 0$ locus reaches this boundary, is also the intersection of the $\dot{\nu}_t = 0$ locus with the boundary. Its debt-GDP ratio is positive if and only if $A/(1-\theta) > A_{\min}$, by (30) and (A.17).

A.6 Properties of Equilibrium Paths

Two regions in the phase diagram. This subsection combines the dynamics with R&D in Appendix A.4 and those without R&D in Appendix A.5. An equilibrium path may move between these regions, with debt, consumption, and firm value remaining continuous at their common boundary. We therefore draw the two regions in one phase diagram, Figure A.7 in Appendix A.7. The upper half of each panel describes the R&D region in the (D_t, g_t) plane, while the lower half describes the region without R&D in the (D_t, ν_t) plane. The common boundary is given by $g_t = 0$ in the former and $\nu_t = \alpha(1-\theta)/A$ in the latter, with the same D_t. P1 is the point where the $\dot{g}_t = 0$ and $\dot{\nu}_t = 0$ loci reach this boundary, as established in Appendix A.5.

Definition of an equilibrium path. An equilibrium path is a path along which households and firms optimize with perfect foresight, the government budget constraint holds, and all markets clear for all $t \geq 0$. At time $t = 0$, D_0 is predetermined. In addition, once the initial growth rate $g_0 \in [0, \overline{g}(A)]$ is given (if $g_0 = 0$, the initial value of firms ν_0 must also be specified), the equilibrium conditions determine the entire path. However, this path may not continue forever. In finite time, the implied growth rate may exceed $\overline{g}(A)$, implying negative consumption; the value of firms may become negative, which contradicts the usual assumption of limited shareholder liability; or the amount of government debt may become infinite, which means there is no continuing path of D_t after that. We can rule out such a path as an equilibrium path because it is incompatible with households' optimization. Once they know their plan will fail in a finite time, they should revise the consumption plan backward from that time, thereby changing the initial consumption level (which determines g_0). Therefore, the initial value g_0 was not compatible with rational expectations. If there is no g_0 that is compatible with rational expectations, then the economy has no equilibrium.

Linear representation and the candidate path. The quadratic terms of (28) and (29) are $-s(\theta)g_t^2$ and $-s(\theta)g_t D_t$. Define the three-dimensional vector $\boldsymbol{\gamma}_t = (Y_t/Y_0)^{s(\theta)}(D_t, g_t, 1)$. Then

$$\dot{\boldsymbol{\gamma}}_t = \mathbf{M}\boldsymbol{\gamma}_t, \quad \text{where}$$

$$\mathbf{M} = \begin{pmatrix} \frac{\overline{g}(A)}{1-\theta} & \frac{\theta\alpha}{\overline{g}(A)} & 0 \\ \overline{g}(A)(\rho+\mu)(\delta+\mu) & A+\delta-\rho+\frac{2\overline{g}(A)}{1-\theta} & \overline{g}(A)\Big(\frac{\alpha(1-\theta)(\rho+\mu)(\delta+\mu)}{A} - \frac{\overline{g}(A)}{1-\theta} - \delta + \rho\Big) \\ 0 & s(\theta) & 0 \end{pmatrix}. \tag{A.18}$$

Lemma A.1 (Linear Representation of the Dynamics) *A path of* (28) *and* (29) *yields*

a solution of (A.18). *Conversely, if* $\boldsymbol{\gamma}_t$ *solves* (A.18) *and* $\gamma_{3,t} > 0$*, its component ratios* $(\gamma_{1,t}/\gamma_{3,t}, \gamma_{2,t}/\gamma_{3,t})$ *solve the original system.*

Proof. From $\dot{Y}_t/Y_t = g_t$, the third component satisfies $\dot{\gamma}_{3,t} = s(\theta)\gamma_{2,t}$. Differentiating the first two components and substituting (29) and (28) gives the first two rows of (A.18). The converse follows by differentiating the component ratios.

The general solution is $\boldsymbol{\gamma}_t = e^{\mathbf{M}t}\boldsymbol{\gamma}_0$. An eigenvector can be normalized to $(D, g, 1)$, because the second and third rows of (A.18) rule out a nonzero eigenvector whose third component is zero. For such an eigenvector, the third row gives the eigenvalue $s(\theta)g$, while the first two rows yield the steady-state conditions. Thus, if z is a root of $F(z;\theta) = 0$, the corresponding growth rate is $g = \widetilde{g}(A,\theta) + z/s(\theta)$, and the associated eigenvalue is $s(\theta)g = z + s(\theta)\widetilde{g}(A,\theta)$. One of these roots corresponds to the excluded intersection E. The eigenvalue associated with E has the largest real part, since the three roots of $F(z;\theta) = 0$ sum to $A + \delta - \rho$, whereas the root corresponding to E exceeds both A and $\delta - \rho$ by Appendix A.4.

When both steady states have R&D, the three eigenvalues are distinct, and diagonalizing $\mathbf{M}$ gives

$$\boldsymbol{\gamma}_t = \chi^* e^{s(\theta)g^* t}(D^*, g^*, 1) + \chi_U e^{s(\theta)g_U t}(D_U, g_U, 1) + \chi_E e^{s(\theta)g_E t}(D_E, g_E, 1), \tag{A.19}$$

where the coefficients sum to one and are uniquely determined by (D_0, g_0). These coefficients are the coordinates of the initial point relative to the three vertices in Figure A.5. Since E has the largest eigenvalue, a path with $\chi_E > 0$ eventually approaches $g_E > \overline{g}(A)$ and violates the labor resource constraint. If $\chi_E < 0$, the third component of $\boldsymbol{\gamma}_t$ eventually becomes negative, which is incompatible with its definition. Hence, a path remaining feasible in the R&D region indefinitely must satisfy $\chi_E = 0$ and therefore lie on the straight line through S and U.

The lower end of the candidate path. This line cannot be extended indefinitely toward negative debt without violating the labor resource constraint. Its slope is $-\overline{g}(A)(\rho + \mu)(\delta + \mu)/z_E$, which extends to $\theta = 0$ by continuity. For $\theta > 0$, this follows from (31) at S and U and from the product $z^* z_U z_E$ of the three roots of $F(z;\theta) = 0$, which equals the constant term of $F(z;\theta)$. The line reaches $g_t = \overline{g}(A)$ at

$$\underline{D}(A,\theta) = -\frac{\alpha(1-\theta)}{A} - \frac{\alpha}{z_E - A}, \tag{A.20}$$

which follows from (30) at S, the sum of the three roots, and the value of $F(z;\theta)$ at $z = A$.

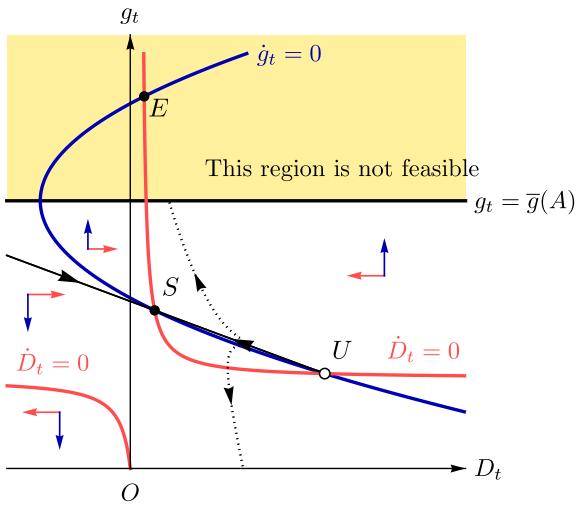


Figure A.5: The steady states S and U, the excluded intersection E, and the straight line through S and U. Paths off the line leave the R&D region through one of its boundaries.

This value is below $-\min\{\alpha/(\delta-\rho), \alpha/(\rho+\mu)\}$, so the initial condition in Proposition 1 lies to its right.[88]

Exclusion of alternative paths. By the definition of an equilibrium path, a path that violates the labor resource constraint, makes firm value negative, or makes debt infinite in finite time is not an equilibrium path. The following lemma shows where paths lead in each region. The proof of Proposition 1 combines its parts to exclude all paths except those that converge to S.

Lemma A.2 (Paths in the Two Regions)

(a) Suppose that both steady states have R&D. The saddle path is a straight line that starts at U and passes through S, which we call the equilibrium line. Unless the initial point is exactly at U, all the paths that start from the equilibrium line converge to S.

(b) A path with R&D that does not converge to a steady state in that region leaves the region in finite time. Either g_t exceeds $\overline{g}(A)$, implying negative consumption, or g_t falls to zero and the path enters the region without R&D. If both steady states have R&D, this occurs for every path with R&D that starts neither on the equilibrium line nor at U.

(c) Along a path without R&D that does not start at U, one of the following occurs in

[88] If the growth rates at both other intersections are at least $\widetilde{g}(A,\theta)$, their sum implies $z_E - A \leq \delta - \rho$, giving the first bound. Otherwise, the gap at S and D^* are both nonpositive. The production labor share at S is then at least α, by (A.13). At E, the condition $F(z;\theta) = 0$ implies that $(g-r)(g-r-\delta+\rho)$ exceeds $(\rho+\mu)(\delta+\mu)$. The positive solution at equality is $\delta+\mu$, so z_E exceeds $\delta+\mu$. Multiplying this quantity by the production labor share and dividing by $(\rho+\mu)(\delta+\mu)$ gives the distance from D^* to the lower end of the line. The second bound follows.

finite time: the path enters the R&D region at P1 or to its right, firm value becomes negative, or debt becomes infinite.

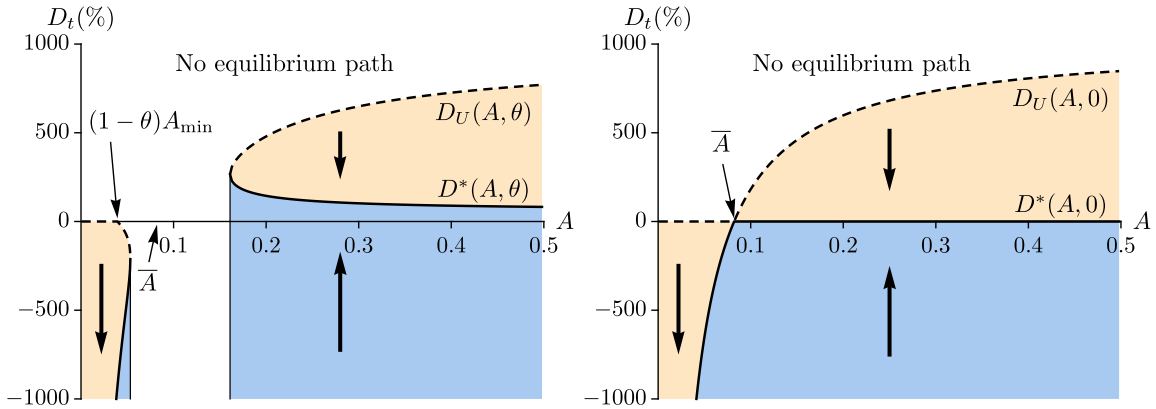
Proof. Part (a) follows from (A.19). On the straight line through S and U, χ_E is zero, and the coefficient χ^* of the higher-growth component is positive precisely when $D_t < D_U$, by $g^* > g_U$ and $D^* < D_U$. A path that starts on the equilibrium line other than at U therefore moves monotonically toward S.

For part (b), the linear representation shows that a path remaining in the R&D region indefinitely must exclude the eigenvector associated with E. If the other two eigenvalues are complex, the third component of their real combination changes sign, so the path cannot remain feasible. If they are distinct and real, the path converges to the higher-growth solution unless it stays at the lower-growth solution or reaches a boundary. For a path reaching $g_t = \overline{g}(A)$ from below, $D_t \geq -\alpha(1-\theta)/A$ must hold at that point by (28). If the inequality is strict, $\dot{g}_t$ is positive, and the labor resource constraint is immediately violated. At equality, $\dot{D}_t = \alpha > 0$ holds, so $\dot{g}_t$ becomes positive immediately afterward. If both steady states have R&D, a path that starts off the straight line through S and U does not converge to a steady state, because its coefficient χ_E is not zero. A path that starts on the extension of the equilibrium line beyond U does not converge to a steady state either, because its coefficient χ^* is negative, so that the third component of $\boldsymbol{\gamma}_t$ eventually becomes negative.

For part (c), differentiating (A.14) with respect to time and substituting (A.15) and (A.16) give

$$\dot{r}_t = r_t(r_t + \delta - \rho) - (1-\alpha)(\rho+\mu)(\delta+\mu).$$

The positive stationary interest rate r_U satisfies $r_U(r_U + \delta - \rho) = (1-\alpha)(\rho+\mu)(\delta+\mu)$. If $r_t < r_U$, the interest rate becomes negative in finite time and stays negative. Then $\dot{\nu}_t$ is smaller than $-(1-\alpha)$ by (A.16), so ν_t becomes negative in finite time unless the path leaves the region earlier. If $r_t = r_U$, the interest rate remains constant, and $D_t + \nu_t$ stays at ν_U. When $D_t > 0$, debt increases and firm value becomes negative, whereas $D_t < 0$ implies that firm value rises until the free-entry boundary is reached. If $D_t = 0$, the path starts at $U = (0, \nu_U)$ and stays there. If $r_t > r_U$, the interest rate rises and becomes infinite in finite time, and so does $D_t + \nu_t$ by (A.14). When $D_t \leq 0$, debt does not increase, so firm value rises until the free-entry boundary is reached. When $D_t > 0$, the ratio ν_t/D_t satisfies $d(\nu_t/D_t)/dt = -(1-\alpha)/D_t$ by (A.15) and (A.16). As long as the path stays in the region, ν_t is bounded, so that debt becomes infinite at that time and ν_t/D_t becomes zero. Since debt increases, integrating the last equation shows that ν_t is at most $1-\alpha$ times the



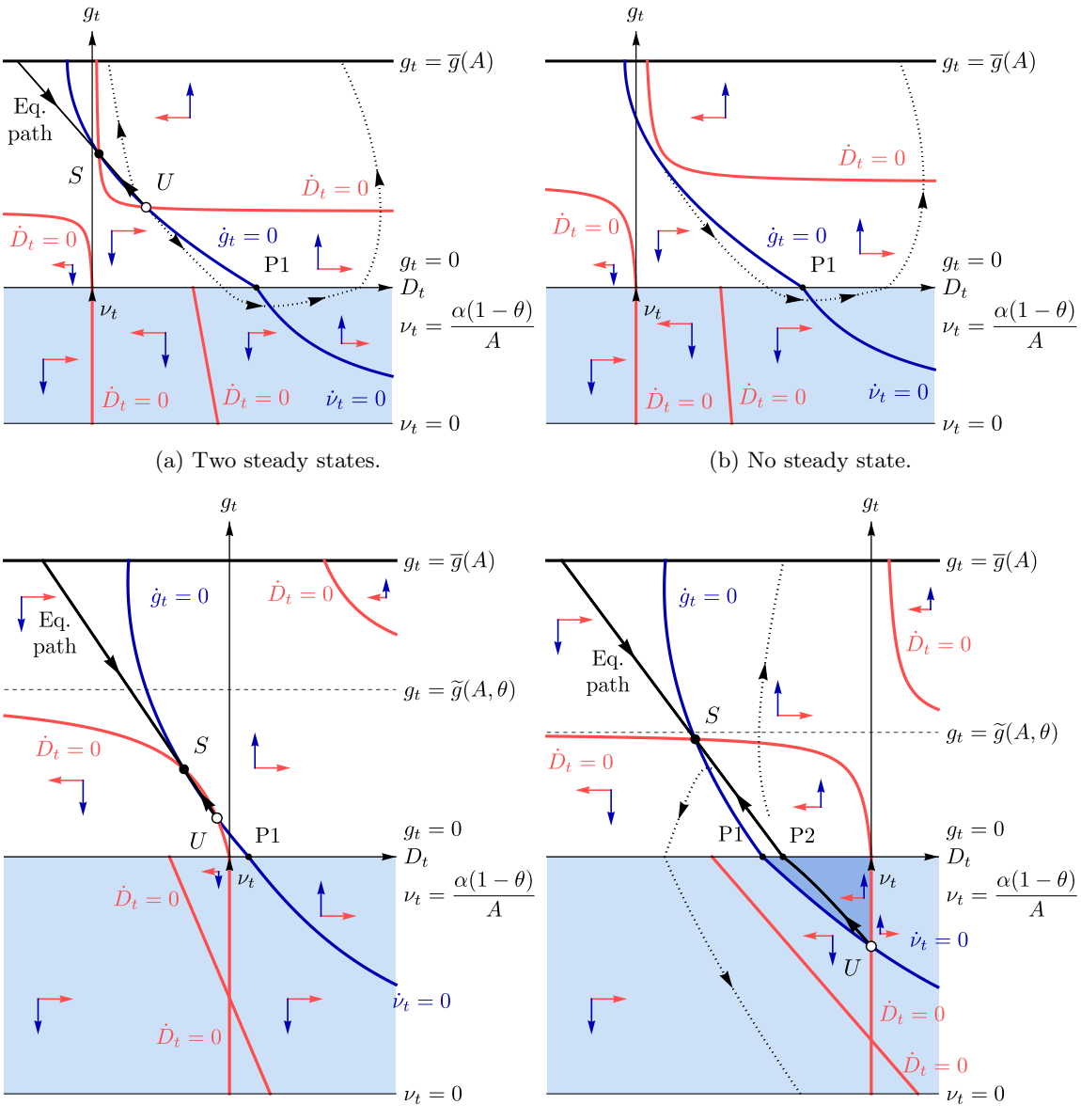
(a) $\theta = 5\%$ (b) $\theta = 0$

Figure A.6: The long-term BGP $D^*(A,\theta)$ (solid) and the unsustainable threshold $D_U(A,\theta)$ (dashed) against research productivity. D_t falls if $D^*(A,\theta) < D_t < D_U(A,\theta)$ and rises if $D_t < D^*(A,\theta)$. Given A and θ, there is no equilibrium path if D_t lies in the white area. The other parameters are those in Table 1.

time remaining until debt becomes infinite. Hence, unless the path reaches the free-entry boundary, firm value becomes negative before debt becomes infinite, or it becomes zero at that time. At the free-entry boundary, $\dot{\nu}_t$ has the same sign as $\dot{g}_t$ at $g_t = 0$, so entry into the R&D region is possible only at P1 or to its right.

A.7 Proof of Proposition 1

Figure A.6(a) illustrates the initial debt-GDP ratios that admit an equilibrium path for each level of R&D productivity A, holding $\theta = 5\%$ fixed. For sufficiently large A, the economy corresponds to Figure A.7(a), where $0 < D^* < D_U$ and equilibrium paths exist for initial debt-GDP ratios in the shaded area. As A decreases, D^* and D_U merge when $\overline{\theta}(A) = \theta$, and no steady state exists for intermediate values of A, including $\overline{A}$, as in panel (b). When A decreases further, two steady states reappear with $D^* < D_U \leq 0$, so no equilibrium path exists for $D_0 \geq 0$. Both steady states have R&D if $A > (1-\theta)A_{\min}$, as in panel (c), whereas U lies in the region without R&D and $D_U = 0$ if $A < (1-\theta)A_{\min}$, as in panel (d). Figure A.6(b) shows the limit as θ tends to zero. Both ends of the white area converge to $\overline{A}$, so the two parts of the shaded area join. For $A > \overline{A}$, D_U rises and D^* falls to zero. For $A < \overline{A}$, D_U converges to zero, even when $A > A_{\min}$ and U has R&D. The rest of this subsection proves what the arrows in Figure A.6 show. Given A and θ, the economy has a unique equilibrium path that converges to D^* from any initial debt-GDP ratio below D_U. It has no equilibrium path if the initial ratio is above D_U, or if there is no steady state. We can see where each path leads in Figure A.7 from the directions in which paths can cross the loci and the boundaries.



(a) Two steady states.

(b) No steady state.

(c) Both steady states have R&D.

(d) U is in the region without R&D.

Figure A.7: Equilibrium and excluded paths in the two regions. In panels (c) and (d), $A < \overline{A}$, and negative debt represents government net assets. In panel (d), the path from U enters the R&D region at P2. In the darker set, D_t falls and ν_t rises.

