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# On the Spherical-Dirichlet Distribution: Corrections and Results

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# On the Spherical-Dirichlet Distribution: Corrections and Results

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**Abstract**

This article revisits the spherical-Dirichlet distribution (SDD) introduced by Guardiola (2020) and clarifies the correction to its density discussed in Guardiola (2025). The principal issue concerns Equation (3) of the original publication, which expresses the density in local Euclidean coordinates rather than with respect to spherical surface-area measure. We derive the corrected intrinsic density and revisit its moments, covariance matrix, mode, and limiting properties. We show that the two density representations describe the same probability distribution, preserving the moment formulas and likelihood-based parameter estimation. However, the mode and uniformity conditions depend on the reference measure and require corresponding clarification. Reported simulation and text mining estimates are revisited and updated. The resulting presentation reconciles the original construction with its intrinsic spherical properties.

**Keywords:** Spherical-Dirichlet distribution; Dirichlet distribution; hypersphere; positive orthant; surface-area measure; maximum likelihood.

## 1 Introduction

The Spherical-Dirichlet distribution (SDD) was introduced by Guardiola (2020) for modeling nonnegative directional data on the unit hypersphere. It is obtained by applying a coordinatewise square-root transformation to a Dirichlet random vector.

A subsequent correction preprint (Guardiola, 2025) addressed the mathematical formulation of the SDD. The main issue concerns Equation (3) of the original publication: its density is expressed relative to a coordinate measure rather than spherical surface-area measure.

This article clarifies that distinction and presents a consistent formulation of the density, moments, covariance structure, mode, and limiting properties. For completeness, we also revisit parameter estimation and the original simulation and text mining applications.

## 2 Basic Properties

### 2.1 Density and Normalizing Constant

Throughout, let $p \geq 2$ be an integer. Let $\boldsymbol{Z} \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_p)$, where $\alpha_i > 0$ and $\alpha_0 = \sum_{i=1}^{p} \alpha_i$. Define $X_i = \sqrt{Z_i}$. Then $\boldsymbol{X}$ lies on

$$\mathbb{S}_+^{p-1} = \left\{ \mathbf{x} \in \mathbb{R}^p : x_i \geq 0,\ \sum_{i=1}^{p} x_i^2 = 1 \right\}.$$

Write $\mathbb{S}_{+,\circ}^{p-1} = \{\mathbf{x} \in \mathbb{S}_{+}^{p-1} : x_i > 0 \text{ for all } i\}$. The random vector lies in this relative interior almost surely; its topological support is the closed orthant. All density formulas below are specified on the relative interior. Boundary values can be assigned arbitrarily without changing the probability law.

The Dirichlet density, with respect to $dz_1 \cdots dz_{p-1}$ on $z_i > 0$, $\sum_{i<p} z_i < 1$, and $z_p = 1 - \sum_{i<p} z_i$, is

$$f_{\mathrm{Dir}}(\boldsymbol{z}; \boldsymbol{\alpha}) = \frac{\Gamma(\alpha_0)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \prod_{i=1}^{p} z_i^{\alpha_i - 1}.$$

Using $z_i = x_i^2$ for $i = 1, \ldots, p-1$, the coordinate Jacobian is

$$\left| \frac{\partial(z_1, \ldots, z_{p-1})}{\partial(x_1, \ldots, x_{p-1})} \right| = 2^{p-1} \prod_{i=1}^{p-1} x_i.$$

Consequently, Equation (3) of Guardiola (2020) can be written as

$$g(\mathbf{x}; \boldsymbol{\alpha}) = \frac{2^{p-1}\Gamma(\alpha_0)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \left( \prod_{i=1}^{p} x_i^{2\alpha_i - 1} \right) x_p^{-1}. \tag{1}$$

Here $x_p = (1 - \sum_{i<p} x_i^2)^{1/2}$, and this expression is a density on $D = \{(x_1, \ldots, x_{p-1}) : x_i > 0, \sum_{i<p} x_i^2 < 1\}$ with respect to $dx_1 \cdots dx_{p-1}$; it is not a density with respect to $p$-dimensional Lebesgue measure. However, the spherical surface-area element satisfies

$$d\omega_{p-1}(\mathbf{x}) = \frac{dx_1 \cdots dx_{p-1}}{x_p} = \frac{dz_1 \cdots dz_{p-1}}{2^{p-1} \prod_{i=1}^{p} x_i}.$$

Indeed, the graph map $F(u) = (u, \sqrt{1 - \|u\|^2})$ satisfies $\det\{(DF)^T DF\} = \det(I + uu^T/x_p^2) = 1/x_p^2$. This proves the surface-area element and its extra factor $1/x_p$.

