Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Radiation of sine-Gordon equation solitons in a model with an extended attractive impurity, damping and a periodic external force

Daniil Kabanov

danya.kabanov.95@mail.ru

Ufa University of Science and Technology

Evgeniy Ekomasov

Ufa University of Science and Technology

Research Article

Keywords:

Posted Date: November 18th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11649364/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

Radiation of sine-Gordon equation solitons in a model with an extended attractive impurity, damping and a periodic external force

Kabanov D.K.[1*] and Ekomasov E.G.[1]

[1]Ufa University of Science and Technology, Zaki Validi St., 32/1, Ufa, 450076, Republic of Bashkortostan, Russian Federation.

*Corresponding author(s). E-mail(s): danya.kabanov.95@mail.ru;
Contributing authors: ekomasoveg@gmail.com;

Abstract

The radiation of sine-Gordon equation solitons in the presence of an impurity, damping, and an external force has been studied by numerical methods. The case of an extended impurity with a rectangular profile is considered; it acts as a potential well for a moving topological soliton (kink) and as a generator of localized oscillations. The finite difference method was used to solve the initial-boundary value problems. Localized waves of soliton type excited at the impurity, even in the absence of an external force and damping, are accompanied by a gradual decrease in their amplitude due to energy loss to wave radiation. Fourier spectral analysis of the emitted waves revealed the presence of two main frequencies at all observation points studied: the first corresponds to the lower boundary of the linear phonon spectrum, and the second is related to the natural oscillation frequency of the soliton. It is shown that the impurity parameters and the magnitude of the external force directly affect the characteristics of the solitons themselves and of the waves they emit, which makes it possible to control their amplitude and frequency. The dependences of the amplitude and frequencies of radiation on the system parameters, the distance from the soliton, and the frequency and amplitude of the periodic external force have been found. It is shown that for the sine-Gordon equation, the use of a variable external force of small amplitude leads to the radiation of stable quasi-monochromatic waves, while the choice of the external force frequency leads to a change in the type of soliton dynamics and in the frequency characteristics of the radiation.

Keywords: kink, soliton, sine-Gordon equation, external force, damping, impurity, radiation.

1 Introduction

The sine-Gordon equation (SGE), which has spatially localized soliton solutions, is used to describe many applied problems [1–6]. However, for correct use in various applications, additional terms have to be introduced into the sine-Gordon equation to account for the presence of external forces and damping in the system and for inhomogeneities of the parameters of the material medium [7–11]. Spatial modulation of the periodic potential (or an impurity) is often taken into account. To date, the SGE in a model with a single impurity has been studied in detail. It was found that both point and extended impurities having different spatial profiles are capable of scattering or capturing kinks, as well as generating a localized wave (or impurity mode) [3, 11–13]. A significant quantitative influence of the impurity profile on kink scattering has been shown, as well as qualitative agreement between numerical results and analytical ones obtained using perturbation theory for solitons. Many interesting results (see, for example, [14–17]) have been obtained in studying the process of excitation of impurity-localized waves, their structure, and their evolution. It has been found how, by means of the action of an external variable force, even in the presence of damping, one can achieve an increase in the amplitude of undamped oscillations of a soliton and a breather [18, 19]. It is shown that oscillations of impurity-localized waves of soliton and breather type are accompanied by radiation. However, a detailed consideration of this issue has been carried out only for the case of impurity-localized waves of breather type [20, 21]. In this work, the dependence of the structure and properties of the radiation of extended-impurity-localized waves of soliton type on the frequency and amplitude of the periodic external force will be studied, taking into account the presence of damping in the system.

2 Basic equations

Let us consider the modified sine-Gordon equation (MSGE) of the following form:

$$u_{tt} - u_{xx} + K(x)\sin(u) = -2h(t)\sin\left(\frac{u}{2}\right) - \alpha u_t. \tag{1}$$

Here α is the damping parameter. The term $2h(t)\sin\left(\frac{u}{2}\right)$ determines the interaction with the periodic external force. The magnitude of this force will be specified by a function of the form [20, 21]:

$$h(t) = \begin{cases} 0, & t^h_{on} < t \geq t^h_{off} \\ h_0 \sin(\omega_h(t - t^h_{on})), & t^h_{on} \geq t < t^h_{off} \end{cases},$$

where h_0 is the amplitude of the external variable force, ω_h is the frequency of the force; t^h_{on} and t^h_{off} are the times of switching the force on and off, respectively. The value of t^h_{off} was set equal to the computation time (the field was not switched off), so t^h_{off} will not be mentioned in what follows.

