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Research Article

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Finite endpoint correction to the Forgacs and Mason fiber buckling model

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Abstract: We revisit the hinged-end model of shear-induced fiber buckling introduced by Forgacs and Mason in 1959. The Gaussian solution selected in their analysis satisfies the governing differential equation but fails the prescribed displacement conditions at the finite endpoints. Solving the boundary-value problem with Kummer's confluent hypergeometric function increases the predicted critical shear stress by a factor of approximately 2.83 when the original hydrodynamic force law is retained. The bending modulus inferred from a measured buckling threshold is consequently reduced by the same factor. Combining this correction with a later drag-law correction substantially reduces the discrepancy between inferred bending and measured tensile moduli for Dacron and humidity-adjusted Nylon, although the rayon data remain sensitive to the suspending medium. Over the experimental aspect-ratio range, the corrected threshold is close to a published prediction for freely suspended fibers when the respective drag approximations are retained. This proximity supports the corrected model as a useful reference calculation, while establishing neither physical equivalence to free-fiber models nor a bound on their buckling thresholds.

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1. Introduction

Forgacs and Mason (1959) investigated the onset of bending in fibers suspended in viscous shear flow. Their analysis, together with the later account by Goldsmith and Mason (1967), remains a historical reference for studies of flexible fibers in fluids. Subsequent work has examined fiber motion and deformation using immersed-boundary methods, slender-body hydrodynamics, and particle-based simulations (Stockie and Green 1998; Tornberg and Shelley 2004; Słowicka et al. 2013). Broader accounts cover elastic fibers in flow and related biological

systems (Lindner and Shelley 2015; Shelley 2016; du Roure et al. 2019). More recent studies address compressional buckling and three-dimensional stability in shear flow (Słowicka et al. 2022; Liu et al. 2024).

These studies differ in their loading, dynamics, and boundary conditions. Here we address a specific inconsistency within the reduced model of Forgacs and Mason (1959): the selected solution satisfies the differential equation but not the hinged-end displacement conditions of the stated problem. We solve the finite-interval problem, reassess the bending moduli inferred from the original experiments, and compare the corrected threshold with the free-fiber prediction of Becker and Shelley (2001).

2. The original model and its boundary mismatch

The model considers a slender circular rod with half-length a, radius b, and aspect ratio $r = a/b$. Let $\tau = \eta\dot{\gamma}$ denote the positive magnitude of the imposed shear stress, where η is the fluid viscosity and $\dot{\gamma}$ is the shear-rate magnitude. At the orientation of maximum compression, the axial compressive-force distribution assumed by Forgacs and Mason (1959, Eqs. 14 and 15) is:

$$P(x) = \frac{p}{2}\left(a^2 - x^2\right), \; p = \frac{\pi\tau}{\ln 2r - 1.75} \tag{1}$$

where x is measured along the rod centerline from its midpoint, and p is the coefficient specifying the axial loading.

The compressive force is largest at the midpoint and vanishes at $x = \pm a$. Forgacs and Mason (1959) use the bending relation $E_b I \dfrac{d^2 w}{dx^2} + P(x) w = 0$, where w is the transverse displacement and $E_b I$ is the bending stiffness ($I = \pi b^4/4$). Substitution of Eq. (1) gives their Eq. (17):

$$\frac{d^2 w}{dx^2} + c\left(a^2 - x^2\right) w = 0 \tag{2}$$

where $c = \dfrac{p}{2E_b I}$ has dimensions of inverse length to the fourth power. Forgacs and Mason (1959) model the rod as hinged at both ends, requiring $w(a) = w(-a) = 0$.

Introducing $z = c^{1/4}x$ transforms Eq. (2) into:

$$\frac{d^2 w}{dz^2} + \left(\lambda - z^2\right) w = 0 \tag{3}$$

where $\lambda = c^{1/2}a^2$. The substitution $w = ye^{-\frac{z^2}{2}}$ then gives:

$$\frac{d^2 y}{dz^2} - 2z\frac{dy}{dz} + (\lambda - 1)\, y = 0 \tag{4}$$

When $\lambda = 2n + 1$ for a non-negative integer n, Eq. (4) admits the polynomial solution $y = H_n(z)$, in the physicists' Hermite convention (Abramowitz and Stegun 1964). For $n = 0$, $H_0 = 1$, and the deflection is $w(z) = w(0)e^{-\frac{z^2}{2}}$. The amplitude $w(0)$ of the deflection cannot be determined by the linear buckling analysis (Timoshenko and Gere 1961, p.48). In the physical coordinate x, the deflection is:

$$w(x) = w(0)e^{-\frac{\sqrt{c}x^2}{2}} \tag{5}$$

This solution, corresponding to $\lambda = 1$, gives the following critical shear stress for buckling:

$$\tau_{FM} = \frac{E_b\left(\ln 2r - 1.75\right)}{2r^4} \tag{6}$$

Although Eq. (5) satisfies Eq. (2), it fails the hinged-end displacement conditions because $w(\pm a) = e^{-1/2}w(0) \approx 0.6065w(0)$.