Thus, the corrected SDD density with respect to spherical surface-area measure is

$$f_{\mathrm{SDir}}(\mathbf{x}; \boldsymbol{\alpha}) = \frac{2^{p-1}\Gamma(\alpha_0)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \prod_{i=1}^{p} x_i^{2\alpha_i - 1}, \qquad \mathbf{x} \in \mathbb{S}_{+,\circ}^{p-1}. \tag{2}$$

In particular,

$$\int_{\mathbb{S}_{+}^{p-1}} \prod_{i} x_i^{2\alpha_i - 1} \, d\omega_{p-1}(x) = 2^{1-p} \frac{\prod_i \Gamma(\alpha_i)}{\Gamma(\alpha_0)},$$

which verifies the normalizing constant. We write $\boldsymbol{X} \sim \mathrm{SDD}(\boldsymbol{\alpha})$.

## 2.2 Moments

The moments follow directly from the Dirichlet distribution. For real $r_i$ satisfying $\alpha_i + r_i/2 > 0$ for every $i$,

$$\mathrm{E}\left( \prod_{i=1}^{p} X_i^{r_i} \right) = \frac{\Gamma(\alpha_0)}{\Gamma\left(\alpha_0 + \frac{1}{2}\sum_{i=1}^{p} r_i\right)} \prod_{i=1}^{p} \frac{\Gamma(\alpha_i + r_i/2)}{\Gamma(\alpha_i)}. \tag{3}$$

This follows by substituting $X_i^{r_i} = Z_i^{r_i/2}$ and taking the ratio of the Dirichlet normalizing integrals. The stated conditions are necessary and sufficient for finiteness of these mixed moments.

Define

$$m_i = \frac{\Gamma(\alpha_i + 1/2)}{\Gamma(\alpha_i)}, \qquad m_0 = \frac{\Gamma(\alpha_0 + 1/2)}{\Gamma(\alpha_0)},$$

and let $\boldsymbol{m} = (m_1, \ldots, m_p)^T$. Then

$$\mathrm{E}(X_i) = \frac{m_i}{m_0}, \qquad \mathrm{E}(X_i^2) = \frac{\alpha_i}{\alpha_0}, \qquad \mathrm{E}(X_i X_j) = \frac{m_i m_j}{\alpha_0}, \quad i \neq j. \tag{4}$$

Consequently,

$$\mathrm{Var}(X_i) = \frac{\alpha_i}{\alpha_0} - \frac{m_i^2}{m_0^2}, \qquad \mathrm{Cov}(X_i, X_j) = m_i m_j \left( \frac{1}{\alpha_0} - \frac{1}{m_0^2} \right), \quad i \neq j. \tag{5}$$

The covariance matrix is

$$\boldsymbol{\Sigma} = \frac{1}{\alpha_0} \operatorname{diag}(\alpha_1 - m_1^2, \ldots, \alpha_p - m_p^2) + \left( \frac{1}{\alpha_0} - \frac{1}{m_0^2} \right) \boldsymbol{m}\boldsymbol{m}^T. \tag{6}$$

These moment expressions remain valid under the corrected density. They also give $\mathrm{tr}(\boldsymbol{\Sigma}) = 1 - \| \mathrm{E}\, \boldsymbol{X} \|^2$. All off-diagonal covariances are strictly negative: if $G \sim \mathrm{Gamma}(\alpha_0, 1)$, strict Jensen inequality gives $m_0 = \mathrm{E}\sqrt{G} < \sqrt{\alpha_0}$, so $1/\alpha_0 - 1/m_0^2 < 0$. Although $\|\boldsymbol{X}\| = 1$, its covariance matrix is positive definite. For a nonzero vector $a$, $a^T \boldsymbol{X}$ is not constant on an open spherical orthant, which has positive density, so $\mathrm{Var}(a^T \boldsymbol{X}) > 0$. The nonlinear norm constraint does not imply a singular covariance matrix.

