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Fractional Bernstein--Kantorovich Operators Based on the Left Sided--Riemann--Liouville Integral

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Abstract

In this paper, we introduce a new family of fractional Bernstein–Kantorovich type operators based on the left-sided form of the Riemann–Liouville fractional integral. We first investigate the fundamental properties of the proposed operators by deriving their moments and establishing Korovkin-type approximation theorems. Quantitative estimates of convergence are obtained by means of the modulus of continuity, while the asymptotic behavior of the operators is investigated through a Voronovskaya-type theorem. Furthermore, a Grüss–Voronovskaya-type theorem is established. By employing the tensor product technique, we construct the corresponding bivariate operators and investigate their approximation behavior. For the bivariate case, Korovkin-type convergence results are established, and the rate of approximation is analyzed by using both the partial and the total moduli of continuity. Numerical examples are also presented to illustrate the approximation properties of the proposed bivariate operators.

1 Introduction

Approximation theory is one of the fundamental branches of mathematical analysis that deals with representing functions by means of simpler mathematical structures. In particular, positive linear operators constitute one of the principal tools in approximation theory due to their positivity properties and their ability to provide systematic methods for studying convergence, approximation rates, and asymptotic behavior of approximating sequences. Among the most classical and extensively studied examples of positive linear operators are the Bernstein operators.

The classical Bernstein polynomials were introduced by Bernstein [1] in 1912 as a constructive proof of the Weierstrass approximation theorem. For $f \in C[0,1]$, the Bernstein operators are defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{1}$$

Due to their positivity, linearity, and approximation properties, Bernstein operators have played a central role in the development of approximation theory. Lorentz [43] presented a comprehensive study of Bernstein polynomials, including convergence properties, approximation estimates, and shape-preserving characteristics. Since then, Bernstein operators have been generalized in numerous directions, giving rise to a broad family of positive linear operators with improved approximation properties. (See [3]-[13])

A significant advancement in the evolution of Bernstein-type operators came with the work of Kantorovich [16] in 1930, who introduced the Bernstein-Kantorovich operators to approximate integrable functions rather than just continuous ones. This modification replaced the point evaluation $f(k/n)$ with an integral mean over a small interval, making the operators applicable to functions in L_p spaces and preserving the convergence properties of the original Bernstein polynomials while extending their utility. The Bernstein–Kantorovich operators are given by

$$K_n(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt, \tag{2}$$

where

$$b_{n,k}(x) = \quad = \quad b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Since their introduction, Bernstein–Kantorovich operators have become a cornerstone of approximation theory, attracting considerable attention because of their remarkable theoretical properties and applications in numerical analysis, computer-aided geometric design, and applied mathematics, which have led to numerous generalizations and modifications in the literature.(See [27]-[42])

Alongside these developments in approximation theory, fractional calculus has emerged as an active research area because it extends classical integral and differential operators to arbitrary orders, and its associated fractional integral operators have found wide applications in mathematical analysis, differential equations, and mathematical modeling. One of the most widely used fractional operators is the right -sided form of Riemann–Liouville fractional integral, defined for $\Re(\alpha) > 0$ by

$$(I_{a^+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

where $\Gamma(\alpha)$ denotes the Gamma function. In recent years, fractional integrals have been incorporated into approximation operators to improve convergence rates and approximation properties.

Based on the Riemann–Liouville fractional integral (3), several new approximation operators have been introduced in the literature. In [14] and [22], the authors studied the approximation properties of the following fractional Bernstein–Kantorovich operators.

$$K_n^{\alpha}(f;x) = \alpha \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 (1-t)^{\alpha-1} f\left(\frac{k+t}{n+1}\right) dt. \qquad \alpha > 0, \tag{4}$$

In particular, Kadak [14] proved that, for $\alpha > 1$, the fractional Bernstein–Kantorovich operators provide a better approximation than the classical Bernstein–Kantorovich operators whenever $x < \frac{\alpha^2+3\alpha-4}{3(\alpha^2+\alpha-2)}$.

Motivated by fractional integration techniques, several fractional-type positive linear operators have been introduced in approximation theory, and their approximation properties have been extensively investigated(See [17]-[26]).

The successful use of the Riemann–Liouville fractional integral in the construction of approximation operators has inspired further investigations of operators generated by its left-sided form of Riemann–Liouville fractional integral. Left-sided form of Riemann–Liouville fractional integral defined for $\Re(\alpha) > 0$ by

$$(I_{b^-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt. \tag{5}$$

In this paper, we introduce a new fractional Bernstein–Kantorovich type operators based on left-sided form of Riemann–Liouville fractional integral. We first investigate the fundamental properties of the proposed operators by deriving their moments and establishing Korovkin-type approximation theorems. Quantitative estimates of convergence are then obtained by means of the modulus of continuity, and the asymptotic behavior of the operators is examined through a Voronovskaya-type theorem. Furthermore, by employing the tensor product technique, we construct the corresponding bivariate operators and investigate their approximation behavior. For the bivariate case, Korovkin-type convergence results are established, and the rate of approximation is analyzed by using both the partial and the total moduli of continuity.

2 Construction of the new operators

We introduce a new family of fractional Bernstein–Kantorovich operators as follows:

Definition 1 *Let $f \in C[0,1]$ and $\alpha, \mu > 0$. The new family of fractional Bernstein-Kantorovich operators are defined by*

$$L_n^{(\theta,\alpha,\mu)}(f;x) = \frac{\alpha}{\mu} \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 t^{\frac{\alpha}{\mu}-1} f\left(\frac{k+\theta t}{n+1}\right) dt, \quad x \in [0,1], \theta \in (0,1], n \in \mathbb{N}. \tag{6}$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

In particular, for $\frac{\alpha}{\mu}=1$ and $\theta=1$, the operators (6) reduce to the classical Bernstein-Kantorovich operators defined in (2). We first establish several auxiliary lemmas concerning the moments of the operators defined in (1). Throughout the paper, $e_j(t)=t^j, t\in[0,1], j=0,1,2,\ldots$.

Lemma 2 *The fractional Bernstein-Kantorovich operators $L_n^{(\theta,\alpha,\mu)}$ defined in (1) satisfy the following moment relations:*

$$L_n^{(\theta,\alpha,\mu)}(e_i;x)=\frac{\alpha}{(n+1)^i}\sum_{m=0}^{i}\binom{i}{m}\frac{n^m\theta^{i-m}}{\alpha+(i-m)\mu}B_n(e_m;x). \tag{7}$$

Proof. *Using the definition of $L_n^{(\theta,\alpha,\mu)}$ in (6), we obtain*

$$\begin{aligned}
L_n^{(\theta,\alpha,\mu)}(e_i;x) &= \frac{\alpha}{\mu}\frac{1}{(n+1)^i}\sum_{k=0}^{n}b_{n,k}(x)\int_0^1 t^{\frac{\alpha}{\mu}-1}(k+\theta t)^i dt\\
&= \frac{\alpha}{\mu}\frac{1}{(n+1)^i}\sum_{k=0}^{n}b_{n,k}(x)\sum_{m=0}^{i}\binom{i}{m}k^m\theta^{i-m}\int_0^1 t^{\frac{\alpha}{\mu}+i-m-1}dt\\
&= \frac{\alpha}{(n+1)^i}\sum_{m=0}^{i}\binom{i}{m}\frac{n^m\theta^{i-m}}{\alpha+(i-m)\mu}\sum_{k=0}^{n}b_{n,k}(x)\frac{k^m}{n^m}\\
&= \frac{\alpha}{(n+1)^i}\sum_{m=0}^{i}\binom{i}{m}\frac{n^m\theta^{i-m}}{\alpha+(i-m)\mu}B_n(e_m;x),
\end{aligned}$$

which completes the proof. ■

3 Auxiliary results

Moments of positive linear operators constitute one of the principal analytical tools in approximation theory, since they determine the reproduction properties and approximation behavior of the operators. Furthermore, they are indispensable in deriving convergence results, quantitative estimates, and asymptotic expansions for positive linear operators.

Lemma 3 *For $n\in\mathbb{N}$, $\alpha,\mu>0$, and $\theta\in(0,1]$, the first five moments of the operator $L_n^{(\theta,\alpha,\mu)}$ are given by, for every $x\in[0,1]$,*

$$L_n^{(\theta,\alpha,\mu)}(e_0;x)=1, \tag{8}$$

$$L_n^{(\theta,\alpha,\mu)}(e_1;x)=\frac{n}{n+1}x+\frac{\theta}{n+1}\frac{\alpha}{\alpha+\mu}, \tag{9}$$

$$L_n^{(\theta,\alpha,\mu)}(e_2;x)=\frac{n^2}{(n+1)^2}x^2+\frac{n}{(n+1)^2}\left(x(1-x)+\frac{2\theta\alpha}{\alpha+\mu}x\right)+\frac{1}{(n+1)^2}\frac{\theta^2\alpha}{\alpha+2\mu}, \tag{10}$$

$$\begin{aligned}
L_n^{(\theta,\alpha,\mu)}(e_3;x) &= \frac{n^3}{(n+1)^3}x^3+\frac{3n^2}{(n+1)^3}\left(x^2(1-x)+\frac{\theta\alpha}{\alpha+\mu}x^2\right)\\
&\quad+\frac{n}{(n+1)^3}\left(x(1-x)(1-2x)+\frac{3\alpha\theta}{\alpha+\mu}x(1-x)+\frac{3\theta^2\alpha}{\alpha+2\mu}x\right)+\frac{1}{(n+1)^3}\frac{\theta^3\alpha}{\alpha+3\mu},
\end{aligned} \tag{11}$$

$$L_n^{(\theta,\alpha,\mu)}(e_4;x) = \frac{n^4}{(n+1)^4}x^4 + \frac{n^3}{(n+1)^4}\left(6x^3(1-x) + \frac{4\alpha\theta}{\alpha+\mu}x^3\right) \tag{12}$$
$$+ \frac{n^2}{(n+1)^4}\left(x^2(x-1)(11x-7) + \frac{12\alpha\theta}{\alpha+\mu}x^2(1-x) + \frac{6\alpha\theta^2}{\alpha+2\mu}x^2\right)$$
$$+ \frac{n}{(n+1)^4}\left(x(1-x)(1-6x+6x^2) + \frac{4\alpha\theta}{\alpha+\mu}x(1-x)(1-2x) + \frac{6\alpha\theta^2}{\alpha+2\mu}x(1-x) + \frac{4\alpha\theta^3}{\alpha+3\mu}x\right)$$
$$+ \frac{1}{(n+1)^4}\frac{\alpha\theta^4}{\alpha+4\mu}.$$

Central moments play a fundamental role in the study of approximation properties of positive linear operators, as they are closely related to the estimation of the rate of convergence and the derivation of asymptotic formulas. As an immediate consequence of Lemma 3, the following central moments of $L_n^{(\theta)}$ are obtained.

