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**Som Subhro Nath**

nathsomsubhro7@gmail.com

Independent Researcher https://orcid.org/0009-0007-6842-0019

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# Basis-Cardinality Invariance in Fuzzy Closure Systems: Computational Evidence and Algebraic Characterisation

Som Subhro Nath
Independent Researcher
Kolkata, West Bengal, India
E-mail ID: nathsomsubhro7@gmail.com
ORCID: 0009-0007-6842-0019

## Abstract

Basis-cardinality invariance is a structural property of closure systems whose dependence on the underlying algebraic order remains insufficiently characterised in the fuzzy setting. This paper examines the phenomenon in finite fuzzy closure systems over integral commutative residuated lattices through exhaustive computational enumeration and algebraic analysis.

Three complementary investigations are reported. An exhaustive enumeration across nine finite algebra–universe configurations identifies 61 closure systems with unequal-sized inclusion-minimal bases among 173 systems examined. A separate investigation of six finite Heyting algebras identifies 23 such systems among 72. More significantly, a singleton-generator analysis across 15 finite algebras produces 30 witnesses in all 12 tested non-chain cases, while none of the three chain controls exhibits this behaviour. Independent verification of 346 explicit bases yields no discrepancies.

Beyond enumeration, a sufficient algebraic criterion is established for the existence of singleton-generated closure systems admitting minimal bases of cardinalities one and two. A further algebraic identity, satisfied by all 30 experimentally identified singleton-generator witnesses, provides a common representation of their generation mechanism. The results establish a consistent computational distinction between the tested chain and non-chain structures and identify an algebraic mechanism underlying the observed failure of basis-cardinality invariance. They do not establish the universal converse for finite non-chain residuated lattices. Instead, they provide a verified computational foundation and a precise structural framework for its investigation.

**Keywords:** Fuzzy closure systems; Residuated lattices; Basis-cardinality invariance; Minimal bases; Finite Heyting algebras; Computational algebra; Algebraic characterisation; Exhaustive enumeration.

# 1. Introduction

The cardinality of inclusion-minimal generating sets is a fundamental structural property of closure systems. Although a closure system may admit several distinct minimal bases, their cardinalities are not necessarily equal. The distinction between uniqueness of a generating set and invariance of its cardinality is mathematically significant: the former concerns representation, whereas the latter concerns the underlying structure of generation.

In fuzzy closure theory, this problem acquires an additional algebraic dimension. Residuated lattices provide the framework for representing graded membership, with lattice operations and residuated implication governing the generation and closure of fuzzy sets. Consequently, the cardinalities of minimal bases are influenced not only by the organisation of the underlying lattice but also by the interaction between its algebraic operations. The conditions under which these interactions preserve basis-cardinality invariance therefore constitute a natural problem in the structural study of fuzzy closure systems.

A particularly relevant case arises when the underlying residuated lattice is linearly ordered. In this setting, basis-cardinality invariance is established in the existing theory. The converse, however, requires separate consideration. Specifically, it is not immediate whether the absence of linear ordering necessarily permits a closure system with inclusion-minimal bases of unequal cardinalities. The distinction between these two directions motivates the present investigation.

The central objective is to examine basis-cardinality variation in finite fuzzy closure systems over integral commutative residuated lattices through exhaustive computational enumeration and algebraic analysis. The investigation considers multiple finite algebraic structures, including Gödel and Łukasiewicz algebras, Boolean lattices, their finite products, and Heyting algebras. Both general closure systems and singleton-generated closure systems are considered.

Singleton-generated systems provide a particularly restrictive setting for this investigation. For an element (x) of a residuated lattice $(L)$, consider the closure system $[\mathcal{S}_x = a \to x \mid a \in L.]$By construction, $(x)$ is a generating base of $(\mathcal{S}_x)$. If the same closure system admits an inclusion-minimal base containing more than one element, basis-cardinality invariance fails. This formulation reduces the search for certain counterexamples to the identification of alternative minimal generating families for a single algebraic element.

The computational investigation examines nine finite algebra–universe configurations, comprising 173 closure systems, of which 61 exhibit unequal-sized inclusion-minimal bases. A complementary study of six finite Heyting algebras examines 72 closure systems and identifies 23 further instances of basis-cardinality variation. In a separate singleton-generator analysis spanning 15 finite algebras, 30 witnesses are identified across all 12 tested non-chain structures, whereas none of the three chain controls exhibits such a witness. An independent verification of 346 explicit bases produces no discrepancies.

The computational observations are accompanied by an algebraic analysis of the generation mechanisms underlying the identified witnesses. A sufficient condition is established for a singleton-generated closure system to possess inclusion-minimal bases of cardinalities one and two. In addition, a recurring identity involving residuated multiplication, lattice joins, and implication is identified across the experimentally obtained singleton-generator witnesses. These results provide an algebraic account of the observed computational phenomenon and distinguish the demonstrated sufficient condition from the broader structural pattern suggested by the experiments.

The contribution of this investigation lies in the combination of exhaustive finite enumeration, independent verification, and algebraic analysis of unequal-sized minimal bases. The

results establish basis-cardinality variation across the investigated finite structures and provide a sufficient algebraic criterion for a specific class of counterexamples. They do not establish the universal converse for all finite non-chain integral commutative residuated lattices. Accordingly, the conclusions are restricted to the verified computational cases and the mathematical statements proved herein.

The remainder of the paper is organised as follows. Section 2 reviews the relevant literature on fuzzy closure systems, residuated lattices, and basis-cardinality invariance. Section 3 establishes the algebraic preliminaries and formal definitions. Section 4 presents the algebraic results concerning singleton-generated closure systems and unequal-sized minimal bases. Section 5 describes the computational experiments and their results. Section 6 discusses the mathematical and computational findings, while Section 7 concludes the paper. References and research declarations are provided in the subsequent sections.

# 2. Literature Review

## 2.1. Algebraic Foundations of Fuzzy Closure Systems

The study of fuzzy closure systems requires a precise distinction between classical closure structures equipped with fuzzy properties and closure systems whose elements are themselves lattice-valued. This distinction is not merely terminological; it determines the algebraic operations under which closure, generation, and minimality are defined.

Ojeda-Hernández et al. [2] addressed this foundational issue by examining alternative formulations of fuzzy closure systems and their correspondence with closure operators. Their analysis established a definition of fuzzy closure systems that admits a one-to-one correspondence with fuzzy closure operators. This contribution provides an important formal foundation for subsequent investigations. However, establishing the correspondence between closure systems and closure operators does not determine the structure or cardinalities of their minimal generating families.

Gao et al. [3] approached the subject from an algebraic and categorical perspective. Working with complete residuated lattices, they introduced algebraic fuzzy closure operators and algebraic fuzzy closure L-systems on fuzzy complete lattices. Their principal results establish correspondences between these structures under appropriate conditions and demonstrate their categorical relationships. This provides a systematic account of algebraic fuzzy closure structures. Nevertheless, categorical equivalence and algebraic characterisation do not, by themselves, resolve the combinatorial properties of inclusion-minimal bases.

