Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Isolated Notations or Integrated Magnitudes? Rational Numbers in Spanish Elementary School Textbooks

Santiago Vicente

sanvicente@usal.es

University of Salamanca

Paula Muñoz

University of Salamanca

Natalia Calvo

University of Salamanca

Josetxu Orrantia

University of Salamanca

Research Article

Keywords: Rational number, Mathematics education, Textbook, Elementary education, Content analysis, Cognitive demand

Posted Date: November 19th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11551157/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: Competing interest reported. Financial interests: Santiago Vicente previously held a research transfer contract with Grupo Editorial SM to design instructional models for primary mathematics textbooks. This contract has expired and has in no way influenced the design, analysis, or interpretation of the results of this study. The remaining authors declare they have no relevant financial or non-financial interests to disclose.

Isolated Notations or Integrated Magnitudes? Rational Numbers in Spanish Elementary School Textbooks

Santiago Vicente[1] ✉, Paula Muñoz[1], Natalia Calvo[1], and Josetxu Orrantia[1]

[1] Department of Developmental and Educational Psychology, University of Salamanca, Salamanca, Spain.

Corresponding author: Santiago Vicente

Authors and affiliations:

Santiago Vicente. Department of Developmental and Educational Psychology, University of Salamanca, Salamanca, Spain. Paseo Canalejas 169, 37008 Salamanca, Spain. +34 923 29 46 30, ext. 5735. sanvicente@usal.es. ORCID: 0000-0002-2072-8133

Paula Muñoz. Department of Developmental and Educational Psychology, University of Salamanca, Salamanca, Spain. Paseo Canalejas 169, 37008 Salamanca, Spain. +34 923 29 46 30, ext. 5739. paulamunozmorenodevega@usal.es. ORCID: 0009-0001-3191-8346

Natalia Calvo. Department of Developmental and Educational Psychology, University of Salamanca, Salamanca, Spain. Avda. Príncipe de Asturias, s/n. 49029-Zamora (+34) 980 545 000. nataliacblaz@usal.es. ORCID: 0000-0002-5452-8535

Josetxu Orrantia. Departamento de Psicología Evolutiva y de la Educación, Universidad de Salamanca, Salamanca, Spain. Avenida del Príncipe de Asturias, s/n. 49029. Zamora. +34 923 294 500. Ext. 5739. orrantia@usal.es. ORCID: 0000-0002-9305-8871

Isolated Notations or Integrated Magnitudes? Rational Numbers in Spanish Elementary School Textbooks

Abstract

According to the Integrated Theory of Numerical Development, understanding rational numbers requires the cognitive integration of fractions, decimals, and percentages. To determine whether Spanish elementary school textbooks promote this integration, we analyzed 8,304 rational number items from the five most widely used publishers in Spain. All items were coded according to their symbolic notation, and multiple-notation items were further classified by their cognitive demand (procedural, conceptual, or mixed), categorizing the underlying mathematical dimension (magnitude, equivalence, estimation, transcoding, density, or arithmetic). The results revealed a high degree of fragmentation, as 94.4% of the tasks presented notations in isolation. Furthermore, the few valid multiple-notation tasks identified focused heavily on procedural skills—particularly transcoding—which accounted for 78.9% of multiple-notation items, to the detriment of the conceptual understanding of cross-notation magnitude, estimation, and density. Significant differences also emerged across textbook series, revealing a trade-off between the frequency and cognitive quality of cross-notation tasks: while some textbooks presented a higher volume of integrated items but reduced them to mechanical transcoding, others offered greater cognitive diversity but severely restricted the overall frequency of cross-notation tasks. This study demonstrates that the Spanish textbook market exhibits a structural instructional bias that may hinder cognitive flexibility by presenting a compartmentalized approach to instruction, failing to support students' development of an integrated number sense.

Keywords: Rational number, Mathematics education, Textbook, Elementary education, Content analysis, Cognitive demand.