Lemma 4 *Let $n \in \mathbb{N}, \alpha, \mu > 0$ and $\theta \in (0,1]$. Then, for every $x \in [0,1]$, $L_n^{(\theta,\alpha,\mu)}$ hold:*

$$L_n^{(\theta,\alpha,\mu)}((t-x);x) = \frac{\theta\alpha - x(\alpha+\mu)}{(n+1)(\alpha+\mu)} := A_n(x), \tag{13}$$

$$L_n^{(\theta,\alpha,\mu)}((t-x)^2;x) = \frac{1}{(n+1)^2}\left(x^2 - x\left(\frac{2\theta\alpha}{\alpha+\mu}\right) + \frac{\theta^2\alpha}{\alpha+2\mu}\right) + \frac{n}{(n+1)^2}x(1-x) := B_n(x), \tag{14}$$

$$L_n^{(\theta,\alpha,\mu)}((t-x)^3;x) = \frac{1}{(n+1)^3}\left(-x^3 + \frac{\theta^3\alpha}{\alpha+3\mu} - \frac{3x\theta^2\alpha}{\alpha+2\mu} + \frac{3x^2\theta\alpha}{\alpha+\mu}\right) + \frac{nx(x-1)}{(n+1)^3}\left((5x-1) - \frac{3\theta\alpha}{\alpha+\mu}\right), \tag{15}$$

$$L_n^{(\theta,\alpha,\mu)}\big((t-x)^4;x\big) = \frac{1}{(n+1)^4}\left(\frac{6x^2\theta^2\alpha}{\alpha+2\mu} - \frac{4x^3\theta\alpha}{\alpha+\mu} - \frac{4x\theta^3\alpha}{\alpha+3\mu} + x^4 + \frac{\theta^4\alpha}{\alpha+4\mu}\right)$$
$$+ \frac{n}{(n+1)^4}\Big(-20x^4 + \frac{30x^3(\alpha+\mu)+2\theta\alpha}{\alpha+\mu} \tag{16}$$
$$-x^2\frac{11(\alpha^2+2\mu^2)+6\theta\alpha^2(\theta+4)+3\alpha\mu(2\theta^2+11)+48\theta\alpha\mu}{(\alpha+\mu)(\alpha+2\mu)}$$
$$+x\frac{\alpha^2+2\mu^2+2\theta\alpha^2(3\theta+2)+6\theta^2\alpha\mu+\alpha\mu(8\theta+3)}{(\alpha+\mu)(\alpha+2\mu)}\Big) + \frac{n^2}{(n+1)^4}3x^2(x-1)^2.$$

Proof. Since the proofs of the central moments follow the same line of argument, we present only the proof of (14). By the linearity of the operator $L_n^{(\theta,\alpha,\mu)}$ we have

$$L_n^{(\theta,\alpha,\mu)}((t-x)^2;x) = L_n^{(\theta,\alpha,\mu)}\left(t^2;x\right) - 2xL_n^{(\theta,\alpha,\mu)}\left(t;x\right) + x^2L_n^{(\theta)}\left(1;x\right).$$

Applying the identities obtained in Lemma 3 we obtain

$$L_n^{(\theta,\alpha,\mu)}\left((t-x)^2;x\right) = \frac{n^2}{(n+1)^2}x^2 + \frac{n}{(n+1)^2}\left(x(1-x) + \frac{2\theta\alpha}{\alpha+\mu}x\right) + \frac{\theta^2\alpha}{\alpha+2\mu}$$
$$- 2\left(\frac{n}{n+1}x + \frac{\theta}{n+1}\frac{\alpha}{\alpha+\mu}\right) + x^2$$
$$= \frac{1}{(n+1)^2}\left(x^2 - x\left(\frac{2\theta\alpha}{\alpha+\mu}\right) + \frac{\theta^2\alpha}{\alpha+2\mu}\right) + \frac{n}{(n+1)^2}x(1-x).$$

■

Lemma 5 *Let $n \in \mathbb{N}, \alpha, \mu > 0$ and $\theta \in (0,1]$. For all $x \in [0,1]$, there holds*

(i) $\lim_{n\to\infty} nL_n^{(\theta,\alpha,\mu)}((t-x);x) = \frac{\theta\alpha - x(\alpha+\mu)}{(\alpha+\mu)}$,

(ii) $\lim_{n\to\infty} nL_n^{(\theta,\alpha,\mu)}((t-x)^2;x) = x(1-x)$,

(iii) $\lim_{n\to\infty} n^2 L_n^{(\theta,\alpha,\mu)}((t-x)^4;x) = 3x^2(x-1)^2$.

Lemma 6 *([16]) For the operators $K_n(f;x)$, we have*

(i) $K_n((t-x);x) = \frac{-x}{n+1} + \frac{1}{2(n+1)}$,

(ii) $K_n((t-x)^2;x) = \frac{x^2}{(n+1)^2} + \frac{nx(1-x)}{(n+1)^2} - \frac{x}{(n+1)^2} + \frac{1}{3(n+1)^2}$.

4 Main Results

4.1 Korovkin-type theorem

Theorem 7 *Let $f \in C[0,1], n \in \mathbb{N}, \alpha, \mu > 0$ and $\theta \in (0,1]$. Then, $L_n^{(\theta,\alpha,\mu)}(f;x)$ uniformly converges to the function $f(x)$ on $[0,1]$.*

Proof. Let $f \in C[0,1]$. From (3), we obtain

$$\begin{aligned} \lim_{n\to\infty} L_n^{(\theta,\alpha,\mu)}(1;x) &= 1, \\ \lim_{n\to\infty} L_n^{(\theta,\alpha,\mu)}(t;x) &= x, \end{aligned}$$

and

$$\lim_{n\to\infty} L_n^{(\theta,\alpha,\mu)}(t^2;x) = x^2.$$

Therefore, operator $L_n^{(\theta,\alpha,\mu)}(f;x)$ satisfies the Bohman–Korovkin [15] conditions. ■

4.2 Local approximation

Positive linear operators $L_n^{(\theta,\alpha,\mu)}(f;x)$ and $K_n(f;x)$ on $C[0,1]$ preserve the constant function, that is

$$L_n^{(\theta,\alpha,\mu)}(1;x) = 1 \quad \text{and} \quad K_n(1;x) = 1,$$

for all $n \in \mathbb{N}$.
Recall that any sequence of positive linear operators satisfying the property $T_n(1;x) = 1$ also satisfies the inequality

$$|T_n(f;x) - f(x)| \le \varepsilon + \frac{2\|f\|}{\delta^2} T_n((t-x)^2;x), \tag{17}$$

where δ is determined by the uniform continuity of the function f which is being approximated. Applying inequality (17) to the operators $L_n^{(\theta,\alpha,\mu)}$ and K_n, we obtain

$$|L_n^{(\theta,\alpha,\mu)}(f;x) - f(x)| \le \varepsilon + \frac{2\|f\|}{\delta^2} L_n^{(\theta)}((t-x)^2;x),$$

and

$$|K_n(f;x) - f(x)| \le \varepsilon + \frac{2\|f\|}{\delta^2} K_n((t-x)^2;x).$$

These estimates show that the order of approximation of the operators $L_n^{(\theta)}$ and K_n to a function f is mainly determined by the quantities

$$L_n^{(\theta,\alpha,\mu)}((t-x)^2;x) \quad \text{and} \quad K_n((t-x)^2;x).$$

Therefore, in order to compare the approximation behavior of these operators, it is sufficient to investigate the inequality

$$L_n^{(\theta,\alpha,\mu)}((t-x)^2;x) < K_n((t-x)^2;x). \tag{18}$$

From the comparison of the second central moments of the operators $L_n^{(\theta,\alpha,\mu)}$ and K_n, we arrive at Lemma 8 and Theorem 9, which characterizes the situations where the operator $L_n^{(\theta,\alpha,\mu)}$ at least as good as the order of approximation by $K_n(f;x)$.

Lemma 8 *Let $\alpha,\mu>0$ and $\theta\in(0,1]$, then Equation (18) holds the following:*

(i) for all $x<\frac{\frac{1}{3}-\frac{\theta^2\alpha}{\alpha+2\mu}}{1-\frac{2\theta\alpha}{\alpha+\mu}}$, if $\frac{\theta\alpha}{\alpha+\mu}<\frac{1}{2}$ and $\frac{\theta^2\alpha}{\alpha+2\mu}<\frac{1}{3}$.

(ii) for all $x>\frac{\frac{1}{3}-\frac{\theta^2\alpha}{\alpha+2\mu}}{1-\frac{2\theta\alpha}{\alpha+\mu}}$, if $\frac{\theta\alpha}{\alpha+\mu}>\frac{1}{2}$ and $\frac{\theta^2\alpha}{\alpha+2\mu}>\frac{1}{3}$.

(iii) for all $x>\frac{\frac{1}{3}-\frac{\theta^2\alpha}{\alpha+2\mu}}{1-\frac{2\theta\alpha}{\alpha+\mu}}$, if $\frac{\theta\alpha}{\alpha+\mu}>\frac{1}{2}$ and $\frac{\theta^2\alpha}{\alpha+2\mu}<\frac{1}{3}$.

Proof. Utilizing Lemma 4 and inequality (3), we can write

$$\left(1-\frac{2\theta\alpha}{\alpha+\mu}\right)x+\left(\frac{\theta^2\alpha}{\alpha+2\mu}-\frac{1}{3}\right) \quad < \quad 0. \tag{19}$$

The solution to the inequality, given in (19) completes the proof. ■

Theorem 9 *Let $\alpha,\mu>0$ and $\theta\in(0,1]$. Then, in the following cases, the approximation order of f by $L_n^{(\theta)}(f;x)$ is at least as good as that $K_n(f;x)$.*

(i) if $\frac{\theta\alpha}{\alpha+\mu}<\frac{1}{2}$ and $\frac{\theta^2\alpha}{\alpha+2\mu}<\frac{1}{3}$, then for all $x\in\left[0,\frac{\frac{1}{3}-\frac{\theta^2\alpha}{\alpha+2\mu}}{1-\frac{2\theta\alpha}{\alpha+\mu}}\right)\bigcap[0,1]$,

(ii) if $\frac{\theta\alpha}{\alpha+\mu}>\frac{1}{2}$ and $\frac{\theta^2\alpha}{\alpha+2\mu}>\frac{1}{3}$, then for all $x\in\left(\frac{\frac{1}{3}-\frac{\theta^2\alpha}{\alpha+2\mu}}{1-\frac{2\theta\alpha}{\alpha+\mu}},\infty\right)\bigcap[0,1]$,

(iii) if $\frac{\theta\alpha}{\alpha+\mu}>\frac{1}{2}$ and $\frac{\theta^2\alpha}{\alpha+2\mu}<\frac{1}{3}$, then for all $x\in[0,1]$.