### 2.3 Mode and Relationship with the Mean

Maximizing the logarithm of Equation (2) subject to $\sum_{i=1}^p x_i^2 = 1$ gives

$$x_i^{\mathrm{mode}} = \sqrt{\frac{2\alpha_i - 1}{2\alpha_0 - p}}, \qquad i = 1, \ldots, p, \tag{7}$$

provided $\alpha_i > 1/2$ for all $i$.

For completeness, put $b_i = 2\alpha_i - 1$. On writing $z_i = x_i^2$, maximization is equivalent to maximizing $\frac{1}{2} \sum_i b_i \log z_i$ on the simplex. If all $b_i > 0$, strict concavity proves that Equation (7) is the unique global mode. If all $b_i \geq 0$ and at least one is positive, the canonical continuous extension of the density has the unique maximizer

$$x_i^* = \begin{cases} \sqrt{b_i / \sum_j b_j}, & b_i > 0, \\ 0, & b_i = 0. \end{cases}$$

To see this, any mass assigned to a zero-exponent coordinate can be moved proportionately to the positive-exponent coordinates to increase the product; strict concavity then applies on that face. For example, $\boldsymbol{\alpha} = (1/2, 1/2, 2)$ has its maximum at $(0, 0, 1)$. If all $b_i = 0$, the density is constant and every point is a maximizer of its continuous extension. If some $b_i < 0$, let the corresponding coordinate tend to zero while keeping all remaining coordinates positive. The density diverges, and there is no finite global density maximum. We therefore describe boundary singularities rather than asserting a uniquely defined boundary mode. Boundary maximizers in this discussion refer to the specified continuous extension, not arbitrary values of an almost-everywhere density.

For the symmetric SDD, $\alpha_i = \alpha > 1/2$,

$$\boldsymbol{x}^{\mathrm{mode}} = \frac{1}{\sqrt{p}} \mathbf{1}. \tag{8}$$

The mean vector is

$$\mathrm{E}(\boldsymbol{X}) = \frac{\Gamma(\alpha+1/2)\Gamma(p\alpha)}{\Gamma(\alpha)\Gamma(p\alpha+1/2)}\mathbf{1}. \tag{9}$$

The mean and mode have the same direction, although the mean has norm less than one for finite $\alpha$. Using the asymptotic gamma ratio,

$$\frac{\Gamma(t+1/2)}{\Gamma(t)} \sim t^{1/2}, \qquad t \to \infty,$$

we obtain

$$\mathrm{E}(\boldsymbol{X}) \longrightarrow \frac{1}{\sqrt{p}}\mathbf{1} \qquad \text{as } \alpha \to \infty.$$

The mode is defined with respect to spherical surface-area measure. Under this formulation, all coordinates and their corresponding parameters enter symmetrically. The unique interior mode is

$$x_i^{\text{mode}} = \sqrt{\frac{2\alpha_i - 1}{2\alpha_0 - p}}, \qquad i = 1, \ldots, p,$$

provided that $\alpha_i > 1/2$ for every $i$, where $\alpha_0 = \sum_{j=1}^{p} \alpha_j$. The apparent asymmetry in the original coordinate density arises solely from the choice of local coordinates; it is absent from the intrinsic surface-area density.