Proof of part (a). We count the steady states in the two regions. On the $\dot{g}_t = 0$ locus in the R&D region, let $z_t = g_t - r_t = s(\theta)(g_t - \widetilde{g}(A, \theta))$. Substituting (30) into (29) shows that $F(z_t; \theta)$ equals $s(\theta)\overline{g}(A)(\rho + \mu)(\delta + \mu)\dot{D}_t$, so F has the sign of $\dot{D}_t$ along the locus. At P1, where $g_t = 0$, the interest rate r_t equals $\overline{g}(A)/(1-\theta)$, which is positive, and the debt dynamics (29) reduce to $\dot{D}_t = r_t D_t$. The sign of F at P1 therefore coincides with that of D_t. At the upper boundary $g_t = \overline{g}(A)$, we have $z_t = A$ and $F(A; \theta) > 0$. Exactly one root of $F(z; \theta) = 0$ lies above $z = A$ and is excluded (Appendix A.4). Hence, apart from the knife-edge case in which two solutions coincide, the R&D region contains zero or two steady states when P1 has positive debt, and exactly one when P1 has negative debt.

By Appendix A.5, P1 has positive debt if and only if $A/(1-\theta) > A_{\min}$, and the region without R&D then has no steady state. If $A/(1-\theta) < A_{\min}$, P1 has negative debt, so the R&D region contains one steady state, S, and the region without R&D contains the other, $U = (0, \nu_U)$. Therefore, except for knife-edge cases, the economy has two steady states or none.

Proof of part (b). First suppose both steady states have R&D, as in Figure A.7(a) or (c). For $D_0 < D_U$, the equilibrium line selects a unique initial growth rate, and the resulting path converges to S. By (A.20) and the initial condition of the proposition, the labor resource constraint holds along this path. The transversality condition and non-Ponzi game condition of households hold along this path, because the path converges to S, where the gap $g^* - r^*$ is below $\delta - \rho$ (Appendix A.4), so that the return on annuities $r^* + \mu$ exceeds the growth rate $g^* - \delta$ of each individual's labor income. By Lemma A.2, along every other path, the labor resource constraint is violated, firm value becomes negative, or debt becomes infinite, unless the path enters the R&D region at P1 or to its right. As illustrated in Figure A.7(a) and (c), such a returning path crosses the upper labor boundary, which violates the resource constraint. Hence, the selected path is the unique equilibrium for $D_0 < D_U$, and no equilibrium exists for $D_0 > D_U$.

Next suppose $U = (0, \nu_U)$ lies in the region without R&D, as in Figure A.7(d). This occurs when $A < (1-\theta)A_{\min}$, so that D_U is zero. In this case, the smallest root of $F(z;\theta) = 0$ corresponds to a negative growth rate, and the straight line through the associated intersection and S reaches $g_t = 0$ at P2. Along this line, g_t rises toward S, so $\dot{g}_t$ is positive at P2, and P2 lies to the right of P1. P2 also has negative debt, by $\dot{D}_t \geq 0$ at $D_t = 0$ in the R&D region, whereas debt falls along the selected line toward S. Hence, P2 lies on the upper edge of the darker set in Figure A.7(d). No path enters this set from outside, so the path through P2 originates from U and eventually converges to S.

Along this path, debt decreases from zero to the debt ratio at P2, so each initial debt ratio between these values selects exactly one point. For lower initial debt ratios satisfying the proposition's lower bound, the economy begins directly on the R&D segment of the same equilibrium path. As in the first case, the transversality condition and non-Ponzi game condition of households hold along this path, because the path converges to S. All other paths violate firm valuation, or enter the R&D region away from P2 and therefore cannot remain feasible, as shown by Lemma A.2 and Figure A.7(d). No equilibrium exists for $D_0 > 0$. This completes the proof of part (b).

Proof of part (c). Suppose the economy has no steady state, as in Figure A.7(b). Then $\dot{D}_t$ is positive along the entire $\dot{g}_t = 0$ locus, as at P1. By Lemma A.2, a path with R&D

must cross the upper labor boundary or enter the region without R&D. In the latter case, firm value becomes negative, debt becomes infinite in finite time, or the path returns to the R&D region at P1 or to its right. Such a returning path eventually crosses the upper labor boundary, as illustrated in Figure A.7(b). Hence, no path can satisfy all equilibrium conditions indefinitely, proving part (c).

A.8 Proof of Proposition 2

We use the stationary condition (A.12) to determine when a long-term BGP with $g^* > r^*$ exists and to derive the upper bound on the subsidy rate. Since θ enters $F(z;\theta)$ only through its constant term, an increase in θ shifts the cubic upward without changing the locations of its extrema.

The condition on research productivity. We first show that a long-term BGP with $g^* > r^*$ exists for some $\theta \geq 0$ if and only if $A > \overline{A}$. When $\theta = 0$, $F(z;0) = 0$ has a root at $z = 0$, and the other two roots satisfy $(A - z)(\delta - \rho - z) = (\rho + \mu)(\delta + \mu)$. The largest root lies above A and is infeasible, as shown in Appendix A.4. The smaller root is positive if and only if $A > \overline{A}$, because the left-hand side is strictly decreasing on $0 < z < \min\{A, \delta - \rho\}$ and exceeds the right-hand side at $z = 0$ precisely when $A > \overline{A}$. Thus, when $A > \overline{A}$, the long-term BGP has $g^* > r^*$ without subsidies, and this property persists for sufficiently small positive subsidies.

When $A < \overline{A}$, the smaller root is negative. As illustrated in Figure A.2(b), increasing θ shifts $F(z;\theta)$ upward, so the two remaining real roots, when they exist, move toward each other in the negative-z region Hence, no positive subsidy can generate a long-term BGP with $g^* > r^*$.

The upper bound on the subsidy. Suppose that $A > \overline{A}$, as illustrated in Figure A.2(a). When $\theta = 0$, the two feasible roots are $z = 0$ and a positive root corresponding to S. As θ increases, the upward shift of $F(z;\theta)$ moves the smaller root rightward and the larger root leftward. They eventually merge at the local minimum of $F(z;\theta)$ when $\theta = \overline{\theta}(A)$.

Since $F_z(z;\theta)$ does not depend on θ, the location of this minimum is the smaller solution of

$$3z^2 - 2(A + \delta - \rho)z + (\delta - \rho)(A - \overline{A}) = 0. \tag{A.21}$$

We denote this solution by $\underline{z}(A)$. It is independent of θ and lies between 0 and $(\delta - \rho)/2$, because the left-hand side of (A.21) is positive at $z = 0$ and negative at $z = (\delta - \rho)/2$. The upper bound is determined by $F(\underline{z}(A); \overline{\theta}(A)) = 0$. Using (A.21) to simplify this condition and solving for $\overline{\theta}(A)$ yields (34). For $\theta < \overline{\theta}(A)$, the two feasible roots remain distinct, and

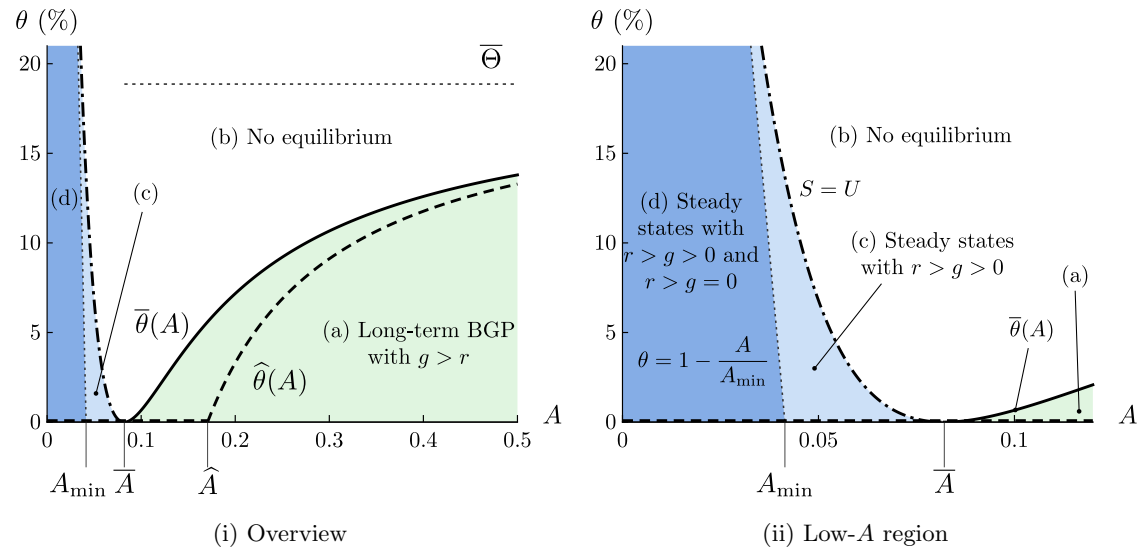


Figure A.8: Steady states in the (A, θ) plane, with the low-A range enlarged.

no feasible root remains when θ exceeds $\overline{\theta}(A)$. At $\theta = \overline{\theta}(A)$, the two roots coincide. This is the knife-edge case excluded in Proposition 1(a), so the coincident steady state is not a long-term BGP.

Comparative statics and the limiting upper bound. We next show that $\overline{\theta}(A)$ is strictly increasing in A. At the contact, $F(\underline{z}(A); \overline{\theta}(A)) = 0$ and $F_z(\underline{z}(A); \overline{\theta}(A)) = 0$. Totally differentiating the first condition with respect to A gives $F_A + F_z \underline{z}'(A) + F_\theta \overline{\theta}'(A) = 0$. From $F_z = 0$ at the local minimum, we obtain $\overline{\theta}'(A) = -F_A/F_\theta$. By (A.12), $F_\theta = (1-\alpha)(\rho+\mu)(\delta+\mu)A/(1-\overline{\theta}(A))^2 > 0$. Using $F = 0$, we also obtain $F_A = -\underline{z}(A)\big((\rho+\mu)(\delta+\mu) + \underline{z}(A)(\delta-\rho-\underline{z}(A))\big)/A < 0$. Therefore, $\overline{\theta}'(A) > 0$.

To find the limit, divide (A.21) by A. Since $\underline{z}(A)$ stays between 0 and $(\delta-\rho)/2$, the terms divided by A vanish as $A \to \infty$, yielding $\underline{z}(A) \to (\delta-\rho)/2$. Substituting this limit into $F(\underline{z}(A); \overline{\theta}(A)) = 0$ after dividing by A gives

$$\lim_{A\to\infty} \overline{\theta}(A) = \overline{\Theta} \equiv \frac{\delta-\rho}{\delta-\rho+4(1-\alpha)\overline{A}}. \tag{A.22}$$

When research productivity is lower. If $A < \overline{A}$, no steady state with $g^* > r^*$ exists for any $\theta \geq 0$, as established above. When R&D is conducted and $\theta > 0$, any steady state must have $g^* < r^*$, which implies a negative debt-GDP ratio by (29). When both steady states have R&D, $D^* < D_U < 0$, as in Figure A.7(c). If $A/(1-\theta) < A_{\min}$, U is instead the steady state without R&D and $D_U = 0$, as in panel (d). The two steady states with R&D may also merge when the subsidy rate reaches their contact condition. These configurations are summarized in Figure A.8. By Proposition 1, an economy with positive initial debt cannot follow an equilibrium path in these cases.

A.9 Proof of Proposition 3 and Corollary 1

Proof of Corollary 1. As illustrated in Figure A.2(a), an increase in θ shifts $F(z;\theta)$ upward, reducing the larger feasible root z^*. By (A.13), long-run growth increases with θ when $z^* > (\delta-\rho)/2$ and decreases when $z^* < (\delta-\rho)/2$.

Proof of Proposition 3. By (A.13), the subsidy affects long-run growth only through $z^* = g^* - r^*$. Growth is increasing in $z^*(\delta-\rho-z^*)$ and is highest at $z^* = (\delta-\rho)/2$. Evaluating (A.13) at this gap gives $\widehat{g}(A)$ in (36). The supply-side relation (27) then yields $\widehat{\theta}(A)$ in (35). By (33) and (35), this subsidy rate is positive if and only if $A > \widehat{A}$. If $A \leq \widehat{A}$, the gap reaches $(\delta-\rho)/2$ only at $\theta = \widehat{\theta}(A) \leq 0$, so that no positive subsidy raises long-run growth by Corollary 1.

We next verify that $\widehat{\theta}(A)$ is feasible when $A > \widehat{A}$. By Appendix A.8, $\underline{z}(A)$ lies below $(\delta-\rho)/2$. Moreover, by (A.12), the derivative $F_z((\delta-\rho)/2;\theta)$ equals $(\rho+\mu)(\delta+\mu)+(\delta-\rho)^2/4$, which is positive. Thus, $F(z;\theta)$ is increasing at $z=(\delta-\rho)/2$, so this gap is the larger of the two feasible roots at the subsidy rate $\widehat{\theta}(A)$, and the two roots are distinct at this rate. Hence, $\widehat{\theta}(A) < \overline{\theta}(A)$, and the growth-maximizing subsidy is attainable on the long-term BGP. Finally, differentiating (35) shows that $\widehat{\theta}(A)$ is strictly increasing in A. As $A \to \infty$, it converges to $\overline{\Theta}$ in (A.22).

A.10 Estimation of A

This appendix derives the estimator for A. From (25) and (26),

$$A = \frac{\alpha}{1-\alpha}\frac{g_t}{L_t^R/L} = \frac{\alpha}{1-\alpha}\frac{w_t L/\mathrm{GDP}_t}{w_t L_t^R/\mathrm{GDP}_t} g_t. \tag{A.23}$$

Using (20), (22), and (23), we obtain

$$\frac{w_t L}{\mathrm{GDP}_t} = \frac{\alpha A}{A-\alpha g_t}. \tag{A.24}$$

Substituting (A.24) into (A.23) and solving for A yields (37).

A.11 Proof of Proposition 4

Under each subsidy rate, the part of an equilibrium path with positive R&D lies on a straight line (Appendix A.7).

Let $\theta_1 < \theta_2$ be two subsidy rates. Since debt does not jump, it suffices to compare the lines with θ_1 and θ_2 at the debt ratio when the rate changes, as in Figure A.9.

The slope of each line, given in Appendix A.6, is inversely proportional to the gap z_E

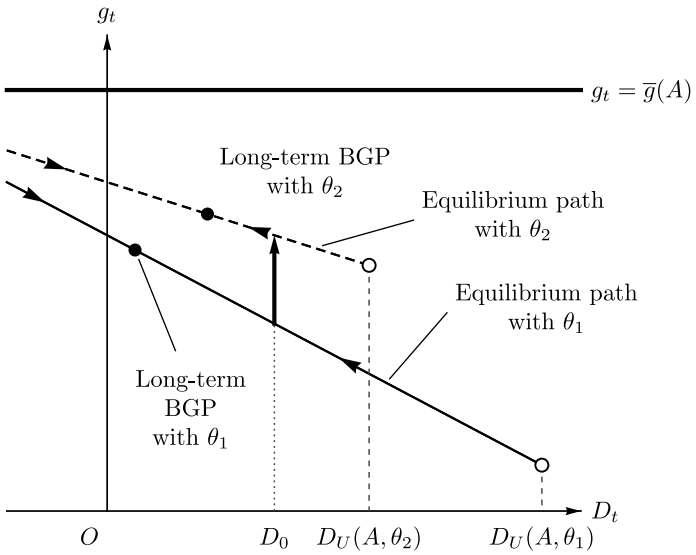


Figure A.9: The equilibrium lines with two subsidy rates. The line with $\theta_2 > \theta_1$ lies above the line with θ_1.

at E. As Figure A.2(a) shows, a higher subsidy shifts $F(z;\theta)$ upward and raises z_E, so that the line with θ_2 is flatter. Each line reaches $g_t = \overline{g}(A)$ at $\underline{D}(A,\theta)$ in (A.20), whose two terms rise toward zero with the subsidy. The line with θ_2 therefore reaches the upper labor boundary closer to the vertical axis. Since it is also flatter, it lies above the line with θ_1 wherever it is below this boundary. This range includes the debt ratio when the rate changes, because the economy has an equilibrium path under θ_2. An unanticipated increase in the subsidy therefore raises g_t, and the new path has positive R&D. The same comparison proves the result for a decrease, in which g_t may fall to zero.

A.12 Transition Paths and Deviations from the Balanced Growth Path without Subsidies

When the subsidy changes permanently and unexpectedly at $t = 0$, D_t does not jump, whereas g_t jumps to the new equilibrium path. We first consider the case in which both S and U have R&D, as in the simulations in Section 4.5. Omitting the arguments (A,θ) of their coordinates, the line through them is

$$g_t = g^* + \frac{g_U - g^*}{D_U - D^*}(D_t - D^*). \tag{A.25}$$

On the line, $\chi_E = 0$ in (A.19), and the initial weights are fixed by D_0. Taking component ratios gives

$$D_t - D^* = \frac{(D_U - D^*)(D_0 - D^*)}{(D_U - D_0)e^{s(\theta)(g^* - g_U)t} + D_0 - D^*}, \tag{A.26}$$

$$\frac{Y_t}{Y_0} = e^{g^* t}\left(\frac{D_U - D_0 + (D_0 - D^*)e^{-s(\theta)(g^* - g_U)t}}{D_U - D^*}\right)^{1/s(\theta)}. \tag{A.27}$$

The second expression follows from $\gamma_{3,t} = (Y_t/Y_0)^{s(\theta)}$. Thus, convergence eventually occurs at the rate $s(\theta)(g^* - g_U)$, which tends to zero as the two steady states coincide. Under the parameters of Section 4.5, this rate is 0.0027 for $A = 0.399, \theta = 11.74\%$, and 0.0007 for $A = 0.120, \theta = 2\%$. If U is in the region without R&D, the matrix solution in Appendix A.6 gives the segment with research and is joined to the path in (D, ν) at P2.

We derive the percentage deviations shown in Figure 9. The economy starts from $D_0 = 0$, so Y_t is given by (A.27) with this initial condition. Without subsidies, $Y_t^{\theta=0} = Y_0 e^{g^*(A,0)t}$. Hence,

$$\frac{Y_t - Y_t^{\theta=0}}{Y_t^{\theta=0}} = e^{(g^*(A,\theta) - g^*(A,0))t}\left(\frac{D_U(A,\theta) - D^*(A,\theta)e^{-s(\theta)(g^* - g_U)t}}{D_U(A,\theta) - D^*(A,\theta)}\right)^{1/s(\theta)} - 1, \tag{A.28}$$

where g^* and g_U in the exponent are evaluated under the subsidy θ.

Some labor is used for research rather than consumption. By (A.7), consumption is $C_t = (1 - g_t/\overline{g}(A))Y_t$, so

$$\frac{C_t - C_t^{\theta=0}}{C_t^{\theta=0}} = \frac{\overline{g}(A) - g_t}{\overline{g}(A) - g^*(A,0)}\frac{Y_t}{Y_t^{\theta=0}} - 1. \tag{A.29}$$

Similarly, GDP is $\mathrm{GDP}_t = \big(1 - (1-\alpha)g_t/\overline{g}(A)\big)Y_t$ by (22), so

$$\frac{\mathrm{GDP}_t - \mathrm{GDP}_t^{\theta=0}}{\mathrm{GDP}_t^{\theta=0}} = \frac{\overline{g}(A) - (1-\alpha)g_t}{\overline{g}(A) - (1-\alpha)g^*(A,0)}\frac{Y_t}{Y_t^{\theta=0}} - 1. \tag{A.30}$$

The factor in front of $Y_t/Y_t^{\theta=0}$ in this last expression lies between one and the corresponding factor for consumption. Thus, the GDP deviation lies between those of full-production GDP and consumption.

B Social Rate of Return of R&D and Dynamic Efficiency

B.1 Proof of Proposition 5

Here, we derive the social rate of return of R&D. As explained in Definition 1, we discretize time in the model with a step size of Δt. We later take the limit of $\Delta t \to 0$. We assume that the economy is on the BGP before time t. The state variable n_t at time t is also on the BGP because it is predetermined by time t. Then, we marginally increase L_t^R by $dL_t^R > 0$,

which reduces the production of consumption goods at time t (denoted by $dX_t < 0$) but increases the number of goods at $t + \Delta t$ (denoted by $dn_{t+\Delta t} > 0$). Then, at $t + \Delta t$, we marginally decrease $L^R_{t+\Delta t}$ by $dL^R_{t+\Delta t} < 0$ so that $n_{t+2\Delta t}$ returns to the original BGP. The production of consumption goods at $t+\Delta t$ will be higher than that on the BGP. We denote the difference by $dX_{t+\Delta t} > 0$.