Proof. Since the domain of the given operators is [0,1], the proof is an obvious consequence of Lemma 8. ■

To demonstrate these theoretical findings, we present graphical illustrations for each case stated in Theorem 9. In the following figures, the regions highlighted in green represent the intervals on which the operator $L_n^{(\theta,\alpha,\mu)}$ provides a better approximation compared to the operator K_n; this observation is consistent with the results obtained in the Theorem 9.

Case(i): Let $f(x)=1-cos(\pi x)$. To demonstrate the applicability of the theorem, we choose the parameters $\alpha=0.5$, $\theta=0.3$, $\mu=1$ and $n=40$. As demonstrated in the following figure, the operator $L_n^{(\theta,\alpha,\mu)}$ provides better approximation than the operator K_n on the interval $[0,0.03942)$.

Case(ii): Let $f(x)=1-cos(\pi x)$. To verify the applicability of the theorem, we employ the parameter set $\alpha=6$, $\theta=0.95$, $\mu=0.5$ and $n=40$. As illustrated in the following figure, the operator $L_n^{(\theta,\alpha,\mu)}$ delivers a better approximation than the operator K_n on the interval $[0.5840,1)$.

Case(iii): Let $f(x)=1-cos(\pi x)$. To verify the applicability of the theorem, we employ the parameter set $\alpha=4$, $\theta=0.75$, $\mu=2$ and $n=40$. As illustrated in the following figure, the operator $L_n^{(\theta,\alpha,\mu)}$ delivers a better approximation than the operator K_n on the interval $[0,1)$.

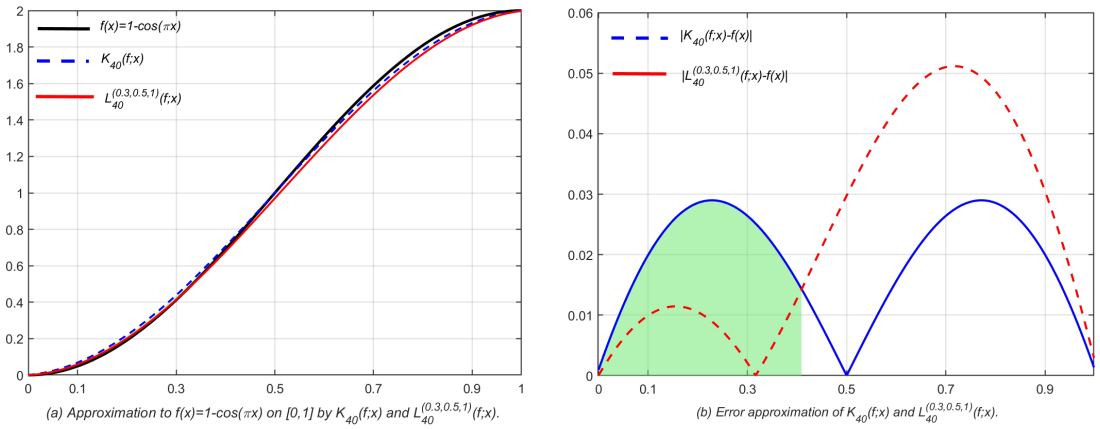


Figure 1: Comparision of the operators $L_n^{(\theta,\alpha,\mu)}$ and K_n in approximation $f(x) = 1 - cos(\pi x)$ with the parameter set $\alpha = 0.5$, $\theta = 0.3$, $\mu = 1$ and $n = 40$.

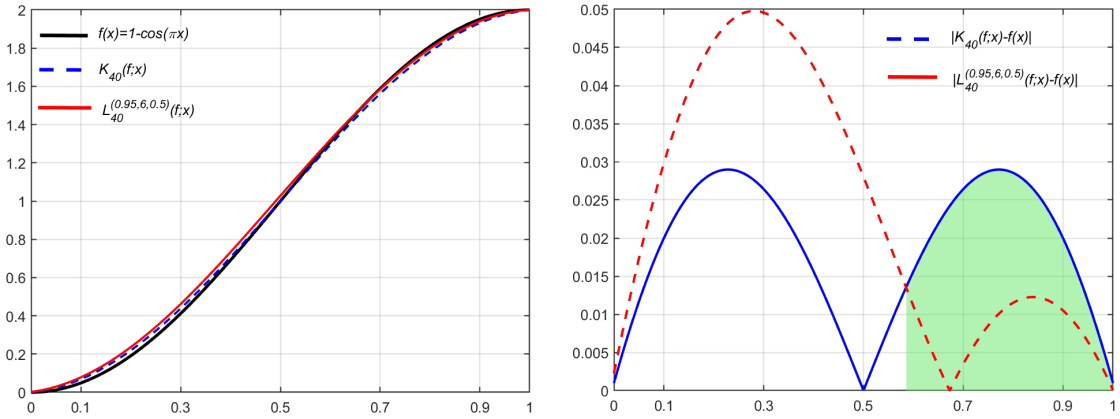


Figure 2: Comparision of the operators $L_n^{(\theta,\alpha,\mu)}$ and K_n in approximation $f(x) = 1 - cos(\pi x)$ with the parameter set $\alpha = 6$, $\theta = 0.95$, $\mu = 0.5$ and $n = 40$.

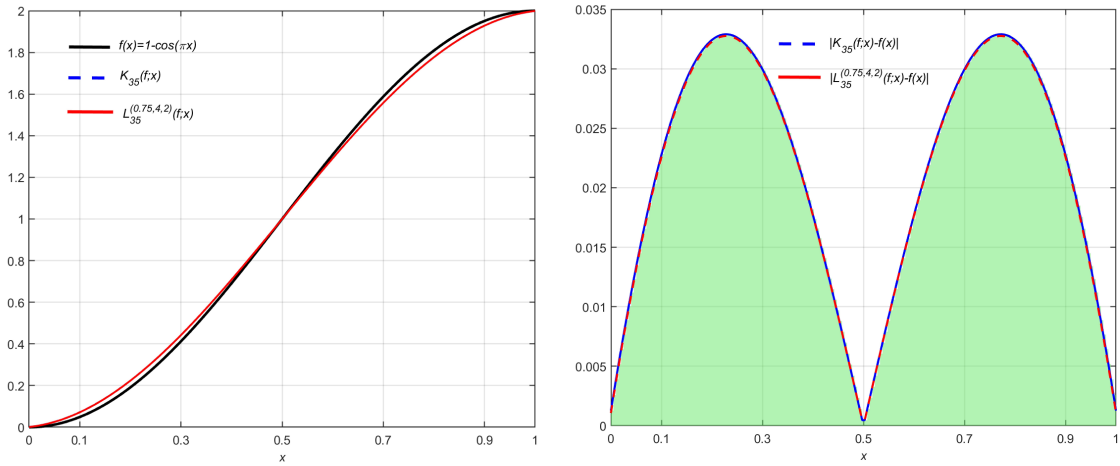


Figure 3: Comparision of the operators $L_n^{(\theta,\alpha,\mu)}$ and K_n in approximation $f(x) = 1 - cos(\pi x)$ with the parameter set $\alpha = 4$, $\theta = 0.75$, $\mu = 2$ and $n = 40$.

In this work, $C[0,1]$ denote the space of all bounded real valued continuous function on $[0,1]$. This space is equipped with the following norm:

$$\|f\| = \sup_{x\in[0,1]} |f(x)|.$$

Lemma 10 *Let $\alpha,\mu>0$ and $\theta\in(0,1]$. For each function $f\in C[0,1]$ and $x\in[0,1]$, we have*

$$\left|L_n^{(\theta,\alpha,\mu)}(f;x)\right| \le \|f\|.$$

Proof. By using the definition of $L_n^{(\theta,\alpha,\mu)}$ and Lemma 3, we have

$$\begin{aligned}
L_n^{(\theta,\alpha,\mu)}(f;x) &= \frac{\alpha}{\mu}\sum_{k=0}^{n} b_{n,k}(x)\left|\int_0^1 t^{\frac{\alpha}{\mu}-1} f\left(\frac{k+\theta t}{n+1}\right)dt\right|,\\
&\le \frac{\alpha}{\mu}\sum_{k=0}^{n} b_{n,k}(x)\int_0^1 t^{\frac{\alpha}{\mu}-1}\left|f\left(\frac{k+\theta t}{n+1}\right)dt\right|\\
&\le \frac{\alpha}{\mu}\sum_{k=0}^{n} b_{n,k}(x)\int_0^1 t^{\frac{\alpha}{\mu}-1}\left|f\left(\frac{k+\theta t}{n+1}\right)\right|dt\\
&\le \|f\|\frac{\alpha}{\mu}\sum_{k=0}^{n} b_{n,k}(x)\int_0^1 t^{\frac{\alpha}{\mu}-1}dt\\
&= \|f\|L_n^{(\theta,\alpha,\mu)}(e_0;x)\\
&= \|f\|.
\end{aligned}$$

■

In order to establish the fundamental theorems of local approximation, we first recall the definitions of the moduli of smoothness and the K-functional. For $f\in C[0,1]$, the modulus of continuity is given by:

$$\omega(f,\delta) = \sup_{0<h\le\delta}\ \sup_{x,x+h\in[0,1]} |f(x+h)-f(x)|.$$

Modulus of continuity has the following property:

$$|f(t)-f(x)| \le \omega(f;|t-x|) \tag{20}$$

and

$$\omega(f;\lambda\delta) \le (1+\lambda)\,\omega(f;\delta), \tag{21}$$

where $\lambda>0$.