The algebraic framework underlying these developments is supplied by residuated lattices. Their defining adjoint relationship,

$$[a \otimes b \leq c \quad \Leftrightarrow \quad a \leq b \rightarrow c,]$$

connects conjunction and implication and permits the extension of classical set operations to lattice-valued structures. The resulting generation process is more intricate than ordinary intersection-based closure. In particular, the interaction between scalar shifts and lattice meets introduces structural constraints that have no direct counterpart in the Boolean setting.

## 2.2. Bases of Closure Systems over Residuated Lattices

The most directly relevant contribution is the work of Bělohlávek and Konečný [1], *Bases of closure systems over residuated lattices*. The authors investigated the generation of L-closure systems and established a decomposition theorem for general closure operators. Their analysis demonstrated that bases need not be unique and may possess different cardinalities. They further obtained a constructive

description of the size of a largest base and established that all bases have equal cardinality when the underlying residuated lattice is linearly ordered.

This result provides the principal mathematical starting point for the present investigation. It establishes linear ordering as a sufficient condition for basis-cardinality invariance and confirms that unequal-sized bases can occur in the general residuated-lattice setting. However, the chain condition does not establish a necessary-and-sufficient characterisation. In particular, the implication

$$[L \text{ is a chain} \quad \Longrightarrow \quad \mathrm{BCI}(L)]$$

does not, by itself, establish its converse,

$$[\mathrm{BCI}(L) \quad \Longrightarrow \quad L \text{ is a chain.}]$$

The latter implication requires independent mathematical justification. This distinction is central to the present investigation.

## 2.3. Fuzzy Closure Operators and Related Algebraic Structures

The broader theory of fuzzy closure operators provides additional context for the present problem. Bělohlávek [4] developed the theory of fuzzy closure operators over lattice-valued sets, establishing foundational properties of closure in the many-valued setting. His subsequent work on concept lattices and order in fuzzy logic [5] connected fuzzy Galois structures with order-theoretic representations.

Bělohlávek, Funioková, and Vychodil [6] examined fuzzy closure operators with truth stressers, demonstrating that the formulation of monotonicity and closure requires careful treatment in many-valued logic. These contributions are relevant to the formal specification of the closure operations used in the present work. However, their principal focus is the algebraic behaviour of closure operators rather than the cardinality distribution of their inclusion-minimal generating families.

The relationship between fuzzy closure systems and relational structures has also been investigated. Ojeda-Hernández et al. [7] studied fuzzy closure relations and their correspondence with closure systems in complete fuzzy lattices. Similarly, Ojeda-Hernández et al. [8] investigated fuzzy closure structures through formal concepts and fuzzy Galois connections.

These studies provide important representational and order-theoretic developments. Nevertheless, the existence of a correspondence between closure operators, relational structures, and formal concepts does not imply that all minimal bases of the corresponding closure system possess equal cardinality. Representation and basis-cardinality invariance remain distinct mathematical questions.

## 2.4. Minimal Representations and Implication-Based Approaches

Minimality in fuzzy formal concept analysis has also been studied through attribute implications. Bělohlávek and Vychodil [9] investigated the computation of non-redundant bases of fuzzy attribute implications using maximal independent sets. Their work provides an algorithmic approach to reducing implication systems while preserving their logical consequences.

Bělohlávek and Vychodil [10] further developed fuzzy Horn logic through its proof-theoretic foundations. These contributions are relevant to the broader study of dependencies, generation, and minimal representation in fuzzy systems. However, an implication basis and a closure-system generating base are not identical mathematical objects. The former concerns a minimal representation of logical dependencies, whereas the latter concerns an inclusion-minimal family of elements generating an entire closure system under specified operations.

This distinction is particularly important in the present setting. Results concerning the reduction of implication systems cannot be directly interpreted as results concerning the equality of cardinalities

of arbitrary minimal bases. The underlying generation operations and the corresponding notions of minimality must be examined independently.

## 2.5. Structural Developments in Fuzzy Closure Theory

Further developments have examined the internal structure of fuzzy closure systems and their associated elements. Ojeda-Hernández et al. [11] investigated closure systems as fuzzy extensions of meet-subsemilattices, providing an additional perspective on the relationship between classical and lattice-valued closure structures.

Ojeda-Hernández et al. [12] subsequently examined quasi-closed elements in complete fuzzy lattices. Their investigation extends the characterisation of quasi-closed elements to fuzzy closure systems and connects these elements with the construction of implication bases. The study is relevant to the analysis of minimal representations, although quasi-closedness addresses a different structural property from the cardinality invariance considered here.

Other work on L-fuzzy closure systems and their algebraic properties has contributed to the development of the subject. In particular, the relationship between fuzzy closure operators, fuzzy complete lattices, and their categorical representations has been investigated in several settings. These results establish a broad foundation for the study of fuzzy closure structures, but they do not provide a general criterion for determining when all inclusion-minimal bases of a finite fuzzy closure system have equal cardinality.

## 2.6. Research Gap and Positioning of the Present Study

The existing literature establishes the algebraic foundations of fuzzy closure systems, their correspondence with closure operators, their categorical structure, and the existence of unequal-sized bases over residuated lattices. Most importantly, Bělohlávek and Konečný [1] established basis-cardinality invariance when the underlying residuated lattice is linearly ordered.

The structural relationship between non-chain algebraic structures and the failure of basis-cardinality invariance remains the central question examined in this investigation. The established chain result does not provide a universal converse, and the cited foundational and categorical studies do not directly supply such a characterisation.

The present study approaches this problem through exhaustive computational enumeration of selected finite residuated lattices and finite Heyting algebras, together with an algebraic analysis of the resulting witnesses. Particular attention is given to singleton-generated closure systems,

$$[\mathcal{S}_x = a \to x \mid a \in L,]$$

for which the singleton $(x)$ provides an immediate generating base. The existence of an alternative inclusion-minimal generating family with more than one element therefore constitutes a direct witness of basis-cardinality variation.

The investigation further establishes a sufficient algebraic condition for the existence of singleton-generated closure systems with minimal bases of cardinalities one and two. In addition, a recurring algebraic identity is identified across the experimentally obtained singleton-generator witnesses, providing a common representation of their generation mechanism.

The contribution is directed towards the computational verification and algebraic explanation of basis-cardinality variation in finite fuzzy closure systems. The findings establish the phenomenon in the investigated structures and provide a sufficient mathematical criterion for a particular class of witnesses. They do not establish the universal converse for all finite non-chain integral commutative residuated lattices.

# 3. Preliminaries and Mathematical Framework

## 3.1. Integral Commutative Residuated Lattices

Throughout this paper, $\big(L = (L,\wedge,\vee,\otimes,\rightarrow,0,1)\big)$ denotes an integral commutative residuated lattice. Thus, $((L,\wedge,\vee,0,1))$ is a bounded lattice, $\big((L,\otimes,1)\big)$is a commutative monoid, and the operations $(\otimes)$and $(\rightarrow)$ satisfy the residuation property

$$[a \otimes b \leq c \quad \Leftrightarrow \quad a \leq b \rightarrow c, \qquad a,b,c \in L.]$$

Integrality means that the monoidal identity coincides with the greatest element of the lattice. Commutativity ensures that $(a \otimes b = b \otimes a)$ for all $(a,b \in L)$.