In the following, we derive the relationships among $dn_{t+\Delta t}$, dL^R_t, and $dL^R_{t+\Delta t}$. The discrete version of (15) at the time between t and $t + \Delta t$ is

$$n_{t+\Delta t} = n_t + aL^R_t n_t \Delta t, \tag{B.1}$$

where $a = A/L$. Taking the total derivative of Equation (B.1) while keeping n_t constant yields

$$dn_{t+\Delta t} = an_t \Delta t \, dL^R_t. \tag{B.2}$$

Similar to (B.1), the discrete version of (15) from time $t + \Delta t$ to time $t + 2\Delta t$ is

$$n_{t+2\Delta t} = n_{t+\Delta t} + aL^R_{t+\Delta t} n_{t+\Delta t} \Delta t. \tag{B.3}$$

By taking the total derivative of (B.3), while keeping $n_{t+2\Delta t}$ unchanged from the BGP, we obtain

$$0 = \left(1 + aL^R_{t+\Delta t}\Delta t\right) dn_{t+\Delta t} + an_{t+\Delta t}\Delta t \, dL^R_{t+\Delta t}. \tag{B.4}$$

We substitute $n_{t+\Delta t}$ in (B.1) and $dn_{t+\Delta t}$ in (B.2) into (B.4), which yields

$$0 = (1 + aL^R_{t+\Delta t}\Delta t)dL^R_t + (1 + aL^R_t \Delta t)dL^R_{t+\Delta t}. \tag{B.5}$$

Recall that the economy is on the BGP before the manipulation of L^R_t, where L^R_t is constant for all t. Let this constant value be L^{R*}. Then, $L^R_t = L^{R*} + dL^R_t$ and $L^R_{t+\Delta t} = L^{R*} + dL^R_{t+\Delta t}$. Substituting these values into (B.5) gives

$$0 = (1 + aL^{R*}\Delta t)(dL^R_t + dL^R_{t+\Delta t}) + 2a\Delta t \, dL^R_t \, dL^R_{t+\Delta t}. \tag{B.6}$$

This equation implies that the ratio of the changes in L^R_t and $L^R_{t+\Delta t}$ is

$$-\frac{dL^R_{t+\Delta t}}{dL^R_t} = 1 + \frac{2a\Delta t \, dL^R_{t+\Delta t}}{1 + aL^{R*}\Delta t}. \tag{B.7}$$

Now, we consider the changes in the final output X_t in periods t and $t + \Delta t$. Note that in period t, n_t is still on the BGP. Therefore, from (14), the changes in X_t and L^R_t are

related by

$$dX_t = \frac{\partial X_t}{\partial L_t^R} dL_t^R = n_t^{\frac{1-\alpha}{\alpha}} \left(-dL_t^R\right). \tag{B.8}$$

In period $t + \Delta t$, $n_{t+\Delta t}$ deviates from the BGP by $dn_{t+\Delta t}$. Therefore, the change in $X_{t+\Delta t}$ can be expressed as

$$\begin{aligned} dX_{t+\Delta t} &= \frac{\partial X_{t+\Delta t}}{\partial n_{t+\Delta t}} dn_{t+\Delta t} + \frac{\partial X_{t+\Delta t}}{\partial L_{t+\Delta t}^R} dL_{t+\Delta t}^R \\ &= \frac{1-\alpha}{\alpha} n_{t+\Delta t}^{\frac{1-\alpha}{\alpha}-1} (L - L_{t+\Delta t}^R) dn_{t+\Delta t} - n_{t+\Delta t}^{\frac{1-\alpha}{\alpha}} dL_{t+\Delta t}^R \\ &= n_{t+\Delta t}^{\frac{1-\alpha}{\alpha}} \left[\frac{1-\alpha}{\alpha} (L - L_{t+\Delta t}^R) \frac{a\Delta t \, dL_t^R}{1 + aL_t^R \Delta t} - dL_{t+\Delta t}^R \right]. \end{aligned} \tag{B.9}$$

where we use (B.1) and (B.2) in the third line.

We obtain the social return of R&D investment by substituting (B.8) and (B.9) into Definition 1:[89]

$$\begin{aligned} r^s \Delta t &= \frac{dX_{t+\Delta t}}{-dX_t} - 1 \\ &= \left(\frac{n_{t+\Delta t}}{n_t} \right)^{\frac{1-\alpha}{\alpha}} \left(\frac{1-\alpha}{\alpha} (L - L_{t+\Delta t}^R) \frac{a\Delta t}{1 + aL_t^R \Delta t} - \frac{dL_{t+\Delta t}^R}{dL_t^R} \right) - 1. \end{aligned} \tag{B.10}$$

On the RHS of the above equation, we can use the Taylor expansion as follows:

$$\left(\frac{n_{t+\Delta t}}{n_t} \right)^{\frac{1-\alpha}{\alpha}} = (1 + a\, L_t^R \Delta t)^{\frac{1-\alpha}{\alpha}} = 1 + \frac{1-\alpha}{\alpha} aL_t^R \Delta t + O((\Delta t)^2),$$

where $O((\Delta t)^2)$ is the collection of terms that are of order $(\Delta t)^2$ and higher. We can also eliminate $dL_{t+\Delta t}^R / dL_t^R$ via (B.7). Applying both to (B.10), we obtain

$$r^s = \frac{1-\alpha}{\alpha} aL_t^R + \frac{\frac{1-\alpha}{\alpha} a(L - L_{t+\Delta t}^R) + 2a\, dL_{t+\Delta t}^R}{1 + aL_t^R \Delta t} + O(\Delta t). \tag{B.11}$$

Recall that Δt is infinitesimally small. In the continuous-time limit, $\Delta t \to 0$. Additionally, we consider marginal perturbations of dL_t^R and $dL_{t+\Delta t}^R$ from the BGP. Therefore, $dL_{t+\Delta t}^R \to 0$. Applying these to (B.11) and $aL = A$, we obtain

$$r^s = \frac{1-\alpha}{\alpha} A = \overline{g}(A).$$

Since L_t^R cannot exceed L, the growth rate g_t in (25) is at most $\overline{g}(A)$, which equals r^s.

[89] In Definition 1, r^s is defined as the limit of the first line of (B.10) as $\Delta t \to 0$. We calculate the limit after we rewrite (B.10) as (B.11).

B.2 Proof of Proposition 6

Let g^* be the growth rate in aggregate consumption on the original BGP, and let L^{R*} and L^{P*} be the constant amounts of R&D labor and production labor, respectively. Suppose that from this BGP, the path of aggregate consumption, $C_t = X_t$, is increased by dX_t from time t to time $t + \Delta t$, where Δt is a small time interval. In the following, we show that the aggregate consumption from $t + \Delta t$ cannot be maintained on the original BGP.

As explained in the proof of Proposition 5 in Appendix B.1, the only way to increase aggregate consumption at time t is to reduce R&D labor and increase production labor. Then, at $t + \Delta t$, $n_{t+\Delta t}$ should be lower than the BGP value, which means that the production technology of the final goods is inferior to that of the original BGP case. To keep $X_{t+\Delta t}$ at the original BGP value despite a smaller $n_{t+\Delta t}$, we need to allocate more workers to production, which means that $L^P_{t+\Delta t} > L^{P*}$.

Below, we examine the dynamics of the economy from $t + \Delta t$ in continuous time. For $t' \geq t + \Delta t$, the aggregate consumption should be the same as that of the original BGP, which is growing at the rate of g^*. By taking the log of (14) and then differentiating with respect to t', we obtain

$$\frac{\dot{X}_{t'}}{X_{t'}} = \frac{1-\alpha}{\alpha}\frac{\dot{n}_{t'}}{n_{t'}} + \frac{\dot{L}^P_{t'}}{L^P_{t'}} = g^* \text{ for all } t' \geq t + \Delta t.$$

Using (15), (25) and (23), the above equation can be rewritten as

$$\begin{aligned}\frac{\dot{L^P_{t'}}}{L^P_{t'}} &= g^* - \frac{1-\alpha}{\alpha}A\frac{L^R_{t'}}{L} \\ &= \frac{1-\alpha}{\alpha}A\frac{L^{R*} - L^R_{t'}}{L} \\ &= \frac{1-\alpha}{\alpha}A\frac{L^P_{t'} - L^{P*}}{L}.\end{aligned}$$

Therefore, we have the following autonomous differential equation for $L^P_{t'}$ for $t' \geq t+\Delta t$:

$$\dot{L^P_{t'}} = \frac{1-\alpha}{\alpha}A\frac{L^P_{t'} - L^{P*}}{L}L^P_{t'}. \tag{B.12}$$

As shown in Figure B.1, the RHS of (B.12) is positive when $L^P_{t'} > L^{P*}$. Recall that $L^P_{t+\Delta t} > L^{P*}$. Therefore, $\dot{L^P_{t'}} > 0$ for all $t' \geq t + \Delta t$, and $L^P_{t'}$ exceeds the total labor supply L in a finite period. Therefore, such a path is not feasible.

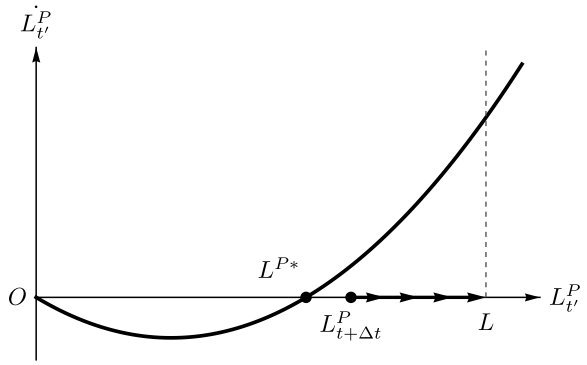


Figure B.1: Movement of L^P_t after $t' = t + \Delta t$

C Analysis of the Second-Generation Model

This appendix provides the details of the analysis of the second-generation model in Section 6. We first prove Lemma 1 and derive the equilibrium dynamics. We then examine the steady states and prove Propositions 7 and 8. Finally, we describe the details of the calibration in Section 6.4 and show the relationship with the baseline model discussed in Section 6.5.

C.1 Proof of Lemma 1

Preliminaries

We first derive three expressions used in the proof. First, the price of the good of industry j is $p_s(j) = (w_s - f v_s^{\text{new}})/\alpha$, and its demand is $x_s(j) = q_s(j)^{\alpha/(1-\alpha)} p_s(j)^{-1/(1-\alpha)} X_s$ (Section 6.1). Hence, the profit of the leader is

$$\pi_s(j) = (1-\alpha)\left(\frac{w_s - f v_s^{\text{new}}}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}} X_s\, q_s(j)^{\frac{\alpha}{1-\alpha}}. \tag{C.1}$$

Second, from (43), the quality at $s \geq t$ is

$$q_s(j) = q_t(j) \exp\left(\int_t^s \left(\omega_{\text{in}} \ell_u^R(j) + \omega_{\text{out}} \frac{L_u^R}{n_u}\right) du\right), \tag{C.2}$$

where the spillovers from other industries are common to all industries. Third, the leader loses its position when an outsider obtains a new core technology, which occurs at the rate $a\ell_u^R(j)$, or when the industry becomes obsolete, which occurs at the rate ε. Therefore, the value of the leader is

$$v_t(j) = \int_t^\infty \pi_s(j) \exp\left(-\int_t^s \left(r_u + a\ell_u^R(j) + \varepsilon\right) du\right) ds. \tag{C.3}$$

Proof of Part (a)

Suppose that the leader of industry j uses a unit of effective labor for research. This research does not change the rate at which the leader is replaced, because this rate depends only on the research of outsiders by (C.3). It raises the quality of the good at the rate ω_{in}, and by (C.1) and (C.2), all future profits rise in the same proportion. Hence, the value of the leader increases at most at the rate $\alpha\omega_{\text{in}}v_t(j)/(1-\alpha)$, and it increases less if outsiders respond by conducting more research in the industry. The cost of this research is $(1-\theta)w_t - fv_t^{\text{new}}$, because the leader also obtains the ideas of the worker. An outsider who pays the same cost obtains $av_t(j)$, so the free-entry condition (42) implies $av_t(j) \leq (1-\theta)w_t - fv_t^{\text{new}}$. By the assumption of the lemma, $\alpha\omega_{\text{in}}/(1-\alpha)$ is less than a, so the benefit of the research is smaller than its cost. Therefore, the leader does not conduct research, regardless of whether outsiders conduct research in the industry.

Proof of Part (b)

By part (a), only outsiders conduct research in each industry, and their research $\ell_u^R(j)$ appears both in the rate of replacement in (C.3) and in the growth of quality in (C.2). Substituting (C.1) and (C.2) into (C.3) gives

$$\begin{aligned} v_t(j) = q_t(j)^{\frac{\alpha}{1-\alpha}} \int_t^\infty (1-\alpha) \left(\frac{w_s - fv_s^{\text{new}}}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}} X_s \\ \times \exp\left(\int_t^s \Big(\frac{\alpha\omega_{\text{out}}}{1-\alpha}\frac{L_u^R}{n_u} - r_u - \varepsilon\Big)du\right) \\ \times \exp\left(-\Big(a - \frac{\alpha\omega_{\text{in}}}{1-\alpha}\Big)\int_t^s \ell_u^R(j)du\right) ds. \end{aligned} \tag{C.4}$$

Only $q_t(j)$ and the last exponential factor differ across industries. In the last exponential factor, the research of outsiders lowers the value of the leader through replacement at the rate a and raises it through quality growth at the rate $\alpha\omega_{\text{in}}/(1-\alpha)$. Under the assumption $\omega_{\text{in}} < a(1-\alpha)/\alpha$, the former effect dominates, so that this factor is at most one.

We use this property to compare two industries j and j' with $q_t(j) \geq q_t(j')$. We show that $v_t(j)/q_t(j)^{\alpha/(1-\alpha)} \leq v_t(j')/q_t(j')^{\alpha/(1-\alpha)}$, with strict inequality if outsiders conduct research in industry j before they start research in industry j'. Consider the first date at or after t at which outsiders conduct research in industry j'. Until this date, $q_s(j) \geq q_s(j')$ continues to hold by (C.2), and the last exponential factor in (C.4) is one for j' and at most one for j. If there is no such date, the inequality follows from (C.4). Otherwise, at this date, the free-entry condition (42) holds with equality in industry j', so that $v_s(j') \geq v_s(j)$, and hence the inequality holds at this date. Splitting the integral in (C.4) at this date, we obtain the inequality at t. In particular, if $q_t(j) = q_t(j')$, the values of the two leaders are

equal.

Now suppose that $q_0(j)$ is the same for all industries and that two industries j and j' have different qualities at some date. Consider the last date until which their qualities are equal. Right after this date, outsiders conduct research in both industries, in only one of them, or in neither.[90] If they conduct research in neither industry, the qualities remain equal by (C.2). If they conduct research in only one of them, the strict inequality above contradicts the equality of the values at this date. If they conduct research in both, the free-entry condition (42) holds with equality in both industries, and the values of the two leaders and their time derivatives are equal. Since the value of each leader satisfies $\dot{v}_t(j) = \big(r_t + a\ell_t^R(j) + \varepsilon\big)v_t(j) - \pi_t(j)$ by (C.3), we obtain $a\big(\ell_t^R(j) - \ell_t^R(j')\big)v_t(j) = \pi_t(j) - \pi_t(j')$. Hence, by (C.1) and (C.2), the ratio $q_t(j)/q_t(j')$ follows a differential equation whose right-hand side is zero when the ratio is one, so that the ratio remains one. In the other two cases, the qualities remain equal for a while after this date, which contradicts the definition of this date. Therefore, $\ell_t^R(j)$ and $q_t(j)$ are the same for all industries at every t, including new industries, which start with the average quality of the existing industries.

C.2 Derivation of Equilibrium Dynamics

First suppose vertical R&D is conducted. The free-entry condition (44) and the effective subsidy rate (52) give $av_t = (1-\theta^e)(w_t - fv_t)$. By (46), $v_t = \alpha(1-\theta^e)n_t^{(1-\alpha)/\alpha}q_t/a$, and its growth rate is $g_t - g_L$. The profit-value ratio is $(1-\alpha)aL_t^P/((1-\theta^e)\alpha n_t)$. Substituting these expressions and $aL_t^P/n_t = \kappa(g_L+\varepsilon) - g_{q,t}/\lambda$ into (50) yields the supply-side relation (51).

Differentiating $D_t = B_t/Y_t$ gives $\dot{D}_t = (r_t - g_t)D_t + \theta w_t L_t^R/Y_t$. Since the subsidy $\theta w_t L_t^R$ equals θ^e times the cost of vertical R&D, $(w_t - fv_t)L_t^R$, the expenditure ratio is $\alpha\theta^e g_{q,t}/\overline{g_q}(\kappa, g_L)$. Using (51) then gives (54). For the growth rate, the goods market and the production equations imply $C_t/Y_t = 1 - g_{q,t}/\overline{g_q}(\kappa, g_L)$. Hence, $\dot{g}_{q,t} = -(\overline{g_q} - g_{q,t})(\dot{C}_t/C_t - g_t)$. The aggregate Euler equation is unchanged, except that $\beta - \mu = g_L$. In addition, the asset market gives

$$\frac{K_t}{Y_t} = D_t + \frac{\alpha(1-\theta^e)}{\kappa(g_L+\varepsilon)}. \tag{C.5}$$

[90] We consider equilibria in which, for each industry, the set of dates with research consists of finitely many intervals in any bounded period.

Substituting these expressions yields (53). For reference, the two stationary loci are

$$\dot{D}_t = 0: \quad D_t = \frac{\alpha\theta^e}{\overline{g_q}(\kappa, g_L)} \frac{g_{q,t}}{g_t - r_t}, \tag{C.6}$$

$$\dot{g}_{q,t} = 0: \quad D_t = \frac{\overline{g_q}(\kappa, g_L) - g_{q,t}}{\overline{g_q}(\kappa, g_L)(\rho + \mu)(\delta + \beta)} \big(\delta - \rho + g_L - (g_t - r_t)\big) - \frac{\alpha(1 - \theta^e)}{\kappa(g_L + \varepsilon)}, \tag{C.7}$$

where $g_t - r_t$ is the affine function of $g_{q,t}$ in (51). The first expression is used when $g_t \neq r_t$. At $\theta = 0$, the debt locus consists of $D_t = 0$ and $g_t = r_t$.

Next suppose vertical R&D is absent. Then $g_{q,t} = 0$, $g_t = g_L/\alpha$, and $C_t = Y_t$. Define $\nu_t = n_t v_t / Y_t$ as in Appendix A.5. The free-entry condition requires

$$0 < \nu_t \leq \frac{\alpha(1 - \theta^e)}{\kappa(g_L + \varepsilon)}, \tag{C.8}$$

with equality on the boundary. The aggregate Euler equation gives $g_t - r_t = \delta - \rho + g_L - (\rho + \mu)(\delta + \beta)(D_t + \nu_t)$. Substituting this into the government budget and the valuation of firms gives

$$\dot{D}_t = \big((\rho + \mu)(\delta + \beta)(D_t + \nu_t) - (\delta - \rho + g_L)\big) D_t, \tag{C.9}$$

$$\dot{\nu}_t = \big((\rho + \mu)(\delta + \beta)(D_t + \nu_t) - \delta + \rho + \varepsilon\big)\nu_t - (1 - \alpha). \tag{C.10}$$

The subsidy affects only the boundary, because none is paid inside this region. The R&D dynamics have the same common quadratic term as in Appendix A.6, with coefficient $(1-\alpha)s(\theta^e)/(\alpha\lambda)$ in place of $s(\theta)$. Thus, the linear representation applies to $(D_t, g_{q,t})$. The dynamics without vertical R&D must instead be described in (D_t, ν_t), because $g_{q,t}$ is fixed at zero.