The function $K(x)$ determines the spatial inhomogeneity of the potential and is specified in the form [18]:

$$K(x) = \begin{cases} 1, & -W/2 > x > W/2 \\ 1 - \Delta K\ , & -W/2 \leq x \leq W/2 \end{cases},$$

where the parameters are: W is a quantity determining the width of the inhomogeneity; ΔK is a quantity determining the depth of the potential well if $\Delta K\ > 0$. As shown in [18], for certain positive values of ΔK , a localized wave of soliton type can be generated in the impurity region.

To solve equation 1, the finite difference method (FDM) [12, 21, 22] with an explicit integration scheme and a "cross"-type approximation stencil was used. The boundary conditions have the form:

$$u(-L, t) = 0; \quad u(L, t) = 2\pi,$$

where L is the value of x for the boundary of the coordinate region. In the calculations, we will set the value of L equal to the computation time in order to prevent reflection of the emitted waves from the boundary of the computational region. The stability condition of the numerical scheme is the inequality:

$$\frac{\Delta t^2}{\Delta x^2} \leq 1,$$

where Δt is the time discretization step of the scheme, and Δx is the coordinate discretization step of the scheme.

Let at the initial time $t_0 = 0$ and at the subsequent time $t = \Delta t$ there be a kink specified by the following equations:

$$u(x, t = 0) = 4 \operatorname{arctg}(\exp(\delta \cdot (x - x_0)));$$

$$u(x, t = \Delta t) = 4 \operatorname{arctg}(\exp(\delta (x - x_0 - v_0 \Delta t))).$$

Here $\delta = \frac{1}{\sqrt{1-v^2}}$ is the inverse width of the kink, x_0 is the initial position of the geometric center of the kink, and v_0 is the initial velocity of the kink. The kink starts moving with a constant velocity $v_0 < 1$. We consider only cases where v_0 is greater than a certain critical velocity v_{krit} [23] at which the kink passes through the impurity region, generates a localized wave, and moves on. Since the radiation of localized waves is being studied, the kink will be removed from the system as it approaches the edge of the computational region. Also, in order to exclude the influence of the kink's radiation, we will set the switching-on time parameter t^h_{on} such that the external force is switched on after the kink has been removed. The value $t^h_{on} = 150.0$ satisfies the above condition for all systems that will be considered in what follows. To analyze the frequency characteristics of the waves under study, the fast Fourier transform (FFT) was used.

3 Case of inertial dynamics

Let us consider, as an example, the results of numerical calculations in the absence of an external force and damping for a system with parameters $W = 2.5, \Delta K = 2.0,$

when a localized wave of soliton type is excited at the impurity. Figure 1a shows the dependence $u(t)$ for the soliton at $v_0 = 0.75$, and Figure 1b shows the Fourier spectrum of this dependence for the time slice $t \in [500, 1000]$, from which one can single out the natural oscillation frequency of the soliton, $\omega_{sol} = 0.72$. Also, the spectrum contains another frequency, numerically equal to $\omega = 1.44$, which corresponds to the second harmonic of ω_{sol}. Figures 2a and 3b show the dependences $u(t)$ of the soliton radiation at the point $x_1 = -25$ for $t \in [0, 1000]$ and $t \in [500, 1000]$ (steady-state oscillations). Figures 2b and 3a present the Fourier spectra for the above dependences. From the spectrum of steady-state oscillations, four main frequencies can be singled out. The first $\omega_1 = 1.0$ is the frequency of ordinary linear waves, and the frequency ω_4 is its second harmonic. The frequency $\omega_2 = 1.44$ is the second harmonic of ω_{sol}. The frequency $\omega_3 = 1.714$ is difficult to relate directly to the value of ω_{sol}, but its value can be compared with the sum $\omega_1 + \omega_2$. That is, the waves emitted by the soliton in the case of inertial dissipationless dynamics have more lines in the spectrum than the radiation spectrum of a breather [21].