Several standard consequences of residuation are used throughout the subsequent analysis. In particular,

$$[a \otimes b \leq a, \qquad a \otimes b \leq b,]$$

and

$$[a \rightarrow 1 = 1, \qquad 1 \rightarrow a = a, \qquad 0 \rightarrow a = 1.]$$

Moreover, residuation implies that the operation $\big(a \rightarrow (\cdot)\big)$preserves arbitrary meets whenever they exist. Consequently,

$$(a \rightarrow b) \wedge (a \rightarrow c).$$

Commutativity and residuation also yield the identity

$$(a \otimes b) \rightarrow c.$$

These identities are essential for analysing the interaction between scalar shifts and meet-based generation.

A residuated lattice is called a chain if its underlying lattice order is linear. Otherwise, it is referred to as a non-chain residuated lattice. The distinction between these two classes is central to the investigation of basis-cardinality invariance.

## 3.2. Fuzzy Sets and Scalar Shifts

Let $(U)$ be a nonempty universe. An (L)-fuzzy set on (U) is a mapping

$$[A \rightarrow L.]$$

The collection of all (L)-fuzzy sets on (U) is denoted by (L^U). It is equipped with the pointwise ordering

$$[A \leq B \quad \Leftrightarrow \quad A(u) \leq B(u) \quad \text{for every } u \in U.]$$

The meet and join of fuzzy sets are defined pointwise:

$$[(A \wedge B)(u) = A(u) \wedge B(u),]$$

$$[(A \vee B)(u) = A(u) \vee B(u).]$$

For $(a \in L)$ and $(A \in L^U)$, the scalar shift of (A) by (a) is defined as

$$[(a \rightarrow A)(u) = a \rightarrow A(u), \qquad u \in U.]$$

The scalar shift is an essential operation in the present framework. Unlike ordinary set-theoretic closure, generation is not determined exclusively by intersections. The interaction between scalar shifts and pointwise meets is responsible for several of the algebraic properties investigated subsequently.

In particular, equation (5) gives

$$(a \otimes b) \to A.$$

Thus, successive scalar shifts can be represented by a single shift determined by the monoidal product of their coefficients.

## 3.3. Fuzzy Closure Systems

A nonempty family ($\mathcal{S} \subseteq L^U$) is called an (L)-fuzzy closure system if it satisfies the following conditions:

1. It is closed under arbitrary pointwise meets. That is, for every indexed family $\left(A_{i_{i\in I}} \subseteq \mathcal{S}\right)$,

$$\left[\bigwedge_{i\in I} A_i \in \mathcal{S}.\right]$$

2. It is closed under scalar shifts. That is, for every $(A \in \mathcal{S})$ and every $(a \in L)$,

$$[a \to A \in \mathcal{S}.]$$

The empty meet is interpreted as the greatest fuzzy set, namely the constant function $(1_U)$. Hence every such closure system contains $(1_U)$.

The closure-system convention adopted here explicitly includes both arbitrary meets and scalar shifts. This specification is necessary because alternative formulations of fuzzy closure systems may employ different closure operations. All subsequent definitions and computational results refer to the convention stated above.

## 3.4. Generation and Minimal Bases

Let ($\mathcal{S} \subseteq L^U$) be an (L)-fuzzy closure system, and let ($X \subseteq \mathcal{S}$). The closure system generated by $(X)$, denoted by $([X]_{\mathcal{S}})$, is the smallest (L)-fuzzy closure system containing $(X)$. Equivalently,

$$\bigcap\{\mathcal{T}: X \subseteq \mathcal{T} \subseteq \mathcal{S},; \mathcal{T} \text{ is an } L\text{-fuzzy closure system}\}.$$

A subset ($B \subseteq \mathcal{S}$) is called a base of $(\mathcal{S})$ if

$$[[B]_{\mathcal{S}} = \mathcal{S}.]$$

A base (B) is inclusion-minimal if

$$[[B \setminus A]_{\mathcal{S}} \neq \mathcal{S} \qquad \text{for every } A \in B.]$$

The collection of all inclusion-minimal bases of $(\mathcal{S})$ is denoted by

$$[\mathrm{MinBas}(\mathcal{S})\,.]$$

For finite closure systems, every base contains an inclusion-minimal base. The present investigation concerns the cardinalities of these inclusion-minimal generating families, rather than uniqueness of the bases themselves.

## 3.5. Basis-Cardinality Invariance

The property of interest is defined at the level of the underlying residuated lattice.

**Definition 1 (Basis-cardinality invariance).** A finite integral commutative residuated lattice (L) satisfies basis-cardinality invariance, denoted by $\big(\mathrm{BCI}(L)\big)$, if, for every finite universe (U), every (L)-fuzzy closure system ($\mathcal{S} \subseteq L^U$), and every pair of inclusion-minimal bases ($B_1, B_2 \in \mathrm{MinBas}(\mathcal{S})$),

$$[|B_1| = |B_2|.]$$

Failure of this property is witnessed by a finite fuzzy closure system possessing two inclusion-minimal bases of different cardinalities. Formally, (L) fails BCI if there exist a finite universe (U), an (L)-fuzzy closure system($\mathcal{S} \subseteq L^U$), and bases ($B_1, B_2 \in \mathrm{MinBas}(\mathcal{S})$) such that

$$[|B_1| \neq |B_2|.]$$

This definition distinguishes basis-cardinality invariance from uniqueness. A closure system may admit several distinct inclusion-minimal bases while satisfying BCI, provided that all such bases have equal cardinality.

## 3.6. Singleton-Generated Closure Systems

A particularly useful class of closure systems arises from a single fuzzy set. For ($x \in L^U$), define

$$a \to x \mid a \in L.$$

When (U) consists of a single element, fuzzy sets on (U) can be identified with elements of (L). Accordingly, for $(x \in L)$, equation (19) reduces to

$$a \to x \mid a \in L \subseteq L.$$

The family $(\mathcal{S}_x)$ is an$(L)$-fuzzy closure system on a singleton universe. Indeed, it is closed under scalar shifts because

$$(c \otimes a) \to x \in \mathcal{S}_x.$$

It is also closed under arbitrary meets. For any family$\big(a_{i\,i\in I} \subseteq L\big)$,

$$\left(\bigvee_{i\in I} a_i\right) \to x,$$

where the join exists because $(L)$ is complete. In the finite setting considered here, completeness is automatic.

The element $(x)$ belongs to $(\mathcal{S}_x)$, since

$$[1 \to x = x.]$$

Furthermore, $(\mathcal{S}_x)$ is generated by $(x)$, because scalar shifts of $(x)$ produce precisely its elements. Therefore,

$$\big[[x] = \mathcal{S}_x.\big]$$

The singleton $(x)$ is consequently an inclusion-minimal base of $(\mathcal{S}_x)$.