1. Introduction

Understanding rational numbers represents a fundamental milestone in mathematical learning (Siegler et al., 2012; Siegler & Tian, 2022). Mastery of rational numbers during elementary school predicts

subsequent achievement in high school algebra and science (Bailey et al., 2012; Siegler et al., 2012). Likewise, the proportional reasoning inherent to this domain is essential for analyzing scientific phenomena, meaning that deficient learning compromises long-term academic progress (Braithwaite et al., 2022; Lamon, 2020; Schiller et al., 2024). Beyond the school setting, proficiency with rational numbers shapes future career trajectories and decision-making in adulthood (Lamon, 2020; Powell & Nelson, 2021; Thompson et al., 2023).

Mastering rational numbers poses one of the greatest cognitive challenges during formal schooling (Behr et al., 1983; Ni & Zhou, 2005) and even into adulthood (Lortie-Forgues et al., 2015). Understanding these numbers requires restructuring prior knowledge of whole numbers and assimilating novel dimensions, such as numerical density (Braithwaite & Siegler, 2018; Vamvakoussi & Vosniadou, 2010) and the fact that the same quantity can be represented across three different notations: fractions, decimals, and percentages. Each of these notations activates a distinct cognitive model. Whereas fractions, with their bipartite structure (a/b), emphasize multiplicative relations between quantities and the counting of discrete entities, decimals feature a unidimensional, base-10 positional structure that directly maps magnitude onto a continuum (Liu et al., 2025; Qiu & Wang, 2021). Finally, percentages have a hybrid nature: although they represent a rational magnitude, their implicit constant denominator of 100 allows them to be processed as intuitively as whole numbers. This standardization avoids both the positional uncertainty of decimals and the denominator variability of fractions, making percentages a powerful "conceptual bridge" for magnitude estimation (Tian et al., 2020).

According to the Integrated Theory of Numerical Development (Siegler et al., 2011; Siegler & Tian, 2022), deep learning of rational numbers requires coordinating these different notations within a single cognitive structure. Developing such cross-notation knowledge (e.g., understanding that $1/2$, 0.5, and 50% represent the same magnitude) predicts later mathematics achievement to a greater extent than isolated mastery of each notation (Braithwaite et al., 2022; Ching et al., 2024; Schiller & Siegler, 2023). Indeed, interventions that explicitly target this conversion—and particularly those that integrate percentages by leveraging their standardized scale (Schiller et al., 2024; Siegler & Tian, 2022)— promote the development of a number sense that connects procedures to actual magnitudes. Conversely, teaching these notations in a segregated manner fosters mechanical understanding and consolidates persistent biases, such as the whole number bias (Ni & Zhou, 2005; Obersteiner et al., 2013) or the "percentages-

are-larger" bias (Schiller & Siegler, 2023; Schiller, Siegler, et al., 2025b). This representational disconnection risks fostering impulsive calculation (Schiller, 2020) and constrains the development of advanced mathematical reasoning (Braithwaite & Siegler, 2018; Siegler et al., 2011).

A second cognitive challenge in rational number learning stems from the requirement to articulate an interconnected network spanning six fundamental dimensions, which can be grouped into conceptual dimensions (magnitude, equivalence, estimation, and density) and procedural dimensions (transcoding and arithmetic; Behr et al., 1983; Kieren, 1980; Lamon, 2020; Tian et al., 2020).

Conceptual mastery of *magnitude* involves accessing the quantitative meaning and value of a number by locating it on a common mental number line across all three notations (Siegler et al., 2011), a skill that stands as one of the strongest predictors of mathematical and algebraic achievement (Bailey et al., 2012; Powell & Nelson, 2021; Siegler et al., 2012). *Equivalence* entails recognizing that values with different symbolic representations—both within notations (e.g., $2/4$ and $1/2$) and across notations (e.g., $1/2$, 0.5, and 50%)—represent the exact same magnitude and occupy the same position on the number line (Kamii & Clark, 1995; Schiller & Siegler, 2023). Understanding equivalence provides the foundation for a unified number sense (Braithwaite et al., 2022), whereas its absence isolates notations and hinders cognitive flexibility (Ching et al., 2024; Tian et al., 2020). *Estimation* involves using numerical benchmarks and magnitude approximations to anticipate and evaluate the plausibility of cross-notation quantities without relying on exact algorithmic computation (Tian et al., 2020). Finally, *density* requires understanding the existence of infinitely many quantities between any two rational numbers, demanding a profound conceptual shift away from whole numbers (McMullen & Van Hoof, 2020; Vamvakoussi & Vosniadou, 2010).