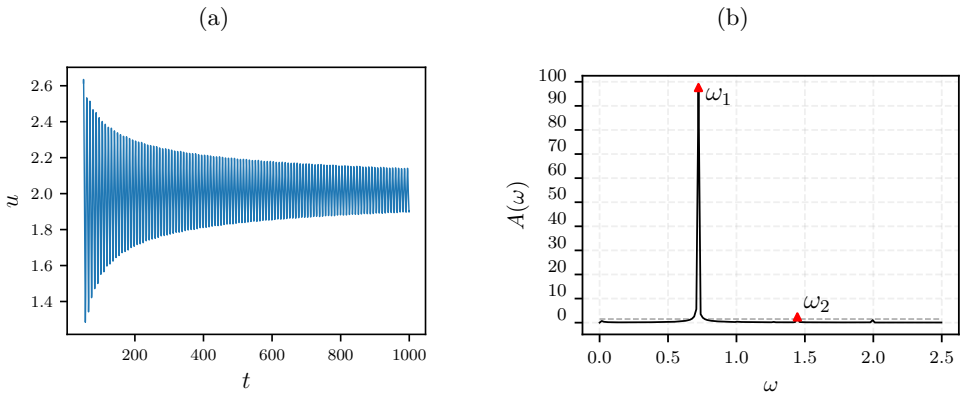


Fig. 1: (a) Dependence $u(x = 0, t)$ at $W = 2.5, \Delta K = 2.0, v_0 = 0.75$; (b) Fourier spectrum of the dependence $u(t)$ of the soliton in the system with parameters for $t \in [500, 1000]$, where $\omega_1 = 0.72, \omega_2 = 1.44$

4 Case with an external force and damping

Let us next find the function $u(t)$ for various values of x taking into account the external force and damping. The damping parameter α will be set equal to 0.001. The choice of the external force frequency in this case leads to more significant changes in the dynamics than in the case of a breather [21]. If the frequency is set to $\omega_h = 2\omega_{sol} = 1.44$, then the character of the soliton oscillations does not change - the center of the soliton oscillations and the main oscillation frequency are preserved (Figure 4). It is important to note that the external force frequency appears in the

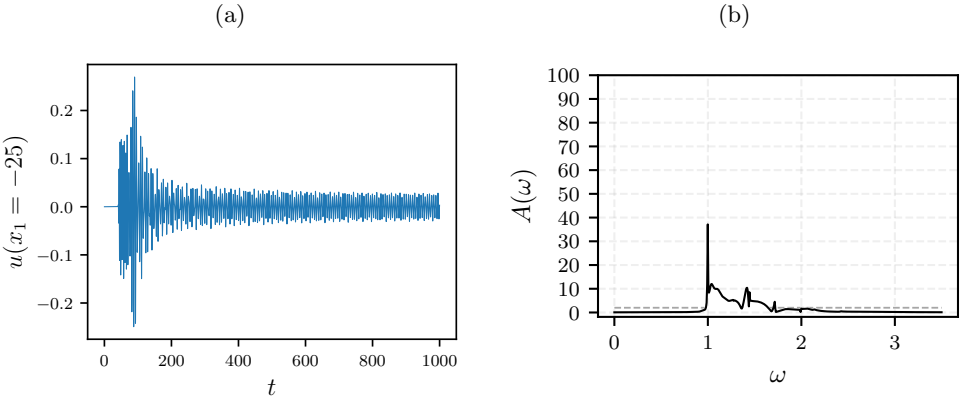


Fig. 2: For the system $W = 2.5, \Delta K = 2.0, v_0 = 0.75$: (a) dependence $u(t)$ of the radiation at the point $x_1 = -25$; (b) Fourier spectrum of this dependence for the time slice $t \in [0, 1000]$

