Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Asymptotic Analysis of Periodic Solutions of Generalized Burgers Equations at Large Times

Pallabi Kundu Pallabi Kundu
Techno India University
Dr.Smriti Nath Smriti Nath
smriti.n@technoindiaeducation.com

Techno India University

Research Article

Keywords: Nonlinear equations, partial differential equations, Burgers equations, asymptotic periodic solutions, large time

Posted Date: November 23rd, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11664095/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

Asymptotic Analysis of Periodic Solutions of Generalized Burgers Equations at Large Times

Pallabi Kundu and Dr. Smriti Nath(Corresponding Author)

Abstract. In this paper, we investigate the large-time asymptotic behavior of periodic solutions to a generalized Burgers equation with nonlinear convection and non-planar effect. A perturbative method is applied to this equation to derive analytical solutions. This equation is solved under periodic,bounded and continuous initial conditions on $\mathbb{R}$. Numerical simulations based on a mixed Godunov method are carried out under periodic boundary conditions to validate the analytical results.

Keywords. Nonlinear equations, partial differential equations, Burgers equations, asymptotic periodic solutions, large time.

1. Introduction

Among nonlinear partial differential equations, the Burgers equation occupies a fundamental position because of its mathematical significance and wide range of applications in applied mathematics, physics and engineering.The Burgers equation incorporates the essential features of both the first-order nonlinear wave equation and the heat equation. Owing to its simple mathematical structure, it provides an effective framework for investigating the interaction between nonlinear convection and diffusion.
The homogeneous form of the Burgers equation was first presented by Bateman[1], who derived its steady-state solution.
In this work, we investigate a family of nonlinear parabolic equations known as generalized Burgers equations (GBEs), which extend the classical Burgers model by incorporating additional nonlinear and time-dependent effects,

$$u_t + uu_x = \epsilon u_{xx}. \tag{1.1}$$

The classical Burgers equation (1.1equation.1.1) possesses a remarkable property: it can be mapped to the linear heat equation through the Hopf-Cole

transformation [2,3] . Specifically, by introducing the dependent-variable transformation

$$u = -2\epsilon \frac{\psi_x}{\psi},$$

the nonlinear equation is converted into a linear diffusion equation. The following nonlinear partial differential equations,

$$u_t + u^\alpha u_x + \frac{j}{2t} u = \epsilon u_{xx}, \tag{1.2}$$

$$u_t + (au + bu^2) u_x = \epsilon u_{xx}, \tag{1.3}$$

are some generalizations of classical Burgers equation and are collectively referred to as generalized Burgers equations (GBEs). In general,GBEs are not exactly linearizable via Hopf-Cole transformation.So,these equations can be solved asymptotically.We can study asymptotic behavior of these GBEs.GBE is typically characterized by two types of solution profiles : N-wave profiles and periodic solutions.

Earlier investigations have established the presence of N-wave profiles and periodic solutions in various forms of generalized Burgers equations.Sachdev et al. [4] established composite N-wave solutions of non-planar Burgers equations,

$$u_t + u\, u_x + \frac{ju}{2t} = \frac{\delta}{2} u_{xx}, \quad \text{where } j = \frac{m}{n},\ 0 < j < 2,\ m, n \in \mathbb{N},\ (m, n) = 1.$$

Rao and Satyanarayana [5] derived an improved version of large- time asymptotic N-wave solution of this equation for j $\geq$ 0,$\delta > 0$.

In their work, Sachdev et al. [6] established the long time asymptotic structure of periodic solutions associated with the modified Burgers equation.

$$u_t + u^n u_x = \frac{\delta}{2} u_{xx}, \quad \delta > 0, \quad \text{for } n = 2 \text{ and } n = 3.$$

The work of Nath [8] on the large-time asymptotic analysis of solutions to certain generalized Burgers equations serves as an important motivation for the present investigation. Nath and Rao [9] extended the asymptotic theory by obtaining large-time periodic solutions for two different generalized Burgers equations.

$$(a) \quad u_t + u^2 u_x + \frac{ju}{2t} = \epsilon u_{xx}, \quad x \in \mathbb{R},\ t > 0,$$

$$(b) \quad u_t + (au + bu^2) u_x = \epsilon u_{xx}, \quad x \in \mathbb{R},\ t > 0,$$

where $\epsilon > 0$ is small, and $j \geq 0$, $a > 0$, $b > 0$ are constants, obtained via a perturbative approach. Samanta and Rao[10] constructed large time asymptotics to solution of a generalized Burgers equation

$$u_t + u^p u_x + \lambda u = u_{xx}, \quad x > 0,\ t > 0,$$
$$u(0, t) = \tilde{u}_0, \quad t > 0,$$
$$u(x, 0) = 0, \quad x > 0,$$

where $p \geq 1$ is an integer, and λ and $\tilde{u}_0 > 0$ are constants, with linear damping on half-line. Samanta and Chidella[11] investigated a relevant special solution of the following Burgers equation

$$\begin{aligned} u_t + uu_x - u_{xx} &= 0, \qquad x > 0,\ t > 0, \\ u(x,0) &= \hat{u}_0 + u_0(x), \qquad x \geq 0, \\ u(0,t) &= \tilde{u}_0, \qquad t > 0, \end{aligned}$$

where $\tilde{u}_0 > 0$, $\hat{u}_0$ are real numbers with $\tilde{u}_0 > \hat{u}_0$,$u_0(x)$ is an integrable function on $[0,\infty]$ and studied the exact large-time asymptotic behavior of the solution u of this IBVP. Samanta and Paswan[12] constructed the large-time asymptotic behavior on the infinite line for the following generalized Burgers equation:

$$V_t + (H(V))_x + K(t)V_x = V_{xx}, \qquad -\infty < x < \infty,\ t > 0,$$

with initial condition

$$V(x,0) = \begin{cases} v_+, & x \geq 0, \\ v_-, & x < 0, \end{cases}$$

where $v_- > v_+$ and $v_- > 0$. The coefficient $K(t)$ is written as K(t)=k_0+$k_e(t)$, and the flux function $H \in C^2$ satisfies H(0)=0, together with the convexity condition $H''(u) > 0$ on the domain of interest.