On the procedural plane, *transcoding* involves translating quantities from one notation to another and selecting the most appropriate notation for a given task (Tian et al., 2020). The final dimension, *arithmetic*, entails executing computational algorithms and, specifically, flexibly switching between notations to capitalize on their cognitive affordances depending on the situation, such as using fractions for multiplicative relations and decimals for addition and subtraction (Braithwaite et al., 2022; Liu & Braithwaite, 2023; Tian et al., 2020).

In sum, rational numbers must be learned by integrating their notations and meanings. When instruction reduces them to a set of disconnected rules, students tend to develop a merely instrumental understanding—knowing what to do, but not why (Lemonidis & Pilianidis, 2020)—which encourages rote memorization of mechanically applied procedures and confusion across operations or notations (Powell & Nelson, 2021). This approach promotes "impulsive calculation," defined as operating on symbols without attending to numerical magnitude or the plausibility of the result (Schiller, 2020; Schiller & Siegler, 2023), thereby generating superficial associations rather than structural comprehension (Braithwaite & Siegler, 2018).

1.1. Textbooks and the Learning of Mathematics and Rational Numbers

Textbooks exert a decisive influence on classroom instruction; indeed, they are considered the "potentially implemented curriculum" (Valverde et al., 2002) because they define which mathematical contents, representations, and strategies are enacted in classroom instruction (González-Peñaloza et al., 2023). Textbook choice directly influences student achievement, with significant learning differences emerging across publishing programs (Agodini et al., 2009), as students better assimilate the arithmetic principles and problem-solving strategies to which textbooks devote greater space and practice frequency (Sievert et al., 2021; Törnroos, 2005). Likewise, students better assimilate mathematical notions and strategies when working with textbooks aligned with empirically grounded learning principles (Sievert et al., 2021; Törnroos, 2005).

Regarding rational number learning, empirical evidence indicates that textbooks reproduce a fragmented view through three structural deficiencies. First, they present notations in isolation, hindering "cross-notation knowledge" (Siegler & Tian, 2022). Indeed, the simultaneous co-occurrence of multiple notations within a single task is severely limited (Siegler & Oppenzato, 2021). Second, their distribution generates spurious associations between superficial problem features and algorithmic procedures (Braithwaite & Siegler, 2018). Third, textbooks reinforce cognitive biases by enforcing an artificial semantic specialization, such as linking fractions to discrete contexts and decimals to continuous ones (Qiu & Wang, 2021).

Consequently, students' difficulties with rational numbers in school may be, to a large extent, the result of this instructional fragmentation (Schiller & Siegler, 2023; Siegler et al., 2011). However, prior research

has evaluated this disconnection from a purely quantitative perspective, limiting its scope to frequency counts by notation (e.g., Siegler & Oppenzato, 2021). This approach leaves a fundamental question unresolved: merely observing that two notations co-occur in a task is insufficient if its true cognitive demand is ignored. At present, it remains unknown whether such problems require a deep conceptual connection—such as integrating magnitudes on a mental number line (Siegler et al., 2011)—or whether they are reduced to a mere procedural routine of mechanical transcoding (Tian et al., 2020). Furthermore, the literature tends to treat school textbooks within a given country as a uniform block (Siegler & Tian, 2022), overlooking how pedagogical design differences across publishers shape students' opportunities to learn (Törnroos, 2005).

1.2. The present study

To address this empirical gap, the present study analyzed the treatment of rational numbers in the most widely used elementary textbooks in Spain to determine the extent to which they provide learning opportunities that support an integrated number sense. Specifically, the study addressed three primary objectives: (a) to analyze the overall distribution and longitudinal progression across grade levels of each notation and of multiple-notation tasks (including their specific cross-notation combinations) in order to determine whether the textbook curriculum fosters notational compartmentalization; (b) to evaluate the cognitive demand of multiple-notation tasks to determine whether they emphasize the conceptual understanding of magnitude or are restricted to procedural mechanical transcoding; and (c) to compare different textbook publishers to determine whether they converge in their curricular treatment of rational numbers or exhibit substantial instructional divergences.