The Peetre's K-functional is given by

$$K_2(f,\delta) = \inf_{g\in C^2[0,1]}\{\|f-g\|+\delta\|g''\|\} \qquad (\delta>0), \tag{22}$$

where $C^2[0,1]=\left\{g\in C[0,1]: g',g''\in C[0,1]\right\}$. For $C>0$,

$$K_2(f,\delta) \le C\omega_2\left(f;\sqrt{\delta}\right),$$

where the modulus of smoothness of order two ([44]) is defined as

$$\omega_2(f,\sqrt{\delta}) := \sup_{0<h\le\delta}\ \sup_{x\pm h\in[0,1]} |f(x+h)-2f(x)+f(x-h)|$$

Theorem 11 *Let $\alpha,\mu>0$ and $\theta\in(0,1]$. For each function $f\in C[0,1]$ and $x\in[0,1]$, we have*

$$\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| \le 2\omega\left(f,\sqrt{\frac{1}{(n+1)^2}\left(x^2-x\left(\frac{2\theta\alpha}{\alpha+\mu}\right)+\frac{\theta^2\alpha}{\alpha+2\mu}\right)+\frac{n}{(n+1)^2}x(1-x)}\right).$$

Proof. Since $L^{(\theta,\alpha,\mu)}(1;x)=1$ and $b_{n,k}(x)\geq 0$ on $[0,1]$, we are able to write

$$\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| \leq \frac{\alpha}{\mu}\sum_{k=0}^{n} b_{n,k}(x)\int_0^1 t^{\frac{\alpha}{\mu}-1}\left|f\left(\frac{k+\theta t}{n+1}\right)-f(x)\right| dt \tag{23}$$

Using the properties of modulus of continuity (21) we obtain

$$\left|f\left(\frac{k+\theta t}{n+1}\right)-f\left(x\right)\right| \leq \left(1+\frac{\left|\frac{k+\theta t}{n+1}-x\right|}{\delta}\right)\omega(f,\delta). \tag{24}$$

Using (24) in (23), we obtain

$$\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| \leq \frac{\alpha}{\mu}\sum_{k=0}^{n} b_{n,k}(x)\int_0^1 t^{\frac{\alpha}{\mu}-1}\left(1+\frac{\left|\frac{k+\theta t}{n+1}-x\right|}{\delta}\right)\omega(f,\delta), dt \tag{25}$$

By applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| &\leq \omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{\alpha}{\mu}\sum_{k=0}^{n} b_{n,k}(x)\int_0^1 t^{\frac{\alpha}{\mu}-1}\left(\frac{k+\theta t}{n+1}-x\right)^2}\right)\\ &= \omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{1}{(n+1)^2}\left(x^2-x\left(\frac{2\theta\alpha}{\alpha+\mu}\right)+\frac{\theta^2\alpha}{\alpha+2\mu}\right)+\frac{n}{(n+1)^2}x(1-x)}\right).\end{aligned}$$

Taking $\delta=\sqrt{\frac{1}{(n+1)^2}\left(x^2-x\left(\frac{2\theta\alpha}{\alpha+\mu}\right)+\frac{\theta^2\alpha}{\alpha+2\mu}\right)+\frac{n}{(n+1)^2}x(1-x)}$, we get the desired result. ■

Theorem 12 *Let $f\in C[0,1]$. Then for all $x\in[0,1]$, there exist absolute constant $C>0$ such that*

$$\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| \leq Cw_2(f;\sqrt{\delta_n(x)})+w\left(f;\theta_n(x)\right),$$

where $\theta_n(x)=A_n(x)$ and $\delta_n(x)=B_n(x)+A_n^2(x)$.

Proof. Firstly,we define the auxiliary operatoras

$$L_n^{(\theta,\alpha,\mu)*}(f;x)=L_n^{(\theta,\alpha,\mu)}(f;x)+f(x)-f(\mu_n(x)). \tag{26}$$

where $\mu_n(x)=\frac{n}{n+1}x+\frac{\theta}{n+1}\frac{\alpha}{\alpha+\mu}$. Note that, from Lemma 3, we have

$$L_n^{(\theta,\alpha,\mu)*}(1;x)=1 \text{ and } L_n^{(\theta,\alpha,\mu)*}((t-x);x)=0. \tag{27}$$

Now for any given function $g\in C^2[0,1]$, we write Taylor's expansion formula as follows:

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t (t-s)g''(s)ds.$$

Now, applying the linear operator $L_n^{(\theta,\alpha,\mu)*}$ on both sides of the above equation and from (27), we get

$$\begin{aligned}L_n^{(\theta,\alpha,\mu)*}(g;x)-g(x) &= L_n^{(\theta,\alpha,\mu)*}((t-x)g'(x);x)+L_n^{(\theta,\alpha,\mu)*}\left(\int_x^t (t-s)g''(s)ds;x\right)\\ &= g'(x)L_n^{(\theta,\alpha,\mu)*}((t-x);x)+L_n^{(\theta,\alpha,\mu)}\left(\int_x^t (t-s)g''(s)ds;x\right)-\int_x^{\mu_n(x)}\left(\mu_n(x)-s\right)g''(s)ds\\ &= L_n^{(\theta,\alpha,\mu)}\left(\int_x^t (t-s)g''(s)ds;x\right)-\int_x^{\mu_n(x)}\left(\mu_n(x)-s\right)g''(s)ds.\end{aligned}$$

On the other hand,

$$\left|\int_x^t (t-s)g''(s)ds\right| \leq \int_x^t |t-s|\,|g''(s)|\,ds$$
$$\leq \|g''\| \int_x^t |t-s|\,ds \leq \|g''\|\,(t-s)^2$$

and

$$\left|\int_x^{\mu_n(x)} (\mu_n(x)-s)\,g''(s)ds\right| \leq \int_\tau^{\mu_n(x)} |\mu_n(x)-s|\,|g''(s)|\,ds$$
$$\leq \|g''\|\,(\mu_n(x)-x)^2$$
$$= \|g''\| \left(L_n^{(\theta,\alpha,\mu)}(t-x;x)\right)^2,$$

which implies

$$\left|L_n^{(\theta,\alpha,\mu)*}(g;x)-g(x)\right| \leq \left|L_n^{(\theta,\alpha,\mu)}\left(\int_x^t (t-s)g''(s)ds;x\right)\right| + \left|\int_x^{\mu_n(x)} (\mu_n(x)-s)\,g''(s)ds\right|$$
$$\leq \|g''\| \left\{L_n^{(\theta,\alpha,\mu)}((t-x)^2;x) + \left(L_n^{(\theta,\alpha,\mu)}(t-x);x\right)^2\right\}$$
$$= \|g''\|\,\delta_n(x). \tag{28}$$

Using Lemma 3 and (26), we get

$$\left|\mathrm{L}_n^{(\theta,\alpha,\mu)*}(f;x)\right| \leq \left|L_n^{(\theta,\alpha,\mu)}(f;x)\right| + |f(x)| + |f(\mu_n(x))| \leq \|f\|\,L_n^{(\theta,\alpha,\mu)}(1;x) + 2\|f\| \leq 3\|f\|.$$

Using (28) and the uniform boundedness of $L^{(\theta,\alpha,\mu)*}$, we get

$$\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| \leq \left|L_n^{(\theta,\alpha,\mu)*}(f-g;x)-(f-g)(x)\right| + \left|L_n^{(\theta,\alpha,\mu)*}(g;x)-g(x)\right| + |f(x)-f(\mu_n(x))|$$
$$\leq 4\|f-g\| + \|g''\|\,\delta_n(x) + \omega(f,\theta_n(x)).$$

Taking infimum on the right hand side over all $g \in C^2[0,1]$, we obtain

$$\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| \leq 4K_2(f;\delta_n(x)) + \omega(f,\theta_n(x)),$$

which together with (22) gives the proof of the theorem. ■

We now proceed to analyze the rate of convergence for the operators $L_n^{(\theta,\alpha,\mu)}$ using the Lipschitz class $\mathrm{Lip}_M(c)$,where $M > 0$ and $0 < c \leq 1$. Recall that a function f belongs to $Lip_M(c)$ if it satisfies the condition:

$$|f(t)-f(x)| \leq M|t-x|^c \qquad (t,x \in \mathbb{R}).$$

We have the following theorem.

Theorem 13 *Let $\alpha,\mu > 0, x \in [0,1]$ and $\theta \in (0,1]$. For every $f \in Lip_M(c)$, we have*

$$\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| \leq M\left[L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)\right]^{\frac{c}{2}}.$$

where $L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)$ is given in Lemma 4.

Proof. Since $L_n^{(\theta,\alpha,\mu)}(f;x)$ are linear positive operators and $f \in Lip_M(c)$, we have

$$\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| \leq L_n^{(\theta,\alpha,\mu)}(|f(t)-f(x)|;x)$$
$$= \sum_{k=0}^n b_{n,k}^{(\alpha)}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right|$$
$$\leq M\sum_{k=0}^n b_{n,k}(x)\left|\frac{k}{n}-x\right|^c.$$

By taking $p=\frac{2}{c}$ and $q=\frac{2}{2-c}$ in the Hölder's inequality, we get

$$\begin{aligned}\left|L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right| &\leq M\left[\sum_{\rho=0}^{s} b_{n,k}^{(\alpha)}(x)\left(\frac{k}{n}-x\right)^2\right]^{\frac{c}{2}}\left[\sum_{k=0}^{s} b_{n,k}^{(\alpha)}(x)\right]^{\frac{2-c}{2}}\\ &= M\left[L_n^{(\theta,\alpha,\mu)}\left((t-x)^2;x\right)\right]^{\frac{c}{2}}.\end{aligned}$$

■

5 Asymptotic approximation by $L_n^{(\theta,\alpha,\mu)}$

The Voronovskaja theorem is a fundamental result in approximation theory that addresses the convergence behavior of specific approximation operators. It provides an estimate for the rate at which a sequence of such operators converges to a given function. Named after the Soviet mathematician Tamara Voronovskaja, the theorem typically expresses the difference between a function and its approximation through a remainder term. This result plays a crucial role in evaluating the accuracy and efficiency of approximation methods in mathematical analysis. In this section, we establish a Voronovskaja-type theorem for $L_n^{(\theta,\alpha,\mu)}(f;x)$.

Theorem 14 *Let $\alpha,\mu>0, x\in[0,1]$ and $\theta\in(0,1]$. If f' and $f''\in C[0,1]$, then*

$$\lim_{n\to\infty} n\left(L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right)=\frac{\theta\alpha-x(\alpha+\mu)}{(\alpha+\mu)}f'(x)+\frac{1}{2}x(x-1)f''(x). \tag{29}$$

Proof. By using Taylor formula, we have

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t;x)(t-x)^2, \tag{30}$$

where $r(t;x)\to 0$ as $t\to x$ and $r(t;x)\in C[0,1]$.

Applying $L_n^{(\theta,\alpha,\mu)}$ to (30) we obtain,

$$\begin{aligned}n\left(L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right) &= nf'(x)L_n^{(\theta,\alpha,\mu)}(t-x;x) \\ +\frac{1}{2}nf''(x)L_n^{(\theta,\alpha,\mu)}\left((t-x)^2;x\right) &+ nL_n^{(\theta,\alpha,\mu)}\left(r(t;x)(t-x)^2;x\right).\end{aligned} \tag{31}$$

Utilizing the Cauchy-Schwartz inequality to last part of (31) we get,

$$nL_n^{(\theta,\alpha,\mu)}\left(r(t;x)(t-x)^2;x\right)\leq\sqrt{L_n^{(\theta,\alpha,\mu)}\left(r^2(t;x);x\right)}\sqrt{n^2L_n^{(\theta,\alpha,\mu)}\left((t-x)^4;x\right)}.$$

From Lemma 5, we have

$$\lim_{n\to\infty} n^2L_n^{(\theta,\alpha,\mu)}\left((t-x)^4;x\right)=3x^2(x-1)^2.$$

Since $r^2(x;x)=0$ and $r^2(.,x)\in C[0,1]$, by Theorem 7, we get

$$\lim_{n\to\infty} L_n^{(\theta,\alpha,\mu)}\left(r^2(t;x);x\right)=r^2(x;x)=0$$

uniformly on $[0,1]$. Therefore,

$$\lim_{s\to\infty} L_n^{(\theta,\alpha,\mu)}\left(r(t;x)(t-x)^2;x\right)=0. \tag{32}$$

Hence, from (32) and Lemma 5 we get immediately

$$\lim_{n\to\infty} n\left(L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right)=\frac{\theta\alpha-x(\alpha+\mu)}{(\alpha+\mu)}f'(x)+\frac{1}{2}x(x-1)f''(x).$$

■

6 Quantitative Voronovskaja-Type Theorem

In this section, we present a quantitative Voronovskaya theorem for operators (6) involving the second-order modulus of smoothness which gives a higher order of approximation than the one in terms of the first-order modulus of continuity (see [45]).