3. Solution on the finite interval

Equation (3) is invariant under reflection of z, and the endpoint conditions are $w\left(\sqrt{\lambda}\right) = w\left(-\sqrt{\lambda}\right) = 0$. In the original coordinate, Eq. (2) is a weighted eigenvalue problem with a weight that is positive in the interior. Its lowest eigenfunction has no interior node and can be

chosen positive; symmetry then makes it even. The even solution can be expressed using Kummer's confluent hypergeometric function (Abramowitz and Stegun 1964):

$$w(z) = w(0)e^{-\frac{z^2}{2}}\,{}_1F_1\left(\frac{1-\lambda}{4};\frac{1}{2};z^2\right) \tag{7}$$

The endpoint conditions ($w\left(\sqrt{\lambda}\right) = w\left(-\sqrt{\lambda}\right) = 0$) require:

$${}_1F_1\left(\frac{1-\lambda}{4};\frac{1}{2};\lambda\right) = 0 \tag{8}$$

Numerical calculation yields the smallest positive root $\lambda \approx 1.6816$, giving the following critical shear stress for buckling:

$$\tau_{crit} = \frac{2.8278E_b\left(\ln 2\mathrm{r} - 1.75\right)}{2r^4} \tag{9}$$

4. Discussion and conclusions

The expression for the coefficient p used by Forgacs and Mason (1959) comes from Burgers' (1938) approximate theory for rigid rods. Goldsmith and Mason (1967, p. 138) derive a revised expression for p , which, in the notation used here, is as follows:

$$p = \frac{\pi\tau}{\ln 2\mathrm{r} - 1.5} \tag{10}$$

Jiang (2023) uses an equivalent-inclusion approach to investigate the stress field in an elastic prolate object in viscous flow and obtains a refined longitudinal-stress expression (Jiang 2023, p. 337, Eq. 14.36). Evaluated at maximum compression in simple shear flow and expressed in the present notation, the expression is:

$$p = \frac{\pi\tau A(r)}{\ln 2\mathrm{r} - 1.5} \tag{11}$$

where $A(r) = 1 + \left(1 - \ln 2\mathrm{r}\right)/\mathrm{r}^2$. The departure of $A(r)$ from unity is less than 0.05% for r > 100 and less than 0.017% over the experimental aspect-ratio range. We therefore use Eq. (10) in the following comparisons.

Combining the finite-endpoint correction with the revised loading coefficient gives the buckling threshold:

$$\tau_{correct} = \frac{2.828 E_b \left(\ln 2\mathrm{r} - 1.5 \right)}{2r^4} \tag{12}$$

We use this relation to reexamine the experimental data of Forgacs and Mason (1959), which they interpreted using Eq. (6).

The bending moduli inferred from a measured critical shear stress using the original and corrected relations are, respectively:

$$E_{b,\,\mathrm{FM}} = \frac{2r^4 \tau_{crit}}{\ln 2\mathrm{r} - 1.75} \tag{13a}$$

$$E_{b,\,\mathrm{corrected}} = \frac{2r^4 \tau_{crit}}{2.828\left(\ln 2\mathrm{r} - 1.5 \right)} \tag{13b}$$

Forgacs and Mason (1959, Table IV, p. 469) compared inferred bending moduli with independently measured tensile moduli. They reported a bending-to-tensile modulus ratio of approximately 4 for Dacron and, after a humidity adjustment, approximately 3.2 for Nylon. Table 1 applies the correction $E_{b,\,\mathrm{corrected}} = \frac{\ln 2\mathrm{r} - 1.75}{2.8278\left(\ln 2\mathrm{r} - 1.5 \right)} E_{b,\,\mathrm{FM}}$ to their tabulated bending moduli.

