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Information consistent regret and finite population corrections for routing games

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Information consistent regret and finite population corrections for routing games

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Abstract

Finite-player evaluations of routing policies depend on when a deviating traveller learns the other travellers' route choices. Optimizing expected travel time before those choices are revealed and averaging the best deviation after their realization produce different quantities. We give an exact decomposition of this difference and a two-link example in which ex-ante exploitability decays as the inverse population size, while hindsight regret decays as its inverse square root. We then derive a type-dependent correction for static unsplittable routing games with independent route randomization, fixed type counts, and equal player weights. The correction combines a tagged traveller's own contribution to congestion with the curvature effect of random link occupancy. For four-times continuously differentiable latency functions, its uniform cost error is of order N^{-2}. A profile with corrected maximum type regret at most η consequently has true finite-player exploitability at most $\eta + 2LK/N^2$, where the constants are explicit. Deterministic calculations on an affine Pigou example, a quartic two-link example, and a network with two traveller types illustrate the distinction between information effects and finite-population effects. These results provide evaluation and calibration tools for static routing benchmarks; they do not establish approximation rates for dynamic traffic with queues or spillback.

Keywords: routing games; finite populations; exploitability; information; mean field approximation; heterogeneous travellers

1 Introduction

A routing policy is often evaluated by the travel time a traveller could save through a unilateral deviation. Such a statement is incomplete until the information available at the time of deviation is specified. A route chosen before other travellers act is different from a route selected after their choices have been revealed. This distinction matters even in a two-link network, and it can change the apparent rate at which a mean field policy approximates a finite-player equilibrium.

Cabannes et al. [1] use mean field policies to approximate finite-player dynamic routing games. Their full preprint motivates the present question because the policy-based deviation definition and the realized-count calculation in its Pigou appendix appear to use different information sets. The analysis below is self-contained and restricts attention to simultaneous static route choice. It does not revise the dynamic simulator or establish that the other reported experiments use the same estimator.

The second issue is the accuracy of a nonatomic policy when deployed by a finite population. A traveller who selects an edge adds its own mass to that edge. Moreover, nonlinear travel times depend on the distribution of the other travellers' choices, rather than only their expected flow. We retain both effects in a corrected cost map and translate its approximation error directly into an ex-ante regret certificate.

The ingredients have established precedents. Binomial expected resource costs appear in Ashlagi, Monderer and Tennenholtz [2]. Cominetti et al. [3] develop convergence and quantitative bounds for large atomic congestion games. Refined mean field methods explicitly retain finite-size terms [4,5], and second-order approximation accuracy in heterogeneous stochastic systems is already known [5]. We therefore do not claim that binomial averaging, finite-size corrections, or the order N^{-2} are new ideas. Our focus is their use in information-consistent routing evaluation, with a type-dependent correction, an explicit uniform remainder, and a solver-residual-to-exploitability bound for the model specified below.

The contributions are an exact comparison of two evaluation criteria, a finite-population correction for multiple route-choice populations, and a reproducible benchmark protocol. The mathematical argument does not require uniqueness of equilibrium or independence of occupancies across edges. It does require independent randomization across travellers and static, separable link costs.

2 Information available to a deviating traveller

Consider a traveller with a finite route set $\mathcal{A}$. Let Z denote all opponents' random actions. Write $C(a, Z)$ for the cost of selecting route a, including the traveller's own effect on congestion. A prescribed mixed action A is independent of Z. Define its baseline expected cost and ex-ante exploitability by

$$J = \mathbb{E}[C(A, Z)], \qquad R = J - \min_{a \in \mathcal{A}} \mathbb{E}[C(a, Z)].$$

The minimizing route is chosen before observing Z. Randomizing the deviation cannot improve upon a pure minimizing route because expected cost is linear in the deviating distribution. Now define hindsight regret by

$$H = J - \mathbb{E}[\min_{a \in \mathcal{A}} C(a, Z)].$$

Here the hypothetical deviator observes the opponents' actions, holds those actions fixed, and then chooses a best response. This is an information benchmark. It need not be implementable by a traveller in a simultaneous game.