Lemma 15 *Let $\alpha, \mu > 0$ and $\theta \in (0,1]$. For each $x \in [0,1]$, we have*

$$\frac{\left|L_n^{(\theta,\alpha,\mu)}((t-x)^3;x)\right|}{L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)} \leq \frac{C}{n+1}, \tag{33}$$

and

$$\frac{\left|L_n^{(\theta,\alpha,\mu)}((t-x)^4;x)\right|}{L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)} \leq \frac{K}{n+1}, \tag{34}$$

where $C, K > 0$.

Proof. From Lemma 4, we have,

$$L_n^{(\theta,\alpha,\mu)}((t-x)^2;x) = \frac{1}{(n+1)^2}\left(nx(1-x)+R(x)\right),$$

where $R(x) = x^2 - \frac{2\theta\alpha}{\alpha+\mu}x + \frac{\theta^2\alpha}{\alpha+2\mu}$ and $R(x) \geq m, m := \frac{\theta^2\alpha\mu^2}{(\alpha+2\mu)(\alpha+\mu)^2}$.
Therefore:

$$L_n^{(\theta,\alpha,\mu)}((t-x)^2;x) \geq \frac{nx(1-x)+m}{(n+1)^2}.$$

Also, From Lemma 4, we have,

$$L_n^{(\theta,\alpha,\mu)}((t-x)^3;x) = \frac{A(x)}{(n+1)^3} + \frac{nx(x-1)}{(n+1)s}B(x)$$

where $A(x) = -x^3 + \frac{\theta^3\alpha}{\alpha+3\mu} - \frac{3x\theta^2\alpha}{\alpha+2\mu} + \frac{3x^2\theta\alpha}{\alpha+\mu}$ and $B(x) = (5x-1) - \frac{3\theta\alpha}{\alpha+\mu}$.
By the absolute value property of the sum,

$$\left|L_n^{(\theta,\alpha,\mu)}((t-x)^3;x)\right| \leq \frac{|A(x)|}{(n+1)^3} + \frac{nx(1-x)}{(n+1)^3}|B(x)|$$

For every $\alpha, \mu > 0, \theta, x \in [0,1]$, we get

$$|A(x)| \leq 8 \text{ and } |B(x)| \leq 4. \tag{35}$$

and

$$\left|L_n^{(\theta,\alpha,\mu)}((t-x)^3;x)\right| \leq \frac{8+4nx(1-x)}{(n+1)^3}$$

So,

$$\frac{\left|L_n^{(\theta,\alpha,\mu)}((t-x)^3;x)\right|}{L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)} \leq F\frac{1}{n+1},$$

where $F = \frac{8+4nx(1-x)}{nx(1-x)+m}$ and $F \leq \max(4, \frac{8}{m})$.
Therefore,

$$\frac{\left|L_n^{(\theta,\alpha,\mu)}((t-x)^3;x)\right|}{L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)} \quad \leq \quad \frac{C}{n+1},$$

where $C = \max(4, \frac{8}{m})$ and $m = \frac{\theta^2\alpha\mu^2}{(\alpha+2\mu)(\alpha+\mu)^2}$.
Using the same method, the second inequality can also be proven. ■

Theorem 16 *Let $\alpha, \mu > 0, x \in [0,1]$ and $\theta \in (0,1]$. For any $f \in C^2[0,1]$, the following condition holds:*

$$\begin{aligned}&\left|n\left[L_n^{(\theta,\alpha,\mu)}(f;x) - f(x)\right] - \frac{\theta\alpha - x(\alpha+\mu)}{(\alpha+\mu)}f'(x) - \frac{1}{2}x(1-x)f''(x)\right| \\ &\leq \frac{M}{3}\left\{\omega_1\left(f'';\frac{1}{\sqrt{n+1}}\right) + \omega_2\left(f'';\frac{1}{\sqrt{n+1}}\right)\right\} + \frac{4}{(n+1)}\left(\|f'\| + \|f''\|\right).\end{aligned}$$

Proof. From Gonska and Rasa [45] (Theorem 3), we have

$$\begin{aligned}&\left|L_n^{(\theta,\alpha,\mu)}(f;x) - f(x) - L_n^{(\theta,\alpha,\mu)}(t-x;x)f'(x) - \frac{1}{2}L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)f''(x)\right| \\ &\leq L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)\left\{\frac{\left|L_n^{(\theta,\alpha,\mu)}((t-x)^3;x)\right|}{L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)}\frac{5}{6h}\omega_1\left(f'';h\right) + \left(\frac{3}{4} + \frac{1}{16h^2}\frac{\left|L_n^{(\theta,\alpha,\mu)}((t-x)^4;x)\right|}{L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)}\right)\omega_2\left(f'';h\right)\right\}.\end{aligned}$$

Using Lemma 15 we obtain, the following inequalities hold:

$$\begin{aligned}&\left|L_n^{(\theta,\alpha,\mu)}(f;x) - f(x) - \frac{\theta\alpha - x(\alpha+\mu)}{(n+1)(\alpha+\mu)}f'(x)\right. \\ &\left.-\frac{1}{2}\left(\frac{1}{(n+1)^2}\left(x^2 - x\left(\frac{2\theta\alpha}{\alpha+\mu}\right) + \frac{\theta^2\alpha}{\alpha+2\mu}\right) + \frac{n}{(n+1)^2}x(1-x)\right)f''(x)\right| \\ &\leq \left[\frac{1}{(n+1)^2}\left(x^2 - x\left(\frac{2\theta\alpha}{\alpha+\mu}\right) + \frac{\theta^2\alpha}{\alpha+2\mu}\right) + \frac{n}{(n+1)^2}x(1-x)\right] \\ &\quad\times\left\{\frac{5}{6h}\frac{C}{(n+1)}\omega_1(f'';h) + \left(\frac{3}{4} + \frac{K}{(n+1)}\frac{1}{16h^2}\right)\omega_2(f'';h)\right\}.\end{aligned}$$

Now, multiplying both side with n and using $h = \frac{1}{\sqrt{n+1}}$, we get

$$\begin{aligned}&\left|n\left[L_n^{(\theta,\alpha,\mu)}(f;x) - f(x)\right] - \frac{n(\theta\alpha - x(\alpha+\mu))}{(n+1)(\alpha+\mu)}f'(x)\right. \\ &\left.-\frac{1}{2}\left(\frac{n}{(n+1)^2}\left(x^2 - x\left(\frac{2\theta\alpha}{\alpha+\mu}\right) + \frac{\theta^2\alpha}{\alpha+2\mu}\right) + \frac{n^2}{(n+1)^2}x(1-x)\right)f''(x)\right| \\ &\leq \left[\frac{n}{(n+1)^2}\left(x^2 - x\left(\frac{2\theta\alpha}{\alpha+\mu}\right) + \frac{\theta^2\alpha}{\alpha+2\mu}\right) + \frac{n^2}{(n+1)^2}x(1-x)\right] \\ &\quad\times\left\{\frac{5}{6}\frac{C}{\sqrt{n+1}}\omega_1\left(f'';\frac{1}{\sqrt{n+1}}\right) + \left(\frac{12+K}{16}\right)\omega_2\left(f'';\frac{1}{\sqrt{n+1}}\right)\right\} \\ &\leq \frac{1}{3}\left\{\frac{5C}{6}\omega_1\left(f'';\frac{1}{\sqrt{n+1}}\right) + \left(\frac{12+K}{16}\right)\omega_2\left(f'';\frac{1}{\sqrt{n+1}}\right)\right\} \\ &\leq \frac{M}{3}\left\{\omega_1\left(f'';\frac{1}{\sqrt{n+1}}\right) + \omega_2\left(f'';\frac{1}{\sqrt{n+1}}\right)\right\},\end{aligned}$$

where $M=\max\left\{\frac{5C}{6},\frac{12+K}{16}\right\}$.
Now, we write,

$$\begin{aligned}
&\left|n\left[L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right]-\frac{\theta\alpha-x(\alpha+\mu)}{(\alpha+\mu)}f'(x)-\frac{1}{2}x(1-x)f''(x)\right|\\
&\leq\left|n\left[L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)\right]-\frac{n(\theta\alpha-x(\alpha+\mu))}{(n+1)(\alpha+\mu)}f'(x)\right.\\
&\left.-\frac{1}{2}\left(\frac{n}{(n+1)^2}\left(x^2-x\left(\frac{2\theta\alpha}{\alpha+\mu}\right)+\frac{\theta^2\alpha}{\alpha+2\mu}\right)+\frac{n^2}{(n+1)^2}x(1-x)\right)f''(x)\right|\\
&+\left|-\frac{\theta\alpha-x(\alpha+\mu)}{(n+1)(\alpha+\mu)}f'(x)\right.\\
&\left.+\frac{1}{2}\left(\frac{3n+1}{(n+1)^2}x^2-x\left(\frac{1+2n}{(n+1)^2}+\frac{2\theta\alpha}{\alpha+\mu}\frac{n}{(n+1)^2}\right)+\frac{n}{(n+1)^2}\frac{\theta^2\alpha}{\alpha+2\mu}\right)f''(x)\right|\\
&\leq\frac{M}{3}\left\{\omega_1\left(f'';\frac{1}{\sqrt{n+1}}\right)+\omega_2\left(f'';\frac{1}{\sqrt{n+1}}\right)\right\}+\frac{4}{(n+1)}\left(\|f'\|+\|f''\|\right)
\end{aligned}$$

■

7 Grüss-Voronovskaya-Type Theorem

Theorem 17 *Let $\alpha,\mu>0, x\in[0,1]$ and $\theta\in(0,1]$. For $f,g\in C^2[0,1]$, there holds the following equality:*

$$\lim_{s\to\infty}n\left[L_n^{(\theta,\alpha,\mu)}(fg;x)-L_n^{(\theta,\alpha,\mu)}(f;x)L_n^{(\theta,\alpha,\mu)}(g;x)\right]=f'(x)g'(x)x(1-x).$$