For Dacron, the arithmetic mean of the four corrected ratios is 1.332, with individual values from 1.173 to 1.479. For Nylon, the two corrected ratios average 0.794 before humidity adjustment. Applying the original authors' proposed 20% increase in bending modulus gives ratios of 0.990 and 0.916, with a mean of 0.953. The historical ratio of 3.2 is quoted as reported; the original tabulated Nylon values give a mean ratio of 2.381, or 2.857 after the same humidity adjustment. Our recalculation uses the tabulated values.

For rayon, the corrected ratios are 0.751 in castor oil and 0.197 and 0.118 in corn syrup. The dependence on suspending medium remains consistent with the original discussion of water-induced plasticization. The correction brings the Dacron and humidity-adjusted Nylon ratios closer to unity, but it moves the corn-syrup rayon ratios farther from unity. These arithmetic means weight each tabulated entry equally and provide descriptive comparisons, not fitted

estimates or statistical tests. The results do not establish agreement for every material or exclude specimen heterogeneity, differences in testing conditions, or limitations of the reduced model.

Table 1: Bending-to-tensile modulus ratios before and after the eigenvalue and drag-law corrections. Data from Forgacs and Mason (1959, Table IV, p. 469); corrected values calculated here.

Material	**Medium**	r	$E_{b,\,\mathrm{FM}}/E$	$E_{b,\,\mathrm{corrected}}/E$
Dacron	Corn syrup	180	4.435	1.479
Dacron	Corn syrup	228	4.380	1.465
Dacron	Corn syrup	268	3.499	1.173
Dacron	Corn syrup	310	3.609	1.211
Rayon	Corn syrup	190	0.591	0.197
Rayon	Corn syrup	281	0.353	0.118
Rayon	Castor oil	241	2.242	0.751
Nylon	Corn syrup	170	2.476	0.825
Nylon	Corn syrup	194	2.286	0.763

The aspect ratio $r = a/b$ is the rod half-length divided by its radius. E denotes the measured tensile modulus. Values are rounded to three decimal places; the Nylon entries do not include the humidity adjustment.

Freely suspended fiber has no physical hinges. Free-fiber models impose zero endpoint force and moment and couple bending to hydrodynamic drag and fiber motion (Becker and Shelley 2001; Tornberg and Shelley 2004). Becker and Shelley (2001) obtain a minimum free-fiber instability threshold Zc ≈ 153.2 at maximum compression. In the present notation, $Z = \dfrac{128\tau r^4}{E_b \ln 2r}$, giving:

$$\tau_{BS} = \frac{153.2 E_b \ln 2r}{128 r^4} \tag{14}$$

Becker and Shelley (2001) state that the earlier treatment underestimates the required flow strength by about 2.4. Our conversion gives $\tau_{BS}/\tau_{FM} \approx 2.39\left(\dfrac{\ln 2r}{\ln 2\mathrm{r} - 1.75}\right)$ and recovers their stated factor at leading logarithmic order. Using the corrected threshold in Eq. (12) instead

gives $\tau_{BS}/\tau_{correct} \approx 0.846\left(\frac{\ln 2r}{\ln 2r - 1.5}\right)$. Over $r = 170 \square 310$, this yields $\tau_{BS}/\tau_{correct} \approx 1.14 \square 1.10$.

Thus, with the respective drag approximations retained, the free-fiber prediction is approximately 10–14% higher for the same assumed bending stiffness. These are comparisons of predicted thresholds, not comparisons of moduli fitted independently to the data.

The numerical proximity should be interpreted with the drag conventions in mind. At common leading-logarithmic order, the ratio of the free-fiber threshold to the corrected hinged-model threshold is approximately 0.8465, so their ordering reverses. The comparison therefore concerns the complete model predictions and cannot isolate the effect of the endpoint conditions. It also compares ideal instability criteria; visibly detectable bending during rotation can depend on the growth of disturbances and the time spent under compression (Słowicka et al. 2022).

The Gaussian selected by Forgacs and Mason fails the finite-end displacement conditions of their stated hinged model. It cannot be a solution for freely suspended fiber either. Although Eq. (2) ensures zero bending moment at the endpoints, as required for a freely suspended fiber, the Gaussian solution fails the zero-transverse-force condition. This failure concerns the boundary conditions, not the local validity of the Hermite solutions: decay at infinity and zero displacement at finite endpoints define different spectral problems. Enforcing the finite-end conditions changes the inferred bending stiffness substantially and supports a reassessment of the original experimental comparisons. The corrected solution provides a verification case for numerical solvers of the same reduced equation and boundary conditions. Its proximity to a free-fiber threshold does not establish physical equivalence or a rigorous upper or lower bound on free-fiber buckling.

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