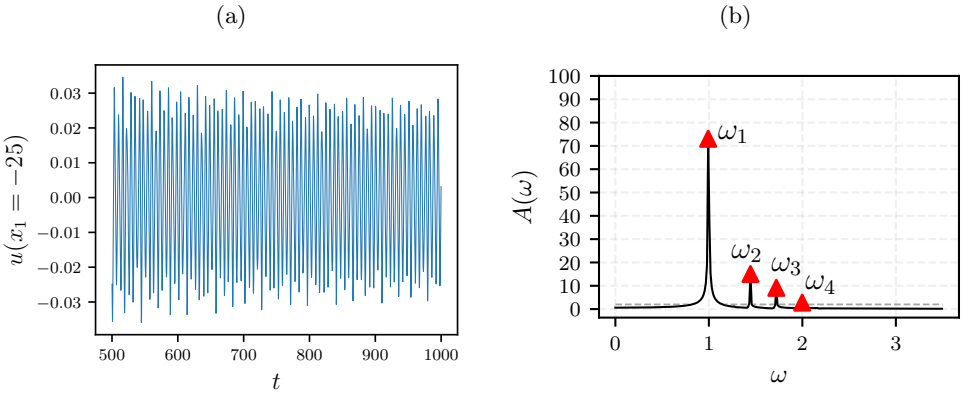


Fig. 3: For the case $W = 2.5, \Delta K = 2.0, v_0 = 0.75$: (a) dependence $u(t)$ of the radiation at the point $x_1 = -25$ in the time range $t \in [500, 1000]$; (b) Fourier spectrum of this dependence, where $\omega_1 = 1.0, \omega_2 = 1.44, \omega_3 = 1.714, \omega_4 = 2.0$

spectrum - the second harmonic of the natural oscillation frequency of the soliton, $\omega_2 = 1.44$. The dependences $u(t)$ of the radiation at the points $x_1 = -25, x_2 = -50$, and $x_3 = -100$ and their spectra are shown in Figure 5. It can be seen that the main frequency of the radiation has become $\omega_1 = 2\omega_{sol} = 1.44$ – the second harmonic of the natural frequency of the soliton, while the side frequencies $\omega_2 = 3\omega_{sol} = 2.17$ and $\omega_3 = 4\omega_{sol} = 2.88$ lead to small beats. An increase in the amplitude of the external

force also leads to an increase in the amplitudes of the soliton oscillations and its radiation, as shown in Figure 6.

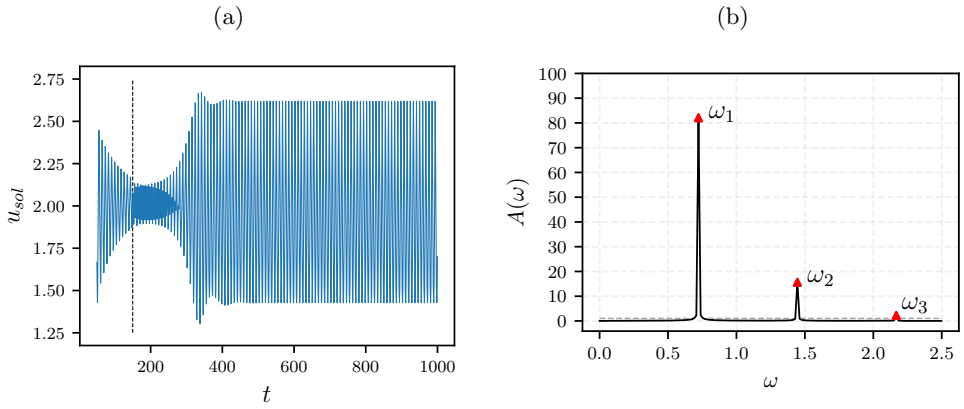


Fig. 4: For the case $W = 2.5, \Delta K = 2.0, \omega_{sol} = 0.72, h_0 = 0.1, \omega_h = 2\omega_{sol}$: (a) time dependence of u_{sol}, where the switching-on time of the external force is shown by the black dashed line; (b) Fourier spectrum of this dependence for the time slice $t \in [600, 1000]$, where $\omega_1 = 0.72, \omega_2 = 1.44 = 2\omega_1, \omega_3 = 2.17 = 3\omega_1$