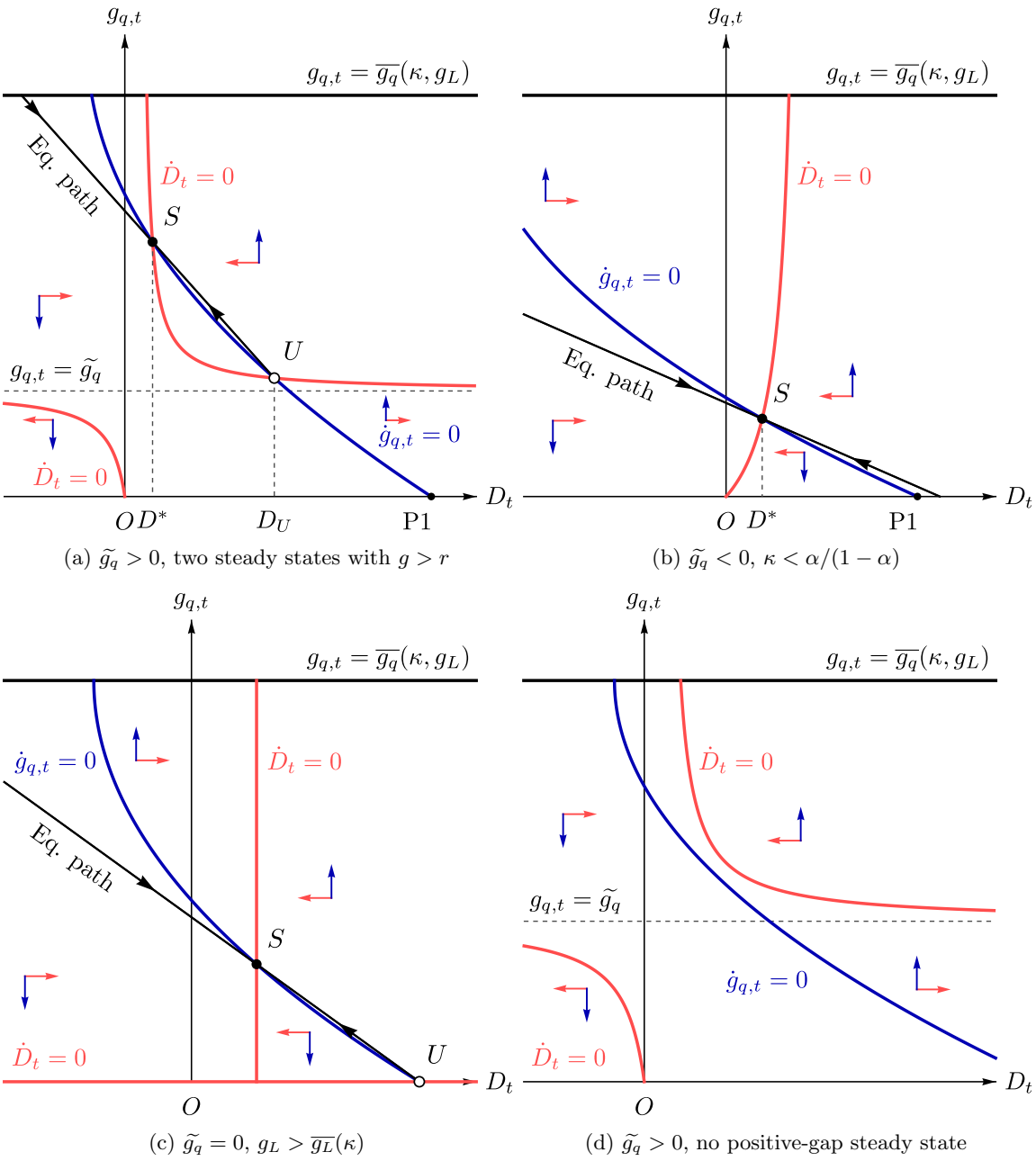
C.3 Steady States with Vertical R&D

Figure C.1 shows the stationary loci in the (D, g_q) plane. The $\dot{g}_q = 0$ locus opens to the right, and the $\dot{D} = 0$ locus is a hyperbola unless its asymptote is at $g_q = 0$. We first examine the economy without subsidies, where the debt locus consists of $D = 0$ and $g_q = \widetilde{g_q}(\kappa, g_L, 0)$.

The steady state on $D = 0$ satisfies $g > r$ if the growth locus crosses the axis above the $g = r$ line. Substituting this line into (C.7) gives the condition (55). Equivalently,

$$\kappa > \overline{\kappa}(g_L) \equiv \frac{(\rho + \mu)(\delta + \mu + g_L)}{(g_L + \varepsilon)(\delta - \rho + g_L)} - 1. \tag{C.11}$$

This threshold is strictly decreasing in g_L, and its inverse is $\overline{g_L}(\kappa)$.



(a) $\widetilde{g}_q > 0$, two steady states with $g > r$

(b) $\widetilde{g}_q < 0$, $\kappa < \alpha/(1-\alpha)$

(c) $\widetilde{g}_q = 0$, $g_L > \overline{g_L}(\kappa)$

(d) $\widetilde{g}_q > 0$, no positive-gap steady state

Figure C.1: The stationary loci with vertical R&D. Panels (a) and (d) show two intersections and no intersection when $\widetilde{g}_q > 0$. In panel (b), $\widetilde{g}_q < 0$, and the second economic steady state is in the region without vertical R&D. In panel (c), $\widetilde{g}_q = 0$ and U is on the boundary.

As in Figure A.7 for the baseline model, let P1 denote the point where the $\dot{g}_{q,t} = 0$ locus meets the boundary $g_{q,t} = 0$. A positive-debt intersection with vertical R&D also requires P1 to have positive debt. Evaluating (C.7) at P1 gives

$$\frac{\kappa}{1-\theta^e} > \underline{\kappa}(g_L), \tag{C.12}$$

$$\underline{\kappa}(g_L) = \alpha \frac{\sqrt{(\delta-\rho-\varepsilon)^2 + 4(1-\alpha)(\rho+\mu)(\delta+\mu+g_L)} - (\delta-\rho-\varepsilon)}{2(1-\alpha)(g_L+\varepsilon)}. \tag{C.13}$$

The threshold $\underline{\kappa}(g_L)$ is decreasing in g_L. The two thresholds evaluate the growth locus on the $g = r$ line and at $g_q = 0$. These two lines coincide when $\kappa/(1-\theta^e) = \alpha/(1-\alpha)$, as in Figure C.1(c), and the value of g_L at which $\overline{\kappa}(g_L) = \alpha/(1-\alpha)$ is $\widetilde{g_L}$ in (56). Hence, at $g_L = \widetilde{g_L}$, the two thresholds coincide at $\alpha/(1-\alpha)$. Below this population-growth rate, $\overline{\kappa} > \underline{\kappa}$, and above it the ordering is reversed. Thus, horizontal innovation may sustain $g > r$ without vertical R&D. Appendix C.4 examines that possibility.

The shape of the $\dot{D} = 0$ locus depends on the sign of $\widetilde{g_q}$, which, by (51), satisfies

$$\widetilde{g_q}(\kappa, g_L, \theta) < 0 \quad \Longleftrightarrow \quad \frac{\kappa}{1-\theta^e} < \frac{\alpha}{1-\alpha}. \tag{C.14}$$

When $\widetilde{g_q} < 0$, $g > r$ even at $g_q = 0$, giving the upward-sloping debt locus in Figure C.1(b). When $\kappa < \alpha/(1-\alpha)$, a higher subsidy brings $\widetilde{g_q}$ to zero at $\theta = \alpha - (1-\alpha)\kappa$. If $g_L > \overline{g_L}(\kappa)$, the higher-growth intersection survives, as in Figure C.1(c), and eventually disappears at a contact inside the research region. If $\widetilde{g_L} < g_L \leq \overline{g_L}(\kappa)$, it instead reaches the boundary with $g_q = 0$ and $g = r$. These are the two ways in which the upper subsidy bound is reached.

To check these intersections, we use the gap $z \equiv g - r$ at a candidate steady state, which is an increasing affine function of g_q by (51). Eliminating D from the two stationary loci gives $F^{2G}(z; \theta^e) = 0$, where

$$\begin{aligned} F^{2G}(z;\theta^e) \equiv & -\alpha(\rho+\mu)(\delta+\beta)\theta^e z_{\min}(\kappa, g_L, \theta^e) \\ & - z\Big(\big((1+\kappa)(g_L+\varepsilon) - z\big)\big(\delta - \rho + g_L - z\big) - (\rho+\mu)(\delta+\beta)\Big), \end{aligned} \tag{C.15}$$

and

$$z_{\min}(\kappa, g_L, \theta^e) \equiv (g_L + \varepsilon)\left(1 - \frac{(1-\alpha)\kappa}{\alpha(1-\theta^e)}\right). \tag{C.16}$$

By (51), $z_{\min}(\kappa, g_L, \theta^e)$ is the gap at $g_q = 0$, whose sign is opposite to that of $\widetilde{g_q}(\kappa, g_L, \theta)$ by (C.14). Hence, the region with positive vertical R&D corresponds to $z_{\min}(\kappa, g_L, \theta^e) < z \leq (1+\kappa)(g_L + \varepsilon)$.

As in the baseline model, the graph of $F^{2G}(z; \theta^e)$ has the qualitative shape illustrated in Figure A.2, with A and $\delta - \rho$ replaced by $(1+\kappa)(g_L+\varepsilon)$ and $\delta - \rho + g_L$, respectively. Unlike the baseline case, its constant term can be negative. Nevertheless, at the upper labor boundary $z = (1+\kappa)(g_L+\varepsilon)$, (C.15) gives $F^{2G}(z;\theta^e) = (\rho+\mu)(\delta+\beta)\big((1+\kappa)(g_L + \varepsilon) - \alpha\theta^e z_{\min}(\kappa, g_L, \theta^e)\big) > 0$. If $\delta - \rho + g_L$ exceeds this boundary, F^{2G} is also positive there. Hence, as in the baseline model, exactly one root z_E exceeds both $(1+\kappa)(g_L+\varepsilon)$ and $\delta - \rho + g_L$. The corresponding intersection E is excluded by the labor resource constraint.

The other two roots, when they exist, represent steady states with vertical R&D only if they exceed $z_{\min}(\kappa, g_L, \theta^e)$. As in the baseline model, we denote the higher-growth steady state by S and the other by U when both are feasible. If the smaller root does not exceed $z_{\min}(\kappa, g_L, \theta^e)$, it is excluded from the region with vertical R&D. Steady states without vertical R&D are examined in Appendix C.4.

C.4 Steady States without Vertical R&D

The dynamics (C.9) and (C.10) do not depend on the subsidy. The policy changes only the boundary (C.8). A steady state in this region has either $D = 0$ or $g = r$.

At $D = 0$, the firm-value ratio is the positive solution of $(\rho + \mu)(\delta + \beta)\nu^2 - (\delta - \rho - \varepsilon)\nu - (1 - \alpha) = 0$. When $g_L > \widetilde{g_L}$, this point is $S = (0, \nu^*)$, where

$$\nu^* = \frac{\delta - \rho - \varepsilon + \sqrt{(\delta - \rho - \varepsilon)^2 + 4(1 - \alpha)(\rho + \mu)(\delta + \beta)}}{2(\rho + \mu)(\delta + \beta)}. \tag{C.17}$$

The gap there is $g_L + \varepsilon - (1 - \alpha)/\nu^*$. It is positive if and only if $g_L > \widetilde{g_L}$.[91] The point lies in the region without vertical R&D if and only if $\kappa/(1 - \theta^e) < \underline{\kappa}(g_L)$, by (C.8). Thus, the same threshold that locates P1 relative to $D = 0$ also determines whether this steady state conducts vertical research.

At $g = r$, the other steady state is

$$\nu_U = \frac{1 - \alpha}{g_L + \varepsilon}, \qquad D_U = \frac{\delta - \rho + g_L}{(\rho + \mu)(\delta + \beta)} - \frac{1 - \alpha}{g_L + \varepsilon}. \tag{C.18}$$

Its debt ratio is positive if and only if $g_L > \widetilde{g_L}$. It lies in the region without vertical R&D if $\kappa/(1 - \theta^e) < \alpha/(1 - \alpha)$. When $g_L > \widetilde{g_L}$, the steady state U remains in this region after S begins vertical research, because firm value is lower at U than at S, as in Figure C.2(b).

If $g_L < \widetilde{g_L}$, the roles are reversed. The steady state at $g = r$ has negative debt and is S, whereas the steady state on $D = 0$ is U. We then denote the positive value in (C.17) by ν_U. Neither point sustains a positive gap in this case.

The two regions meet at $g_q = 0$ and the upper boundary of ν. If a steady state is on this boundary, the stationary debt condition requires $D = 0$ or $g = r$. The former occurs at $\kappa/(1 - \theta^e) = \underline{\kappa}(g_L)$, and the latter at $\kappa/(1 - \theta^e) = \alpha/(1 - \alpha)$. At these values, the two descriptions give the same steady state, so that a steady state moves continuously from one region to the other as θ rises. The first value determines $\underline{\theta}(\kappa, g_L)$ in Appendix C.6, and

[91] Since ν^* is the positive solution of $(\rho + \mu)(\delta + \beta)\nu^2 - (\delta - \rho - \varepsilon)\nu - (1 - \alpha) = 0$, the gap is positive if and only if the left-hand side is negative at $\nu = (1 - \alpha)/(g_L + \varepsilon)$. This condition is equivalent to $(g_L + \varepsilon)(\delta - \rho + g_L) > (1 - \alpha)(\rho + \mu)(\delta + \mu + g_L)$, which holds if and only if $g_L > \widetilde{g_L}$ by (56).

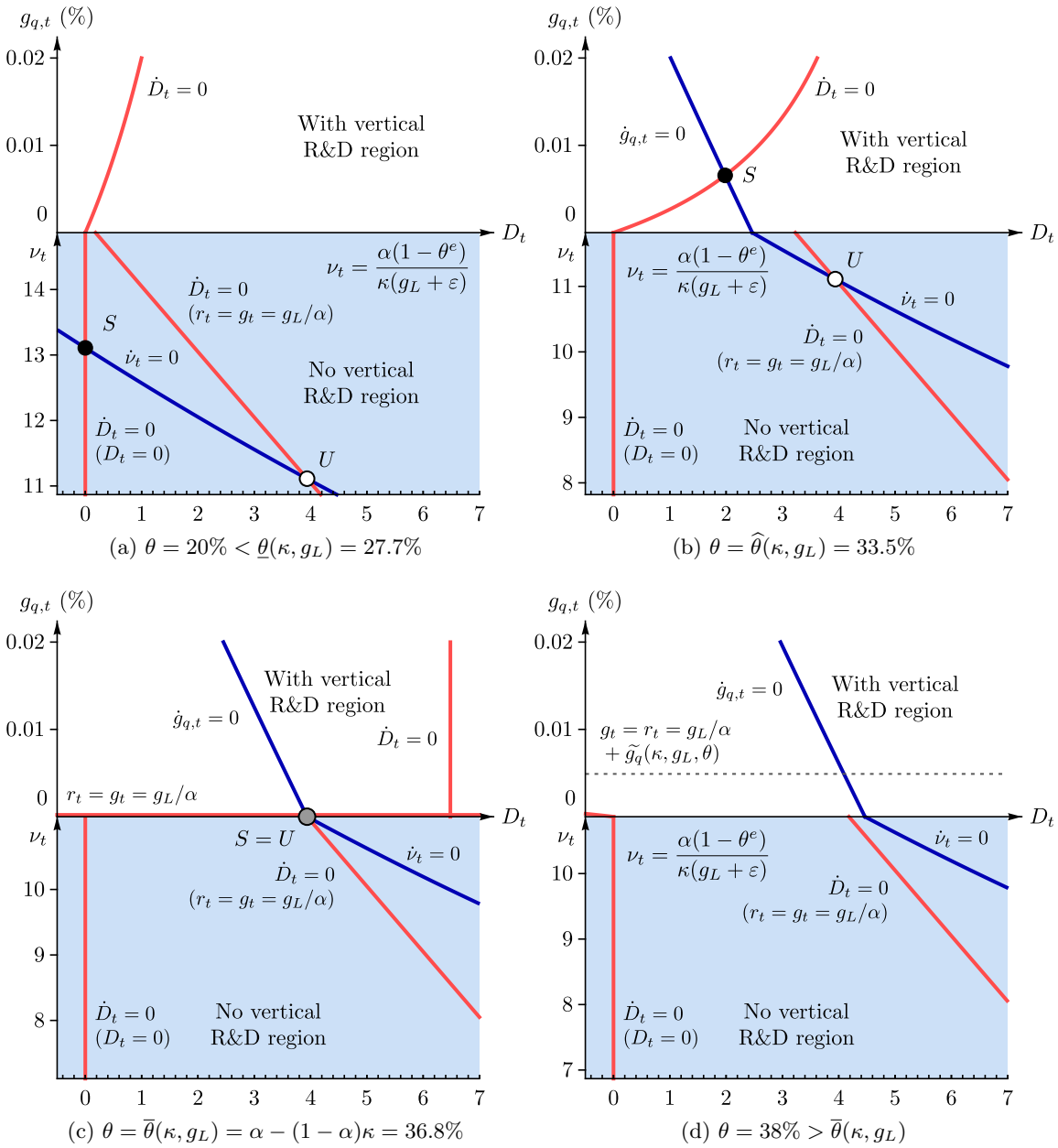


Figure C.2: Phase diagrams when small subsidies have no effect. The parameters are those in Table 1 with $\varepsilon = 0.01$ and $\lambda = (1-\alpha)/\alpha$. The upper and lower halves of each panel show the regions with and without vertical R&D, respectively. Black circles indicate the long-term BGP S, white circles the steady state U at the unsustainable threshold, and gray circles coincident steady states.

the second determines the upper bound $\alpha - (1-\alpha)\kappa$ in Proposition 7.

C.5 Uniqueness and Nonexistence of Equilibrium

We show that Proposition 1 holds for $D_0 \geq 0$, with g_t replaced by $g_{q,t}$ in the R&D region. The lower-debt steady state is S, and the debt ratio at U is the unsustainable threshold, whichever region contains these points.

Except for knife-edge cases, the count across both regions is two or zero, as in Proposi-

tion 1(a). At P1, $\dot{D} = (r - g)D$, where $r - g$ has the sign of $\widetilde{g_q}$ by (51). Together with the sign of the debt ratio at P1 and the excluded upper solution of (C.15), this gives the number of positive-g_q intersections. If $\widetilde{g_q} > 0$, the count is the same as in the baseline model (Appendix A.7). If $\widetilde{g_q} < 0$ and both steady states without vertical R&D lie inside their region, there is no positive-g_q intersection: the growth locus has negative debt throughout the feasible range, whereas the stationary budget with $g > r$ requires nonnegative debt. Once the no-debt steady state crosses the boundary, there is one intersection with vertical R&D and the other steady state remains without it.

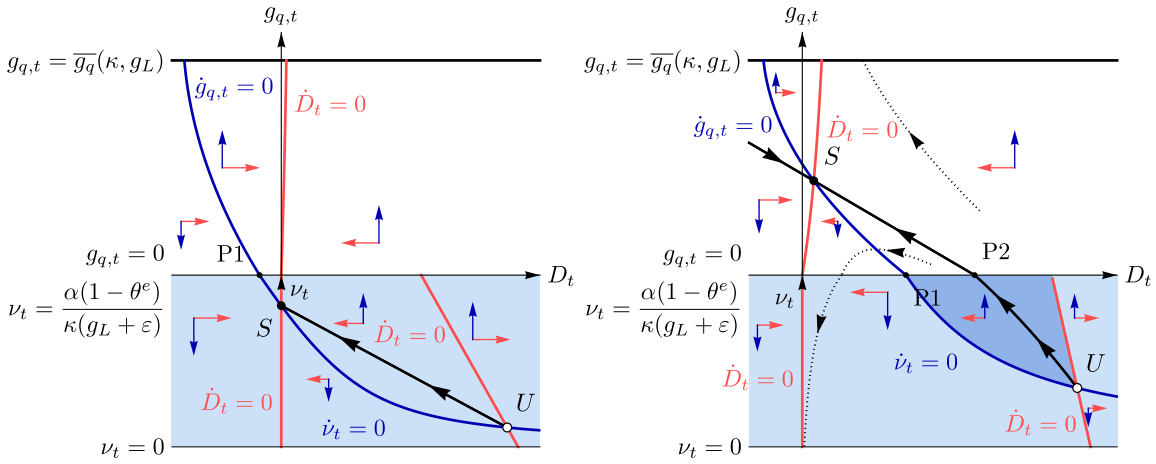
First suppose $\kappa/(1-\theta^e) > \alpha/(1-\alpha)$, so that $\widetilde{g_q} > 0$. The configurations in Figure C.1(a) and (d) are the same as those in Figure 5. The common quadratic terms of (53) and (54) give the linear representation of Lemma A.1. The excluded intersection E has the largest eigenvalue, as in Appendix A.6, because its gap exceeds both $(1+\kappa)(g_L+\varepsilon)$ and $\delta-\rho+g_L$ (Appendix C.3). The same eigenvalue argument therefore excludes paths off the equilibrium line.