Proof. The second derivative of the product of two functions f and g is given by

$$(fg)''(x)=f''(x)g(x)+2f'(x)g'(x)+g''(x)f(x).$$

With an appropriate rearrangement, we obtain

$$\begin{aligned}
&n\left\{L_n^{(\theta,\alpha,\mu)}((fg);x)-L_n^{(\theta,\alpha,\mu)}(f;x)L_n^{(\theta,\alpha,\mu)}(x;x)\right\}\\
&=n\left\{L_n^{(\theta,\alpha,\mu)}((fg);x)-f(x)g(x)-(fg)'L_n^{(\theta,\alpha,\mu)}(t-x;x)\right.\\
&-\frac{(fg(x))''}{2!}L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)\\
&-g(x)\left[L_n^{(\theta,\alpha,\mu)}(f;x)-f(x)-\frac{f'(x)}{1!}L_n^{(\theta,\alpha,\mu)}(t-x;x)-\frac{f''(x)}{2!}L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)\right]\\
&-L_n^{(\theta,\alpha,\mu)}(f;x)\left[L_n^{(\theta,\alpha,\mu)}(g;x)-g(x)-\frac{g'(x)}{1!}L_n^{(\theta,\alpha,\mu)}(t-x;x)-\frac{g''(x)}{2!}L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)\right]\\
&+2\frac{f'(x)g'(x)}{2!}L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)+\frac{g''(x)}{2!}L_n^{(\theta,\alpha,\mu)}((t-x)^2;x)\left(f(x)-L_n^{(\theta,\alpha,\mu)}(f;x)\right)\\
&\left.+\frac{g'(x)}{1!}L_n^{(\theta,\alpha,\mu)}(t-x;x)\left[f(x)-L_n^{(\theta,\alpha,\mu)}(f;x)\right]\right\}.
\end{aligned}$$

Using values $\lim_{n\to\infty}nL_n^{(\theta,\alpha,\mu)}(t-x;x)=\frac{\theta\alpha-x(\alpha+\mu)}{(\alpha+\mu)}$, $\lim_{n\to\infty}nL_n^{(\theta,\alpha,\mu)}((t-x)^2;x)=x(1-x)$, Theorem 7 and Lemma 4, we obtain our results. ■

8 Shape Preserving Properties

Recall that, for any positive integer $m<n$, the m^{th} order derivative of $b_{n,k}$ is as follows:

$$b_{n,k}^{(m)}(x)=n(n-1)\cdots(n-m+1)b_{n-m,k}(x),$$

for $k = 0, 1, 2, \cdots n - m$. Moreover,

$$\Delta_s^m f(t) := \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(t + is), \tag{36}$$

where $t \in [0, 1]$.
The following theorem shows that, our operators has shape preserving properties.

Theorem 18 *Let $\alpha, \mu > 0, x \in [0, 1]$ and $\theta \in (0, 1]$. For each $n \in \mathbb{N}$ we have,*

i) If f is a non-decreasing (non-increasing) function on the interval $[0, 1]$ then so is $L_n^{(\theta,\alpha,\mu)}(f; x)$.

ii) If f is a convex (concave) function on the interval $[0, 1]$ then so is $L_n^{(\theta,\alpha,\mu)}(f; x)$.

Proof.

i) Assume that f is a non-decreasing function on $[0, 1]$. Then

$$L_n^{(\theta,\alpha,\mu)}(f; x) = \frac{\alpha}{\mu} \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 t^{\frac{\alpha}{\mu}-1} f\left(\frac{k + \theta t}{n + 1}\right) dt.$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx} L_n^{(\theta,\alpha,\mu)}(f; x) = \frac{\alpha}{\mu} \sum_{k=0}^{n} b'_{n,k}(x) \int_0^1 t^{\frac{\alpha}{\mu}-1} f\left(\frac{k + \theta t}{n + 1}\right) dt.$$

Using the identity

$$b'_{n,k}(x) = n\Big(b_{n-1,k-1}(x) - b_{n-1,k}(x)\Big),$$

we get

$$\frac{d}{dx} L_n^{(\theta,\alpha,\mu)}(f; x) = \frac{\alpha n}{\mu} \sum_{k=0}^{n} \Big(b_{n-1,k-1}(x) - b_{n-1,k}(x)\Big) \times \int_0^1 t^{\frac{\alpha}{\mu}-1} f\left(\frac{k + \theta t}{n + 1}\right) dt.$$

After shifting the indices, we obtain

$$\frac{d}{dx} L_n^{(\theta,\alpha,\mu)}(f; x) = \frac{\alpha n}{\mu} \sum_{k=0}^{n-1} b_{n-1,k}(x) \int_0^1 t^{\frac{\alpha}{\mu}-1} \times \left[f\left(\frac{k + 1 + \theta t}{n + 1}\right) - f\left(\frac{k + \theta t}{n + 1}\right)\right] dt.$$

Since

$$\frac{k + 1 + \theta t}{n + 1} > \frac{k + \theta t}{n + 1},$$

and f is non-decreasing on $[0, 1]$, it follows that

$$f\left(\frac{k + 1 + \theta t}{n + 1}\right) - f\left(\frac{k + \theta t}{n + 1}\right) \geq 0.$$

Moreover,

$$b_{n-1,k}(x) \geq 0 \quad \text{and} \quad t^{\frac{\alpha}{\mu}-1} \geq 0.$$

Hence,

$$\frac{d}{dx} L_n^{(\theta,\alpha,\mu)}(f; x) \geq 0,$$

which proves that $L_n^{(\theta,\alpha,\mu)}(f; x)$ is also non-decreasing.
The proof for the non-increasing case is similar.

ii) Assume that f is a convex function on $[0,1]$. Then

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x) \geq 0,$$

whenever the points belong to $[0,1]$.

Now,

$$L_n^{(\theta,\alpha,\mu)}(f;x) = \frac{\alpha}{\mu} \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 t^{\frac{\alpha}{\mu}-1} f\left(\frac{k+\theta t}{n+1}\right) dt.$$

Differentiating twice with respect to x, we obtain

$$\frac{d^2}{dx^2} L_n^{(\theta,\alpha,\mu)}(f;x) = \frac{\alpha}{\mu} \sum_{k=0}^{n} b''_{n,k}(x) \int_0^1 t^{\frac{\alpha}{\mu}-1} f\left(\frac{k+\theta t}{n+1}\right) dt.$$

Using

$$b''_{n,k}(x) = n(n-1)\left(b_{n-2,k-2}(x) - 2b_{n-2,k-1}(x) + b_{n-2,k}(x)\right),$$

we get

$$\frac{d^2}{dx^2} L_n^{(\theta,\alpha,\mu)}(f;x) = \frac{\alpha n(n-1)}{\mu} \sum_{k=0}^{n} \left(b_{n-2,k-2}(x) - 2b_{n-2,k-1}(x) + b_{n-2,k}(x)\right) \times \int_0^1 t^{\frac{\alpha}{\mu}-1} f\left(\frac{k+\theta t}{n+1}\right) dt.$$

Changing the indices in the sums, we obtain

$$\frac{d^2}{dx^2} L_n^{(\theta,\alpha,\mu)}(f;x) = \frac{\alpha n(n-1)}{\mu} \sum_{k=0}^{n-2} b_{n-2,k}(x) \int_0^1 t^{\frac{\alpha}{\mu}-1} \times \left[f\left(\frac{k+2+\theta t}{n+1}\right) - 2f\left(\frac{k+1+\theta t}{n+1}\right) + f\left(\frac{k+\theta t}{n+1}\right) \right] dt.$$

Since f is convex on $[0,1]$, we have

$$f\left(\frac{k+2+\theta t}{n+1}\right) - 2f\left(\frac{k+1+\theta t}{n+1}\right) + f\left(\frac{k+\theta t}{n+1}\right) \geq 0.$$

Also,

$$b_{n-2,k}(x) \geq 0 \quad \text{and} \quad t^{\frac{\alpha}{\mu}-1} \geq 0.$$

Hence,

$$\frac{d^2}{dx^2} L_n^{(\theta,\alpha,\mu)}(f;x) \geq 0.$$

Therefore, $L_n^{(\theta,\alpha,\mu)}(f;x)$ is convex. The proof for the concave case is similar.

■

9 Bivariate case of $L_n^{(\theta,\alpha,\mu)}(f;x)$

In this section, we construct a bivariate case for the operators defined by (1) in the following manner:

Definition 19 *Let $f \in C(I^2)$, $\alpha, \mu > 0$, and $\theta \in (0,1]$. The bivariate Kantorovich-type Bernstein operators are defined by*

$$L_{n,m}^{(\theta,\alpha,\mu)}(f;x,y) = \left(\frac{\alpha}{\mu}\right)^2 \sum_{k=0}^{n}\sum_{j=0}^{m} b_{n,k}(x)b_{m,j}(y)\int_0^1\int_0^1 t^{\frac{\alpha}{\mu}-1}s^{\frac{\alpha}{\mu}-1} f\left(\frac{k+\theta t}{n+1}, \frac{j+\theta s}{m+1}\right) dt\, ds, \tag{37}$$

where $I = [0,1], b_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ and $b_{m,j}(y) = \binom{m}{j}y^j(1-y)^{m-j}$.

If $\frac{\alpha}{\mu} = 1$ and $\theta = 1$, the operators (37) reduce to reduce to the bivariate case of the operators considered in (2). For $f \in C(I^2)$, the space of all real valued continuous functions on I^2, we define the norm

$$\|f\|_{C(I^2)} \quad = \quad \sup_{(x,y)\in I^2} |f(x,y)|.$$

Lemma 20 *Let $e_{t,s} = t^i s^j, (i,j) \in I^2, i,j \in \mathbb{N}\cup 0$ with $i+j \le 4$, be the 2-dimensional test functions. Then the following identities hold:*

(i) $L_{n,m}^{(\theta,\alpha,\mu)}(e_{0,0};x,y) = 1,$

(ii) $L_{n,m}^{(\theta,\alpha,\mu)}(e_{1,0};x,y) = \frac{n}{n+1}x + \frac{\theta}{n+1}\frac{\alpha}{\alpha+\mu},$

(iii) $L_{n,m}^{(\theta,\alpha,\mu)}(e_{2,0};x,y) = \frac{n^2}{(n+1)^2}x^2 + \frac{n}{(n+1)^2}\left(x(1-x) + \frac{2\theta\alpha}{\alpha+\mu}x\right) + \frac{1}{(n+1)^2}\frac{\theta^2\alpha}{\alpha+2\mu},$

(iv) $L_{n,m}^{(\theta,\alpha,\mu)}(e_{0,1};x,y) = \frac{m}{m+1}y + \frac{\theta}{m+1}\frac{\alpha}{\alpha+\mu},$

(v) $L_{n,m}^{(\theta,\alpha,\mu)}(e_{0,2};x,y) = \frac{m^2}{(m+1)^2}y^2 + \frac{m}{(m+1)^2}\left(y(1-y) + \frac{2\theta\alpha}{\alpha+\mu}y\right) + \frac{1}{(n+1)^2}\frac{\theta^2\alpha}{\alpha+2\mu}.$