In all the above works, the non-planar term and the nonlinear term have been treated separately. The combined effect of both terms on periodic solutions has not been studied. In this paper, we study and derive a fourth-order asymptotic expansion with explicit coefficients, obtain the zero-shift up to third order of the GBE and validate the results numerically, including a spatial grid-refinement study showing the asymptotic solution outperforms the leading-order linear solution. Section 2 introduces the governing equation together with the initial and periodic boundary conditions and outlines the perturbative approach. Section 3 derives the asymptotic expansion and explicit coefficients. Section 4 presents the numerical study and corresponding error tables for two initial profiles. Section 5 represents the grid refinement results. Section 6 discusses the applications of GBEs, and Section 7 concludes the paper.

2. Methods

In this study we investigate the generalized Burgers equation

$$u_t + (au + bu^2)u_x + \frac{ju}{2t} = \epsilon u_{xx} \tag{2.1}$$

under the initial-boundary conditions

$$u(x,t_0) = u_0(x), t > 0, x \in \mathbb{R} \tag{2.2}$$

$$u_0(0) = u_0(2\pi) \tag{2.3}$$

$$u(x,t) = u(x+2\pi,t), x \in \mathbb{R} \tag{2.4}$$

where $\epsilon > 0$ is small and j $\geq$ 0,a $>$ 0,b $>$ 0 are constants[9]. Here,we are using perturbative method to study this equation.

3. Results

The asymptotic behaviour of the periodic solution of the generalized Burgers equation (2.1equation.2.1) for large values of t is approximated by the function

$$u(x,t) = A_1 t^{-j/2} e^{-\epsilon t} \sin(x - x_0), \tag{3.1}$$

which is an exact solution of the corresponding linear equation

$$u_t + \frac{ju}{2t} = \epsilon u_{xx}. \tag{3.2}$$

This linear solution serves as the dominant term in the asymptotic representation of the 2π-periodic solution of (2.1equation.2.1) satisfying the initial and periodic boundary conditions (2.2equation.2.2)–(2.4equation.2.4). To obtain a more accurate approximation of the nonlinear problem, we extend the leading-order solution by incorporating perturbative corrections and derive a higher-order asymptotic expansion. Motivated from the asymptotic expansions proposed in [9], we assume

$$u(x,t) = A_1 t^{-j/2} e^{-\epsilon t} \sin(x - \tilde{x_0}(t)) + U_1(x,t), \tilde{x_0}(t) \to x_0 \; as \; t \to \infty \tag{3.3}$$

$$\tilde{x_0}(t) = x_0 + x_1(t)e^{-\epsilon t} + x_2(t)e^{-2\epsilon t} + x_3(t)e^{-3\epsilon t} + ... \tag{3.4}$$

We note that $\tilde{x_0}$(t) is the zero of the solution u of GBEs((2.1equation.2.1)) satisfying $\tilde{x_0}(t)$=0.In other words,$\tilde{x_0}(t)$ describes the movement of zero $x = 0$ of $u(x, t_0)$ as the initial profile $u(x, t_0)$ evolves under GBEs for time t $> t_0$. Using (3.3equation.3.3)-(3.4equation.3.4) in (2.1equation.2.1) and neglecting the leading order term,we get

$$U_{1,t} + \frac{jU_1}{2t} - \epsilon U_{1,xx} \sim e^{-2\epsilon t}\Big[A_1 t^{\frac{-j}{2}}(x_1' - \epsilon x_1)\cos\big(x - \tilde{x}_0(t)\big) - \frac{aA_1^2 t^{-j}}{2}\sin\big(2(x - \tilde{x}_0(t))\big)\Big], \quad \text{as } t \to \infty, \tag{3.5}$$

$$U_1(x,t) \sim e^{-2\epsilon t}\, c_1(t)\, \sin\big(2(x - \tilde{x}_0(t))\big), \quad \text{as } t \to \infty, \tag{3.6}$$

$$\left(2\epsilon + \frac{j}{2t}\right) c_1(t) + c_1'(t) \sim -\frac{aA_1^2 t^{-j}}{2}, \quad \text{as } t \to \infty. \tag{3.7}$$

This implies that

$$c_1(t) \sim -\frac{aA_1^2 t^{-j}}{2\left(2\epsilon + \frac{j}{2t}\right)} \; as \; t \to \infty \tag{3.8}$$

$$U_1(x,t) \sim -\frac{aA_1^2 t^{-j}}{2\left(2\epsilon + \frac{j}{2t}\right)} e^{-2\epsilon t} \sin\ 2(x - \tilde{x_0}(t))\ as\ t \to \infty \tag{3.9}$$

$$u(x,t) \sim A_1 t^{-j/2} e^{-\epsilon t} \sin(x - \tilde{x_0}(t)) + c_1(t) e^{-2\epsilon t} \sin\ 2(x - \tilde{x_0}(t))\ as\ t \to \infty. \tag{3.10}$$

Here c_1 is given by (3.8equation.3.8) The higher-order terms of the asymptotic expansion are obtained by considering the limit t $\to\infty$.
Let, y=x-$\tilde{x_0}$(t)
By applying the above transformation, the generalized Burgers equation (2.1equation.2.1) is reduced to the following partial differential equations

$$u_t + (-\tilde{x_0'}(t) + au + bu^2)u_y + \frac{ju}{2t} = \epsilon u_{yy} \tag{3.11}$$