# 3 Relationships with Other Distributions

## 3.1 Limiting and Uniform Cases

For the symmetric SDD, $\alpha_i = \alpha$,

$$f_{\text{SDir}}(\mathbf{x}; \alpha) = \frac{2^{p-1}\Gamma(p\alpha)}{\Gamma(\alpha)^p} \prod_{i=1}^{p} x_i^{2\alpha-1}. \tag{10}$$

Since $X_i^2 \sim \text{Beta}(\alpha, (p-1)\alpha)$,

$$\mathrm{E}(X_i^2) = \frac{1}{p}, \qquad \mathrm{Var}(X_i^2) = \frac{p-1}{p^2(p\alpha+1)}.$$

Consequently,

$$\boldsymbol{X} \xrightarrow{\mathbb{P}} \frac{1}{\sqrt{p}}\mathbf{1} \qquad \text{as } \alpha \to \infty. \tag{11}$$

When $\alpha_i = 1/2$ for all $i$, Equation (2) reduces to

$$f_{\text{SDir}}(\mathbf{x}) = \frac{2^{p-1}\Gamma(p/2)}{\pi^{p/2}}, \qquad \mathbf{x} \in \mathbb{S}_+^{p-1}. \tag{12}$$

The area of the positive orthant is $\omega_{p-1}(\mathbb{S}_+^{p-1}) = 2^{1-p}\pi^{p/2}/\Gamma(p/2)$, so the displayed density is exactly its reciprocal. For $p \geq 2$, all $\alpha_i = 1/2$ is also necessary for a constant surface density: $\sum_i (\alpha_i - 1/2) \log z_i$ can be constant throughout the simplex only when all its coefficients vanish.

This differs from the coordinate uniformity condition in the original article, which used $\alpha_i = 1/2$ for $i < p$ and $\alpha_p = 1$.

### 3.2 Comparison with Directional Distributions

The von Mises–Fisher distribution has density

$$f_{\mathrm{vMF}}(\mathbf{x};\boldsymbol{\mu},\kappa) = c_p(\kappa)\exp(\kappa\boldsymbol{\mu}^T\mathbf{x}),$$

where $\mathbf{x}\in\mathbb{S}^{p-1}$, $c_p(\kappa)$ is the normalizing constant with respect to full-sphere surface area, $\|\boldsymbol{\mu}\|=1$, and $\kappa\geq 0$ (Mardia and Jupp, 2000).

More generally, the Fisher–Bingham family has density proportional to

$$f_{\mathrm{FB}}(\mathbf{x}) \propto \exp\left(\boldsymbol{b}^T\mathbf{x}+\mathbf{x}^T A\mathbf{x}\right),$$

where $A$ is symmetric (Kent, 1982; Mardia and Jupp, 2000).

Unlike their usual full-sphere formulations, the SDD is supported only on the positive orthant. Its parameters jointly determine shape and concentration. Truncating the other families to this orthant may yield analytically intractable normalizing constants, requiring numerical approximation.

## 4 Inference for the Spherical-Dirichlet Distribution

Let $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$ be independent observations from $\mathrm{SDD}(\boldsymbol{\alpha})$, with every coordinate strictly positive. Exact observed zeros require a separate observation model or an explicitly justified preprocessing rule; the log-likelihood below does not apply to them directly.

### 4.1 Method of Moments

Define

$$\overline{X}_j = \frac{1}{n}\sum_{k=1}^{n} X_{kj}, \qquad \overline{X_j^2} = \frac{1}{n}\sum_{k=1}^{n} X_{kj}^2.$$

Since $\sum_{j=1}^{p}\overline{X_j^2}=1$, only $p-1$ second-moment equations are independent.

Following the original approach, we use

$$\frac{\Gamma(\alpha_1+1/2)}{\Gamma(\alpha_1)}\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_0+1/2)} = \overline{X}_1 \tag{13}$$

and

$$\frac{\alpha_j}{\alpha_0} = \overline{X_j^2}, \qquad j=2,\ldots,p. \tag{14}$$

Equivalently, $\alpha_j=\alpha_0\overline{X_j^2}$ for all $j$. Substituting into Equation (13) reduces estimation to a scalar nonlinear equation for $\alpha_0$. The MOM estimates are then

$$\widehat{\alpha}_j^{\mathrm{MOM}} = \widehat{\alpha}_0\,\overline{X_j^2}.$$

**Existence, uniqueness, and computation.** Put $q=\overline{X_1^2}\in(0,1)$ and $a=\overline{X}_1$. The scalar equation is

$$h_q(t) = \frac{\Gamma(tq+1/2)\Gamma(t)}{\Gamma(tq)\Gamma(t+1/2)} = a, \qquad t>0.$$