The proof for the region without vertical R&D requires three further checks. By (C.8), $(g_L + \varepsilon)\nu_t$ is smaller than $1 - \alpha$. By (C.10), firm value therefore falls on the line $g_t = r_t$, on which debt is constant, as on the line $r_t = 0$ in Figure A.7. If both steady states have vertical R&D, P1 has positive debt and the region without vertical R&D has no steady state. Finally, at the boundary, $\dot{\nu}_t$ has the sign of $\dot{g}_{q,t}$ at $g_{q,t} = 0$. These properties give the same directions in which paths can cross the loci and the boundaries as in Figure A.7, and hence the same exclusion of paths. If U instead lies in the region without vertical R&D, D_U is zero, and there is no equilibrium path with positive initial debt.

Next suppose $\kappa/(1 - \theta^e) \leq \alpha/(1 - \alpha)$ and $g_L < \widetilde{g_L}$. In this case, S has negative debt and D_U is zero by Appendix C.4. P1 is to the left of the vertical axis, and the growth locus has negative debt throughout the R&D region. A path with nonnegative debt and vertical R&D therefore crosses the labor boundary. Without vertical R&D, a path above U reaches the upper boundary, whereas along a path with positive debt below U, firm value becomes negative or debt becomes infinite in finite time, because $(g_L + \varepsilon)\nu_U$ is smaller than $1 - \alpha$. Hence, an economy with positive initial debt has no equilibrium path.

Finally, suppose $\kappa/(1-\theta^e) < \alpha/(1-\alpha)$ and $g_L > \widetilde{g_L}$. The two configurations are shown in Figure C.3. The steady state S is without vertical R&D when $\kappa/(1 - \theta^e) \leq \underline{\kappa}(g_L)$, and otherwise it is in the R&D region. In both cases, U remains without vertical R&D and its debt ratio is given by (C.18).

In the region without vertical R&D, both quadratic terms are $(\rho + \mu)(\delta + \beta)(D_t + \nu_t)$ times the corresponding variable. Multiplying $(D_t, \nu_t, 1)$ by the integrating factor



(a) $\kappa/(1-\theta^e) < \underline{\kappa}(g_L)$ (b) $\underline{\kappa}(g_L) < \kappa/(1-\theta^e) < \alpha/(1-\alpha)$

Figure C.3: Equilibrium paths when U is in the region without vertical R&D. In panel (a), S is also in that region. In panel (b), the equilibrium path enters the R&D region at P2, and D_t falls and ν_t rises in the darker set.

$\exp(-(\rho+\mu)(\delta+\beta)\int_0^t (D_u+\nu_u)du)$ gives a linear system. Its third stationary solution has negative firm value and is excluded. Among the two feasible points, the eigenvalue of the no-debt point is higher precisely when its gap is positive. Thus, the argument of Lemma A.2 applies in this region as well.

In Figure C.3(a), the equilibrium line joins U and $S = (0, \nu^*)$. Every path on the line with $0 \leq D_0 < D_U$ remains within the region and converges to S. A path with vertical R&D is excluded because P1 is to the left of the axis. Along every other path without vertical R&D, firm value becomes negative, debt becomes infinite in finite time, or the path reaches that R&D region. This proves existence and uniqueness below D_U, and nonexistence above it.

In panel (b), the equilibrium line reaches $g_q = 0$ at P2, to the right of P1. It meets the boundary before the line on which $g = r$.[92] The unique path through P2 in the region without vertical R&D converges to U backward in time, as in Figure A.7(d). Along a path below both loci, firm value becomes negative. A path above enters the R&D region off the equilibrium line, or crosses downward and then makes firm value negative. Thus, the diagram selects exactly one path to P2 for every debt ratio between its value there and

[92] On the boundary, the point where $g = r$ lies to the right of P1 by the gap at $g_q = 0$ divided by $(\rho+\mu)(\delta+\beta)$. This follows from (C.7) at $g_q = 0$ and the gap in the region without vertical R&D (Appendix C.2). Write g_E for the full-output growth rate at the excluded intersection E, so that its vertical growth rate is $g_E - g_L/\alpha$. Substituting the equilibrium line into (53) at $g_q = 0$ shows that the debt ratio at P2 is $1 - \widetilde{g}_q/(g_E - g_L/\alpha)$ times the debt ratio at P1. This factor exceeds one because $\widetilde{g}_q$ is negative. Hence, the distance from P1 to P2 is the debt ratio at P1 times $-\widetilde{g}_q/(g_E - g_L/\alpha)$. This factor equals the gap at $g_q = 0$ divided by the gap at E minus the gap at $g_q = 0$. By (C.7), $(\rho+\mu)(\delta+\beta)$ times the debt ratio at P1 is less than $\delta - \rho + g_L$ minus the gap at $g_q = 0$. This is less than the gap at E minus the gap at $g_q = 0$, because the gap at E exceeds $\delta - \rho + g_L$. Therefore, the distance from P1 to P2 is smaller than the distance from P1 to the point where $g = r$. This gives the ordering drawn in panel (b).

D_U.

As in Appendix A.7, this path, followed by the straight line to S, is the unique equilibrium path for $0 \leq D_0 < D_U$, and there is none for $D_0 > D_U$. The transversality condition and non-Ponzi game condition of households hold as in Appendix A.7, since each candidate path continues to the feasible steady state S.

C.6 When Small Subsidies Have No Effect

The preceding two subsections showed that horizontal innovation alone can sustain a long-term BGP with $g > r$, even without vertical R&D. At such a steady state, the private return to vertical R&D is insufficient to cover its cost, so the free-entry condition holds with strict inequality. A small subsidy therefore need not induce vertical R&D, leaving long-run growth unchanged. This subsection examines how a sufficiently large subsidy can overcome this barrier and induce vertical R&D.

We focus on the case where $\kappa < \alpha/(1-\alpha)$ and $\widetilde{g_L} < g_L < \underline{g_L}(\kappa)$. The threshold $\underline{g_L}(\kappa)$ is defined by $\kappa = \underline{\kappa}(g_L)$, where $\underline{\kappa}(g_L)$ is given in (C.13). Figure C.2 illustrates how the steady states change as the subsidy rate increases, using $\kappa = 2.79$ and $g_L = 0.5\%$.

In Panel (a), the long-term BGP S lies in the region without vertical R&D, with $D^* = 0$ and $g > r$. As shown in Appendix C.4, the dynamics in this region are independent of θ because no subsidies are paid. Thus, increasing θ leaves S unchanged, but shifts the boundary of the region without vertical R&D. The steady state reaches this boundary when $\kappa/(1-\theta^e) = \underline{\kappa}(g_L)$, by (C.8) and (C.13). Using (52), the minimum subsidy rate required to induce vertical R&D is therefore

$$\underline{\theta}(\kappa, g_L) = \frac{\underline{\kappa}(g_L) - \kappa}{1 + \underline{\kappa}(g_L)}. \tag{C.19}$$

Below this threshold, S remains unchanged, so the subsidy has no effect on long-run growth. Once θ exceeds this threshold, S enters the region with vertical R&D, and g_q^* becomes positive.

As θ increases further, g_q^* rises until θ reaches $\widehat{\theta}(\kappa, g_L)$, as shown in Figure C.2(b), and then declines. In this numerical example, S eventually reaches $g_q^* = 0$ and $g^* = r^*$ at $\theta = \alpha - (1-\alpha)\kappa$, as shown in Panel (c). This is the upper bound $\overline{\theta}(\kappa, g_L)$ established in Proposition 7. Beyond this rate, no steady state with $g > r$ exists, as illustrated in Panel (d).

When $\overline{g_L}(\kappa) < g_L < \underline{g_L}(\kappa)$, increasing θ initially induces vertical R&D at S, as in Figure C.2(b). Unlike the numerical example, however, S remains in the region with

positive vertical R&D as the phase diagram moves from Panel (b) through (c) to (a) of Figure C.1, where U also enters this region. With a further increase in θ, the two steady states merge and disappear while $g_q^* > 0$ and $g^* > r^*$, leading to the configuration in Figure C.1(d).

C.7 Proof of Proposition 7

Proof of part (i). Suppose first that $\kappa > \alpha/(1-\alpha)$. Then $z_{\min}(\kappa, g_L, \theta^e) < 0$ for every subsidy rate, so a steady state with $g > r$ must conduct vertical R&D by Appendix C.4. By (C.15), a feasible positive root requires $(1+\kappa)(g_L+\varepsilon)(\delta-\rho+g_L) > (\rho+\mu)(\delta+\beta)$. Conversely, if this inequality holds, $F^{2G}(z;0) = 0$ has a feasible positive root, which persists under sufficiently small positive subsidies. By (55), the condition is equivalent to $g_L > \overline{g_L}(\kappa)$.

Next suppose $\kappa \leq \alpha/(1-\alpha)$. If $g_L \leq \widetilde{g_L}$, neither steady state without vertical R&D has a positive gap by Appendix C.4. A steady state with vertical R&D cannot have one either. If $z_{\min}(\kappa, g_L, \theta^e) \geq 0$, then $\kappa/(1-\theta^e) \leq \alpha/(1-\alpha) \leq \underline{\kappa}(g_L)$, so the necessary condition (C.12) fails. If $z_{\min}(\kappa, g_L, \theta^e) < 0$, the constant term of $F^{2G}(z;\theta^e)$ is positive, while $(1+\kappa)(g_L+\varepsilon)(\delta-\rho+g_L)$ does not exceed $(\rho+\mu)(\delta+\beta)$ by $\kappa \leq \alpha/(1-\alpha)$ and (56). Hence, $F^{2G}(z;\theta^e) > 0$ for every feasible $z > 0$. Conversely, if $g_L > \widetilde{g_L}$, the no-subsidy economy has a long-term BGP with $g > r$, with or without vertical R&D as described in Appendices C.3 and C.4. This proves the two necessary and sufficient population-growth conditions.

Proof of part (ii). Suppose these conditions hold. If $\widetilde{g_L} < g_L \leq \overline{g_L}(\kappa)$, the long-term BGP reaches $g_q = 0$ and $g = r$ at $\theta = \alpha - (1-\alpha)\kappa$, as shown in Figure C.2(c). Beyond this rate, $z_{\min}(\kappa, g_L, \theta^e) < 0$ and the constant term of $F^{2G}(z;\theta^e)$ is positive. Since κ does not exceed $\overline{\kappa}(g_L)$, the equation $F^{2G}(z;\theta^e) = 0$ has no feasible positive root. Hence, this rate is the upper bound.

If $g_L > \overline{g_L}(\kappa)$, the higher-growth intersection remains inside the region with positive vertical R&D when $z_{\min}(\kappa, g_L, \theta^e) = 0$. As the subsidy increases further, the two intersections eventually merge at the local minimum of $F^{2G}(z;\theta^e)$. Since the subsidy affects only the constant term, the location of this local minimum is independent of θ^e. Differentiating (C.15) with respect to z gives the contact condition

$$\begin{aligned} 3z^2 - 2\big((1+\kappa)(g_L+\varepsilon) + \delta - \rho + g_L\big)z \\ + (1+\kappa)(g_L+\varepsilon)(\delta-\rho+g_L) - (\rho+\mu)(\delta+\beta) = 0. \end{aligned} \tag{C.20}$$

Let $\underline{z}(\kappa, g_L)$ denote the smaller solution of (C.20). It lies between 0 and $(\delta-\rho+g_L)/2$,

because the left-hand side of (C.20) is positive at $z = 0$ by $g_L > \overline{g_L}(\kappa)$ and negative at $z = (\delta - \rho + g_L)/2$. The upper subsidy rate $\overline{\theta}(\kappa, g_L)$ is determined by $F^{2G}(\underline{z}(\kappa, g_L); \theta^e) = 0$, together with (52). At this contact, both steady states have positive vertical R&D and $g > r$.

To establish uniqueness and the comparative static, divide (C.15) by $(\rho+\mu)(\delta+\beta)(g_L+\varepsilon)$ to obtain

$$\theta^e \left(\frac{(1-\alpha)\kappa}{1-\theta^e} - \alpha \right) = z \left(\left(1 + \kappa - \frac{z}{g_L + \varepsilon} \right) \frac{\delta - \rho + g_L - z}{(\rho+\mu)(\delta+\beta)} - \frac{1}{g_L + \varepsilon} \right). \tag{C.21}$$

Over the feasible interval $0 < z \leq (1+\kappa)(g_L + \varepsilon)$, the right-hand side of (C.21) attains its positive maximum at $z = \underline{z}(\kappa, g_L)$. The left-hand side is strictly increasing in θ whenever it is positive, because θ^e increases with θ by (52), and it tends to infinity as θ approaches one. Hence, the upper subsidy rate is unique.

For any fixed feasible positive gap below both $(1+\kappa)(g_L+\varepsilon)$ and $\delta - \rho + g_L$, the right-hand side of (C.21) is strictly increasing in g_L. In particular, $(\delta - \rho + g_L - z)/(\delta + \beta)$ increases with g_L, while increasing g_L also raises $(1+\kappa)(g_L+\varepsilon)$. Therefore, its positive maximum increases with g_L. Since the left-hand side does not depend on g_L, $\overline{\theta}(\kappa, g_L)$ is strictly increasing in this case.

C.8 Proof of Proposition 8

For the calculations in this subsection, write $z \equiv g^* - r^*$. Unlike the baseline case, the stationary growth rate depends on the subsidy both through this gap and through the value of the industries created without vertical R&D. Combining (C.6), (C.7), and (51) gives

$$g_q^* = \overline{g_q}(\kappa, g_L) \left(1 - \frac{\alpha(\rho+\mu)(\delta+\beta)(1 + (1-\theta^e)/\kappa)}{z(\delta - \rho + g_L - z) + (\rho+\mu)(\delta+\beta)} \right). \tag{C.22}$$

The term involving $(1-\theta^e)/\kappa$ is the effect of horizontal entry. The gap in this expression is determined by (C.15), with $g - r = z$. These two equalities are sufficient for the comparative statics.

Proof of part (i). We first consider a long-term BGP with positive vertical R&D without subsidies. Since g_L is fixed and θ^e increases with θ, the sign of the growth effect is the sign of $dg_q^*/d\theta^e$. Differentiating (C.22) gives

$$\frac{dg_q^*}{d\theta^e} = \frac{\alpha\lambda(g_L+\varepsilon)(\rho+\mu)(\delta+\beta)}{z(\delta - \rho + g_L - z) + (\rho+\mu)(\delta+\beta)} \left(1 + \frac{(1+\kappa-\theta^e)(\delta - \rho + g_L - 2z)}{z(\delta - \rho + g_L - z) + (\rho+\mu)(\delta+\beta)} \frac{dz}{d\theta^e} \right). \tag{C.23}$$

The first term in parentheses is the positive effect at a given gap. The second is the effect

through the change in the gap. Differentiating the stationary condition (C.15) at $\theta^e = 0$ gives

$$\left.\frac{dz}{d\theta^e}\right|_{\theta^e=0} = -\frac{(\rho+\mu)(\delta+\beta)(g_L+\varepsilon)((1-\alpha)\kappa-\alpha)}{z((1+\kappa)(g_L+\varepsilon)+\delta-\rho+g_L-2z)}.$$

At this rate, the stationary condition also gives $((1+\kappa)(g_L+\varepsilon)-z)(\delta-\rho+g_L-z) = (\rho+\mu)(\delta+\beta)$. Substituting these expressions into (C.23) shows that the initial growth effect is positive if and only if

$$(g_L+\varepsilon)(\delta-\rho+g_L-z)^2 > (1-\alpha)(\rho+\mu)(\delta+\beta)(\delta-\rho+g_L-2z). \tag{C.24}$$

The factor removed from the derivative is positive, by $0 < z < \min\{(1+\kappa)(g_L+\varepsilon), \delta-\rho+g_L\}$. Completing the square in $\delta-\rho+g_L-z$ shows that the inequality (C.24) holds for every feasible positive z when $g_L > \widetilde{g_L}$.

If $g_L < \widetilde{g_L}$, let $\Xi = \sqrt{1-(g_L+\varepsilon)(\delta-\rho+g_L)/((1-\alpha)(\rho+\mu)(\delta+\beta))}$, as in the threshold calculation. Solving the quadratic inequality gives $z > \Xi(\delta-\rho+g_L)/(1+\Xi)$. Along the no-subsidy stationary condition, the gap increases with κ. Substituting the threshold gap gives

$$\widehat{\kappa}(g_L) = (1+\Xi)\overline{\kappa}(g_L) + \Xi\left(1+\frac{\delta-\rho+g_L}{(g_L+\varepsilon)(1+\Xi)}\right). \tag{C.25}$$

Thus, the initial effect is positive if and only if $\kappa > \widehat{\kappa}(g_L)$. This threshold exceeds $\overline{\kappa}(g_L)$ and is strictly decreasing in g_L.[93] Its inverse defines $\widehat{g_L}(\kappa)$ for $\kappa > \alpha/(1-\alpha)$, with $\overline{g_L}(\kappa) < \widehat{g_L}(\kappa) < \widetilde{g_L}$.

It remains to show that a negative initial growth effect cannot become positive at a higher subsidy. Consider any interior rate at which $dg_q^*/d\theta^e = 0$. The expression for g_q^* from (51) increases with both the gap and θ^e, so $dz/d\theta^e < 0$ there. By (C.23), $z < (\delta-\rho+g_L)/2$ must also hold.

At the higher-growth intersection, $F^{2G}(z;\theta^e)$ is increasing in the gap. Its derivative with respect to θ^e is $(\rho+\mu)(\delta+\beta)(g_L+\varepsilon)((1-\alpha)\kappa/(1-\theta^e)^2-\alpha)$, which strictly increases with θ^e. Once the gap starts falling, it therefore continues to fall. Substituting its derivative into (C.23) shows that the marginal growth effect is positive precisely when

$$\left(\frac{1-\theta^e}{1+\kappa-\theta^e}\right)^2\big(z(\delta-\rho+g_L-z)+(\rho+\mu)(\delta+\beta)\big)^2 > (1-\alpha)(g_L+\varepsilon)(\rho+\mu)(\delta+\beta)(\delta-\rho+g_L-2z). \tag{C.26}$$

[93] Both Ξ and $\overline{\kappa}$ decrease with g_L. More precisely, $d\overline{\kappa}/dg_L = -(1+\overline{\kappa})/(g_L+\varepsilon) - (\rho+\mu)^2/((g_L+\varepsilon)(\delta-\rho+g_L)^2)$, while the derivative of $(\delta-\rho+g_L)/(g_L+\varepsilon)$ is smaller than $1/(g_L+\varepsilon)$. In differentiating (C.25), all terms involving $d\Xi/dg_L$ are negative. The remainder is smaller than $(-(1+\Xi)(1+\overline{\kappa})+\Xi/(1+\Xi))/(g_L+\varepsilon) < 0$. As g_L approaches $\widetilde{g_L}$ from below, Ξ tends to zero and $\widehat{\kappa}$ tends to $\alpha/(1-\alpha)$. The defining relation has a unique inverse above this value.

At a stationary growth rate, the two sides are equal. Beyond it, the left-hand side decreases because θ^e rises and z falls below $(\delta - \rho + g_L)/2$, while the right-hand side increases. Hence, every interior stationary point of long-run growth is a maximum, and the marginal effect cannot turn from negative to positive.