Lemma 21 *Let $n,m \in \mathbb{N}, \alpha,\mu > 0$ and $\theta \in (0,1]$. For all $(x,y) \in I^2$, there holds*

(i) $L_{n,m}^{(\theta,\alpha,\mu)}((t-x);x,y) = \frac{\theta\alpha - x(\alpha+\mu)}{(n+1)(\alpha+\mu)},$

(ii) $L_{n,m}^{(\theta,\alpha,\mu)}((t-x)^2;x,y) = \frac{1}{(n+1)^2}\left(x^2 - x\left(\frac{2\theta\alpha}{\alpha+\mu}\right) + \frac{\theta^2\alpha}{\alpha+2\mu}\right) + \frac{n}{(n+1)^2}x(1-x) = \delta_n(x),$

(iii) $L_{n,m}^{(\theta,\alpha,\mu)}((s-y);x,y) = \frac{\theta\alpha - y(\alpha+\mu)}{(m+1)(\alpha+\mu)},$

(iv) $L_{n,m}^{(\theta,\alpha,\mu)}((s-y)^2;x,y) = \frac{1}{(m+1)^2}\left(y^2 - y\left(\frac{2\theta\alpha}{\alpha+\mu}\right) + \frac{\theta^2\alpha}{\alpha+2\mu}\right) + \frac{m}{(m+1)^2}y(1-y) = \delta_m(y).$

For $g \in C(I^2)$ and δ_n, δ_m, the first order complete modulus of continuiy for the bivariate case is defined as follows :

$$\omega(g;\delta_n,\delta_m) \quad = \quad \sup\{|g(t,s) - g(x,y)| : |t-x| \le \delta_n, |s-y| \le \delta_m\}.$$

The function $\omega(g;\delta_n,\delta_m)$ satisfies the following properties:

(i) $\omega(g;\delta_n,\delta_m) \to 0$, if $\delta_n \to 0, \delta_m \to 0,$

(ii) $|g(t,s)-g(x,y)| \leq \omega(g;\delta_n,\delta_m)\left(1+\frac{|t-x|}{\delta_n}\right)\left(1+\frac{|s-y|}{\delta_m}\right)$

Theorem 22 *For $f \in C(I^2)$, we have*

$$\lim_{n,m\to\infty} L_{n,m}^{(\theta,\alpha,\mu)}(f(t,s);x,y) = f(x,y)$$

uniformly in $(x,y) \in I^2$.

Proof. In view of Lemma 20, there follows

$$\lim_{n,m\to\infty} L_{n,m}^{(\theta,\alpha,\mu)}(e_{i,j};x,y) = e_{i,j}(x,y).$$

uniformly in $(x,y) \in I^2$ for $(i,j) \in \{(0,0),(1,0),(0,1)\}$, and

$$L_{n,m}^{(\theta,\alpha,\mu)}(e_{2,0}+e_{0,2};x,y) \quad = \quad e_{2,0}(x,y)+e_{0,2}(x,y),$$

uniformly in $(x,y) \in I^2$. Hence, the theorem is proved by applying the Volkov theorem [46] ■
In the following result, we estimate the rate of convergence of the bivariate operators (37) in terms of the complete modulus of continuity

Theorem 23 *Let $g \in C(I^2)$. Then for all $(x,y) \in I^2$ we have*

$$\left|L_{n,m}^{(\theta,\alpha,\mu)}(g;x,y)-g(x,y)\right| \leq 4\omega\left(g,\sqrt{\delta_n(x)},\sqrt{\delta_m(y)}\right) \tag{38}$$

Proof. By using the linearity and positivity of the operator $L_{n,m}^{(\theta,\alpha,\mu)}(f;x,y)$ and taking into account the property (ii) of modulus of continuity,

$$\begin{aligned}
\left|L_{n,m}^{(\theta,\alpha,\mu)}(g;x,y)-g(x,y)\right| &\leq L_{n,m}^{(\theta,\alpha,\mu)}\left(|(g;x,y)-g(x,y)|;x,y\right) \\
&\leq \omega\left(g;\sqrt{\delta_n(x)},\sqrt{\delta_m(y)}\right)\Bigg(L_{n,m}^{(\theta,\alpha,\mu)}(1;x,y) \\
&+ \frac{1}{\sqrt{\delta_n(x)}}L_{n,m}^{(\theta,\alpha,\mu)}\left(|t-x|;x,y\right) \\
&+ \frac{1}{\sqrt{\delta_m(y)}}L_{n,m}^{(\theta,\alpha,\mu)}\left(|s-y|;x,y\right) \\
&+ \frac{1}{\sqrt{\delta_n(x)}}\frac{1}{\sqrt{\delta_m(y)}}L_{n,m}^{(\theta,\alpha,\mu)}\left(|t-x|\,|s-y|;x,y\right)\Bigg).
\end{aligned}$$

Applying the Cauchy Schwarz inequality and using Lemma 20, we obtain

$$\begin{aligned}
\left|L_{n,m}^{(\theta,\alpha,\mu)}(g;x,y)-g(x,y)\right| &\leq \omega\left(g;\sqrt{\delta_n(x)},\sqrt{\delta_m(y)}\right)\left(1+\frac{\sqrt{L_{n,m}^{(\theta,\alpha,\mu)}\left((t-x)^2;x,y\right)}}{\sqrt{\delta_n(x)}}\right. \\
&= \frac{1}{\sqrt{\delta_m(y)}}\sqrt{L_{n,m}^{(\theta,\alpha,\mu)}\left((s-y)^2;x,y\right)} \\
&= \left.\frac{\sqrt{L_{n,m}^{(\theta,\alpha,\mu)}\left((t-x)^2;x,y\right)}\sqrt{L_{n,m}^{(\theta,\alpha,\mu)}\left((s-y)^2;x,y\right)}}{\sqrt{\delta_n(x)}\sqrt{\delta_m(y)}}\right),
\end{aligned}$$

which yields us the desired result. ■
For $g \in C(I^2)$ and $\delta > 0$, the partial moduli of continuity with respect to the x and y are given by

$$\omega_1(g;\delta) \quad = \quad \sup\left(|g(x_1,y)-g(x_2,y)| : y \in [0,1] \text{and} \, |x_1-x_2| \leq \delta\right)$$

and

$$\omega_2(g;\delta) \quad = \quad \sup\left(|g(x,y_1)-g(x,y_2)| : x\in[0,1] \text{and} \, |y_1-y_2|\leq \delta\right).$$

It is clear that they satisfy the properties of the usual modulus of continuity.

Theorem 24 *Let $g\in C(I^2)$ and $(x,y)\in I^2$ be arbitrary. Then there holds the inequality*

$$\left|L_{n,m}^{(\theta,\alpha,\mu)}(g;x,y)-g(x,y)\right| \quad \leq \quad 2\left(\omega_1\left(g;\sqrt{\delta_n(x)}\right)+\omega_2\left(g;\sqrt{\delta_m(y)}\right)\right).$$

Proof. Using the definition of the partial moduli of continuity, we have

$$\begin{aligned}
\left|L_{n,m}^{(\theta,\alpha,\mu)}(g;x,y)-g(x,y)\right| &\leq L_{n,m}^{(\theta,\alpha,\mu)}\left(|g(t,s)-g(x,y)|;x,y\right)\\
&= L_{n,m}^{(\theta,\alpha,\mu)}\left(|g(t,s)-g(t,y)|;x,y\right)\\
&+ L_{n,m}^{(\theta,\alpha,\mu)}\left(|g(t,y)-g(x,y)|;x,y\right)\\
&\leq \omega_1\left(g;\sqrt{\delta_n(x)}\right)\left(L_{n,m}^{(\theta,\alpha,\mu)}(1;x,y)\right.\\
&+ \left.\frac{1}{\sqrt{\delta_n(x)}}L_{n,m}^{(\theta,\alpha,\mu)}\left(|t-x|;x,y\right)\right)\\
&+ \omega_2\left(g;\sqrt{\delta_m(y)}\right)\left(L_{n,m}^{(\theta,\alpha,\mu)}(1;x,y)\right.\\
&+ \left.\frac{1}{\sqrt{\delta_m(y)}}L_{n,m}^{(\theta,\alpha,\mu)}\left(|s-y|;x,y\right)\right).
\end{aligned}$$

Now, applying the Cauchy Schwarz inequality and Lemma 20

$$\begin{aligned}
\left|L_{n,m}^{(\theta,\alpha,\mu)}(g;x,y)-g(x,y)\right| &\leq \omega_1\left(g;\sqrt{\delta_n(x)}\right)\left(1+\frac{1}{\sqrt{\delta_n(x)}}\sqrt{L_{n,m}^{(\theta,\alpha,\mu)}\left((t-x)^2;x,y\right)}\right)\\
&+ \omega_2\left(g;\sqrt{\delta_m(y)}\right)\left(1+\frac{1}{\sqrt{\delta_m(y)}}\sqrt{L_{n,m}^{(\theta,\alpha,\mu)}\left((s-y)^2;x,y\right)}\right)\\
&= 2\left(\omega_1\left(g;\sqrt{\delta_n(x)}\right)+\omega_2\left(g;\sqrt{\delta_m(y)}\right)\right).
\end{aligned}$$

This completes the proof of the theorem. ■

Now, we study the degree of approximation for the bivariate operators (37) for functions in the Lipschitz class. For $0\leq\lambda_1,\lambda_2\leq 1$, the Lipschitz class $Lip_M(\lambda_1,\lambda_2)$ for a function of two variables is defined as follows:

$$|g(t,s)-g(x,y)| \quad \leq \quad M_g\left|t-x\right|^{\lambda_1}\left|s-y\right|^{\lambda_2},$$

where $(t,s),(x,y)\in I^2$ and M_g is a positive constant.