The expression for u given in (3.10equation.3.10)suggests that u(y,t) be chosen in the following form,inspired by the asymptotic representation introduced in [9]:

$$u(y,t) = e^{-\epsilon t} f_1(y,t) + e^{-2\epsilon t} f_2(y,t) + e^{-3\epsilon t} f_3(y,t) + e^{-4\epsilon t} f_4(y,t) + ...as\ t \to \infty \tag{3.12}$$

Substituting the expression for u given by (3.12equation.3.12)into (3.11equation.3.11) and equating the coefficients of corresponding terms $e^{-\epsilon t}$,$e^{-2\epsilon t}$,$e^{-3\epsilon t}$,$e^{-4\epsilon t}$,we obtained

$$f_{1,t} + \left(\frac{j}{2t} - \epsilon\right) f_1 = \epsilon\, f_{1,yy},$$

$$f_{2,t} - \epsilon f_{2,yy} + \left(\frac{j}{2t} - 2\epsilon\right) f_2 = (x_1' - \epsilon x_1)\, f_{1,y} - a f_1 f_{1,y}.$$

$$\begin{aligned} f_{3,t} - 3\epsilon f_3 - \epsilon f_{3,yy} &= (x_1' - \epsilon x_1) f_{2,y} + f_{1,y}(x_2' - 2\epsilon x_2) \\ &\quad - a\big(f_2 f_{1,y} + f_1 f_{2,y}\big) - b f_1^2 f_{1,y} - \frac{j}{2t} f_3, \end{aligned} \tag{3.13}$$

$$\begin{aligned} f_{4,t} - 4\epsilon f_4 - \epsilon f_{4,yy} &= (x_1' - \epsilon x_1) f_{3,y} + (x_2' - 2\epsilon x_2) f_{2,y} + (x_3' - 3\epsilon x_3) f_{1,y} \\ &\quad - a\big(f_1 f_{3,y} + f_2 f_{2,y} + f_3 f_{1,y}\big) - b\big(f_1^2 f_{2,y} + 2 f_1 f_2 f_{1,y}\big) - \frac{j}{2t} f_4. \end{aligned} \tag{3.14}$$

From (3.10equation.3.10)

$$f_1(y,t) = A_1 t^{\frac{-j}{2}} \sin y, \tag{3.15}$$

$$f_2(y,t) = c_1 \sin(2y), \tag{3.16}$$

$$\text{where} \quad c_1 = -\frac{aA_1^2 t^{-j}}{2\left(2\epsilon + \frac{j}{2t}\right)}.$$

$$x_1 = 0. \tag{3.17}$$

$$\begin{aligned}\therefore f_{3,t} - 3\epsilon f_3 - \epsilon f_{3,yy} + \frac{j}{2t} f_3 = (x_2' - 2\epsilon x_2) A_1 t^{-j/2} \cos y \\ - a t^{-j/2} \left(\frac{3A_1 c_1}{2} \sin(3y) - \frac{A_1 c_1}{2} \sin y \right) \\ + \frac{bA_1^3 t^{-3j/2}}{4} \left(\cos(3y) - \cos y\right).\end{aligned} \tag{3.18}$$

From the form of the right-hand side of the above equation,we assume[7]

$$f_3(y,t) = B_1(t)\cos y + B_2(t)\cos(3y) + B_3(t)\sin y + B_4(t)\sin(3y). \tag{3.19}$$

From (3.18equation.3.18) and (3.19equation.3.19),

$$B_1' - 2\epsilon B_1 + \frac{j}{2t} B_1 = (x_2' - 2\epsilon x_2) A_1 t^{-j/2} - \frac{bA_1^3 t^{-3j/2}}{4}, \tag{3.20}$$

$$B_2' + 6\epsilon B_2 + \frac{j}{2t} B_2 = \frac{bA_1^3 t^{-3j/2}}{4}, \tag{3.21}$$

$$B_3' - 2\epsilon B_3 + \frac{j}{2t} B_3 = \frac{aA_1 c_1 t^{-j/2}}{2} \tag{3.22}$$

$$B_4' + 6\epsilon B_4 + \frac{j}{2t} B_4 = -\frac{3aA_1 c_1 t^{-j/2}}{2}. \tag{3.23}$$

The condition $f_3(0,t) = 0$ gives
$B_2(t) = -B_1(t)$.
Solving (3.20equation.3.20)-(3.23equation.3.23),

$$B_2(t) = -B_1(t) = \frac{bA_1^3 t^{-3j/2}}{24\epsilon + \frac{2j}{t}} \tag{3.24}$$

$$B_3(t) = B_4(t) = -\frac{aA_1 c_1 t^{-j/2}}{2\left(2\epsilon + \frac{j}{2t}\right)} \tag{3.25}$$

Equating the coefficients of corresponding terms on both sides of the equation yields the following ordinary differential equation for x_2:
$(x_2' - 2\epsilon x_2)A_1 = 0 \implies x_2' - 2\epsilon x_2 = 0$

$\implies x_2(t)=0$

$$f_3(y,t)=\frac{bA_1^3t^{-3j/2}}{\left(24\epsilon+\frac{2j}{t}\right)}(-\cos y+\cos 3y)-\frac{aA_1c_1t^{-j/2}}{2\left(2\epsilon+\frac{j}{2t}\right)}(\sin y+\sin 3y) \tag{3.26}$$

$$f_{4,t}-4\epsilon f_4-\epsilon f_{4,yy}=(x_3'-3\epsilon x_3)A_1t^{-j/2}\cos y-\frac{2A_1^4t^{-2j}b}{\left(24\epsilon+\frac{2j}{t}\right)}\cos 4y+\frac{aA_1^4t^{-2j}b}{\left(24\epsilon+\frac{2j}{t}\right)}\cos 2y$$
$$+\frac{3a^2A_1^4t^{-2j}}{8\left(2\epsilon+\frac{2j}{t}\right)^2}\sin 2y+\frac{3a^2A_1^4t^{-2j}}{8\left(2\epsilon+\frac{2j}{t}\right)^2}\cos 2y+\frac{a^2A_1^4t^{-2j}}{2\left(2\epsilon+\frac{2j}{t}\right)^2}\sin 4y$$
$$-2bc_1A_1^2t^{-j}\cos 2y+bc_1A_1^2t^{-j}\cos 4y-\frac{j}{2t}f_4 \tag{3.27}$$

$$\text{Let},f_4(y,t)=R_1(t)\cos y+R_2(t)\cos 2y+R_3(t)\sin 2y+R_4(t)\cos 4y+R_5(t)\sin 4y \tag{3.28}$$