When $\kappa > \alpha/(1-\alpha)$, any long-term BGP with $g > r$ already has vertical research without subsidies. The initial-effect condition therefore gives $g_L > \widehat{g_L}(\kappa)$. When $\kappa \leq \alpha/(1-\alpha)$, such a BGP exists if and only if $g_L > \widetilde{g_L}$. If it already has vertical research, the initial effect is positive. Otherwise, a sufficiently large subsidy induces research at the threshold in Appendix C.6 and raises growth. This proves part (i).

Proof of part (ii). Under the condition of part (i), some subsidy raises growth and there is at most one interior maximum. Growth is not maximized at the upper bound. If $g_L > \overline{g_L}(\kappa)$, the intersections meet at a gap below $(\delta - \rho + g_L)/2$. The derivative of $F^{2G}(z;\theta^e)$ with respect to the gap tends to zero from above, while its derivative with respect to θ^e is positive. Hence, $dz/d\theta^e$ tends to minus infinity, and the negative indirect effect in (C.23) dominates. If $\widetilde{g_L} < g_L \leq \overline{g_L}(\kappa)$, vertical research vanishes at the upper bound, so its growth rate is lower there than at rates immediately below. Thus, the growth-maximizing subsidy lies strictly between zero and the upper bound.

Proof of part (iii). In the stated range, S remains in the region without vertical R&D until θ reaches $\underline{\theta}(\kappa, g_L)$, as shown in Figure C.2(a). Since S has no vertical R&D, the output growth rate remains g_L/α throughout this interval.

C.9 Calibration Details

This subsection describes the data and the three steps of the estimation of κ in Section 6.4. For each of the 24 countries, the growth rates of real GDP and population and the investment-GDP ratio are the averages over 2000–2023 in the Penn World Table version 11.0 (Feenstra et al., 2015). For the history of population, we use the annual population series in the Maddison Project Database (Bolt and van Zanden, 2025) from 1820 through 2022 and extend it to 2023 using the population growth in the Penn World Table.

In the first step, we consider $\xi_t \equiv n_t/(fL_t)$, the number of industries relative to the rate at which new industries are created. By (39), it evolves according to $\dot{\xi}_t = 1 - (g_{L,t} + \varepsilon)\xi_t$, which does not depend on f, where $g_{L,t}$ is the growth rate of the population, and hence of L_t, at time t. If $g_{L,t}$ stays at g_L, ξ_t converges to $1/(g_L + \varepsilon)$, which corresponds to (40). For each country, we start from the first year of the longest uninterrupted population series that ends in 2023, which is 1820 except for Canada and New Zealand (1870), Turkey (1923), and Iceland and Luxembourg (1950). The initial value of ξ_t is the value to which

it would converge if the average population growth rate over the first ten years continued. Using the population growth rate in each year as $g_{L,t}$, we update ξ_t by

$$\xi_{t+1} = e^{-(g_{L,t}+\varepsilon)}\xi_t + \frac{1 - e^{-(g_{L,t}+\varepsilon)}}{g_{L,t} + \varepsilon}. \tag{C.27}$$

We then compute the average level of ξ_t and its average log growth rate over 2000–2023.

In the second step, we compute the growth rate implied by the model for a given value of κ, using the data other than the growth rate of real GDP. By (47), the growth rate of full-production output is the sum of the population growth rate, $(1-\alpha)/\alpha$ times the rate of horizontal innovation, $\dot{n}_t/n_t$, and the rate of vertical innovation. The rate of horizontal innovation is $\dot{n}_t/n_t = g_{L,t} + \dot{\xi}_t/\xi_t$, which is obtained in the first step. By Lemma 1, the rate of vertical innovation is $\lambda a L_t^R/n_t = \lambda\kappa(L_t^R/L_t)/\xi_t$. Hence,

$$g_t = g_{L,t} + \frac{1-\alpha}{\alpha}\Big(g_{L,t} + \frac{\dot{\xi}_t}{\xi_t}\Big) + \frac{\lambda\kappa}{\xi_t}\frac{L_t^R}{L_t}. \tag{C.28}$$

For the last term, we infer the R&D labor share from the investment-GDP ratio, as in Section 4.4. In the current model, GDP is the sum of the wages and the profits, $\mathrm{GDP}_t = w_t L_t + n_t \pi_t$. Since the wage includes the value of new businesses, $f v_t$, per unit of labor, GDP can also be written as $\mathrm{GDP}_t = C_t + w_t L_t^R + f v_t L_t^P$, where the last term is the value of the ideas of production workers. In the model, firms pay wages for two kinds of labor, production labor and research labor. If these expenditures were classified according to the current national accounting standards, which treat R&D expenditure as capital formation, only the cost of research labor, $w_t L_t^R$, would be classified as investment. We therefore regard only $w_t L_t^R$ as investment. Without subsidies, $f v_t/(w_t - f v_t) = 1/\kappa$ holds by the free-entry condition (44). Since labor income net of the value of ideas, $(w_t - f v_t)L_t$, equals αY_t by (46) and (47), we have $w_t L_t = \alpha(1+1/\kappa)Y_t$. Moreover, $n_t\pi_t = (1-\alpha)X_t = (1-\alpha)Y_t(1-L_t^R/L_t)$ by (45) and (47). Hence, the R&D labor share is

$$\frac{L_t^R}{L_t} = \frac{(1+\alpha/\kappa)\, w_t L_t^R/\mathrm{GDP}_t}{\alpha(1+1/\kappa) + (1-\alpha)\, w_t L_t^R/\mathrm{GDP}_t}, \tag{C.29}$$

where $w_t L_t^R/\mathrm{GDP}_t$ is measured by the investment-GDP ratio. Substituting this share into (C.28), we obtain the growth rate implied by the model for a given κ.

In the third step, we choose κ so that the growth rate implied by the model equals the growth rate of real GDP. In this step, we evaluate each term of (C.28) by its average over 2000–2023. For g_t and $g_{L,t}$, we use the average growth rates of real GDP and population.[94]

[94]We measure growth rates by averages of annual log differences, because in the continuous-time model

For $\dot{\xi}_t/\xi_t$ and ξ_t, we use the averages over 2000–2023 of the log growth rate and the level of ξ_t obtained in the first step. For L_t^R/L_t, we use (C.29) with the average investment-GDP ratio. This condition is a quadratic equation in κ, and it has a unique positive solution, because the growth rate of real GDP exceeds the sum of the first two terms on the right-hand side of (C.28) in all 24 countries.

If we also regard the value of the ideas of production workers, $fv_tL_t^P$, as investment, the investment-GDP ratio measures $(w_tL_t^R + fv_tL_t^P)/\text{GDP}_t$. In this alternative specification, the estimates of κ are higher in all 24 countries. The classification of the countries is then the same as in the baseline model (Section 4.4), except that Italy lies in the region where the government cannot sustain the subsidies by rolling over its debt.

C.10 Nesting of the Baseline Model

As in Section 6.5, set $g_L = 0$ and $\lambda = (1-\alpha)/\alpha$, and let f and ε tend to zero while $\varepsilon/f = L$ stays fixed. Then $\kappa = a/f$ tends to infinity, whereas $\kappa\varepsilon = aL$ stays fixed.

First consider the production side. No industries are created or become obsolete in the limit, and $n_t = 1$ by (40). The term fv_t disappears from wages and the private cost of research. Hence, the free-entry condition becomes $av_t \leq (1-\theta)w_t$, and the effective subsidy rate equals θ. The no-arbitrage condition becomes $r_t = (\pi_t + \dot{v}_t)/v_t - aL_t^R$, with $\pi_t = (1-\alpha)X_t$. These are the conditions of the quality-ladder model of Grossman and Helpman (1991b), where aL_t^R is the replacement rate of leaders. Each replacement raises log quality by λ, so $g_t = \lambda aL_t^R$.

The pricing difference comes from the final-goods technology. Our CES aggregator gives the monopoly price w_t/α when fv_t vanishes, whereas their Cobb–Douglas aggregator gives a limit markup equal to the quality step e^λ. The markups coincide approximately when $\lambda = (1-\alpha)/\alpha$ and exactly if $\lambda = \ln(1/\alpha)$. We use the former value to obtain the exact correspondence with the baseline variety-expansion model.

In this limit, $A^* = aL$ by (58). Both models then have $g_t = \overline{g}(A^*)L_t^R/L$. They also have the same aggregate profit and the same value of firms from free entry. Net of the loss from replacement, the change in leader value in the quality-ladder model is the same as the change in firm value in the variety-expansion model. Thus, the interest rate is the same function of research labor, given by (21) with $A = A^*$.

Since households and the government are unchanged, (53) and (54) converge to (28)

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the average growth rate of a variable over a year equals the difference in its logarithm. Section 4.4 uses simple growth rates instead. If we use the simple growth rate of real GDP, keeping the other inputs unchanged, the classification changes only for Greece, which moves to the region where the subsidies are sustainable but do not raise growth.

and (29). The dynamics without vertical research converge to (A.15) and (A.16). The stationary loci, their contacts, and the equilibrium paths therefore coincide in the limit. This is a correspondence between the reduced-form equilibrium conditions, not an identification of horizontal expansion with quality improvement.

Finally, setting $g_L = 0$ in the threshold formulas gives $\varepsilon(1 + \overline{\kappa}(0)) = \overline{A}$ and $\varepsilon\underline{\kappa}(0) \to A_{\min}$. In (C.25), $\Xi \to 1$, so $\varepsilon(1 + \widehat{\kappa}(0)) \to \widehat{A}$. Thus, the conditions for sustainable rollover, the onset of research, and growth-promoting subsidies converge to those of the baseline model. The contact condition (C.20) and the stationary condition (C.15) also converge to their baseline counterparts, giving the subsidy upper bound $\overline{\theta}(A^*)$ when $A^* > \overline{A}$.

In the stationary growth relation (C.22), the term $(1 - \theta^e)/\kappa$ vanishes because no new industry is created without research. The relation becomes (A.13). Hence, for $A^* > \widehat{A}$, the growth-maximizing subsidy and the maximized growth rate converge to $\widehat{\theta}(A^*)$ and $\widehat{g}(A^*)$.

D Analysis of the Basic Research Model

This appendix analyzes the model with basic research in Section 7.1. We derive the equilibrium dynamics, examine the steady states, and prove Proposition 9. Finally, we describe the calibration of L_0^B/L.

D.1 Derivation of Equilibrium Dynamics

Write $b = (L^B - L_0^B)/L_0^B \geq 0$ for the proportional increase in basic research employment, so $A = A_0(1 + b)^\phi$. The initial basic researchers are paid by the existing labor tax. This tax multiplies each individual's labor income by the same constant, and therefore leaves the aggregate consumption equation unchanged. Only the additional wages are financed by debt.

The labor available to private production and research is $L - bL_0^B$. Thus,

$$L_t^P + L_t^R = L - bL_0^B, \tag{D.1}$$

$$\overline{G}(b) = \left(1 - b\frac{L_0^B}{L}\right)\overline{g}\big(A_0(1 + b)^\phi\big). \tag{D.2}$$

The latter is the maximum growth rate when all remaining private labor is used for R&D. The private technologies and free-entry condition are unchanged. Substituting the new labor allocation into (21) gives

$$g_t - r_t = s(\theta)\big(g_t - \widetilde{g}(\theta, b)\big), \qquad \widetilde{g}(\theta, b) = \frac{1 - \alpha}{1 - \alpha\theta}\overline{G}(b). \tag{D.3}$$

The government pays for research subsidies and the additional basic researchers. Dividing its budget by Y_t gives

$$\dot{D}_t = (r_t - g_t)D_t + \frac{\theta\alpha}{\overline{g}(A)}g_t + \alpha b\frac{L_0^B}{L}. \tag{D.4}$$

The value of firms relative to output is still $\alpha(1-\theta)/A$, whereas consumption relative to output is now $(\overline{G}(b) - g_t)/\overline{g}(A)$. Repeating the derivation in Appendix A.2 gives

$$\dot{g}_t = (\overline{G}(b) - g_t)(g_t - r_t - \delta + \rho) + \overline{g}(A)(\rho+\mu)(\delta+\mu)\left(D_t + \frac{\alpha(1-\theta)}{A}\right). \tag{D.5}$$

Together with (D.3), these are two equations in D_t and g_t. Their stationary loci are

$$\dot{D}_t = 0: \quad D_t = \frac{\alpha}{g_t - r_t}\left(b\frac{L_0^B}{L} + \frac{\theta g_t}{\overline{g}(A)}\right), \tag{D.6}$$

$$\dot{g}_t = 0: \quad D_t = \frac{\overline{G}(b) - g_t}{\overline{g}(A)(\rho+\mu)(\delta+\mu)}\big(\delta - \rho - (g_t - r_t)\big) - \frac{\alpha(1-\theta)}{A}, \tag{D.7}$$

with the budget condition used in its undivided form if $g_t = r_t$. At $b = 0$, the model coincides with the baseline model with $A = A_0$.

D.2 Steady States

For a given policy pair (θ, b), steady states are determined by the intersections of (D.6) and (D.7). As in the baseline model, the $\dot{g}_t = 0$ locus opens to the right. However, expenditure on additional basic research shifts the $\dot{D}_t = 0$ locus, which no longer passes through the origin when $b > 0$, even if $\theta = 0$.

To characterize the steady states, write $z \equiv g - r$. Eliminating D from the stationary conditions using (D.3) gives

$$\begin{aligned} z\Big(\big(A(1 - bL_0^B/L) - z\big)(\delta - \rho - z) - (\rho+\mu)(\delta+\mu)\Big) \\ = (\rho+\mu)(\delta+\mu)A\left(\frac{(1-\alpha)\theta}{1-\theta} + b\frac{L_0^B}{L}\right). \end{aligned} \tag{D.8}$$

The labor constraint $g \leq \overline{G}(b)$ is equivalent to $z \leq A(1 - bL_0^B/L)$. As in Appendix A.4, exactly one root lies above this bound and is excluded by the resource constraint. The remaining roots describe steady states with private R&D only if $g > 0$. When two such steady states exist, the higher-growth, lower-debt one is denoted by S, and the other by U.

The quadratic terms of the dynamic equations are unchanged, so the linear representation and the equilibrium-line argument in Appendix A.6 continue to apply. Additional

basic research introduces a constant term in the debt equation, modifying the first-row, third-column entry of the linear system without changing the common quadratic term used in the transformation.

D.3 Uniqueness and Nonexistence of Equilibrium

For $b > 0$, Proposition 1 holds for $D_0 \geq 0$, with $\overline{G}(b)$ as the growth limit. In the region without private research, $C_t/Y_t = 1 - bL_0^B/L$, and the free-entry boundary remains $\nu_t = \alpha(1-\theta)/A$. The debt and firm-value equations are $\dot{D}_t = r_t D_t + \alpha b L_0^B/L$ and $\dot{\nu}_t = r_t \nu_t - (1-\alpha)(1 - bL_0^B/L)$. A steady state with positive firm value requires $r_t > 0$, whereas the stationary debt condition then requires negative debt. Hence, this region contains no steady state with nonnegative debt.

If $\alpha + bL_0^B/L < 1$, there is a steady state U with $D_U = -\alpha b L_0^B/(L r_U)$ and $\nu_U = (1-\alpha)(1 - bL_0^B/L)/r_U$. The interest rate is the positive solution of

$$r_U^2 + (\delta - \rho) r_U = (\rho + \mu)(\delta + \mu) \frac{1 - \alpha - bL_0^B/L}{1 - bL_0^B/L}.$$

This point lies in the region without private research if and only if $r_U \geq \overline{G}(b)/(1-\theta)$, the rate on its boundary at $g = 0$. If $\alpha + bL_0^B/L \geq 1$, there is no positive solution for r_U and hence no steady state in this region. At P1, $\dot{D}$ has the sign of $\overline{G}(b)/(1-\theta) - r_U$ when the positive root exists, and is positive otherwise. The sign test in the proof of Proposition 1(a) in Appendix A.7 therefore gives two intersections with positive growth or none when the steady state without private research lies outside, and one when it lies inside. Except for knife-edge cases, counting both regions gives two steady states or none, as in Proposition 1(a).

The proof of Proposition 1 now follows the geometry of Appendix A.7. The excluded upper intersection has the largest eigenvalue, and the only paths remaining in the R&D region lie on the equilibrium line and converge to S. At $D_t = 0$, debt rises in either region because basic research spending is positive. Without private research, the interest rate again depends only on $D_t + \nu_t$, and its time derivative depends only on the interest rate, vanishing at the solutions of the equation for r_U above. The proof of Lemma A.2(c) therefore applies, and firm value becomes negative or debt becomes infinite in finite time, unless the path enters the R&D region. The transversality condition and non-Ponzi game condition of households hold along the candidate path to S, as in that proof.

D.4 Proof of Proposition 9

We prove the two parts of the proposition separately. In part (a), Proposition 3 shows that R&D subsidies alone cannot increase long-run growth when $A_0 \leq \widehat{A}$. We therefore examine whether a marginal increase in basic research from the initial BGP raises long-run growth. In part (b), where $A_0 > \widehat{A}$, we characterize the growth-maximizing combination of basic research and R&D subsidies. In the following, write $z^* \equiv g^* - r^*$.

Proof of part (a). Suppose $\overline{A} < A_0 \leq \widehat{A}$. We examine whether a marginal increase in basic research from the initial BGP with $\theta = b = 0$ raises long-run growth. With $\theta = 0$, (D.3) implies $g^* = (1-\alpha)(\overline{G}(b) + z^*)$. Differentiating with respect to b at $b = 0$ gives

$$\left.\frac{dg^*}{db}\right|_{b=0} = (1-\alpha)\left(\frac{1-\alpha}{\alpha}A_0\left(\phi - \frac{L_0^B}{L}\right) + \left.\frac{dz^*}{db}\right|_{b=0}\right). \tag{D.9}$$

To determine dz^*/db, let $z_0^*(A_0)$ denote the steady-state gap when $\theta = b = 0$, determined by

$$\big(A_0 - z_0^*(A_0)\big)\big(\delta - \rho - z_0^*(A_0)\big) = (\rho+\mu)(\delta+\mu). \tag{D.10}$$

The relevant solution satisfies $0 < z_0^*(A_0) \leq (\delta-\rho)/2$ for $\overline{A} < A_0 \leq \widehat{A}$, as shown in Appendix A.9. Keeping $\theta = 0$, we differentiate (D.8) with respect to b at $b = 0$. Using $A'(0) = \phi A_0$ and (D.10), we obtain

$$\left.\frac{dz^*}{db}\right|_{b=0} = \frac{A_0\big(\delta - \rho - z_0^*(A_0)\big)\big(\phi z_0^*(A_0) - A_0 L_0^B/L\big)}{z_0^*(A_0)\big(A_0 + \delta - \rho - 2z_0^*(A_0)\big)}. \tag{D.11}$$

We now derive the condition under which a marginal increase in basic research raises long-run growth. Substituting (D.11) into (D.9) and simplifying using (D.10), we obtain $dg^*/db|_{b=0} > 0$ if and only if

$$\phi > \underline{\phi}(A_0) \equiv \frac{L_0^B}{L}\left(1 + \frac{\alpha(\rho+\mu)(\delta+\mu)\big(\delta-\rho-z_0^*(A_0)\big)}{z_0^*(A_0)\big((\delta-\rho-z_0^*(A_0))^2 + (1-\alpha)(\rho+\mu)(\delta+\mu)\big)}\right). \tag{D.12}$$

Since R&D subsidies alone cannot improve growth in this parameter range, this improvement requires additional basic research. Moreover, the threshold $\underline{\phi}(A_0)$ is strictly decreasing in A_0. To see this, the initial gap $z_0^*(A_0)$ is strictly increasing in A_0.[95] The fraction in (D.12) is strictly decreasing in $z_0^*(A_0)$, since its derivative with respect to z has the opposite sign of $(1-\alpha)(\rho+\mu)(\delta+\mu)(\delta-\rho) + (\delta-\rho-z)^2(\delta-\rho-2z)$, which is positive for

[95] Solving (D.10) for A_0 gives $A_0 = (\rho+\mu)(\delta+\mu)/(\delta-\rho-z)+z$. The right-hand side is strictly increasing in z for $0 < z \leq (\delta-\rho)/2$ and equals $\overline{A}$ at $z = 0$. Hence, its inverse $z_0^*(A_0)$ is strictly increasing in A_0 and approaches zero as $A_0 \downarrow \overline{A}$.