This observation provides a restrictive criterion for detecting failure of basis-cardinality invariance. If there exist $(p, q \in \mathcal{S}_x)$ such that

$$[[p,q] = \mathcal{S}_x,]$$

while

$$[[p] \neq \mathcal{S}_x, \qquad [q] \neq \mathcal{S}_x,]$$

then $(p,q)$ is an inclusion-minimal base of cardinality two, whereas $(x)$ is an inclusion-minimal base of cardinality one. Hence,

$$[\mathrm{BCI}(L) \text{ fails.}]$$

The singleton-generated construction is therefore adopted as a principal experimental framework for identifying unequal-cardinality bases.

## 3.7. Notation and Scope

For clarity, the following notation is used throughout the paper:

| Symbol | Meaning |
|---|---|
| $(L)$ | Finite integral commutative residuated lattice |
| $(U)$ | Universe |
| $(L^U)$ | Set of all (L)-fuzzy sets on (U) |
| $(\mathcal{S})$ | (L)-fuzzy closure system |
| $(a \to A)$ | Scalar shift of fuzzy set (A) |
| $([X]_{\mathcal{S}})$ | Closure system generated by (X) |
| $(\mathrm{MinBas}(\mathcal{S}))$ | Inclusion-minimal bases of $(\mathcal{S})$ |
| $(\mathcal{S}_x)$ | Closure system generated by a singleton element (x) |
| $\big(\mathrm{BCI}(L)\big)$ | Basis-cardinality invariance property |

Unless otherwise stated, all residuated lattices considered in the subsequent sections are finite, integral, and commutative. All closure systems are understood in the sense of Definitions and conditions (11)–(12). The mathematical results and computational experiments are formulated relative to this fixed convention.

# 4. Algebraic Conditions for Basis-Cardinality Variation

## 4.1. Generation of Singleton-Generated Closure Systems

Let $\big(L = (L,\wedge,\vee,\otimes,\to,0,1)\big)$ be a finite integral commutative residuated lattice, and let $(x \in L)$. Consider the singleton-generated closure system

$$[\mathcal{S}_x = a \to x \mid a \in L.]$$

By construction, $(x)$ is an inclusion-minimal base of $(\mathcal{S}_x)$. The relevant question is whether another inclusion-minimal base of greater cardinality can generate the same closure system.

The following elementary identity provides the algebraic basis for the subsequent results.

**Lemma 1.** For arbitrary $(a, b, x \in L)$,

$$(a \otimes b) \to x.$$

Furthermore, for any $(a, b \in L)$,

$$(a \vee b) \to x.$$

**Proof.** Equation (28) follows directly from commutativity and residuation. For every $(z \in L)$,

$$\begin{aligned} z \le a \to (b \to x) &\Leftrightarrow z \otimes a \le b \to x \\ &\Leftrightarrow (z \otimes a) \otimes b \le x \\ &\Leftrightarrow z \otimes (a \otimes b) \le x \\ &\Leftrightarrow z \le (a \otimes b) \to x. \end{aligned}$$

Hence the two elements are equal.

For equation (29), residuation gives

$$\begin{aligned} z \le (a \to x) \wedge (b \to x) &\Leftrightarrow z \otimes a \le x \text{ and } z \otimes b \le x \\ &\Leftrightarrow z \otimes (a \vee b) \le x \\ &\Leftrightarrow z \le (a \vee b) \to x. \end{aligned}$$

The equality follows. (□)

**Lemma 2.** For finite (L), the closure system generated by $(p \in L)$on a singleton universe is

$$a \to p \mid a \in L.$$

More generally, for $(p, q \in L)$,

$$\{(a \to p) \wedge (b \to q) \mid a, b \in L\}.$$

**Proof.** The right-hand side of equation (30) contains (p), is closed under scalar shifts by equation (28), and is closed under finite meets by equation (29). It is therefore a closure system containing (p). Every closure system containing (p) must also contain all its scalar shifts, establishing equality.

For equation (31), the displayed family contains both generators, since $(1 \to p = p)$ and $(1 \to q = q)$. It is closed under scalar shifts and finite meets by the same identities. Conversely, every closure system containing (p) and (q) must contain all displayed elements. Thus equality holds. (□)

## 4.2. A Sufficient Condition for Unequal-Sized Minimal Bases

The following theorem establishes a sufficient algebraic condition for the failure of basis-cardinality invariance.

**Theorem 1.** Let (L) be a finite integral commutative residuated lattice. Suppose that there exist $(b, c, x \in L)$satisfying

$$[b < 1, \qquad c < 1, \qquad b \vee c = 1,]$$

and

$$[t \to x = x \quad \Longrightarrow \quad t = 1 \qquad \text{for every } t \in L.]$$

Then the singleton-generated closure system

$$[\mathcal{S}_x = a \to x \mid a \in L]$$

admits two inclusion-minimal bases of different cardinalities, namely

$$[B_1 = x, \qquad B_2 = b \to x,\ c \to x.]$$

Consequently, (L) fails BCI.

**Proof.** Since $(1 \to x = x)$, equation (33) implies $(x \neq 1)$ whenever (L) is nontrivial. By construction,

$$\big[[x] = \mathcal{S}_x.\big]$$

Hence $(B_1 = x)$is an inclusion-minimal base.

Set

$$[p = b \to x, \qquad q = c \to x.]$$

Both (p) and (q) belong to $(\mathcal{S}_x)$. By equation (29),

$$\begin{aligned} p \wedge q &= (b \to x) \wedge (c \to x) \\ &= (b \vee c) \to x \\ &= 1 \to x \\ &= x. \end{aligned}$$

Since the closure system generated by $(p, q)$ is closed under meets, equation (35) implies that

$$[x \in [p, q].]$$

Closure under scalar shifts then yields

$$[\mathcal{S}_x \subseteq [p, q].]$$

The reverse inclusion follows because $(\mathcal{S}_x)$ is itself a closure system containing (p) and (q). Therefore,

$$\big[[p, q] = \mathcal{S}_x.\big]$$

It remains to establish inclusion-minimality. By Lemma 2,

$$(a \otimes b) \to x \mid a \in L.$$

Integrality implies

$$[a \otimes b \leq b < 1.]$$

Consequently, condition (33) gives

$$[(a \otimes b) \to x \neq x \qquad \text{for every } a \in L.]$$

Thus $(x \notin [p])$, and therefore

$$\big[[p] \neq \mathcal{S}_x.\big]$$

The same argument, using (c<1), establishes

$$\big[[q] \neq \mathcal{S}_x.\big]$$

Hence neither generator can be removed from (B_2), making it inclusion-minimal. The two minimal bases have cardinalities

$$[|B_1| = 1, \qquad |B_2| = 2.]$$

Therefore, (L) fails BCI. (□)

**Remark 1.** The theorem establishes a sufficient condition, not a necessary one. The existence of a pair $(b, c < 1)$ satisfying $(b \vee c = 1)$ is not required for every possible instance of unequal-sized minimal bases.