It has a unique finite solution if and only if $q<a<\sqrt{q}$. Indeed, $h_q(0+)=q$ and $h_q(\infty)=\sqrt{q}$. To prove strict monotonicity, put $d(s)=\psi(s+1/2)-\psi(s)$. Its integral representation yields

$$sd(s) = \int_0^\infty \frac{e^{-v}}{1+e^{-v/(2s)}}\,dv,$$

which is strictly decreasing in $s$. Hence $\{\log h_q(t)\}' = qd(tq) - d(t) > 0$. For interior observations, $q < a$ since $X_{k1}^2 < X_{k1}$; $a \leq \sqrt{q}$ follows from Cauchy–Schwarz, with strict inequality exactly when the first coordinate is not constant across the sample. A constant first coordinate therefore gives no finite MOM solution, even if other coordinates vary. Another nonconstant coordinate can be chosen in that case. Different choices generally produce different estimates. A bracketed scalar root finder with log-gamma evaluation avoids unrestricted multivariate optimization and guarantees convergence once a finite root is bracketed.

### 4.2 Maximum Likelihood Estimation

The log-likelihood is

$$\begin{aligned}\ell(\boldsymbol{\alpha}) = n(p-1)\log 2 + n\log\Gamma(\alpha_0) \\ - n\sum_{j=1}^{p}\log\Gamma(\alpha_j) \\ + \sum_{j=1}^{p}(2\alpha_j - 1)\sum_{k=1}^{n}\log X_{kj}.\end{aligned} \tag{15}$$

The score equations are

$$\psi(\alpha_j) - \psi(\alpha_0) = \frac{2}{n}\sum_{k=1}^{n}\log X_{kj}, \qquad j = 1, \ldots, p, \tag{16}$$

where $\psi$ is the digamma function.

For $n \geq 1$, the negative log-likelihood is strictly convex in $\boldsymbol{\alpha} \in (0, \infty)^p$. Its Hessian is

$$H = n\left[\operatorname{diag}\{\psi_1(\alpha_j)\} - \psi_1(\alpha_0)\mathbf{1}\mathbf{1}^T\right],$$

where $\psi_1$ is the trigamma function. This is $n\operatorname{Cov}(\log \boldsymbol{Z})$ under the model, and is positive definite: a nonzero linear combination of $\log Z_j$ cannot be constant on the open simplex. Thus any finite stationary point is the unique global maximum of the log-likelihood. Strict convexity alone does not establish existence of such a point. In particular, if all observations equal an interior vector $x$, the sequence $\alpha_j = tx_j^2$ has $\ell(\boldsymbol{\alpha}) = n(p-1)\log(t)/2 + O(1) \to \infty$, so no finite MLE exists. This includes $n = 1$. Numerical minimization can be performed using L-BFGS-B, subject to $\alpha_j \geq \varepsilon > 0$. A fixed lower bound gives a constrained estimate, which agrees with the unrestricted MLE only when the latter lies in the feasible set. Strict convexity here is in $\boldsymbol{\alpha}$, and is not automatically preserved by a log-parameter reparameterization.

The gradient is

$$\frac{\partial(-\ell)}{\partial\alpha_j} = n\left[\psi(\alpha_j) - \psi(\alpha_0)\right] - 2\sum_{k=1}^{n}\log X_{kj}. \tag{17}$$

Importantly, Equation (1) differs from the corrected density by a factor independent of $\boldsymbol{\alpha}$. Therefore, the correction does not change the likelihood score equations or the maximum likelihood estimator.