$0 < z \leq (\delta - \rho)/2$. Therefore, $\underline{\phi}(A_0)$ is strictly decreasing in A_0 and diverges as $A_0 \downarrow \overline{A}$.

Proof of part (b). Suppose $A_0 > \widehat{A}$. We first derive an upper bound on long-run growth for a given level of basic research and then examine whether this bound can be attained through R&D subsidies. Eliminating D^* and θ from the stationary conditions, as in Appendix A.4, gives

$$g^* = \overline{g}(A)\left(1 - b\frac{L_0^B}{L} - \frac{\alpha(\rho+\mu)(\delta+\mu)}{z^*(\delta-\rho-z^*)+(\rho+\mu)(\delta+\mu)}\right), \qquad A = A_0(1+b)^{\phi}. \quad \text{(D.13)}$$

Compared with (A.13), basic research increases R&D productivity through $A = A_0(1+b)^{\phi}$ but reduces the labor available for private production and R&D by bL_0^B. For a given b, the dependence of growth on z^* is therefore unchanged from the baseline model.

We first fix b and consider how long-run growth depends on the steady-state gap z^*. Equation (D.13) is maximized with respect to z^* at $z^* = (\delta - \rho)/2$, yielding the upper bound

$$\widehat{g}(b) = \overline{g}(A_0)(1+b)^{\phi}\left(1 - b\frac{L_0^B}{L} - 2\alpha\frac{\overline{A}}{\widehat{A}}\right). \quad \text{(D.14)}$$

The remaining question is whether this upper bound can be attained at the saddle-stable steady state S through an R&D subsidy. Substituting $g^* = \widehat{g}(b)$ and $z^* = (\delta - \rho)/2$ into (D.3) and solving for θ yields the candidate subsidy rate

$$\widehat{\theta}(b) = \frac{A(1 - bL_0^B/L - 2\overline{A}/\widehat{A}) - (\delta-\rho)/2}{A(1 - bL_0^B/L - 2\alpha\overline{A}/\widehat{A}) - (\delta-\rho)/2}. \quad \text{(D.15)}$$

The denominator of (D.15) exceeds its numerator by $2(1-\alpha)A\overline{A}/\widehat{A} > 0$. Hence, $\widehat{\theta}(b) \in [0, 1)$ if and only if its numerator is nonnegative. At $z = (\delta - \rho)/2$, the derivative of the left-hand side of (D.8) with respect to z is $-(\delta-\rho)^2/4 - (\rho+\mu)(\delta+\mu) < 0$. Hence, whenever the candidate subsidy rate is feasible, the corresponding steady state is S.

We now determine the level of basic research that maximizes $\widehat{g}(b)$ and examine whether this maximum is attainable. Recall that $\underline{\Phi} \equiv (L_0^B/L)\widehat{A}/(\widehat{A} - 2\alpha\overline{A})$, as defined in Proposition 9. Differentiating (D.14) gives

$$\widehat{g}'(b) = \overline{g}(A_0)(1+b)^{\phi-1}\frac{L_0^B}{L}\left(\frac{\phi - \underline{\Phi}}{\underline{\Phi}} - (1+\phi)b\right). \quad \text{(D.16)}$$

The sign of (D.16) is determined by the term in parentheses, which is strictly decreasing in b. If $\phi \leq \underline{\Phi}$, $\widehat{g}(b)$ is maximized at $b = 0$. Since $A_0 > \widehat{A}$, the subsidy rate $\widehat{\theta}(0)$ is positive and less than one, so this upper bound is attainable. Hence, long-run growth is maximized without additional basic research, proving part (b-i).

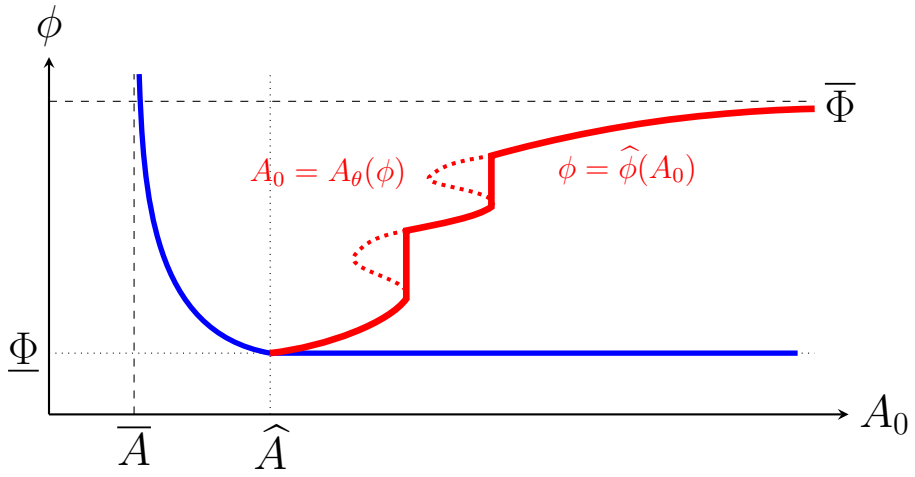


Figure D.1: Construction of the threshold $\widehat{\phi}(A_0)$ from the boundary $A_\theta(\phi)$ (schematic). For each A_0, $\widehat{\phi}(A_0)$ is the smallest value of ϕ on the boundary. The dotted part of the boundary is therefore not used.

If $\phi > \underline{\Phi}$, (D.16) implies that a small increase in b from zero raises $\widehat{g}(b)$. Since $\widehat{\theta}(0) \in (0,1)$, this higher growth rate is attainable for sufficiently small $b > 0$. Hence, any growth-maximizing policy requires $b > 0$, proving part (b-ii).

For part (b-iii), we first maximize the upper bound $\widehat{g}(b)$ without imposing its attainability. When $\phi > \underline{\Phi}$, (D.16) gives

$$\widehat{b} = \frac{\phi - \underline{\Phi}}{(1+\phi)\underline{\Phi}} > 0. \tag{D.17}$$

We then determine whether this upper bound can actually be attained at $b = \widehat{b}$. Substituting (D.17) into the numerator of (D.15), we find that $\widehat{\theta}(\widehat{b}) > 0$ if and only if $\phi < \overline{\Phi}$ and $A_0 > A_\theta(\phi)$, where

$$A_\theta(\phi) = \frac{\widehat{A}(\delta-\rho)(1+\phi)}{4(1-\alpha)\overline{A}(\overline{\Phi}-\phi)}(1+\widehat{b})^{-\phi}, \qquad \overline{\Phi} \equiv \frac{\widehat{A}(1+L_0^B/L) - 2\overline{A}}{2(1-\alpha)\overline{A}}. \tag{D.18}$$

At $\phi = \underline{\Phi}$, $\widehat{b} = 0$ and $A_\theta(\underline{\Phi}) = \widehat{A}$. The boundary tends to infinity as ϕ approaches $\overline{\Phi}$. For each $A_0 > \widehat{A}$, let $\widehat{\phi}(A_0)$ be the first value at which this boundary reaches A_0, as shown in Figure D.1. Before this crossing, $A_0 > A_\theta(\phi)$ and $\widehat{\theta}(\widehat{b}) > 0$. Thus, for $\underline{\Phi} < \phi < \widehat{\phi}(A_0)$, growth is maximized by positive basic research expansion and a positive research subsidy.

D.5 Calibration of L_0^B/L in the Basic Research Model

The GDP of this economy is defined as the sum of consumption expenditures $C_t = X_t$, private investment expenditures for R&D, $(1-\theta)w_t L_t^R$, and government expenditures, $\theta w_t L_t^R + w_t L^B$. Since the production of final goods (14) and the wage (20) are unchanged,

GDP becomes $\text{GDP}_t = n_t^{\frac{1-\alpha}{\alpha}}(L_t^P + \alpha L_t^R + \alpha L^B)$. Therefore, the ratio of basic research expenditure to GDP is given by

$$\frac{w_t L^B}{\text{GDP}_t} = \frac{\alpha L^B}{L_t^P + \alpha L_t^R + \alpha L^B} \equiv \sigma_t^B. \tag{D.19}$$

At the benchmark $b = 0$, we have $L^B = L_0^B$ and $L_t^P + L_t^R = L$ from (D.1). Eliminating L_t^P from (D.19) and then solving for L_0^B/L yields $L_0^B/L = (\sigma_t^B/(\alpha(1-\sigma_t^B)))\big(1-(1-\alpha)L_t^R/L\big)$. Using (25), this equation becomes

$$\frac{L_0^B}{L} = \frac{\sigma_t^B}{\alpha(1-\sigma_t^B)}\left(1 - \frac{\alpha}{A_0}g_t\right).$$

For σ_t^B, we use the average value of the basic research expenditures as a percentage of GDP in the 2000–2023 period from the OECD Main Science and Technology Indicators, over the years for which it is available. This series includes basic research funded by firms, and we use it as a proxy for publicly funded basic research. Accordingly, for g_t, we use the average value of the real GDP growth rate in the same period from the Penn World Table version 11.0, as in Section 4.4. Then, we obtain the estimate of the average value of L^B/L in the 2000–2023 period, which we treat as L_0^B/L. This procedure yields $L_0^B/L = 0.545\%$ for the United States and $L_0^B/L = 0.361\%$ for Italy.

E Analysis of the Sustainable and Unsustainable Subsidies

E.1 Conditions at the Termination of the Subsidy

This subsection derives the conditions that connect the equilibrium path before and after the regime change under the unsustainable policy, as discussed in Section 7.2.

Below, we show that this transition creates discontinuities in both g_t and D_t, even though the regime change is fully anticipated. The discontinuities originate from the discontinuous change of θ from $\theta^u > 0$ to 0 at time T. From (16) and (20), the cost of R&D net of the subsidy before T is $(1-\theta^u)\alpha L n_t^{(1-2\alpha)/\alpha}/A$. Therefore, at T, the cost of entry, net of the subsidy, jumps up by a factor of $1/(1-\theta^u)$. As long as there are positive amounts of entry both before and after T, the value of firms equals the cost of entry both before and after T. By writing the value of the firm just before T by v_{T-} (formally, $v_{T-} = \lim_{t\uparrow T} v_t$), the value of the firm before and after the regime change is connected by $v_{T-} = (1-\theta^u)v_T$. Note that the no-arbitrage condition (19) is derived assuming that the firm value is differentiable with respect to time. When the firm value changes discretely,

we need to construct the no-arbitrage condition using the gross interest factor (the inverse of the discount factor), as we explain below.

Let us define the gross interest factor $R(t,t')$ by the amount of final goods obtained at time t' by investing one unit of final goods in government bonds at time $t \leq t'$. $1/R(t,t')$ represents the discount factor between t and t'; i.e., the value at t of one unit of final goods delivered at t'. The gross interest factor $R(t,t')$ has the following properties. Under perfect foresight, $R(t,t'') = R(t,t')R(t',t'')$ holds for $t \leq t' \leq t''$. Also, if the interest rate is finite between t and t', then $R(t,t') = \exp\left(\int_t^{t'} r_{t''}dt''\right)$ holds.

Note that within each policy regime, the interest rate is given by (21), which is finite. Therefore, $R(t,t') = \exp\left(\int_t^{t'} r_{t''}dt''\right)$ holds if $t < t' < T$ or $T \leq t < t'$. However, this relationship may not hold on an interval that spans $t < T$ and $t' \geq T$ because Equation (21) depends on the no-arbitrage condition (19), which is not well defined at T when $\dot{v}_t$ does not exist. In this case, we need to consider the gross interest factor between just before T and at T, denoted by $R(T-,T) \equiv \lim_{t \uparrow T} R(t,T)$.

Given that investors know that the subsidy is only available before T, they are willing to pay a discrete interest premium if they can obtain funds before the subsidy terminates (i.e., $t < T$) and repay them at $t' \geq T$, because funds at $t < T$ can be used to take advantage of the subsidy while those at $t' \geq T$ cannot. What matters is whether they can get funds before T, not how long before T. They are willing to pay this premium even as the duration between $t < T$ and $t' \geq T$ approaches zero. When interpreted in terms of interest rate (which is defined per unit time), there would be an unbounded spike in interest rate at time T. Expressed in terms of the gross interest factor, the spike can be represented as $R(T-,T) > 1$.

We now formally derive the value of $R(T-,T)$. From (20), the cost of developing a new variety before subsidies, $w_t L/(An_t)$, depends only on n_t. Since n_t always follows (15), it is continuous at T. However, the cost borne by firms rises from $(1-\theta^u)w_t L/(An_t)$ to $w_t L/(An_t)$ at T. Since the free entry condition (16) holds with equality both before and after T, the value of firms satisfies $v_{T-} = (1-\theta^u)v_T$. Note that the value of a firm at $t < T$ can be divided into two parts,

$$v_t = \int_t^{T-} \pi_{t'} e^{-\int_t^{t'} r_{t''}dt''} dt' + \frac{v_T}{R(t,T)}, \tag{E.1}$$

where the first term is the present discounted value of the profits before T, and the second term is the value of the firm at T, discounted by $R(t,T)$. Using $v_{T-} = (1-\theta^u)v_T$, we can

rewrite (E.1) as

$$R(t,T) = \frac{1}{1-\theta^u} \frac{v_{T-}}{v_t - \int_t^{T-} \pi_{t'} e^{-\int_t^{t'} r_{t''} dt''} dt'}. \tag{E.2}$$

The integral in the denominator is the present value of the profits paid between t and $T-$. From Appendix A.2, the profit is $\pi_{t'} = (1-\alpha) n_{t'}^{(1-2\alpha)/\alpha} L_{t'}^P$, which is at most $(1-\alpha) L\, n_t^{(1-2\alpha)/\alpha}$ for $t' \geq t$ because the number of goods n_t does not decrease. Also $e^{-\int_t^{t'} r_{t''} dt''}$ is finite because r_t is finite between t and $T-$. This implies that the integral is at most $T-t$ times a bounded value, so that it converges to zero as t approaches T from below. Since v_t converges to $v_{T-} = (1-\theta^u) v_T$, taking the limit of (E.2) gives

$$R(T-,T) = \frac{v_T}{v_{T-}} = \frac{1}{1-\theta^u} > 1. \tag{E.3}$$

In terms of the instantaneous interest rate, we can interpret (E.3) as r_t having an unbounded spike at time T.[96] Because individuals know the subsidy will be terminated at T, there is a huge demand for last-minute R&D just before T. Since the investment is done by borrowing, the interest rate needs to spike at T to maintain equilibrium in the financial market. The government bond also needs to provide the same rate of return. This means that the outstanding debt will be multiplied by $R(T-,T) = 1/(1-\theta^u) > 1$ at the time T.[97] Since $Y_t = n_t^{(1-\alpha)/\alpha} L$ cannot jump, $D_t = B_t/Y_t$ is also multiplied by $1/(1-\theta^u)$:

$$D_T = \frac{D_{T-}}{1-\theta^u}. \tag{E.4}$$

The spike in the interest rate also affects the consumption. The dynamic optimization of individuals implies that the intertemporal marginal rate of substitution between t and t' should always be equal to the gross interest factor between t and t'. Given the logarithmic utility function (3), this requirement reduces to $(c_{s,t'}/c_{s,t}) e^{\rho(t'-t)} = R(t,t')$.[98] Taking the limit $t \uparrow T$ and using (E.3), we obtain $c_{s,T} = c_{s,T-}/(1-\theta^u)$. Intuitively, the individuals who

[96] We can also interpret this as a result, by analogy with the relationship between the distribution function and the density function in statistics. In this analogy, $\ln R(t,t')$ corresponds to the mass of distribution between t and t', and r_t corresponds to the density function at point t. If $R(t,t')$ is continuous and differentiable at t, the derivative of $\ln R(t,t')$ with respect to t would give $-r_t$. However, when $R(t,t')$ is discontinuous, r_t is not well defined. This happens when the distribution has a mass point. In that case, the distribution's density has an unbounded spike at this point. In this analogy, we can think of the interest rate as having an unbounded spike at time T.

[97] In the perfect foresight model of capital accumulation with an investment subsidy studied by Judd (1985), the return on bonds contains the term $-\dot{\theta}/(1-\theta)$, where θ is the subsidy rate. When the subsidy rate falls discretely from θ^u to zero, the integral of this term is $\ln\bigl(1/(1-\theta^u)\bigr)$, which coincides with the mass point obtained here. The same paper also notes that the subsidy rate need not be differentiable if the equilibrium conditions are interpreted in terms of generalized functions.

[98] When $R(t,t')$ is differentiable, this relationship gives the usual Euler equation.

know that there is a spike in the interest rate at T are willing to wait until T to consume more. Thus, individual consumption jumps up at T. Since the number of individuals born between t and T converges to zero as $t \uparrow T$, aggregating over generations by (5) gives $C_T = C_{T-}/(1-\theta^u)$. The aggregate consumption also jumps up by the factor of $1/(1-\theta^u)$.

Since final goods are used only for consumption, $C_t = X_t = n_t^{(1-\alpha)/\alpha} L_t^P$ holds from (14). Because n_t is continuous, the jump in consumption implies $L_T^P = L_{T-}^P/(1-\theta^u)$. Substituting $L_t^P/L = 1 - g_t/\overline{g}(A)$, which follows from (25) and (23), we obtain

$$g_{T-} = \theta^u\, \overline{g}(A) + (1-\theta^u)\, g_T. \tag{E.5}$$

As explained above, investors are willing to do R&D before T, which creates a spike in the interest rate at T. This induces consumers to wait until T to consume more. The labor freed up in the production sector can thus be allocated to R&D. At time T, this frenzy ends, R&D investment falls, and the growth rate jumps down.

E.2 Method of Numerical Simulation

We use the linear system in Appendix A.6 and the closed forms in Appendix A.12. Under the sustainable policy, (A.25), (A.26), and (A.27), evaluated at $\theta = \theta^s$ and $D_0 = 0$, give the equilibrium path.

For the unsustainable policy, first compute the path after termination under $\theta = 0$. The post-switch debt ratio is $D_T = 0.9 D_U(A, 0)$, and g_T is the height of the equilibrium line at this debt ratio. The closed forms apply with initial condition D_T and time $t - T$. The pre-switch point is then determined by (E.4) and (E.5).

Next, fix a candidate subsidy $\theta^u > \overline{\theta}(A)$ and evaluate $\mathbf{M}$ in (A.18) at that rate. Before termination, the solution is obtained from

$$\frac{\boldsymbol{\gamma}_t}{\gamma_{3,T-}} = e^{\mathbf{M}(t-T)}(D_{T-}, g_{T-}, 1), \qquad t < T.$$

Taking component ratios yields (D_t, g_t). This expression applies even though the economy has no feasible steady state under a permanent subsidy at that rate.

The subsidy begins at the latest earlier time at which $D_t = 0$, found by bisection. We check that the third component remains positive and that $0 < g_t < \overline{g}(A)$ throughout this interval. We then adjust θ^u by bisection until the interval has the prescribed length T. For the calibrations reported in the paper, the interval becomes shorter as θ^u rises.

Within either regime, output ratios follow from $Y_t/Y_{t'} = (\gamma_{3,t}/\gamma_{3,t'})^{1/s(\theta)}$. Output is

continuous at termination. The consumption and GDP paths then follow from the relations in Appendix A.12.