Comparing both sides of above two equations,we get

$$R_1'(t)-3\epsilon R_1(t)=A_1t^{-j/2}(x_3'-3\epsilon x_3)-\frac{j}{2t} \tag{3.29}$$

$$R_2'(t)=\frac{A_1^4bt^{-2j}}{\left(24\epsilon+\frac{2j}{t}\right)}+\frac{3a^2A_1^4t^{-2j}}{8\left(2\epsilon+\frac{2j}{t}\right)^2}-2bc_1A_1^2t^{-j}-\frac{j}{2t} \tag{3.30}$$

$$R_3'(t)=\frac{3a^2A_1^4t^{-2j}}{8\left(2\epsilon+\frac{2j}{t}\right)^2}-\frac{j}{2t} \tag{3.31}$$

$$R_4'(t)+12\epsilon R_4(t)=bc_1A_1^2t^{-j}-\frac{2A_1^4bt^{-2j}}{\left(24\epsilon+\frac{2j}{t}\right)}-\frac{j}{2t} \tag{3.32}$$

$$R_5'(t)+12\epsilon R_5(t)=\frac{a^2A_1^4t^{-2j}}{2\left(2\epsilon+\frac{2j}{2t}\right)^2}-\frac{j}{2t} \tag{3.33}$$

From (3.30equation.3.30)-(3.33equation.3.33),we get

$$R_2(t) = -4A_1^4 bjt^{-2j}\left(\frac{1}{(24\epsilon)^2}\log\left(24\epsilon+\frac{2j}{t}\right)\right)+\frac{1}{48\epsilon}\left(\frac{1}{24\epsilon+\frac{2j}{t}}-\frac{1}{2\left(24\epsilon+\frac{2j}{t}\right)^2}\right)$$
$$-\frac{3a^2A_1^4t^{-2j}j}{2}\left(\frac{1}{(24\epsilon)^2\left(24\epsilon+\frac{2j}{t}\right)^2}-\frac{1}{48\epsilon\left(24\epsilon+\frac{2j}{t}\right)^3}+\frac{1}{\left(24\epsilon+\frac{2j}{t}\right)^4}\right)$$
$$-2bc_1A_1^2tt^{-j}+\frac{j}{4t^2} \tag{3.34}$$

$$R_3(t) = -\frac{3a^2A_1^4jt^{-2j}}{2}\left(\frac{1}{(24\epsilon)^2\left(24\epsilon+\frac{2j}{t}\right)^2}-\frac{1}{48\epsilon\left(24\epsilon+\frac{2j}{t}\right)^3}+\frac{1}{\left(24\epsilon+\frac{2j}{t}\right)^4}\right)$$
$$+\frac{j}{4t^2} \tag{3.35}$$

$$R_4(t) = \frac{bc_1A_1^2t^{-j}}{12\epsilon}-\frac{8A_1^4bjt^{-2j}}{12\epsilon}\left(\frac{1}{(24\epsilon)^2}\log\left(24\epsilon+\frac{2j}{t}\right)+\frac{1}{48\epsilon\left(24\epsilon+\frac{2j}{t}\right)}\right. \tag{3.36}$$
$$\left.-\frac{1}{2\left(24\epsilon+\frac{2j}{t}\right)^2}\right)-\frac{j}{48t^2\epsilon}$$

$$R_5(t) = \frac{4a^2A_1^4t^{-2j}}{24\epsilon}\left(\frac{1}{(24\epsilon)^2\left(24\epsilon+\frac{2j}{t}\right)^3}+\frac{1}{\left(24\epsilon+\frac{2j}{t}\right)^4}\right)+\frac{j}{4t^2}. \tag{3.37}$$

We require that $f_4(0,t)=0$

$$R_1(t) = 2bc_1A_1^2tt^{-j}-\frac{bc_1A_1^2t^{-j}}{12\epsilon} \tag{3.38}$$

$$
\begin{aligned}
f_4(y,t) = {} & 2bc_1A_1^2tt^{-j} - \frac{bc_1A_1^2t^{-j}}{12\epsilon}\cos y \\
& + \Bigg[- 4bA_1^4jt^{-2j}\Bigg\{ \frac{1}{(24\epsilon)^2}\log\left(24\epsilon + \frac{2j}{t}\right) \\
& \qquad + \frac{1}{48\epsilon}\left(\frac{1}{24\epsilon + \frac{2j}{t}} - \frac{1}{2\left(24\epsilon + \frac{2j}{t}\right)^2}\right)\Bigg\} \\
& \quad - \frac{3a^2A_1^4jt^{-2j}}{2}\Bigg\{ \frac{1}{(24\epsilon)^2\left(24\epsilon + \frac{2j}{t}\right)^2} \\
& \qquad - \frac{1}{48\epsilon\left(24\epsilon + \frac{2j}{t}\right)^3} + \frac{1}{\left(24\epsilon + \frac{2j}{t}\right)^4}\Bigg\} \\
& \quad - 2bc_1A_1^2tt^{-j} - \frac{j}{4t^2}\Bigg]\cos 2y \\
& + \Bigg[- \frac{3a^2A_1^4jt^{-2j}}{2}\Bigg\{ \frac{1}{(24\epsilon)^2\left(24\epsilon + \frac{2j}{t}\right)^2} \\
& \qquad - \frac{1}{48\epsilon\left(24\epsilon + \frac{2j}{t}\right)^3} + \frac{1}{\left(24\epsilon + \frac{2j}{t}\right)^4}\Bigg\} - \frac{j}{4t^2}\Bigg]\sin 2y \\
& + \Bigg[\frac{bc_1A_1^2t^{-j}}{12\epsilon} - \frac{8bA_1^4jt^{-2j}}{12\epsilon}\Bigg\{ \frac{1}{(24\epsilon)^2}\log\left(24\epsilon + \frac{2j}{t}\right) \\
& \qquad + \frac{1}{48\epsilon\left(24\epsilon + \frac{2j}{t}\right)} - \frac{1}{2\left(24\epsilon + \frac{2j}{t}\right)^2}\Bigg\} - \frac{j}{48\epsilon t^2}\Bigg]\cos 4y \\
& + \Bigg[\frac{4a^2A_1^4t^{-2j}}{24\epsilon}\Bigg\{ \frac{1}{(24\epsilon)^2\left(24\epsilon + \frac{2j}{t}\right)^3} + \frac{1}{\left(24\epsilon + \frac{2j}{t}\right)^4}\Bigg\} \\
& \quad - \frac{j}{4t^2}\Bigg]\sin 4y. \qquad (3.39)
\end{aligned}
$$