Theorem 25 *Let $g\in Lip_M(\lambda_1,\lambda_2)$. Then for all $(x,y)\in I^2$, we have*

$$\left|L_{n,m}^{(\theta,\alpha,\mu)}(g;x,y)-g(x,y)\right| \quad \leq \quad M_g\delta_n^{\frac{\lambda_1}{2}}(x)\delta_m^{\frac{\lambda_2}{2}}(y).$$

Proof. By the hypothesis, we may write

$$\begin{aligned}
\left|L_{n,m}^{(\theta,\alpha,\mu)}(g;x,y)-g(x,y)\right| &\leq L_{n,m}^{(\theta,\alpha,\mu)}\left(|g(t,s)-g(x,y)|;x,y\right)\\
&= M_gL_{n,m}^{(\theta,\alpha,\mu)}\left(\left|t-x\right|^{\lambda_1}\left|s-y\right|^{\lambda_2};x,y\right)
\end{aligned}$$

Now, using the Hölder inequality with $p_1 = \frac{2}{\lambda_1}, q_1 = \frac{2}{2-\lambda_1}$ and $p_2 = \frac{2}{\lambda_2}, q_2 = \frac{2}{2-\lambda_2}$, in view of Lemma 20, we get

$$\begin{aligned}\left|L_{n,m}^{(\theta,\alpha,\mu)}(g;x,y) - g(x,y)\right| &\leq M_g L_{n,m}^{(\theta,\alpha,\mu)}\left((t-x)^2;x,y\right)^{\frac{\lambda_1}{2}} L_{n,m}^{(\theta,\alpha,\mu)}(1;x,y)^{\frac{2-\lambda_1}{2}} \\ &\times L_{n,m}^{(\theta,\alpha,\mu)}\left((s-y)^2;x,y\right)^{\frac{\lambda_2}{2}} L_{n,m}^{(\theta,\alpha,\mu)}(1;x,y)^{\frac{2-\lambda_2}{2}} \\ &= M_g \delta_n^{\frac{\lambda_1}{2}}(x)\delta_m^{\frac{\lambda_2}{2}}(y).\end{aligned}$$

This completes the proof. ■

10 Graphical simulations

We provide numerical results and graphical simulations for $L_n^{(\theta,\alpha,\mu)}(f;x)$ and $L_{n,m}^{(\theta,\alpha,\mu)}(f;x,y)$ generated using Matlab. These simulations align with our theoretical results.

Example 26 *Let $f(x) = e^{0.5x}x^2(1-x)$. We consider $n \in \{50, 100, 150, 200\}$, $\alpha = 2, \theta = 0.8$ and $\mu = 0.5$ for $L_n^{(\theta,\alpha,\mu)}(f;x)$ with $x \in [0,1]$. Figure 4 illustrates the convergence of operators $L_n^{(\theta,\alpha,\mu)}(f;x)$ to $f(x)$ for fixed α, θ and μ as n increasing. Moreover, Table 1 reports the corresponding absolute errors*

$$E_n^{(0.8,2,0.5)}(f;x) = \left|L_n^{(0.8,2,0.5)}(f;x) - f(x)\right|,$$

showing that the approximation error decreases consistently as n increases.

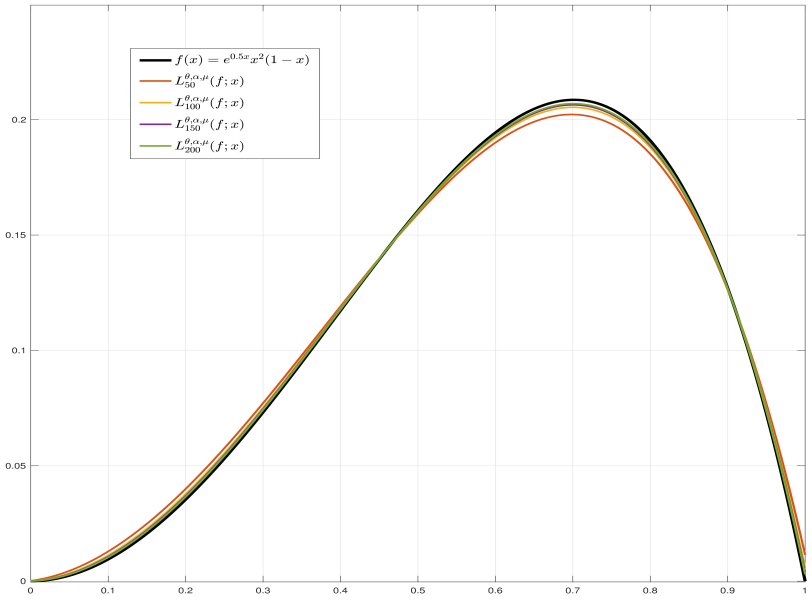


Figure 4: Approximation of $L_n^{(\theta,\alpha,\mu)}$ to $f(x) = e^{0.5x}x^2(1-x)$. with the parameter set $\alpha = 2, \theta = 0.8$ and $\mu = 0.5$.

Table 1: $E_n^{(0.8,2,0.5)}(f;x) = \left|L_n^{(0.8,2,0.5)}(f;x) - f(x)\right|$ for $f(x) = e^{0.5x}x^2(1-x)$.

x	$E_{50}^{(0.8,2,0.5)}(f;x)$	$E_{100}^{(0.8,2,0.5)}(f;x)$	$E_{150}^{(0.8,2,0.5)}(f;x)$	$E_{200}^{(0.8,2,0.5)}(f;x)$
0.0	0.00016293	0.000041683	0.000018670	0.000010543
0.1	0.00341000	0.00171970	0.00114970	0.00086352
0.2	0.00460610	0.00235140	0.00157860	0.00118810
0.3	0.00394840	0.00203280	0.00136860	0.00103150
0.4	0.00180760	0.00094904	0.00064309	0.00048627
0.5	0.00123390	0.00060514	0.00040058	0.00029934
0.6	0.00433750	0.00220190	0.00147540	0.00110930
0.7	0.00635770	0.00325350	0.00218600	0.00164590
0.8	0.00578450	0.00298150	0.00200800	0.00151370
0.9	0.00067768	0.00038123	0.00026393	0.00020167
1.0	0.01140700	0.00581760	0.00390430	0.00293800

Example 27 *Let* $f(x,y) = \sin(2\pi x)\cos(2\pi y) + 0.5\cos(\pi xy) + 0.25e^{x+y}$. *We consider* $n, m \in \{15, 40\}$, $\alpha = 2.8, \theta = 0.75$ *and* $\mu = 1.4$ *for* $L_{n,m}^{(\theta,\alpha,\mu)}(f;x,y)$ *with* $(x,y) \in [0,1]\times[0,1]$. *Figure 5 illustrates the convergence of operators* $L_{n,m}^{(\theta,\alpha,\mu)}(f;x,y)$ *to* $f(x,y)$ *for fixed* α, θ *and* μ *as* n *and* m *increasing. Moreover, Figure 6 displays the corresponding absolute error* $\left|L_{n,m}^{(\theta,\alpha,\mu)}(f;x,y) - f(x,y)\right|$, *demonstrating that the approximation error decreases as the values of* n *and* m *increase.*

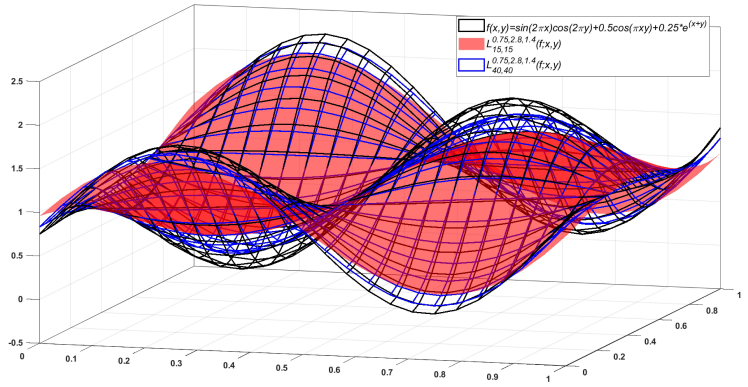


Figure 5: Approximation of $L_{n,m}^{(\theta,\alpha,\mu)}$ to $f(x,y) = \sin(2\pi x)\cos(2\pi y) + 0.5\cos(\pi xy) + 0.25e^{(x+y)}$ with the parameter set $\alpha = 2.8, \theta = 0.75$ and $\mu = 1.4$.

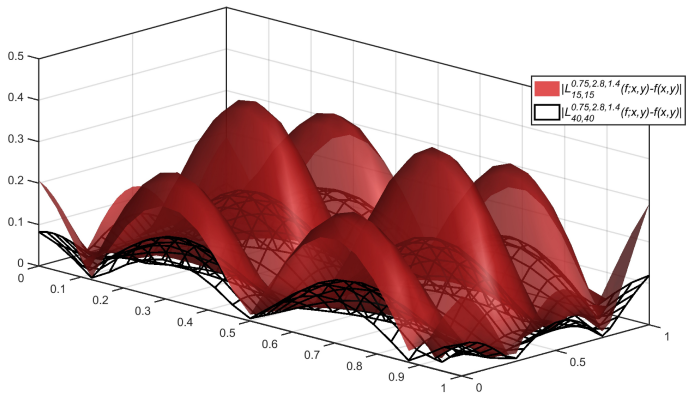


Figure 6: Absolute error of $L_{n,m}^{(\theta,\alpha,\mu)}$ in approximating $f(x,y) = \sin(2\pi x)\cos(2\pi y) + 0.5\cos(\pi xy) + 0.25e^{x+y}$ with the parameter set $\alpha = 2.8$, $\theta = 0.75$, and $\mu = 1.4$.

11 Conclusion

In this paper, we introduced a new family of fractional Bernstein–Kantorovich type operators based on the left-sided form of the Riemann–Liouville fractional integral. The proposed operators generalize the classical Bernstein–Kantorovich operators and provide a more flexible framework for approximating continuous functions through the parameters α, μ, and θ.

The fundamental properties of the operators were investigated by deriving their moments and central moments, which are essential tools for analyzing approximation behavior. Korovkin-type approximation theorems were established, and the uniform convergence of the operators to the identity function was proved under the Bohman–Korovkin conditions. Quantitative estimates of the convergence rate were obtained by means of the modulus of continuity and Peetre's K-functional, and the local approximation properties were examined using Lipschitz-type classes.

The asymptotic behavior of the operators was analyzed through a Voronovskaya-type theorem, which provides an explicit expression for the limit of $n(L_n^{(\theta,\alpha,\mu)}(f;x) - f(x))$ in terms of the first and second derivatives of f. A quantitative Voronovskaya-type theorem was also derived using the second-order modulus of smoothness, yielding a higher order of approximation. In addition, a Grüss–Voronovskaya-type theorem was established to describe the asymptotic behavior of the difference between $L_n^{(\theta,\alpha,\mu)}(fg;x)$ and the product $L_n^{(\theta,\alpha,\mu)}(f;x)L_n^{(\theta,\alpha,\mu)}(g;x)$.

The shape-preserving properties of the operators were studied, and it was shown that the operators preserve monotonicity and convexity of the approximated functions. A bivariate extension of the operators was constructed by means of the tensor product technique. For the bivariate case, Korovkin-type convergence results were established, and the rate of approximation was analyzed using both the complete and partial moduli of continuity, as well as Lipschitz-type classes.

Numerical examples and graphical simulations were presented to illustrate the convergence behavior of the proposed operators for both the univariate and bivariate cases. The results demonstrate that the approximation error decreases as the values of n and m increase, and that the parameters α, μ, and θ provide additional flexibility in improving the approximation accuracy.

These findings suggest that the proposed operators can serve as useful tools in approximation theory, fractional calculus, and related applied fields. Future work may focus on further generalizations of these operators, including their q-analogues, Stancu-type modifications, and applications to other classes of functions and real-world data modeling problems.

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