Proposition 1. For every such game,

$$H = R + V, \qquad V = \min_{a} \mathbb{E}[C(a, Z)] - \mathbb{E}[\min_{a} C(a, Z)] \geq 0.$$

Proof. Add and subtract the minimum expected cost. For every fixed route a, the pointwise minimum is at most $C(a, Z)$. Taking expectations and then minimizing over a proves nonnegativity. Thus V is the value of revealing opponents' realized actions to the deviator. Equality holds if a single route minimizes cost almost surely, and can hold in other degenerate cases. No asymptotic argument is needed.

This identity also explains why averaging a per-realization best response can remain positive at a mixed Nash equilibrium. At that equilibrium, $R = 0$ while V may be positive. Reporting H as a measure of the original policy's Nash exploitability would change the deviation class being tested.

2.1 Exact separation on a Pigou network

There are N travellers, each of weight $1/N$. One route costs $1/2$ and the other costs its total normalized load. All travellers independently choose the congestible route with probability $1/2$. For a tagged traveller, the number of opponents using that route is $M \sim \mathrm{Bin}(N - 1, 1/2)$. The two expected action costs and the prescribed cost are

$$\mathbb{E}[C(0,M)] = \frac{1}{2}, \quad \mathbb{E}[C(1,M)] = \frac{N+1}{2N}, \quad J = \frac{1}{2} + \frac{1}{4N}.$$

Consequently,

$$R_N = \frac{1}{4N}, \qquad H_N = \frac{1}{2}\mathbb{E}\left|\frac{M+1}{N} - \frac{1}{2}\right|.$$

Proposition 2. The two criteria have different asymptotic orders:

$$NR_N = \frac{1}{4}, \qquad \sqrt{N}H_N \longrightarrow \frac{1}{2\sqrt{2\pi}}.$$

Proof. The first identity follows above. Put $Y_N = (M - (N-1)/2)/\sqrt{N}$. The central limit theorem gives convergence to a centered normal variable of variance $1/4$. Its second moments are bounded, so $|Y_N + 1/(2\sqrt{N})|$ is uniformly integrable. Its expectation converges to the absolute first moment of that normal variable, namely $1/\sqrt{2\pi}$. Multiplication by $1/2$ proves the claim.

For $N = 4$, the criteria are $R_N = 0.0625$ and $H_N = 0.09375$. For $N = 100$, they are 0.0025 and approximately 0.0198973. These are different estimands, not different numerical estimators of one estimand. Multiplying all latencies by a time scale T multiplies both quantities by T.

The ex-ante symmetric equilibrium probability is also explicit:

$$p_N = \frac{N-2}{2(N-1)} \quad (N \geq 2), \qquad p_1 = 0.$$

At $N = 2$ the expression equals zero; for $N > 2$ it equalizes the two expected action costs. Its ex-ante regret is zero. This elementary correction is useful as a unit test, rather than as a standalone novelty claim.

3 Static routing with several traveller types

Let E be a finite edge set. Type $k \in \{1, \ldots, m\}$ has $N_k \geq 1$ travellers and a finite nonempty set $\mathcal{P}_k$ of simple admissible routes. Different types may represent different origins, destinations, or access restrictions. All travellers have weight $1/N$, where $N = \sum_k N_k$, and choose routes simultaneously. An edge's travel time is $c_e(z)$ at normalized load $z \in [0,1]$. Route cost is the sum of its edge costs.