Using (3.29equation.3.29) and (3.39equation.3.39),we get

$$x_3' - 3\epsilon x_3 = bc_1A_1t^{-j/2}(2t - \frac{1}{12\epsilon})$$

$$\Longrightarrow$$

$$x_3(t) = -\frac{bc_1A_1t^{-j/2}}{3\epsilon}\left(2t - \frac{1}{12\epsilon}\right) \qquad (3.40)$$

$$u(x,t) = e^{-\epsilon t}f_1(y,t) + e^{-2\epsilon t}f_2(y,t) + e^{-3\epsilon t}f_3(y,t) + e^{-4\epsilon t}f_4(y,t) + \cdots \qquad (3.41)$$

$$\tilde{x_0}(t) = x_0 + x_1(t)e^{-\epsilon t} + x_2(t)e^{-2\epsilon t} + x_3(t)e^{-3\epsilon t} + O(e^{-4\epsilon t}) \quad (3.42)$$

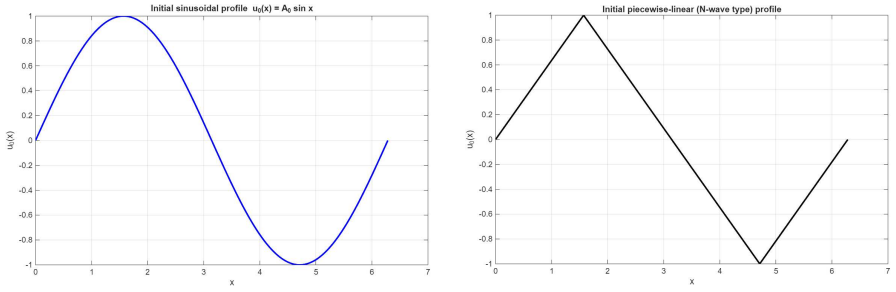
where, f_i, i= 1, 2, 3, 4 *and* x_i, i= 1, 2, 3 are given above.

4. Numerical Study

This section presents a comparison between the large-time asymptotic solutions and the corresponding numerical solutions of the generalized Burgers equation(2.1equation.2.1) under the prescribed initial and periodic boundary conditions. The numerical computations are carried out using the finite difference scheme developed by Dawson [13]. The results demonstrate the accuracy of the asymptotic solutions in describing the long-time behaviour of the equation.Motivated from [9],the numerical simulations are performed on the spatial domain [0,2π],which is discretized using 2000 uniform spatial points with spatial step size δx=2π/2000 and time step is chosen as δt=10^{-4}.
The constants A_1 and x_0 are obtained by matching the numerical solution of the generalized Burgers equation with the solutions of associated linear equation so that the maximum-norm error remains of order $O(10^{-4})$.We consider two different initial profiles,as proposed in[9]

$$u_0(x) = A_0 \sin x. \quad (4.1)$$

$$u_0(x) = \begin{cases} \dfrac{2x}{\pi}, & x \in \left[0, \dfrac{\pi}{2}\right], \\ -\dfrac{2x}{\pi} + 2, & x \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right], \\ \dfrac{2x}{\pi} - 4, & x \in \left[\dfrac{3\pi}{2}, 2\pi\right]. \end{cases} \quad (4.2)$$



(a) (b)

FIGURE 1. Initial profiles at $t = 1$.