**Remark 2.** The condition in equation (33) is an algebraic separation condition on (x). It prevents any proper element of (L) from producing (x) through residuated implication. This condition is precisely what ensures that the two-element generating family is inclusion-minimal.

## 4.3. A Sufficient Algebraic Condition for Joint Generation

The preceding theorem employs the identity $(b \vee c = 1)$ to obtain a two-element generating family. The computational investigation also indicates a more general mechanism in which the two shifted generators need not originate from coefficients whose join is the greatest element.

The following proposition records the corresponding algebraic condition.

**Proposition 1.** Let $(a, b, c, d, x \in L)$, and define

$$[p = b \to x, \qquad q = c \to x.]$$

If

$$\big[\big((a \otimes b) \vee (d \otimes c)\big) \to x = x,\big]$$

then

$$[(a \to p) \wedge (d \to q) = x.]$$

Consequently,

$$[x \in [p, q].]$$

**Proof.** By equation (28),

$a \to (b \to x)(a \otimes b) \to x$,

and similarly,

$$(d \otimes c) \to x.$$

Applying equation (29),

$$\begin{aligned}(a \to p) \wedge (d \to q) &= \big((a \otimes b) \to x\big) \wedge \big((d \otimes c) \to x\big) \\ &= \big((a \otimes b) \vee (d \otimes c)\big) \to x \\ &= x.\end{aligned}$$

Since the generated closure system is closed under scalar shifts and meets, it contains the left-hand side of equation (40). Therefore, it contains (x). (□)

**Remark 3.** Equation (39) is a sufficient algebraic representation of joint generation. It does not, independently, guarantee that either (p) or (q) fails to generate $(\mathcal{S}_x)$. The latter must be verified separately to establish inclusion-minimality of the two-element base.

## 4.4. A General Criterion for Singleton-Generator Witnesses

The preceding results can be consolidated into a direct criterion.

**Theorem 2.** Let (L) be a finite integral commutative residuated lattice. Suppose that there exist $(x, p, q \in L)$ such that

$$[p, q \in \mathcal{S}_x, ]$$

$$[x \in [p, q], ]$$

and

$$[x \notin [p], \qquad x \notin [q].]$$

Then (L) fails BCI.

**Proof.** Since $(p, q \in \mathcal{S}_x)$ and $(\mathcal{S}_x)$ is a closure system,

$$\left[[p, q] \subseteq \mathcal{S}_x.\right]$$

Condition (42) implies that the generated closure system contains (x). By closure under scalar shifts, it therefore contains every element of $(\mathcal{S}_x)$. Thus,

$$\left[[p, q] = \mathcal{S}_x.\right]$$

Conditions (43) imply that neither singleton subset generates $(\mathcal{S}_x)$. Hence $(p, q)$ is an inclusion-minimal base of cardinality two. Since$(x)$ is an inclusion-minimal base of cardinality one, BCI fails. (□)

## 4.5. Structural Interpretation and Scope of the Results

The results establish an explicit algebraic mechanism for basis-cardinality variation in finite fuzzy closure systems. Theorem 1 identifies a sufficient condition under which two scalar shifts jointly recover the original generator through a lattice meet, while neither shifted element can independently recover it. Proposition 1 extends the algebraic description of this joint-generation mechanism by allowing the recovery of (x) through two further scalar shifts.

The distinction between these results is important. Theorem 1 provides a complete sufficient condition, including the verification of minimality. Proposition 1 provides an algebraic identity that guarantees joint generation but requires separate verification of the individual generation conditions. Theorem 2 isolates the exact closure-theoretic requirements for a singleton-generator witness.

The computational investigation reported subsequently identifies instances of the more general mechanism across the tested non-chain algebras. These observations are consistent with the algebraic framework developed here. They do not establish that every finite non-chain integral commutative residuated lattice admits such a witness.

Accordingly, the mathematical conclusions of this section are restricted to the sufficient conditions and implications explicitly proved above. The universal characterisation of basis-cardinality invariance remains separate from these results.

# 5. Computational Experiments and Results

## 5.1. Experimental design

The computational investigation was conducted to examine basis-cardinality invariance in finite fuzzy closure systems and to identify algebraic configurations associated with its failure. Three distinct

experiments were performed. The first exhaustively enumerated closure systems over nine finite algebra–universe configurations. The second examined closure systems induced by six finite Heyting algebras obtained from finite posets. The third investigated singleton-generated closure systems across fifteen finite algebras, testing for the existence of generators that produce minimal bases of unequal cardinalities.

The experiments address different aspects of the problem. The first measures the occurrence of unequal-sized inclusion-minimal bases in explicitly enumerated closure systems. The second tests whether the observed behaviour persists across a broader collection of finite Heyting algebras. The third isolates the algebraic mechanism associated with singleton generation and examines whether the resulting witnesses occur systematically in the tested non-chain structures.

All reported counts refer to the finite structures and configurations explicitly enumerated. They are not estimates of the frequency of basis-cardinality variation across arbitrary finite residuated lattices.

## 5.2. Exhaustive enumeration over nine algebra–universe configurations

The first experiment examined nine configurations involving finite Gödel, Boolean, and Łukasiewicz algebras. For each configuration, the closure systems satisfying the stipulated closure conditions were exhaustively enumerated. Each resulting system was examined for inclusion-minimal bases, and the cardinalities of those bases were compared.

Table 1 summarises the enumeration.

**Table 1. Exhaustive enumeration of finite fuzzy closure systems**

| **Algebra** | **Universe size** | **Closure systems** | **Systems with unequal-sized minimal bases** |
|---|---|---|---|
| Gödel chain ($C_2$) | 2 | 7 | 0 |
| Gödel chain ($C_3$) | 2 | 62 | 0 |
| Boolean lattice ($B_2$) | 2 | 49 | 36 |
| Gödel product ($C_2 \times C_3$) | 1 | 8 | 3 |
| Gödel product ($C_3 \times C_3$) | 1 | 16 | 9 |
| Boolean lattice ($B_3$) | 1 | 8 | 4 |
| Łukasiewicz product ($C_3 \times C_3$) | 1 | 9 | 4 |
| Mixed Gödel–Łukasiewicz product ($C_2 \times C_3$) | 1 | 6 | 2 |
| Łukasiewicz product ($C_2 \times C_4$) | 1 | 8 | 3 |
| **Total** | — | **173** | **61** |

Unequal-sized inclusion-minimal bases occurred in 61 of the 173 enumerated closure systems. The distribution was not uniform across the tested algebraic structures. Neither of the two Gödel-chain configurations produced a system with unequal-sized minimal bases. In contrast, the Boolean and product-algebra configurations exhibited such systems in multiple instances.