All travellers of type k use the same route distribution q_k, with independent draws across travellers. Define the edge-use probability, population share, mean edge load, and normalized occupancy variance by

$$p_{ke} = \sum_{P \in \mathcal{P}_k : e \in P} q_{kP}, \quad \lambda_k = \frac{N_k}{N}, \quad x_e = \sum_k \lambda_k p_{ke}, \quad v_e = \sum_k \lambda_k p_{ke}(1 - p_{ke}).$$

For a tagged type-k traveller who chooses a route containing edge e, let Y_{ke} count opponents on that edge. Its distribution is a sum of independent binomials: the type-k term has $N_k - 1$ trials and every other type has N_j trials. The exact expected edge cost is

$$g^N_{ke}(q) = \mathbb{E}\left[c_e\left(\frac{1 + Y_{ke}}{N}\right)\right], \qquad G^N_{kP}(q) = \sum_{e \in P} g^N_{ke}(q).$$

The additional one is essential: it represents the deviating traveller. Changing a route changes which edges receive this additional mass. For shared edges, it is counted only once. Correlation between different edges' occupancies does not alter the expectation of an additive route cost.

Type-specific exploitability and its population average are

$$R_k^N(q) = \sum_{P \in \mathcal{P}_k} q_{kP} G_{kP}^N(q) - \min_{P \in \mathcal{P}_k} G_{kP}^N(q), \qquad \overline{R}^N = \sum_k \lambda_k R_k^N.$$

Proposition 3. The independent, symmetric-within-type profile induced by q is a mixed Nash equilibrium if and only if $R_k^N(q) = 0$ for every type. Equivalently, every route used by a type minimizes that type's exact expected route cost.

Proof. Hold all other travellers' mixed actions fixed. The tagged traveller's expected cost for a pure route is $G_{kP}^N(q)$. Its best mixed deviation is a minimum-cost pure route. Its current expected cost is the q_k-weighted average. The difference is exactly $R_k^N(q)$, and all travellers of a type face the same distribution of opponents.

In the single-type case, $g_e^N(p_e)$ is a scalar binomial transform of c_e. In the multiple-type case, the map depends on all type-specific edge probabilities. Treating it as a single common latency function of total mean flow generally loses information.

4 A correction with an explicit regret certificate

Suppose each c_e is four-times continuously differentiable on $[0, 1]$. Write

$$M_r = \max_e \sup_{z \in [0,1]} |c_e^{(r)}(z)|, \qquad r = 2, 3, 4,$$

and set

$$K = \frac{M_2}{2} + \frac{M_3}{6} + \frac{M_4}{6}.$$

Define the corrected expected edge cost by

$$\widetilde{g}_{ke}^N(q) = c_e(x_e) + \frac{(1 - p_{ke})c_e'(x_e) + \frac{1}{2} v_e c_e''(x_e)}{N}.$$

The first correction is the mean load change caused by conditioning on the tagged traveller's use of the edge. The second captures nonlinear averaging over opponents' occupancy fluctuations. Their type dependence comes from the tagged traveller, even when all types experience the same physical latency function.

Theorem 4. Uniformly over all profiles, all type counts, all edges and all tagged types,

$$|g_{ke}^N(q) - \widetilde{g}_{ke}^N(q)| \le \frac{K}{N^2}.$$

Proof. Fix an edge and type and suppress their indices. Let $p = p_{ke}$, $a = 1 - p$, $Z = Y - \mathbb{E}Y$, and $\Delta = (1 + Y)/N - x = (Z + a)/N$. Direct calculation gives

$$\mathbb{E}\Delta = \frac{a}{N}, \qquad \mathbb{E}\Delta^2 = \frac{v}{N} + \frac{a(1 - 2p)}{N^2}.$$

The variable Y is a sum of $N-1$ independent Bernoulli variables, with variance $S \le (N-1)/4$. Independence and the centered Bernoulli moments imply

$$|\mathbb{E}Z^3| \le S, \qquad \mathbb{E}Z^4 \le 3S^2 + S.$$

Since $0 \le a \le 1$, expansion of the third and fourth powers yields

$$|\mathbb{E}\Delta^3| \le \frac{4S + 1}{N^3} \le \frac{1}{N^2}, \qquad \mathbb{E}\Delta^4 \le \frac{3S^2 + 11S + 1}{N^4} \le \frac{4}{N^2}.$$

Expand $c(x + \Delta)$ through its cubic term. The fourth-order Taylor remainder in absolute value is at most $M_4\Delta^4/24$. Subtract $c(x) + ac'(x)/N + vc''(x)/(2N)$ and take expectations.