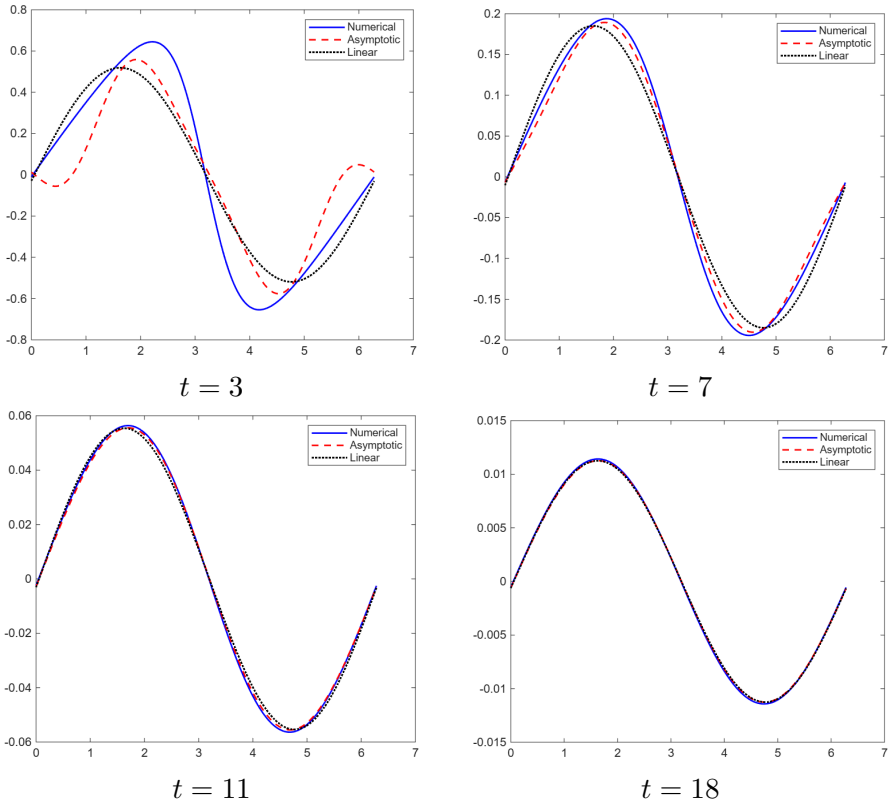
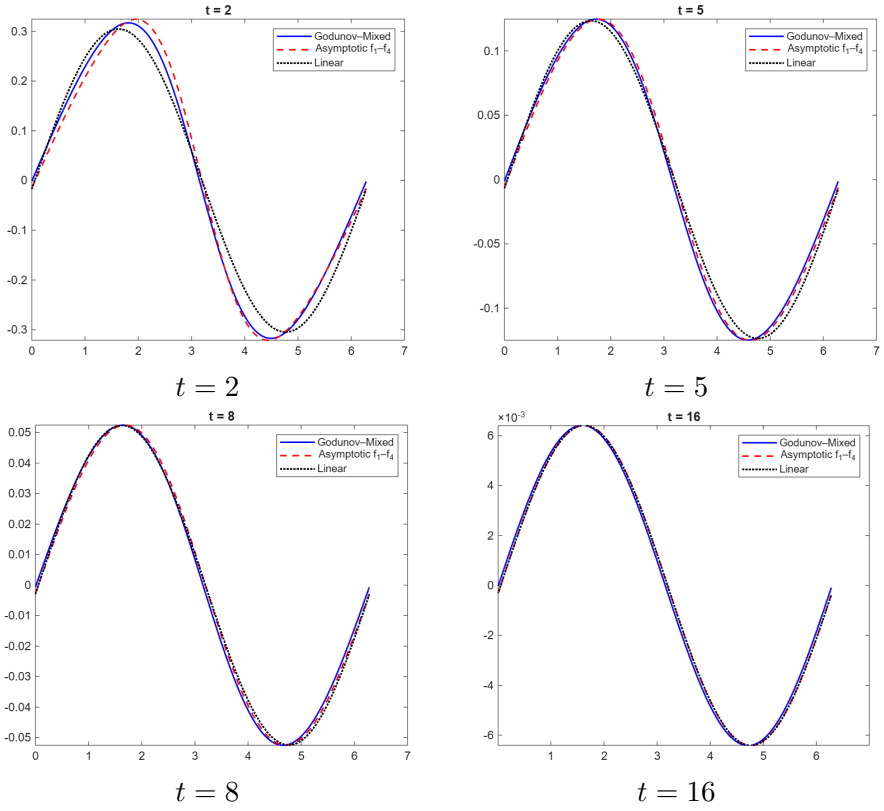


FIGURE 2. Comparison of the numerical solution(u_{num}), the higher-order large-time asymptotic solution(u_{asy}), and the leading-order linear solution (u_{lin}) for the generalized Burgers equation (2.1equation.2.1) with the initial condition $u_0(x)$=$2sinx$ (4.1equation.4.1). The higher-order asymptotic solution is obtained from equation (3.41equation.3.41),whereas the leading-order linear solution corresponds to the associated linear equation (3.2equation.3.2)with solution (3.1equation.3.1). The computations are carried out using the parameter values.Here the constant A_1=0.9756,x_0=0.0552,ϵ=0.25, a=0.7, b=0.07, j=0.1 and t=3,t=7,t=11,t=18

.



$t = 2$ $t = 5$

$t = 8$ $t = 16$

FIGURE 3. Comparison of the numerical solution(u_{num}), the higher-order large-time asymptotic solution(u_{asy}), and the leading-order linear solution (u_{lin}) for the generalized Burgers equation (2.1equation.2.1) with piecewise-linear initial profile (4.2equation.4.2). The higher-order asymptotic solution is obtained from equation (3.41equation.3.41),whereas the leading-order linear solution corresponds to the associated linear equation (3.2equation.3.2)with solution (3.1equation.3.1).Here the constant A_1 =0.4757, x_0=0.0552, $\epsilon = 0.25$, $a = 0.6$, $b = 0.09$, $j = 0.15$ and $t = 2$, $t = 5$, $t = 8$, $t = 16$

.

TABLE 1. Maximum norm errors $\|u_{num} - u_{asy}\|_\infty$ and $\|u_{num} - u_{lin}\|_\infty$ at the different times where, (u_{num}) denotes the numerical solution of((2.1equation.2.1))with initial profile $u_0(x)$=$2sinx$ (4.1equation.4.1),(u_{asy}) represents the large time asymptotic solution obtained from (3.41equation.3.41) and(u_{lin}) denotes the leading order solution of associated linear equation (3.1equation.3.1)given by (3.2equation.3.2) at several times when $\epsilon = 0.25$, $j = 0.1$.

t	$\|u_{num} - u_{asy}\|_\infty$	$\|u_{num} - u_{lin}\|_\infty$
1	0.7740	0.5177
2	0.3624	0.3424
3	0.1787	0.1975
4	0.0861	0.1124
5	0.0419	0.0647
6	0.0209	0.0378
7	0.0108	0.0224
8	0.0058	0.0135
9	0.0033	0.0083
10	0.0020	0.0052
11	0.0013	0.0033
12	0.0009	0.0021
13	0.0007	0.0014
14	0.0005	0.0009
15	0.0004	0.0006
16	0.0003	0.0004
17	0.0002	0.0003
18	0.0002	0.0002