E.3 Expected Lifetime Utility of Each Cohort

This subsection explains how we compute the expected lifetime utility of the individuals alive at $t = 0$, reported in Section 7.2. Before $t = 0$, the economy is on the balanced growth path without subsidies, on which $r_t = r^*(A, 0)$ and the wage grows at $g^*(A, 0)$. For an individual born at $s < 0$, human wealth at birth is $h_{s,s} = w_s/\big(r^*(A,0)+\mu+\delta-g^*(A,0)\big)$ by (A.3), and consumption is $c_{s,t} = (\rho+\mu)h_{s,s}e^{(r^*(A,0)-\rho)(t-s)}$ by (A.2) and the Euler equation for individuals. Substituting these into (4) and integrating from $k_{s,s} = 0$ gives the asset holding just before $t = 0$,

$$k_{s,0-} = \frac{w_0\big(e^{-(r^*(A,0)-\rho-g^*(A,0))s} - e^{\delta s}\big)}{r^*(A,0) + \mu + \delta - g^*(A,0)}. \tag{E.6}$$

At $t = 0$, the subsidy rate rises from zero to θ^s or θ^u unexpectedly. Since the government has no debt at $t = 0$, all assets are shares of firms, and the value of a firm falls from $w_0L/(An_0)$ to $(1-\theta)w_0L/(An_0)$ by (16), where θ is the new subsidy rate. Hence, $k_{s,0} = (1-\theta)k_{s,0-}$. Human wealth at $t = 0$ is $h_{s,0} = e^{\delta s}h_{0,0}$ by (A.3), and $h_{0,0} = \big((\beta+\delta)/\beta\big)\overline{h}_0$ by (A.6), where $\overline{h}_0 = C_0/\big((\rho+\mu)N_0\big) - K_0/N_0$ by (A.4), with $C_0/Y_0 = 1 - g_0/\overline{g}(A)$ and $K_0/Y_0 = \alpha(1-\theta)/A$ from (A.7). Consumption at $t = 0$ is $c_{s,0} = (\rho+\mu)(k_{s,0} + h_{s,0})$ by (A.2). For $t > 0$, $c_{s,t} = c_{s,0}\exp\big(\int_0^t (r_{t'} - \rho)dt'\big)$ by the Euler equation for individuals, multiplied by $1/(1-\theta^u)$ for $t \geq T$ under the unsustainable policy (Appendix E.1). The interest rate is $r_t = \overline{g}(A)/(1-\theta) - \big(s(\theta)-1\big)g_t$ by (27), and $\int_0^t g_{t'}dt' = \ln(\gamma_{3,t}/\gamma_{3,0})/s(\theta)$ holds within each regime by the definition of $\boldsymbol{\gamma}_t$, where g_t and $\gamma_{3,t}$ are given by the closed forms in Appendices A.12 and E.2. By (3), the expected utility of the individual from $t = 0$ onward is proportional to $\int_0^\infty e^{-(\rho+\mu)t}\ln c_{s,t}\,dt$, which equals $\ln c_{s,0}/(\rho+\mu)$ plus a term that is common to all cohorts under a given policy. We evaluate the integral numerically using the closed form of r_t. An individual supports the unsustainable policy if this expected utility is higher than under the sustainable policy, and the boundary age reported in Section 7.2 is the age at which the two are equal.

Figure E.1 shows the gain from each policy relative to the case without subsidies, by the age of the individual at $t = 0$. We measure the gain by the proportional increase in consumption at all remaining ages that would give the same expected utility as the policy. Since utility is logarithmic, this proportion is the exponential of $\rho + \mu$ times the difference in $\int_0^\infty e^{-(\rho+\mu)t}\ln c_{s,t}\,dt$, minus one. Under a positive subsidy, the gain falls with

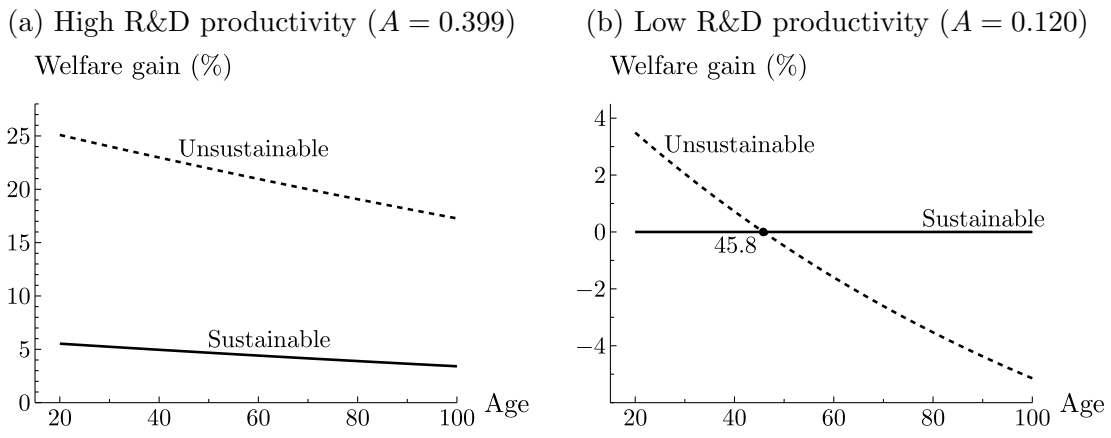


Figure E.1: Welfare gains from the two policies by the age of the individuals alive at the introduction of the policy. The gain is the percentage increase in consumption at all remaining ages that is equivalent to the policy, relative to the case without subsidies. In Panel (b), the sustainable policy is $\theta = 0$, so its gain is zero.

age, because older individuals hold more shares of existing firms, whose value falls when the subsidy is introduced. In Panel (b), the sustainable policy is $\theta = 0$, and the gain from the unsustainable policy turns negative at the age of 45.8.

F Analysis of the Asset Taxation Extension

This appendix analyzes the model with asset taxation in Section 7.3. We derive the equilibrium dynamics, examine the steady states, and prove Proposition 10.

F.1 Derivation of Equilibrium Dynamics

The asset income tax changes the return received by households to $(1-\tau_r)r_t$. Hence, with $\beta = \mu$, the aggregate consumption equation is

$$\dot{C}_t = \big((1-\tau_r)r_t - \rho + \delta\big)C_t - (\rho+\mu)(\delta+\mu)K_t. \tag{F.1}$$

The government receives $\tau_r r_t K_t$ and pays the research subsidy. Its budget is $\dot{B}_t = r_t B_t + \theta w_t L_t^R - \tau_r r_t K_t$. The supply side is unchanged, because the tax applies equally to interest, dividends, and capital gains. Thus, the pre-tax relation (27) still holds, and the after-tax gap is

$$g_t - (1-\tau_r)r_t = (1-\tau_r)s(\theta)\big(g_t - \widetilde{g}(A,\theta)\big) + \tau_r g_t. \tag{F.2}$$

The consumption and asset ratios in (A.7) also remain unchanged. Substituting them into the government budget and the consumption equation gives

$$\dot{D}_t = -\big(g_t - (1-\tau_r)r_t\big)D_t + \frac{\theta\alpha}{\overline{g}(A)}g_t - \tau_r r_t \frac{\alpha(1-\theta)}{A}, \tag{F.3}$$

$$\dot{g}_t = (\overline{g}(A) - g_t)\big(g_t - (1-\tau_r)r_t - \delta + \rho\big) + \overline{g}(A)(\rho+\mu)(\delta+\mu)\left(D_t + \frac{\alpha(1-\theta)}{A}\right). \tag{F.4}$$

The tax on bond interest reduces the effective interest cost of government debt. The last term of (F.3) is the tax revenue from firm shares. Since r_t is determined by (27), the system is again in D_t and g_t.

F.2 Steady States

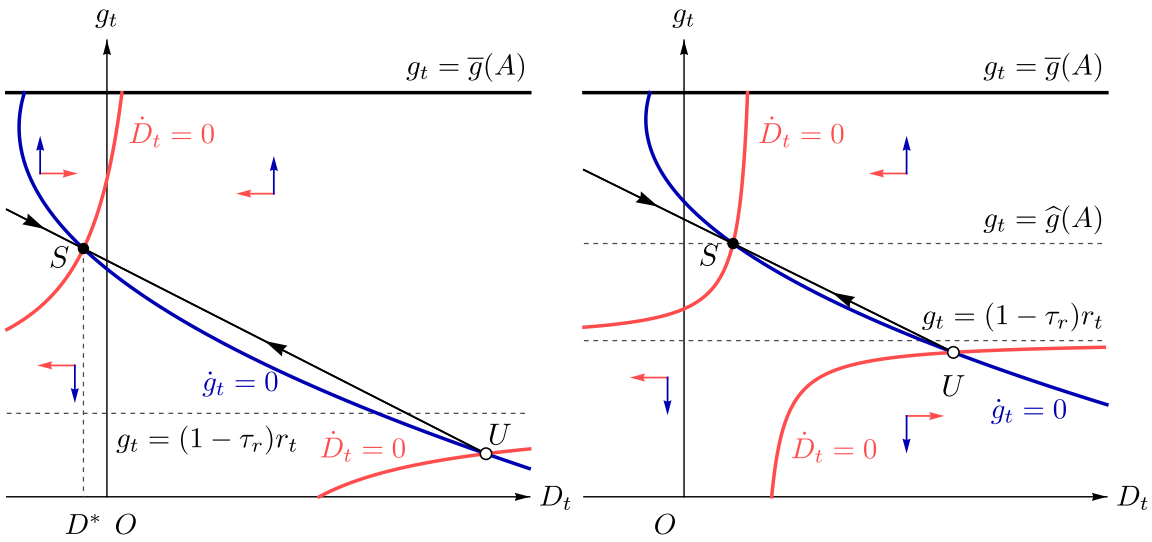
The steady states are the intersections of the two stationary loci obtained from (F.3) and (F.4). Eliminating D gives

$$\begin{aligned}\big(g-(1-\tau_r)r\big)\Big(&\big((1-\tau_r)A + \tau_r\overline{g}(A) - (g-(1-\tau_r)r)\big)\\ &\times\big(\delta-\rho-(g-(1-\tau_r)r)\big) - (\rho+\mu)(\delta+\mu)\Big)\\ = (\rho+\mu)(\delta+\mu)(1-\alpha)A&\left(\frac{(1-\tau_r)\theta}{1-\theta} - \frac{(1-\alpha)\tau_r}{\alpha}\right).\end{aligned} \tag{F.5}$$

The return r is the function of g in (27), so this is a cubic condition in g. There is one extra intersection above $g = \overline{g}(A)$, denoted by E and excluded by the labor constraint. The higher-growth feasible intersection is S, as in the baseline model. The common quadratic coefficient of the dynamic equations is now $(1-\tau_r)s(\theta) + \tau_r$, so the linear representation and the equilibrium-line argument in Appendix A.6 continue to apply.

Unlike the baseline case, the budget may have a surplus without subsidies. With positive pre-tax interest, a positive after-tax gap and $\theta = 0$ require $D < 0$ in (F.3). This is the configuration shown in Figure F.1(a). The subsidy shifts the stationary loci, and the rate that attains $\widehat{g}(A)$ selects the higher-growth intersection in panel (b), as proved below.

At $g = \overline{g}(A)$, the after-tax gap is $(1-\tau_r)A + \tau_r\overline{g}(A)$, and the difference between the two sides of (F.5) is negative. Without subsidies it equals $-(\rho+\mu)(\delta+\mu)(1-\alpha\tau_r)A$, and a higher subsidy makes it smaller. Hence, the excluded intersection is unique by the argument in Appendix C.3.



(a) $\theta = 0$ (b) $\theta = \widehat{\theta}(A, \tau_r)$

Figure F.1: Stationary loci with asset taxation. Panel (a) shows the no-subsidy economy, and panel (b) shows the subsidy that attains the highest long-run growth rate.

F.3 Proof of Proposition 10

For the calculations here, write g^*, r^*, and D^* for $g^*(A, \theta, \tau_r)$, $r^*(A, \theta, \tau_r)$, and $D^*(A, \theta, \tau_r)$, and $z \equiv g^* - (1-\tau_r)r^*$ for the after-tax gap. The stationary budget gives $zD^* = \theta\alpha g^*/\overline{g}(A) - \tau_r r^*\alpha(1-\theta)/A$. Substituting this into the growth condition and using the pre-tax supply relation (27) yields

$$\big(\overline{g}(A) - g^*\big)z(\delta - \rho - z) = (\rho + \mu)(\delta + \mu)\big(g^* - (1-\alpha)\overline{g}(A)\big).$$

Thus, the growth rate is given by (A.13) with this after-tax gap. Neither tax nor subsidy appears separately in that relation.

Proof of part (a). Suppose a long-term BGP with positive research has $r^* \leq 0$. By (27), this requires $g^* \geq (1-\alpha)\overline{g}(A)/(\alpha - (2\alpha - 1)\theta)$, a threshold increasing in θ. Since the no-policy BGP has positive interest, $g^* > g^*(A, 0, 0)$. The after-tax gap is at least g^*, so it exceeds the original no-policy gap, which is above $(\delta - \rho)/2$ by $A > \widehat{A}$. But (A.13) is decreasing in the gap above this value, implying $g^* < g^*(A, 0, 0)$, a contradiction. If research is absent on the BGP, the stationary valuation of firms gives $r^*\nu^* = 1 - \alpha > 0$. Hence, interest is positive whenever a long-term BGP exists.

Proof of part (b). The highest growth rate allowed by (A.13) is attained at $z = (\delta - \rho)/2$ and equals $\widehat{g}(A)$. Substituting this gap and growth rate into (F.2) gives the required subsidy:

$$\widehat{\theta}(A, \tau_r) = \frac{(1 - \alpha\tau_r)\widehat{g}(A) - (1-\alpha)\big((1-\tau_r)\overline{g}(A) + (\delta - \rho)/2\big)}{(\alpha + \tau_r - 2\alpha\tau_r)\widehat{g}(A) - (1-\alpha)(\delta - \rho)/2}. \tag{F.6}$$

Since the interest rate $r^*(A,0,0)$ is positive, the growth rate $g^*(A,0,0)$ exceeds the gap without policy, which is above $(\delta-\rho)/2$ because A is larger than $\widehat{A}$. Together with $\widehat{g}(A) > g^*(A,0,0)$ by Proposition 3, this gives $\widehat{g}(A) > g^*(A,0,0) > (\delta-\rho)/2$. The denominator is therefore positive, because the coefficient $\alpha+\tau_r-2\alpha\tau_r$ is at least $1-\alpha$. The numerator is linear in τ_r and positive at both zero and one, so it is positive for all tax rates under consideration. The denominator exceeds the numerator by $(1-\alpha)(1-\tau_r)(\overline{g}(A)-\widehat{g}(A)) > 0$. Hence, the required subsidy lies in $(0,1)$.

The resulting intersection is S, as in Figure F.1(b). Its growth rate is below the labor limit, and the derivative of the left-hand side of (F.5) with respect to the after-tax gap at $z=(\delta-\rho)/2$ is $-(\delta-\rho)^2/4-(\rho+\mu)(\delta+\mu) < 0$. It is therefore the higher-growth feasible intersection. The upper growth bound is attained, proving part (b).

Proof of part (c). Setting $\tau_r = 0$ in (F.6) gives $\widehat{\theta}(A,0) = \widehat{\theta}(A)$. Differentiating shows that the derivative with respect to τ_r has the sign of $(\overline{g}(A)-\widehat{g}(A))(2\widehat{g}(A)-\delta+\rho)$, which is positive by the preceding inequalities.

Proof of part (d). At the optimum, the growth rate and after-tax gap do not change with the tax rate. The stationary growth condition therefore gives

$$D^*\big(A,\widehat{\theta}(A,\tau_r),\tau_r\big) = \frac{(\delta-\rho)(\overline{g}(A)-\widehat{g}(A))}{2\overline{g}(A)(\rho+\mu)(\delta+\mu)} - \frac{\alpha(1-\widehat{\theta}(A,\tau_r))}{A}.$$

Only the second term changes with τ_r, and its derivative with respect to τ_r is $(\alpha/A)\,\partial\widehat{\theta}(A,\tau_r)/\partial\tau_r$, which is positive by part (c).

G Analysis of the Small Open Economy Extension

This appendix analyzes the small open economy in Section 7.4. We derive the equilibrium dynamics, examine the steady states, and prove Proposition 11.

G.1 Derivation of Equilibrium Dynamics

The supply side is the same as in the baseline model. Solving (27) for the interest rate gives $r_t = \overline{g}(A)/(1-\theta) - (s(\theta)-1)g_t$. Substituting the country-premium relation (60) yields the growth rate (61). Substituting (60) and (61) into the debt dynamics (29) gives

$$\dot{D}_t = (\eta+\psi_2(\theta))D_t^2 + \left(\overline{r} - \psi_1(A,\theta) - \frac{\theta\alpha\psi_2(\theta)}{\overline{g}(A)}\right)D_t + \frac{\theta\alpha\psi_1(A,\theta)}{\overline{g}(A)}. \tag{G.1}$$

Thus, the system reduces to one differential equation in nonnegative debt.

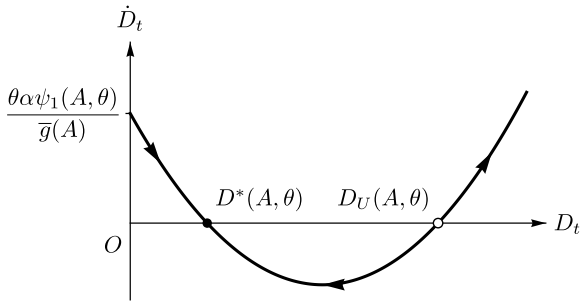


Figure G.1: Debt dynamics in the small open economy with a positive subsidy.

G.2 Steady States

The right-hand side of (G.1) is an upward-opening quadratic, because its leading coefficient $\eta+\psi_2(\theta)$ is positive. At zero debt it equals subsidy expenditure, which is positive for $\theta > 0$ in the research region. When the subsidy is not too large, the quadratic has two positive zeros $D^* < D_U$, and

$$\dot{D}_t = (\eta + \psi_2(\theta))(D_t - D^*)(D_t - D_U).$$

The arrows in Figure G.1 show that debt converges to D^* from every nonnegative initial value below D_U. The lower-debt point is the long-term BGP, and the higher one is the unsustainable threshold.

Above D_U, debt rises at a positive rate that increases with debt. Within finite time it becomes large enough to stop research by (61). Thereafter, output is constant, and $\dot{D}_t = (\overline{r}+\eta D_t)D_t$ exceeds ηD_t^2. Hence, $1/D_t$ falls at least at the rate η, and debt becomes infinite within a period of $1/(\eta D_t)$. By the definition of an equilibrium path in Appendix A.6, there is no equilibrium path from $D_0 > D_U$.

G.3 Proof of Proposition 11

Proof of part (a). When $\theta = 0$, zero debt is a steady state and $r^*(A, 0) = \overline{r}$. By (61), $g^*(A, 0) = ((1-\alpha)/\alpha)(\overline{g}(A) - \overline{r})$. Its gap is positive if and only if $A > \overline{A}^{\text{open}}$. The other zero of (G.1) is $\alpha(g^*(A, 0) - \overline{r})/\eta$. If $A > \overline{A}^{\text{open}}$, this zero is positive, and zero debt is the lower steady state. If $A < \overline{A}^{\text{open}}$, it is negative, and debt rises from every positive initial value. This proves part (a).

Proof of part (b). At the no-subsidy BGP, $D^* = 0$. Differentiating the stationary government budget with respect to the subsidy gives

$$\left.\frac{dD^*}{d\theta}\right|_{\theta=0} = \frac{\alpha g^*(A,0)}{\overline{g}(A)(g^*(A,0) - \overline{r})}.$$

The numerator is additional subsidy spending per unit of the subsidy rate, and the gap in

the denominator is the rate at which growth reduces the debt ratio. Differentiating (61) then gives

$$\left.\frac{dg^*}{d\theta}\right|_{\theta=0} = \frac{1-\alpha}{\alpha}\left(\overline{g}(A) - g^*(A,0) - \eta \left.\frac{dD^*}{d\theta}\right|_{\theta=0}\right).$$

The first effect encourages research directly. The second crowds it out through debt and the country premium. Growth rises if and only if

$$\eta < \widehat{\eta}(A) \equiv \frac{\overline{g}(A)(g^*(A,0) - \overline{r})(\overline{g}(A) - g^*(A,0))}{\alpha g^*(A,0)}.$$

Substituting the no-policy growth rate gives the right-hand side of (62).

To obtain its inverse, write this threshold as $\widehat{\eta}(A) = (\overline{g}(A)/\alpha)(\overline{g}(A) - g^*(A,0))(1 - \overline{r}/g^*(A,0))$. Each factor is positive and increasing when $A > \overline{A}^{\text{open}}$. In particular, the second equals $(1-\alpha)((2\alpha-1)A + \alpha\overline{r})/\alpha^2$, which increases because α exceeds $1/2$. The threshold is zero at $\overline{A}^{\text{open}}$ and tends to infinity with A. Therefore, its inverse $\widehat{A}^{\text{open}}(\eta)$ is well defined and strictly increasing, with $\widehat{A}^{\text{open}}(0) = \overline{A}^{\text{open}}$. This proves part (b).