## 5 Applications to Data

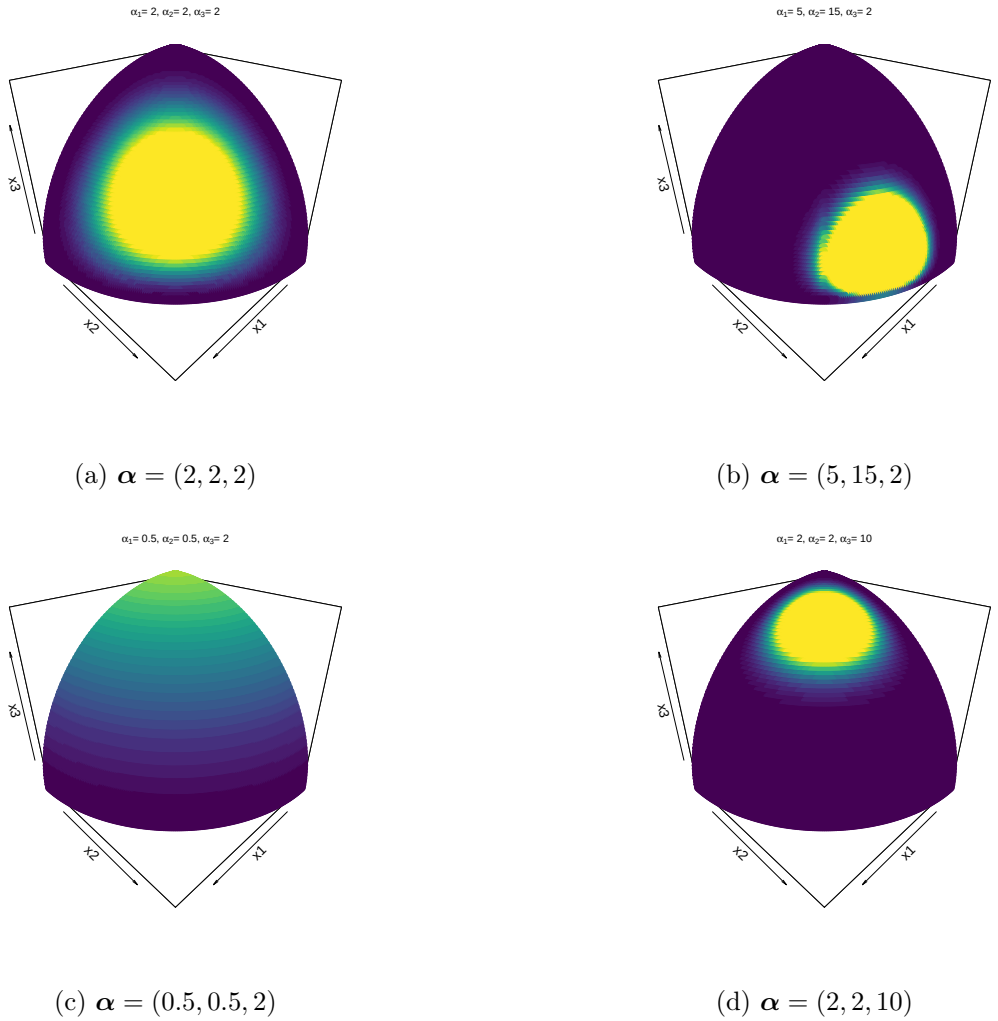
We revisit the simulation and text mining applications from Guardiola (2020). The following numerical results represent updated analyses.

### 5.1 Simulation Example

Four parameter configurations were considered for a three-dimensional SDD:

$$(2, 2, 2), \quad (5, 15, 2), \quad (0.5, 0.5, 2), \quad (2, 2, 10).$$

Each simulation generated 10,000 observations using $X_j = \sqrt{Z_j}$, where $\boldsymbol{Z} \sim \text{Dirichlet}(\boldsymbol{\alpha})$. The corresponding distributions are illustrated in Figure 1.




(a) $\boldsymbol{\alpha} = (2, 2, 2)$ (b) $\boldsymbol{\alpha} = (5, 15, 2)$

(c) $\boldsymbol{\alpha} = (0.5, 0.5, 2)$ (d) $\boldsymbol{\alpha} = (2, 2, 10)$

Figure 1: Spherical-Dirichlet distributions under four concentration parameter configurations.

The estimation error was measured by the relative Euclidean norm $\text{Error}(\%) = 100\|\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\|_2 / \|\boldsymbol{\alpha}\|_2$.

Table 1 reports the parameter estimates, iteration counts, and relative errors.

The displayed estimates have small relative errors. In these four reported samples, the MLE errors are smaller than the MOM errors. This does not establish uniformly better finite-sample performance. The MOM estimates and iteration counts coincide with those in Guardiola (2020), whereas the MLE entries differ slightly. Iteration counts from different algorithms are not directly comparable measures of computational cost.