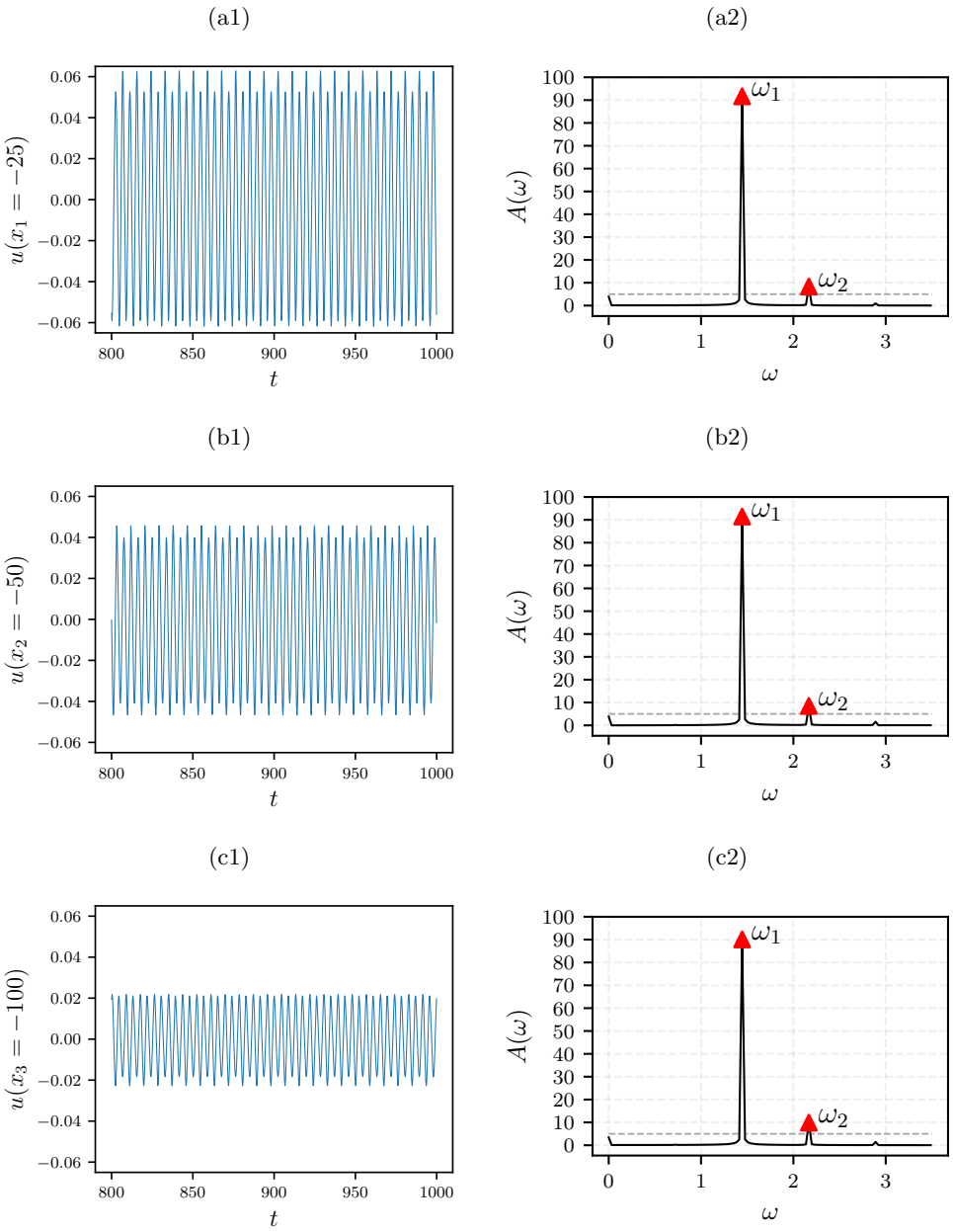


Fig. 5: Dependences $u(t)$ of the radiation and their Fourier spectra at the points: (a1, a2) $x_1 = -25$; (b1, b2) $x_2 = -50$; (c1, c2) $x_3 = -100$ for the system with parameters $W = 2.5, \Delta K = 2.0, \alpha = 0.001, \omega_{sol} = 0.72, h_0 = 0.1, \omega_h = 2\omega_{sol}$

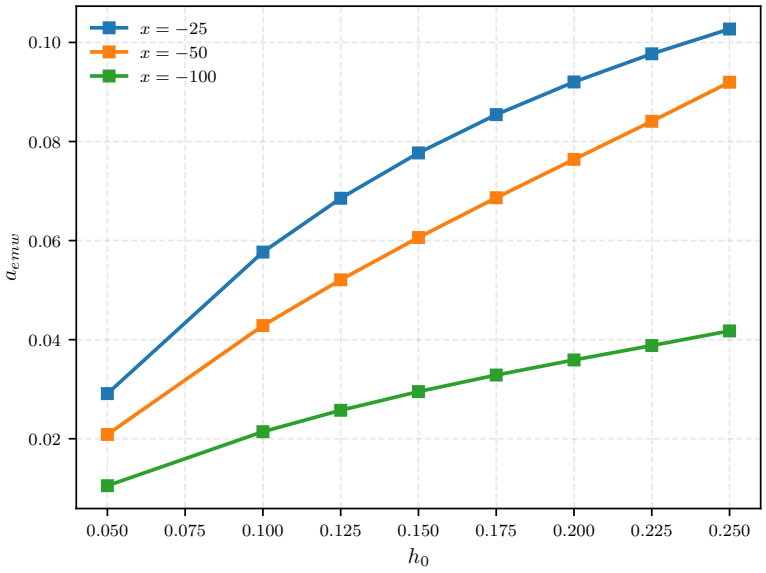


Fig. 6: Dependences of the amplitudes of the radiations at various points on the amplitude of the external force h_0 for the system $W = 2.5, \Delta K = 2.0, \alpha = 0.001, \omega_{sol} = 0.72, \omega_h = 2\omega_{sol}$