The results establish a clear distinction within the tested configurations. The absence of counterexamples in the two chain cases is consistent with basis-cardinality invariance for those particular enumerations. It does not, by itself, establish the property for all finite chains or for all closure systems over arbitrary chain-based residuated lattices. Likewise, the occurrence of counterexamples in the remaining configurations does not establish that every non-chain algebra admits such a counterexample.

The experiment therefore provides finite exhaustive evidence of basis-cardinality variation, rather than a universal classification.

## 5.3. Enumeration over finite Heyting algebras

A second experiment was conducted using finite Heyting algebras represented as lattices of upsets of finite posets. This construction provides a systematic means of obtaining finite distributive lattices and their associated Heyting implication operations.

Six poset configurations were examined. For each resulting Heyting algebra, the closure systems on a singleton universe were exhaustively enumerated and tested for unequal-sized inclusion-minimal bases.

**Table 2. Enumeration over finite Heyting algebras**

| **Poset configuration** | **Algebra cardinality** | **Closure systems** | **Systems with unequal-sized minimal bases** |
|---|---|---|---|
| Three-element chain | 4 | 8 | 0 |
| Three-element fork | 5 | 8 | 2 |
| Inverted three-element fork | 5 | 8 | 2 |
| Four-element diamond | 6 | 16 | 4 |
| N-poset | 7 | 16 | 6 |
| V-poset with an additional point | 10 | 16 | 9 |
| **Total** | — | **72** | **23** |

*Figure 1. Number of closure systems with unequal-sized inclusion-minimal bases across the six tested finite Heyting algebras.*

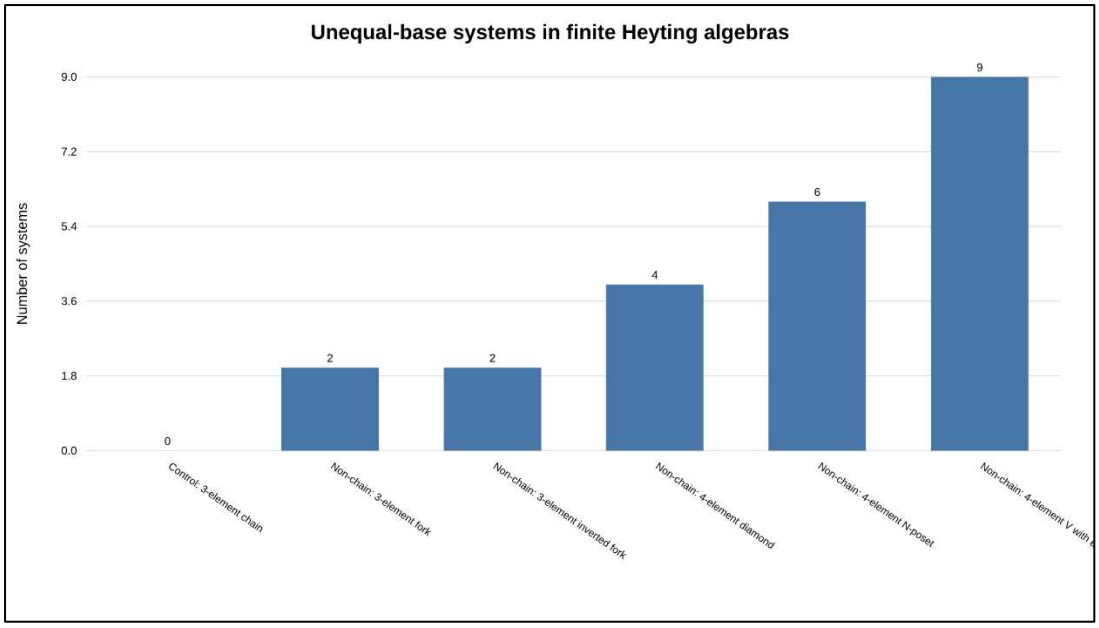
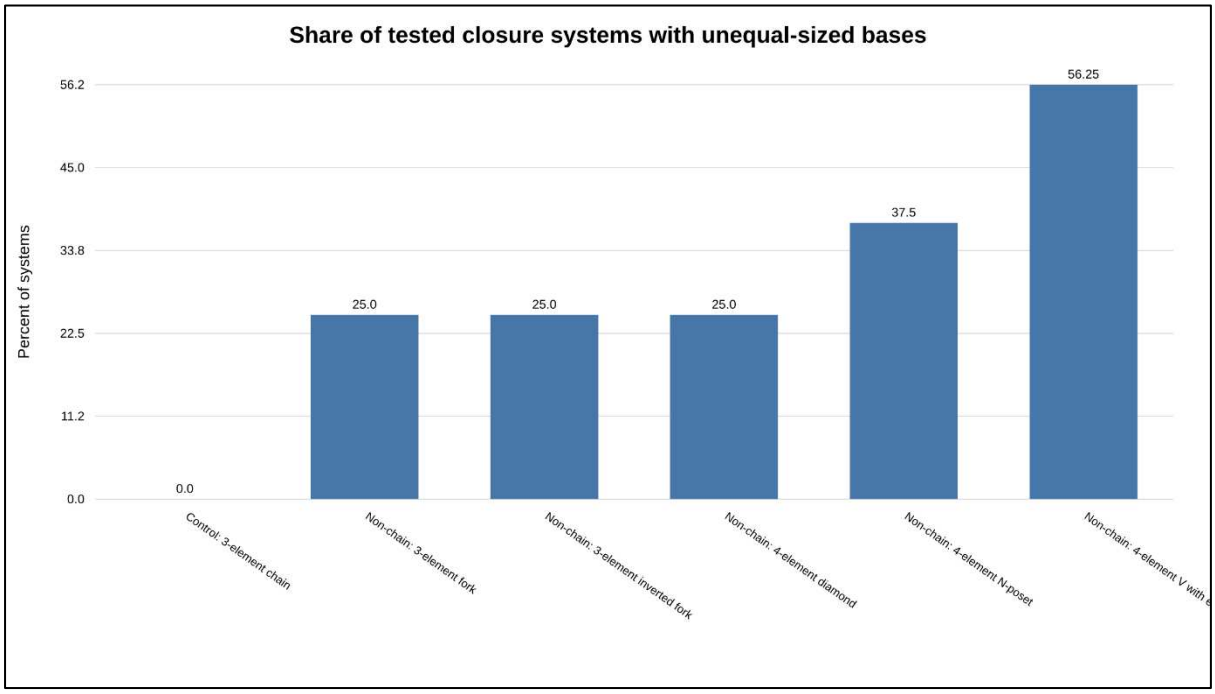



*Figure 2. Percentage of enumerated closure systems exhibiting unequal-sized inclusion-minimal bases across the six tested finite Heyting algebras*.




Unequal-sized minimal bases occurred in 23 of the 72 enumerated closure systems. The chain-derived Heyting algebra produced no counterexample, whereas each of the five non-chain poset configurations produced at least two.

This result is significant in conjunction with the first experiment. The observed distinction is not confined to the particular Gödel and product-algebra configurations considered previously. It also appears in finite Heyting algebras obtained from distinct non-chain order structures.