Table 1: Simulation estimates; errors recomputed from displayed values.

| Method | Iter. | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | Error (%) |
|---|---|---|---|---|---|
| | | $\boldsymbol{\alpha} = (2, 2, 2)$ | | | |
| MOM | 176 | 2.0557 | 2.0983 | 2.0764 | 3.94 |
| MLE | 8 | 2.0538 | 2.0433 | 2.0426 | 2.34 |
| | | $\boldsymbol{\alpha} = (5, 15, 2)$ | | | |
| MOM | 589 | 5.0351 | 14.7148 | 1.9684 | 1.81 |
| MLE | 19 | 5.0584 | 15.2103 | 2.0158 | 1.37 |
| | | $\boldsymbol{\alpha} = (0.5, 0.5, 2)$ | | | |
| MOM | 28 | 0.4964 | 0.4639 | 1.9212 | 4.09 |
| MLE | 13 | 0.4997 | 0.5004 | 2.0091 | 0.43 |
| | | $\boldsymbol{\alpha} = (2, 2, 10)$ | | | |
| MOM | 385 | 1.9349 | 2.0153 | 10.1145 | 1.28 |
| MLE | 14 | 2.0190 | 2.0159 | 10.1229 | 1.21 |

### 5.2 Text Mining Example

The text mining application used 160 documents from the automobile discussion category of the CMU text collection (Lang, 2019).

After preprocessing, nine terms were retained: *auto, write, articl, engin, don, good, time, drive*, and *road*.

Term frequencies were transformed using $x_{\text{transf}} = \log(1.10+x)$ and normalized to unit Euclidean length. The resulting observations lie in the positive orthant of the eight-dimensional unit sphere $\mathbb{S}^8$ embedded in $\mathbb{R}^9$.

Table 2 reports the MOM and MLE estimates.

Table 2: Text mining parameter estimates (nine components).

| Parameter | MOM | MLE |
|---|---|---|
| $\alpha_1$ | 1.2817 | 1.2743 |
| $\alpha_2$ | 0.5575 | 0.5543 |
| $\alpha_3$ | 0.4953 | 0.4924 |
| $\alpha_4$ | 0.3135 | 0.3138 |
| $\alpha_5$ | 0.3624 | 0.3630 |
| $\alpha_6$ | 0.3174 | 0.3177 |
| $\alpha_7$ | 0.3352 | 0.3356 |
| $\alpha_8$ | 0.3100 | 0.3102 |
| $\alpha_9$ | 0.2722 | 0.2721 |

The two methods produced closely agreeing estimates. MOM required 276 iterations, compared with 18 for MLE.

The transformation makes every coordinate positive, including zero counts. It defines a model for the transformed directions, not directly for raw counts. The agreement of two estimators does not necessarily establish goodness of fit. The nine-component specification is an intentional correction to the ten-component description in the 2020 article, as confirmed by the author. The nine terms listed above define the corrected variables.

# 6 Conclusions

This article clarifies the density formulation of the Spherical-Dirichlet distribution introduced by Guardiola (2020) and reconciles it with the subsequent correction (Guardiola, 2025).

The principal distinction is between the original coordinate density and the intrinsic density with respect to spherical surface-area measure.

The correction preserves the moment formulas, covariance structure, and likelihood-based parameter estimation procedures. However, the mode and uniformity conditions must be interpreted relative to the reference measure. In particular, the corrected density yields the intrinsic spherical mode and includes the uniform distribution with respect to spherical surface-area measure when $\alpha_i = 1/2,\ \forall i \in \{1, \ldots, p\}$.

In the four simulation examples considered, both methods produced estimates close to the generating parameters, with MLE yielding smaller relative estimation errors than MOM in each example. In the text mining example, MOM and MLE produced closely agreeing parameter estimates for the nine-component transformed document vectors, illustrating the application of the SDD to strictly positive directional representations of text data.

In summary, the SDD remains a tractable competitive model for directional data with strictly positive components on the unit hypersphere, with an explicit normalizing constant.

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