If the external force frequency is set equal to the natural frequency of the soliton, then at a certain value of h_0 the character of the oscillations changes. The soliton transitions to breather-type oscillations with a large amplitude (Figure 7a). The frequency of such oscillations is $\omega_1 = 0.367$, which is half the external force frequency. Also, the third and fifth harmonics of this frequency are present in the spectrum, and the external force frequency is absent (Figure 7b). Figure 8 shows the dependence $u(t)$ of the radiation at the point $x_1 = -25$ and the corresponding Fourier spectrum. An analysis of the frequency characteristics shows that the main frequency of the radiation in this case is $\omega_2 = 1.08$ – the third harmonic of the soliton oscillation. Also, the soliton frequency $\omega_1 = 0.36$ and its fifth harmonic $\omega_3 = 1.8$ are present in the spectrum.

Thus, if the frequency of the variable external force is equal to the natural frequency of the soliton oscillation, the regime of its oscillations becomes analogous to breather oscillations, and the frequency of such oscillations is equal to half the frequency of the external force. The characteristics of the soliton radiation in this case also become identical to the characteristics of breather radiation – the fundamental radiation frequency is equal to the third harmonic of the soliton oscillation.

5 Conclusion

The radiation of a localized soliton-type impurity mode of the modified sine-Gordon equation has been studied for two cases: inertial motion and motion with a variable

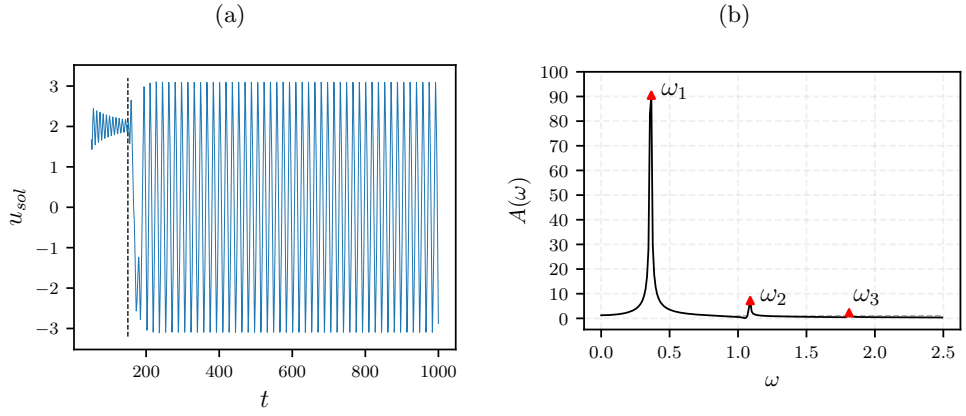


Fig. 7: (a) Dependence $u(t)$ of the soliton for a system with parameters $W = 2.5, \Delta K = 2.0, \alpha = 0.001, \omega_{sol} = 0.72, h_0 = 0.09, \omega_h = \omega_{sol}$; (b) Fourier spectrum of this dependence, where $\omega_1 = 0.367, \omega_2 = 1.08$, and $\omega_3 = 1.8$