The remaining quadratic, cubic, and remainder terms are bounded respectively by $M_2/(2N^2)$, $M_3/(6N^2)$, and $M_4/(6N^2)$. This proves the claimed uniform bound. The constants are deliberately conservative.

Let $L = \max_{k,P\in\mathcal{P}_k} |P|$. Define corrected route costs by summing $\widetilde{g}^N_{ke}$ along each route, and define $\widetilde{R}^N_k$ using the same average-minus-minimum formula as R^N_k.

Corollary 5. If $\max_k \widetilde{R}^N_k(q) \le \eta$, then

$$\max_k R^N_k(q) \le \eta + \frac{2LK}{N^2}.$$

Proof. Each route-cost error is at most LK/N^2. Its weighted average has at most the same error. The minimum of a finite list changes by no more than the largest entrywise perturbation. Adding the two bounds proves the result.

This certificate concerns the largest type-level ex-ante deviation gain. A small population-average regret alone would not provide the same guarantee for a small group. The result bounds exploitability, not distance between route distributions or total system travel time.

4.1 Existence and special cases

A zero-regret profile for the corrected map exists. On the product of the route simplexes, map each profile to the product of distributions supported on its corrected minimum-cost routes. Continuity gives a nonempty, convex, compact-valued, upper hemicontinuous correspondence. Kakutani's fixed-point theorem gives a fixed point. This is an existence argument; it neither proves that a selected numerical algorithm converges nor that the fixed point is unique.

For affine latencies, $c_e(z) = \alpha_e + \beta_e z$, the correction is exact:

$$g^N_{ke}(q) = \alpha_e + \beta_e x_e + \frac{\beta_e(1-p_{ke})}{N}.$$

Thus a zero-regret corrected profile is an exact finite-player mixed equilibrium in this model. For one type, this reduces to the common transformed latency $\alpha_e + \beta_e/N + \beta_e(1-1/N)p_e$. The Pigou correction in Section 2 follows immediately.

4.2 A direct evaluation procedure

Specify the information available to a deviator and whether route choices are independent. Compute edge-use probabilities separately by type, then compute x_e, v_e and the corrected edge costs. Solve the resulting population best-response conditions or another equivalent variational inequality and report the achieved maximum type residual η. Combine that residual with Corollary 5. For benchmarks, also evaluate the exact tagged-player expected costs whenever tractable.

This procedure uses mean occupancies and variances to form the correction. Its cost-map evaluation scales with the number of types and edges rather than the number of action profiles. It does not by itself remove the cost of route enumeration or equilibrium computation. Shortest-path routines may be used where their assumptions, including treatment of possible negative corrected edge costs, are satisfied.

5 Reproducible constructed examples

All examples below are constructed mathematical instances. They use no field observations or calibrated city data. The accompanying Python program uses only the standard library. It evaluates the polynomial expected costs through exact moment identities in floating-point arithmetic; the reported values are not Monte Carlo estimates. Scalar equilibria are found by bisection. The two-type example uses damped Newton iterations for an interior solution, with

maximum equation residual below 2×10^{-14}. All finite-player regrets are then recomputed using the exact expected cost map.

5.1 The effect of information

Table 1 evaluates the uncorrected half-half policy on the affine Pigou network. The corrected policy has zero ex-ante regret in every listed case.