TABLE 2. Maximum norm errors $\|u_{num} - u_{asy}\|_\infty$ and $\|u_{num} - u_{lin}\|_\infty$ at the different times where, (u_{num}) denotes the numerical solution of ((2.1equation.2.1)) with piecewise-linear initial profile (4.2equation.4.2), (u_{asy}) represents the large time asymptotic solution obtained from (3.41equation.3.41) and(u_{lin}) denotes the leading order solution of associated linear equation (3.1equation.3.1)given by (3.2equation.3.2) at several times when $\epsilon = 0.25$, $j = 0.15$.

t	$\|u_{num} - u_{asy}\|_\infty$	$\|u_{num} - u_{lin}\|_\infty$
1	0.0594	0.0618
2	0.0239	0.0495
3	0.0132	0.0330
4	0.0090	0.0212
5	0.0064	0.0135
6	0.0047	0.0087
7	0.0035	0.0057
8	0.0026	0.0039
9	0.0020	0.0027
10	0.0015	0.0019
11	0.0012	0.0013
12	0.0009	0.0010
13	0.0007	0.0007
14	0.0005	0.0006
15	0.0004	0.0004
16	0.0003	0.0003

To examine the sensitivity of the results to the spatial resolution, the computations for the initial profile $u_0(x) = 2sinx$ (4.1equation.4.1) are repeated on three successively refined grids $Nx = 1000, 2000, 3000$. The resulting maximum-norm errors are reported in Table 3table.3 and Table 4table.4 for the asymptotic solution(3.41equation.3.41) and linear solution(3.1equation.3.1) respectively.

TABLE 3. Maximum norm error $\|u_{num} - u_{asy}\|_\infty$ at different times for the initial profile $u_0(x) = 2\sin x$ (4.1equation.4.1), computed on three spatial grids $N_x = 1000$, 2000 and 3000 (with $\delta x = 2\pi/N_x$, $\delta t = 10^{-4}$), where u_{num} denotes the numerical solution of (2.1equation.2.1) with initial profile (4.1equation.4.1) and u_{asy} represents the large-time asymptotic solution obtained from (3.41equation.3.41), when $\epsilon = 0.25$, $j = 0.1$.

t	$N_x = 1000$	$N_x = 2000$	$N_x = 3000$
1	0.7829	0.7740	0.7694
2	0.3832	0.3624	0.3501
3	0.1900	0.1787	0.1720
4	0.0927	0.0861	0.0823
5	0.0460	0.0419	0.0395
6	0.0236	0.0209	0.0193
7	0.0127	0.0108	0.0097
8	0.0072	0.0058	0.0051
9	0.0044	0.0033	0.0028
10	0.0029	0.0020	0.0016
11	0.0020	0.0013	0.0010
12	0.0015	0.0009	0.0007
13	0.0011	0.0007	0.0005
14	0.0008	0.0005	0.0003
15	0.0007	0.0004	0.0002
16	0.0005	0.0003	0.0002
17	0.0004	0.0002	0.0001
18	0.0003	0.0002	0.0001

TABLE 4. Maximum norm error $\|u_{num} - u_{lin}\|_\infty$ at different times for the initial profile $u_0(x) = 2\sin x$ (4.1equation.4.1), computed on three spatial grids $N_x = 1000$, 2000 and 3000 (with $\delta x = 2\pi/N_x$, $\delta t = 10^{-4}$), where u_{lin} denotes the leading-order solution of the associated linear equation (3.1equation.3.1) given by (3.2equation.3.2), when $\epsilon = 0.25$, $j = 0.1$.

t	$N_x = 1000$	$N_x = 2000$	$N_x = 3000$
1	0.5568	0.5177	0.4952
2	0.3642	0.3424	0.3295
3	0.2097	0.1975	0.1902
4	0.1198	0.1124	0.1081
5	0.0694	0.0647	0.0619
6	0.0410	0.0378	0.0359
7	0.0246	0.0224	0.0211
8	0.0151	0.0135	0.0126
9	0.0095	0.0083	0.0076
10	0.0060	0.0052	0.0046
11	0.0039	0.0033	0.0029
12	0.0026	0.0021	0.0018
13	0.0018	0.0014	0.0012
14	0.0012	0.0009	0.0008
15	0.0009	0.0006	0.0005
16	0.0006	0.0004	0.0003
17	0.0005	0.0003	0.0002
18	0.0003	0.0002	0.0002

5. Results of the Grid-Refinement Comparison

Table 3table.3 and Table 4table.4 present the maximum norm errors $\|u_{num}-u_{asy}\|_\infty$ and $\|u_{num}-u_{lin}\|_\infty$ for the initial profile $u_0(x) = 2\sin x$ (4.1equation.4.1) on the three spatial grids $N_x = 1000$, 2000 and 3000, with $\epsilon = 0.25$, $j = 0.1$ and $\delta t = 10^{-4}$. The following observations summarize the results of the comparison.

(i)Sensitivity to the spatial grid. At the sampled times, the errors relative to both the asymptotic approximation and the linear approximation decrease as the spatial grid is refined from $N_x = 1000$ to $N_x = 3000$. The decrease of these errors with increasing time and increasing spatial resolution is observed. However, these comparisons alone do not establish grid independence, since the reported errors include both numerical discretization error and the discrepancy of the analytical approximations.

(ii) Decay in time. On each grid, both errors decrease monotonically and rapidly with t: the asymptotic error $\|u_{num} - u_{asy}\|_\infty$ drops from $O(10^{-1})$ at $t = 1$ to $O(10^{-4})$ by $t = 18$, consistent with the $e^{-\epsilon t}$ decay of the periodic solution and the approach of the solution to its asymptotic regime.