The data nevertheless remain finite and configuration-dependent. They support the existence of a recurring association between non-chain structure and basis-cardinality variation in the tested cases. They do not establish that non-chain order is, by itself, a sufficient condition for failure of basis-cardinality invariance in every finite Heyting algebra.

## 5.4. Singleton-generator witness analysis

The third experiment focused on singleton-generated closure systems. Unlike the preceding enumerations, this experiment was designed to identify explicit algebraic witnesses for the existence of minimal bases of different cardinalities.

For each of fifteen finite algebras, the singleton-generator construction was examined. The search tested whether an element (x) could generate a closure system$(\mathcal{S}_x)$ admitting a second minimal generating set consisting of two elements. The relevant witnesses were checked against the generation and minimality conditions established in Section 4.

**Table 3. Singleton-generator witnesses across fifteen finite algebras**

| **Algebra** | **Cardinality** | **Number of witnesses** |
|---|---|---|
| Chain $(C_2)$ | 2 | 0 |
| Chain $(C_3)$ | 3 | 0 |
| Three-element chain Heyting algebra | 4 | 0 |
| Boolean algebra $(B_2)$ | 4 | 1 |
| Gödel product $(C_2 \times C_3)$ | 6 | 2 |
| Gödel product $(C_3 \times C_3)$ | 9 | 4 |
| Boolean algebra $(B_3)$ | 8 | 4 |
| Łukasiewicz product $(C_3 \times C_3)$ | 9 | 4 |
| Mixed Gödel–Łukasiewicz product | 6 | 2 |
| Łukasiewicz product $(C_2 \times C_4)$ | 8 | 3 |
| Heyting algebra from the fork poset | 5 | 1 |
| Heyting algebra from the inverted fork poset | 5 | 1 |
| Heyting algebra from the diamond poset | 6 | 1 |
| Heyting algebra from the N-poset | 7 | 2 |
| Heyting algebra from the V-poset with an additional point | 10 | 5 |
| **Total** | — | **30** |

The experiment identified 30 witnesses across the twelve tested non-chain algebras. No witness was identified in any of the three chain controls.

Every identified witness satisfied the singleton-generator conditions required for the construction of minimal bases of cardinalities one and two. In particular, the two-element generating set generated the same closure system as the singleton generator, while neither of its individual elements generated that system independently.

The result is more specific than the preceding enumeration. It demonstrates that the observed variation can occur within singleton-generated closure systems, where the generating mechanism admits a direct algebraic description. Consequently, the phenomenon cannot be attributed solely to the complexity of closure systems on larger universes.

The complete set of 30 witnesses also satisfied the algebraic identity examined in Proposition 1. This common identity provides a uniform representation of the joint-generation mechanism observed in the computational data. Its satisfaction by all identified witnesses is an empirical observation over the tested cases; it is not a proof that every failure of basis-cardinality invariance must arise through this identity.

## 5.5. Independent verification

An independent verification was conducted for the nine algebra–universe configurations reported in Section 5.2. The verification reconstructed the corresponding finite structures and examined 346 explicit bases for generation and inclusion-minimality. No discrepancies were detected among the bases examined. This audit provides an independent consistency check of the reported enumeration; it does not constitute a universal verification of the computational procedures or of the results obtained in the other two experiments.

No discrepancy was detected.

This verification supports the internal consistency of the reported enumeration and the classification of the checked bases. It is important, however, to distinguish verification of computational output from proof of a general mathematical statement. The audit establishes agreement between the enumerated results and the independently checked conditions for the examined cases. It does not extend the conclusions beyond those cases, nor does it establish the correctness of every possible implementation of the underlying definitions.

## 5.6. Consolidated observations and limitations

The three experiments yield the following observations.

First, unequal-sized inclusion-minimal bases occur in multiple finite fuzzy closure systems over Boolean, Gödel-product, and Łukasiewicz-product algebras. The phenomenon is therefore not restricted to a single algebraic family.

Second, the finite Heyting-algebra enumeration exhibits the same distinction: the tested chain control produces no counterexample, whereas all five tested non-chain configurations produce at least one.

Third, the singleton-generator experiment identifies 30 explicit witnesses across twelve non-chain algebras. The three chain controls produce none. All identified witnesses satisfy the algebraic conditions associated with the singleton-generator construction developed in Section 4.

Fourth, the independent verification of 346 explicit bases detects no discrepancy in the nine-case exhaustive enumeration.

These findings must be interpreted within their computational scope. The experiments cover a finite collection of algebras, universes, and closure systems. They do not establish a universal equivalence between chain structure and basis-cardinality invariance. In particular, the absence of witnesses in the tested chain algebras is not a proof of invariance for every finite chain, and the presence of witnesses in the tested non-chain algebras is not a proof that every finite non-chain residuated lattice fails the property.

The computational evidence instead establishes a consistent pattern across the examined structures and identifies a concrete algebraic mechanism for a substantial class of counterexamples. The

sufficient criterion and witness identity developed in Section 4 provide the mathematical explanation for the singleton-generated cases. The experiments establish that these conditions are realised repeatedly in the tested finite algebras, while leaving the universal characterisation of basis-cardinality invariance unresolved.

# 6. Discussion

## 6.1. Structural interpretation of the computational results

The computational results establish that basis-cardinality invariance is not preserved across all finite fuzzy closure systems over integral commutative residuated lattices. The occurrence of unequal-sized inclusion-minimal bases in 61 of the 173 systems examined in the first experiment, together with 23 instances in the separate Heyting-algebra enumeration, demonstrates that basis-cardinality variation is realised across several distinct finite algebraic settings.

The significance of these findings lies not merely in the number of counterexamples, but in their distribution. Counterexamples occur in Boolean lattices, Gödel products, Łukasiewicz products, mixed products, and non-chain Heyting algebras. The phenomenon is therefore not attributable to a single implication operation or one particular construction of a finite lattice.

The chain controls exhibit a contrasting behaviour. No unequal-sized minimal bases were identified in the three chain configurations examined in the singleton-generator experiment, nor in the two chain configurations of the first enumeration or the chain-derived Heyting algebra. This consistency is noteworthy, but its logical significance must remain restricted to the tested structures. The computational evidence does not establish chain structure as a necessary and sufficient condition for basis-cardinality invariance.

## 6.2. The singleton-generator mechanism

The most structurally informative result is the identification of 30 singleton-generator witnesses across twelve non-chain algebras. These witnesses demonstrate that basis-cardinality variation can arise even when the underlying closure system is generated by a single element.

Let $[\mathcal{S}_x = a \to x \in L.]$The singleton $(x)$ is a minimal base of $(\mathcal{S}_x)$. Under the sufficient conditions established in Section 4, elements $(b, c \in L)$, with $(b, c < 1)$ and $(b \vee c = 1)$, yield two elements

$$[p = b \to x, \qquad q = c \to x]$$

whose joint generation recovers the same closure system, while neither element generates it independently.