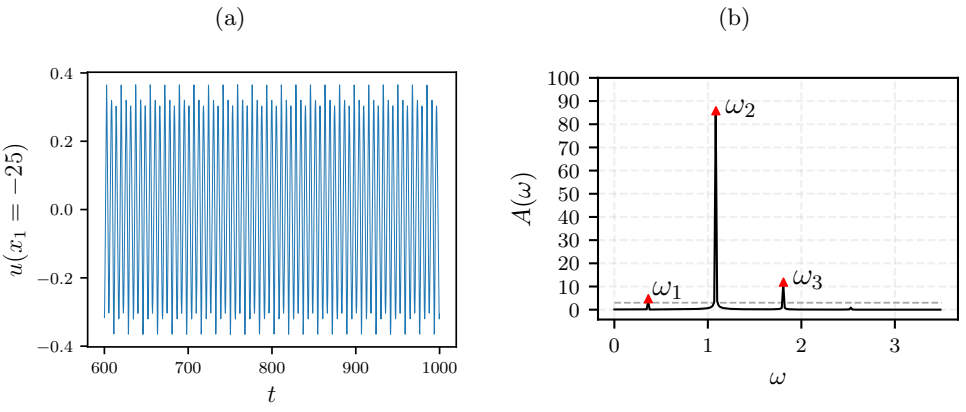


Fig. 8: (a) Dependence $u(t)$ of the radiation for a system with parameters $W = 2.5, \Delta K = 2.0, \alpha = 0.001, \omega_{sol} = 0.72, h_0 = 0.09, \omega_h = \omega_{sol}$, at the point $x_1 = -25$; (b) Fourier spectrum of this dependence, where $\omega_1 = 0.36$, $\omega_2 = 1.08$ and $\omega_3 = 1.8$

external force and damping taken into account. It has been established that the radiation spectrum contains a frequency equal to twice the soliton oscillation frequency. It is shown that in the case of a variable external force of small amplitude with a frequency equal to twice the natural frequency of the soliton, quasi-monochromatic waves are also radiated and their frequency characteristics change. A change in the amplitude of the external force leads to a change in the oscillation amplitude of both the soliton

and its radiation. It is shown that when an external force with a frequency equal to the soliton frequency is applied and at a certain value of the external force amplitude, it is possible to change the type of its oscillations, in which waves are radiated whose characteristics coincide with those of breather radiation.

Supplementary information. Not applicable.

Acknowledgements. Not applicable.

Declarations

- **Funding:** E. G. Ekomasov, D. K. Kabanov acknowledge the support from Ministry of Science and Higher Education of the Russian Federation (№ FRRR-2026-0010);

- **Conflict of interest:** The authors have no competing interests to declare that are relevant to the content of this article;
- **Ethics approval and consent to participate:** Not applicable;
- **Consent for publication:** Not applicable;
- **Data availability:** Not applicable;
- **Materials availability:** Not applicable;
- **Code availability:** Not applicable;
- **Author contribution:** All authors contributed equally to this work. All authors read and approved the final manuscript.

References

[1] Dauxois, T., Peyrard, M.: Physics of Solitons. https://books.google.ru/books?id=YKe1UZc_Qo8C

[2] Dodd, R., Eilbeck, J., Gibbon, J., Morris, H.: Soliton and Nonlinear Wave Equations

[3] The Sine-Gordon Model and Its Applications: From Pendula and Josephson Junctions to Gravity and High-Energy Physics. https://doi.org/10.1007/978-3-319-06722-3

[4] Delev, V.A., Nazarov, V.N., Scaldin, O.A., Batyrshin, E.S., Ekomasov, E.G.: Complex dynamics of the cascade of kink–antikink interactions in a linear defect of the electroconvective structure of a nematic liquid crystal. JETP Letters **110**, 607–612 (2019) https://doi.org/10.1134/S0370274X19210070

[5] Shamsutdinov, M.A., Lomakina, I.Y., Nazarov, V.N., Kharisov, A.T., Shamsutdinov, D.M.: Ferro- i Antiferromagnitodinamika: Nelineinye Kolebaniya, Volny i Solitony, p. 455. Nauka, Moscow (2009)

[6] Chevizovich, D., Michieletto, D., Mvogo, A., Zakiryanov, F., Zdravkovic, S.: A review on nonlinear DNA physics. R. Soc. Open Sci. **7**(11), 200774 (2020) https://doi.org/10.1098/rsos.200774

[7] Saadatmand, D., Dmitriev, S.V., Borisov, D.I., Kevrekidis, P.G.: Interaction of Sine-Gordon kinks and breathers with a parity-time-symmetric defect. Phys. Rev. E **90**(5), 052902 (2014) https://doi.org/10.1103/PhysRevE.90.052902

[8] Dorey, P., Gorina, A., Perapechka, I., Romanczukiewicz, T., Shnir, Y.: Resonance structures in kink–antikink collisions in a deformed sine-Gordon model. Journal of High Energy Physics **2021**(9), 145 (2021) https://doi.org/10.1007/JHEP09(2021)145

[9] Fabian, A.L., Kohl, R., Biswas, A.: Perturbation of topological solitons due to sine-Gordon equation and its type. Communications in Nonlinear Science and Numerical Simulation **14**(4), 1227–1244 (2009) https://doi.org/10.1016/j.cnsns.2008.01.013