(iii) Asymptotic versus leading-order solution. At early times the leading-order linear solution is slightly more accurate; however, a crossover occurs at $t \approx 3$, after which the higher-order asymptotic solution is uniformly more accurate, and its advantage grows with time.This demonstrates that the higher-order corrections f_2, f_3, f_4 in the expansion (3.41equation.3.41) yield a substantial improvement over the leading-order linear solution precisely in the large-time regime the paper is concerned with.

6. Applications of the GBEs

The GBEs are used as a tool in various branches of applied fields where nonlinear convection and diffusion exist simultaneously. In fluid dynamics, the dependent variable represents velocity, where nonlinear convection leads to wave steepening and shock formation while diffusion smooths out discontinuities.

In traffic flow, the same equation governs the evolution of vehicle density or velocity, where nonlinear interactions between vehicles give rise to stop-and-go waves that propagate along highways. The diffusion term captures driver response and adaptation while the time-dependent term introduces long-time modulation in the system. The lo verall study represents the decay, stabilization and long-time behavior of traffic congestive wave.

In nonlinear acoustics, the equation represents finite amplitude sound waves where nonlinearity causes wave steepening and shock formation, the diffusion term represents viscous and thermal dissipation. The time-dependent term introduces long-time modulation of acoustic waves to study the shock structure and long-distance sound propagation.
These are some well-known direct applications of GBEs. Apart from that, the equation is useful in analyzing long-time asymptotic behaviour and stability of solutions in presence of time dependent effects. In finance mathematics, it is used as a tool to study the complex market behaviours and nonlinear price evolution, long-time stabilization effects. In biological field, they are used to study population dynamics, neural activity and biological fluids propagation.

7. Conclusions

In this paper, we have investigated the large-time asymptotic behaviour of periodic solutions of a generalized Burgers equation given by ((2.1equation.2.1)) under periodic boundary conditions, using a perturbative approach. The asymptotic expansion also enables us to determine the time-dependent shift of the zero $\tilde{x_0}(t)$ of the solution, and explicit formulas for the coefficients have been obtained. These results extend earlier works on classical and modified Burgers equations by providing a more general framework that incorporates both nonlinear convection effects and the contribution of non-planar term. We also have solved generalized Burgers equation ((2.1equation.2.1)) numerically with initial boundary conditions using a mixed-Godunov method[13]. The numerical simulations demonstrate excellent agreement with the derived asymptotic approximation in the large-time regime. Furthermore, the computational results validate the accuracy of the asymptotic solution (3.41equation.3.41) for the generalized Burgers equation (2.1equation.2.1) for two different initial profiles ((4.1equation.4.1)) for the set a=0.7, b=0.07, ϵ=0.25, j=0.1 and ((4.2equation.4.2)) for a=0.6, b=0.09, ϵ=0.25, j=0.15 respectively along with error tables. There is a further scope for exploration of composite N-wave and periodic solutions for higher order or multidimensional generalized Burgers equations. The current work addresses non-planar Burgers equations but with relatively simple or periodic boundary conditions. Studies involving variable boundary conditions are scarce.
Funding:
This study did not receive any funding.
Data availability statement :

All data generated or analysed during this study are included in this published article.

Declaration:

Not Applicable

8. References

[1] H. Bateman,Some recent researches on the motion of fluids, *Mon. Weather Rev.*43:163-170(1915).

[2] E. Hopf,The partial differential equation

$$u_t + uu_x = \mu u_{xx}, \textit{Commun. Pure Appl.Math.}3 : 201 - 230(1950).$$

[3] J. D. Cole,On a quasi-linear parabolic equation occurring in aerodynamics, *Quart.Appl.Math.*9:225-236(1951).

[4] P. L. Sachdev, Ch. S. Rao and K. T. Joseph, Analytic and numerical study of N-waves governed by the nonplanar Burgers equation, *Stud. Appl. Math.* 103:89-120 (1999).

[5] Ch. S. Rao and E. Satyanarayana, Asymptotic N-wave solutions of the nonplanar Burgers equation, *Stud. Appl. Math.* 121:199–221 (2008).

[6] P. L. Sachdev, Ch. S. Rao and B. O. Enflo, Large-time asymptotics for periodic solutions of the modified Burgers equation, *Stud. Appl. Math.* 114:307–323 (2005).

[7] Ch. S. Rao and S. Nath, A Study of Separable Solutions of a Generalized Burgers Equation, *Stud. Appl. Math.*134:403-419(2015).

[8] S. Nath.Large time asymptotics to solutions of some generalized Burgers equations,*Ph.D thesis,IIT Madras,India,*2016

[9] S. Nath and Ch. S. Rao, Asymptotic periodic solutions of some generalized Burgers equations, *Appl. Math. J. Chinese Univ.* 33(4): 390-408(2018)

[10] P. Samanta and Ch. S. Rao,Large-Time Asymptotic to Solutions of a Generalized Burgers Equation With Linear Damping on Half-Line, *Q.Jl Mech.Appl.Math* 75(3):275-299(2022).

[11] P. Samanta and S. R. Chidella,Exact and Asymptotic Solutions of Burgers' Equation On Half-line, *Q.Jl Mech.Appl.Math*78(1):1-30(2025).

[12] P. Samanta and R. K. Paswan,Large-Time Asymptotic Behavior of the Generalized Burgers Equation on the Infinite Line, *J.Multi.Discip.Eng.Technol.*19(01):84-95(2025).

[13] C. N. Dawson,Godunov-mixed methods for advective flow problems in one space dimension,SIAM J Numer Anal,1991,28(5): 1282-1309.

Pallabi Kundu
Research Scholar
Techno India University
Kolkata-700091

e-mail: pallabi.kundu020@gmail.com

Dr. Smriti Nath(Corresponding Author)
Associate Professor
Techno India University
Kolkata-700091
e-mail: smriti.n@technoindiaeducation.com