Table 1: Ex-ante and hindsight regret for the same policy with time scale one

N	Ex-ante regret	Hindsight regret	Ratio
4	0.062500	0.093750	1.50
10	0.025000	0.061523	2.46
30	0.008333	0.036116	4.33
100	0.002500	0.019897	7.96
1000	0.000250	0.006306	25.23

The ratio grows with population size because the two quantities have different orders. A sample-by-sample best-response calculation can therefore make an increasingly accurate ex-ante policy appear much less accurate when it is evaluated against additional information.

5.2 Nonlinear travel times

The second network has two routes with costs $1/2$ and z^4. Its nonatomic congestible-route probability is $2^{-1/4}$. We compare that probability with equilibria of the corrected and exact cost maps. Table 2 reports true finite-player ex-ante regret, using the same evaluator for both policies.

Table 2: Ex-ante regret on the quartic two-route network

N	Nonatomic policy	Corrected policy
10	7.535400e-2	7.807669e-4
20	3.872680e-2	2.032777e-4
40	1.962578e-2	5.049927e-5
80	9.878450e-3	1.252517e-5
160	4.955608e-3	3.115772e-6
320	2.481899e-3	7.768308e-7
640	1.241973e-3	1.939335e-7

At $N = 10$, the nonatomic, corrected, and exact probabilities are approximately 0.840896, 0.793701, and 0.795733. The correction reduces exploitability substantially without exactly recovering the equilibrium distribution. Between $N = 320$ and $N = 640$, the uncorrected regret decreases by a factor of approximately 2, and the corrected regret by approximately 4. These numerical ratios illustrate, rather than prove, the asymptotic behavior. Exact-cost equilibrium residuals in this example are below 10^{-12}.

5.3 Two types sharing a congested edge

Let type proportions be $\lambda_1 = 0.4$ and $\lambda_2 = 0.6$, with N a multiple of five. Each type chooses between a route containing a common edge followed by its own private edge and a private constant-cost alternative. The common edge has cost $0.2 + z^4$. Type 1's private edge costs $0.1+0.6z^2$ and its alternative costs 0.5. Type 2's private edge costs $0.15+0.8z^2$ and its alternative

costs 0.65. Private edges are used by only their own type. All loads are normalized by total population N.

The graph has one origin and two destinations. The common edge connects the origin to a branching node, and each type's private edge connects that node to its destination. Direct edges from the origin supply the constant-cost alternatives. Each shared route has two edges.

Table 3: Maximum type-level ex-ante regret on the shared-edge network

N	Nonatomic policy	Corrected policy
10	8.306668e-2	5.015941e-3
20	4.135891e-2	1.093425e-3
40	2.061852e-2	2.353841e-4
80	1.029180e-2	5.323893e-5
160	5.141249e-3	1.256854e-5
320	2.569427e-3	3.047403e-6
640	1.284409e-3	7.498928e-7

The nonatomic shared-route probabilities are approximately

$$(0.581891, 0.678298).$$

At $N = 10$, the corrected probabilities are $(0.412823, 0.629341)$ and the exact ones are $(0.332584, 0.664675)$. The sizable distribution difference at this small population illustrates why a regret certificate should not be described as a policy-distance certificate. All exact-cost solutions have maximum type regret below 10^{-12}.

For these costs, $M_2 = 12$, $M_3 = 24$, $M_4 = 24$, and $K = 14$. With $L = 2$, the theoretical additive certificate is $56/N^2$. It is conservative relative to Table 3, and should not be presented as an empirically tight uncertainty interval.

5.4 Computational checks and reproducibility

The implementation verifies the polynomial moment formula by independently enumerating all opponents' binary choices for a two-type, five-player instance and both tagged types. It also checks the bound in Theorem 4 on 2,000 deterministically seeded polynomial instances with three types, allowing signed polynomial coefficients. The largest observed ratio of cost error to the proved bound is approximately 0.639. This diagnostic does not replace the proof or test every smooth function. The affine formula and the four-player Pigou hindsight value have separate arithmetic checks.