[10] González, J.A., Bellorín, A., García-Nustes, M.A., Guerrero, L.E., Jiménez, S., Vázquez, L.: Arbitrarily large numbers of kink internal modes in inhomogeneous sine-Gordon equations. Phys. Lett. A **381**(24), 1995–1998 (2017) https://doi.org/10.1016/j.physleta.2017.03.042

[11] Di Fresco, G., De Santis, D., Guarcello, C., *et al.*: Effects of correlated noise on the excitation of robust breathers in an ac-driven, lossy sine-Gordon system. Chaos, Solitons & Fractals **189**, 115678 (2024) https://doi.org/10.1016/j.chaos.2024.115678

[12] Ekomasov, E.G., Gumerov, A.M., Kudryavtsev, R.V.: Resonance dynamics of kinks in the sine-Gordon model with impurity, external force and damping. Journal of Computational and Applied Mathematics **312**, 198–208 (2017) https://doi.org/10.1016/j.cam.2016.04.013

[13] Ekomasov, E.G., Gumerov, A.M., Kudryavtsev, R.V., Dmitriev, S.V., Nazarov, V.N.: Multisoliton dynamics in the sine-Gordon model with two-point impurities. Brazilian Journal of Physics **48**(6), 576–584 (2018) https://doi.org/10.1007/s13538-018-0606-4

[14] Ekomasov, E.G., Gumerov, A.M., Murtazin, R.R.: Interaction of sine-Gordon solitons in the model with attracting impurities. Mathematical Methods in the Applied Sciences **40**(17), 6178–6186 (2016) https://doi.org/10.1002/mma.3908

[15] Ekomasov, E.G., Nazarov, V.N., Samsonov, K.Y.: Changing the dynamic parameters of localized breather and soliton waves in the sine-Gordon model with extended impurity, external force, and decay in the autoresonance mode. Russian Journal of Nonlinear Dynamics **18**(2), 217–229 (2022) https://doi.org/10.20537/nd220205

[16] Samsonov, K.Y., Kabanov, D.K., Nazarov, V.N., Ekomasov, E.G.: Localized nonlinear waves of the sine-gordon equation in a model with three extended impurities. Computer Research and Modeling **16**(4), 855–868 (2024) https://doi.org/

10.20537/2076-7633-2024-16-4-855-868

[17] Gumerov, A.M., Ekomasov, E.G., Murtazin, R.R., Nazarov, V.N.: Transformation of sine-gordon solitons in models with variable coefficients and damping. Comput. Math. and Math. Phys. **55**(4), 628–637 (2015) https://doi.org/10.1134/S096554251504003X

[18] Ekomasov, E., Nazarov, V., Samsonov, K., Murtazin, R.: The possibility of controlling the dynamics and structure of a magnetic soliton in a three-layer ferromagnetic structure. Technical Physics Letters **47** (2021) https://doi.org/10.1134/S1063785021050229

[19] Ekomasov, E.G., Nazarov, V.N., Gumerov, A.M., Samsonov, K.Y., Murtazin, R.R.: External magnetic field control of the magnetic breather parameters in a three-layer ferromagnetic structure. Letters on Materials **10**(2), 141–146 (2020) https://doi.org/10.22226/2410-3535-2020-2-141-146

[20] Ekomasov, E.G., Kabanov, D.K., Sharafullin, I.F.: Magnetic breather as a frequency-tunable spin wave generator. Physics of Metals and Metallography **127**(2), 177–184 (2026) https://doi.org/10.7868/S3034621526020032

[21] Ekomasov, E.G., Kabanov, D.K., Fakhretdinov, M.I.: Radiation of localized Klein-Gordon waves in a model with impurity and damping in the presence of a time-dependent external force. Computational Mathematics and Modeling **37**, 1135–1150 (2026) https://doi.org/10.1007/s10598-026-09749-w

[22] Currie, J.P., Trullinger, S.E., Bishop, A.R., Krumhansl, J.A.: Numerical simulation of sine-Gordon soliton dynamics in the presence of perturbations. Phys. Rev. B **15**(12), 5567–5580 (1977) https://doi.org/10.1103/PhysRevB.15.5567

[23] Piette, B., Zakrzewski, W.J.: Scattering of sine-gordon kinks on potential wells. Journal of Physics A: Mathematical and Theoretical **40**(22), 5995 (2007) https://doi.org/10.1088/1751-8113/40/22/016