The resulting minimal bases have cardinalities one and two:

$$[B_1 = x, \qquad B_2 = p, q.]$$

This construction isolates the mechanism responsible for the observed variation. The failure does not require a complicated universe or a large generating family. It can emerge from the interaction between the residuated implication, the lattice join, and the generation properties of individual elements.

The distinction between joint generation and individual generation is essential. The identity established in Proposition 1 guarantees that the relevant joint expression recovers $(x)$ under its stated hypothesis. Minimality, however, requires the additional exclusion of individual generation. These are separate requirements and cannot be treated as interchangeable.

## 6.3. Role of the algebraic identity

The common identity satisfied by all 30 experimentally identified singleton-generator witnesses provides a compact algebraic description of their generation mechanism. Its relevance is that it translates the computational observation into a condition expressed entirely through the operations of the underlying residuated lattice.

In particular, the expression
$\left[\left((a \otimes b) \vee (d \otimes c)\right) \rightarrow x = x\right]$ implies

$$[(a \rightarrow p) \wedge (d \rightarrow q) = x,]$$

where $(p = b \rightarrow x)$ and $(q = c \rightarrow x)$.

The identity therefore supplies a sufficient joint-generation relation for the witnesses under consideration. It does not independently establish that the two generators are individually insufficient. That conclusion depends on the non-generation conditions stated in Section 4.

This distinction prevents an overextension of the computational result. The identity is a verified common feature of the identified witnesses, not a complete characterisation of all possible failures of basis-cardinality invariance.

## 6.4. Relationship between algebraic structure and basis cardinality

The experiments suggest that the order structure of the underlying algebra is relevant to basis-cardinality behaviour. In the tested chain structures, the absence of witnesses contrasts with their repeated occurrence in non-chain structures. The results are consistent with the interpretation that the availability of incomparable algebraic elements can facilitate alternative minimal generating configurations.

However, the experiments do not isolate non-chain order as the sole causal factor. The tested algebras differ not only in their order structures but also in their cardinalities, operations, and algebraic constructions. Consequently, the observed chain–non-chain distinction should be interpreted as an empirical pattern rather than a general structural theorem.

The sufficient criterion of Section 4 provides a more precise statement. It identifies a particular algebraic configuration capable of producing minimal bases of different cardinalities. This criterion is stronger than a purely order-theoretic observation because it incorporates both the join structure and the behaviour of implication relative to the selected element.

The computational results demonstrate repeated realisation of this configuration. They do not establish that it exhausts the possible mechanisms of basis-cardinality variation.

## 6.5. Scope and limitations

Some limitations delimit the conclusions.

First, the experiments concern finite algebras and finite universes. No conclusion is established for infinite residuated lattices or infinite fuzzy closure systems.

Second, the algebraic configurations were selected from specific finite families, including Gödel, Boolean, Łukasiewicz, mixed-product, and finite Heyting structures. The results do not constitute an exhaustive classification of finite integral commutative residuated lattices.

Third, the chain controls are limited in number. Their consistent behaviour provides a useful comparison but cannot establish a general invariance theorem for chain-based algebras.

Fourth, the independent verification covers the nine configurations in the first enumeration. Although no discrepancy was detected among the 346 explicit bases checked, computational verification remains bounded by the structures and conditions actually examined.

Finally, the sufficient algebraic criterion does not establish a converse. No claim is made that every failure of basis-cardinality invariance must arise from the singleton-generator construction or satisfy the identified identity.

These limitations are substantive. They define the boundary between the results established in this paper and broader claims that remain unsupported by the present analysis.

## 6.6. Principal contribution

The principal contribution is the combination of exhaustive finite enumeration with an algebraic construction explaining a recurring class of basis-cardinality failures.

The computational investigations establish the occurrence of unequal-sized inclusion-minimal bases across multiple finite algebraic families. The singleton-generator analysis further demonstrates that such failures can arise in a particularly restricted setting, with explicit witnesses satisfying a common algebraic relation. The sufficient criterion developed in Section 4 converts this observed behaviour into a formal result.

Taken together, these contributions provide more than a collection of isolated counterexamples. They establish a coherent relationship between finite computational evidence, residuated-lattice operations, and the structure of minimal generating sets. The resulting framework is sufficient to demonstrate failure of basis-cardinality invariance in the identified cases, but insufficient to provide a universal characterisation of the property.

# 7. Conclusion

This paper examined basis-cardinality invariance in finite fuzzy closure systems over integral commutative residuated lattices through exhaustive computational enumeration and algebraic analysis. The results establish that inclusion-minimal bases of unequal cardinalities occur across several finite algebraic families, including Boolean, Gödel-product, Łukasiewicz-product, mixed-product, and finite Heyting structures.

The first enumeration identified 61 closure systems with unequal-sized minimal bases among 173 systems examined. A separate investigation of finite Heyting algebras identified 23 such systems among 72. These findings demonstrate that basis-cardinality invariance does not hold universally across the finite structures considered.

The singleton-generator investigation provides the principal algebraic result. Across fifteen finite algebras, 30 witnesses were identified in twelve non-chain cases, while none of the three chain controls exhibited the same behaviour. A sufficient algebraic criterion was established for constructing singleton-generated closure systems with minimal bases of cardinalities one and two. The identified witnesses also satisfy a common algebraic identity describing their joint-generation mechanism.

The independent verification of 346 explicit bases produced no discrepancies in the nine configurations subjected to audit. This supports the reliability of the reported computational results within the examined scope.

The conclusions are deliberately restricted. The experiments do not establish a universal equivalence between chain structure and basis-cardinality invariance, nor do they prove that the singleton-generator construction accounts for every possible failure. In particular, no universal converse to the sufficient algebraic criterion is claimed.

The central result is therefore twofold: basis-cardinality variation is demonstrably present across multiple classes of finite fuzzy closure systems, and a specific residuated-lattice mechanism explains a substantial, computationally verified class of these failures. This provides a rigorous finite foundation for the study of basis-cardinality behaviour without extending the conclusions beyond what the algebraic proofs and computational evidence establish.

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# 9. Code and Data Availability Statement

All source codes, computational datasets, and experimental outputs supporting the findings of this study are publicly available in the Zenodo repository at https://doi.org/10.5281/zenodo.23812995.

The repository provides the computational resources and corresponding data required to inspect and reproduce the reported finite algebraic enumerations and results.

# 10. Statements and Declaration

## 10.1 Conflict of Interests:

The author declares no conflicts of interest relevant to the research presented in this manuscript.

## 10.2 Funding Information:

No funding was received from any government institution, commercial organization, or non-profit entity. The study was conducted out of independent academic interest and was entirely self-funded.

## 10.3 Research Involving Humans and/or Animals:

No experiments involving human participants or animals were conducted in the course of this study.

## 10.4 Informed Consent:

Informed consent was not applicable, as no human participants were involved in the study.

## 10.5 Author Contributions:

The author conceived and designed the study, developed the experimental framework, conducted the experiments and analysis, and wrote and revised the manuscript.

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