Running `python3 reproduce.py` regenerates `results.json` and the three numerical CSV files. Parameters, solver tolerances and the stress-test seed are specified in the program. No external package or private input is required.

6 Interpretation and limits

The information comparison and the finite-population correction address different questions. Proposition 1 determines which deviations an evaluation allows. Theorem 4 approximates the expected cost of an allowed ex-ante deviation. Applying the correction while still minimizing separately within each realized sample would retain the information mismatch.

The static assumptions are substantive. In a dynamic network, a deviation may change arrival times, queue lengths, downstream choices, and the information observed by other travellers. Even the opponent trajectories may change when their policy functions remain fixed. Frozen-trajectory hindsight comparisons and re-simulated unilateral policy deviations can therefore disagree for

additional reasons. The present Taylor calculation cannot establish the dynamic approximation rate claimed by a separate model.

Independence across travellers is also necessary. Correlated recommendations, common shocks and coordinated fleets alter the variance and higher moments used in Theorem 4. Unequal player weights require a weighted tagged-player expansion. Threshold costs and hard capacity rules may violate the smoothness assumption. These cases require separate analysis rather than substitution into the displayed correction.

For static evaluation, average marginal regret remains a useful measure of unilateral improvement [6]. Its value depends on the feasible deviation set and the supplied travel costs. It does not alone measure welfare, emissions, fairness, or the causal effect of navigation use. Population-average and maximum type regret should be accompanied by a separate system objective whenever an intervention is meant to improve total travel time.

The examples are intentionally small enough to permit exact evaluation. They establish that the implementation matches the displayed formulas on these instances. They do not establish runtime advantages on large networks, robustness to uncertain demand, or empirical effectiveness. A direct dynamic extension would need an information-adapted best-response evaluator and control of how a unilateral deviation perturbs the evolving state distribution.

7 Conclusion

Ex-ante exploitability and hindsight regret differ by the value of information revealed to a hypothetical deviator. On a two-link benchmark, this difference changes the observed population-size rate from N^{-1} to $N^{-1/2}$. For independent static routing with several traveller types, a correction accounting for self-congestion and occupancy curvature yields a uniform N^{-2} cost error and a corresponding maximum-regret certificate. The combination provides a reproducible way to distinguish evaluation choices from approximation error in finite-population routing experiments.

References

[1] Cabannes, T., Laurière, M., Perolat, J., Marinier, R., Girgin, S., Perrin, S., Pietquin, O., Bayen, A. M., Goubault, E., and Elie, R. (2021). Solving N-player dynamic routing games with congestion: a mean field approach. arXiv:2110.11943v2. https://arxiv.org/abs/2110.11943v2

[2] Ashlagi, I., Monderer, D., and Tennenholtz, M. (2009). Two-terminal routing games with unknown active players. Artificial Intelligence, 173(15), 1441–1455. https://doi.org/10.1016/j.artint.2009.07.002

[3] Cominetti, R., Scarsini, M., Schröder, M., and Stier-Moses, N. (2023). Approximation and convergence of large atomic congestion games. Mathematics of Operations Research, 48(2), 784–811. Published online 2022. https://doi.org/10.1287/moor.2022.1281

[4] Gast, N., and Van Houdt, B. (2017). A refined mean field approximation. Proceedings of the ACM on Measurement and Analysis of Computing Systems, 1(2), Article 33, 1–28. Author manuscript: https://inria.hal.science/hal-01622054

[5] Allmeier, S., and Gast, N. (2021). Mean field and refined mean field approximations for heterogeneous systems: It works! arXiv:2111.01594. https://arxiv.org/abs/2111.01594

[6] Cabannes, T., Sangiovanni, M., Keimer, A., and Bayen, A. M. (2019). Regrets in routing networks: Measuring the impact of routing apps in traffic. ACM Transactions on Spatial Algorithms and Systems, 5(2), Article 9, 1–19. https://doi.org/10.1145/3325916