2. Method

2.1. Sample

To ensure curricular and ecological representativeness, the sample selection was grounded in an empirical mapping of textbook adoption across two Spanish autonomous communities with contrasting performance profiles in international assessments (TIMSS): Castile and León (high achievement) and Andalusia (low achievement). During the 2024–2025 academic year, educational records, institutional websites, and direct inquiries were used to scrutinize textbook adoption across 2,528 elementary schools (representing 100% of schools in Andalusia and 75.8% in Castile and León).

Based on this market distribution, five commercial series developed under the current national curriculum framework (LOMLOE) were selected: *Operación Mundo* (Anaya, 28.50%), *Construyendo Mundos* (Santillana, 27.71%), *Revuela* (SM, 11.97%), *FanFest* (Edelvives, 8.33%), and *EdebéOn* (Edebé, 1.54%). Together, these five textbook series accounted for 78.05% of the total adoption across the scrutinized schools, ensuring that the analyzed corpus reflects the predominant instructional materials implemented in Spanish elementary classrooms.

The unit of analysis was the instructional item, defined as any theoretical explanation, exercise, or word problem that unequivocally involved a rational number. Theoretical explanations were included alongside exercises and problems because they function as worked examples that explicitly model target mathematical relationships and cognitive demands. In multi-part exercises comprising several sub-tasks (e.g., a, b, c), sub-items were coded as independent units of analysis only if they differed in at least one of the evaluated parameters. When sub-tasks shared identical characteristics, they were coded as a single item to prevent the artificial overrepresentation of repetitive procedural drills. Although the initial screening encompassed all six elementary grades, no rational number items were identified in Grade 1 textbooks; consequently, the analyzed corpus comprised 8,304 items spanning Grades 2 through 6 (Santillana: n = 2,461; SM: n = 2,095; Edebé: n = 1,436; Anaya: n = 1,340; Edelvives: n = 972).

2.2. Measures

2.2.1. Notation

Items were initially classified into four notation categories: *fraction*, *decimal*, *percentage*, and *multiple notation* (simultaneous combinations of two or all three notations). Tasks that did not specify a required notation for the result (e.g., open-ended probability or relative frequency tasks) were excluded. To ensure that the connection across notations was genuine, we used a screening category termed *coexistence* to discard items in which different notations appeared in parallel but operated independently (e.g., a store receipt displaying a 20% discount alongside a €5.50 price tag, where the mathematical task exclusively required decimal computation).

2.2.2. Cognitive Demand of Multiple-Notation Items

Subsequently, valid multiple-notation tasks were classified according to their cognitive demand and underlying mathematical concept (Lamon, 2020), structured into three broad domains: Conceptual Knowledge, Procedural Knowledge, and a Mixed Domain.

Conceptual Knowledge included tasks requiring structural understanding across four dimensions:

- Magnitude: Locating or comparing quantities across notations, either on a continuous spatial model (number line) or through symbolic representations (including quantities expressed across standard metric measurement units), excluding discrete counting models and tasks requiring explicit prior algorithmic transcoding.
- Equivalence: Recognizing or verifying that different symbolic notations represent the exact same quantity when the cross-notation equality or correspondence is explicitly presented for relational inspection. This differs from transcoding tasks, where the student is prompted to generate the missing notation.
- Estimation: Using numerical benchmarks or magnitude approximations to anticipate or evaluate the reasonableness of a cross-notation result without exact computation.
- Density: Identifying the existence of infinitely many intermediate values (infinite divisibility) between two numbers expressed in different notations.

Procedural Knowledge grouped tasks focused on algorithmic execution into two dimensions:

- Transcoding: Translating a quantity by generating an equivalent representation from one notation to another, with or without explicit algorithmic rules.
- Arithmetic: Solving computational operations in which different notations converge within the same equation or problem, excluding whole-number divisions expressed as fractions (which were considered transcoding).

Finally, the Mixed Domain comprised items that require or model the interdependent integration of conceptual and procedural dimensions. In these tasks, algorithmic execution (e.g., transcoding) is not an end in itself, but a tool subordinated to structural understanding (e.g., estimating magnitudes).

Figure 1 illustrates an example of an item categorized under each dimension

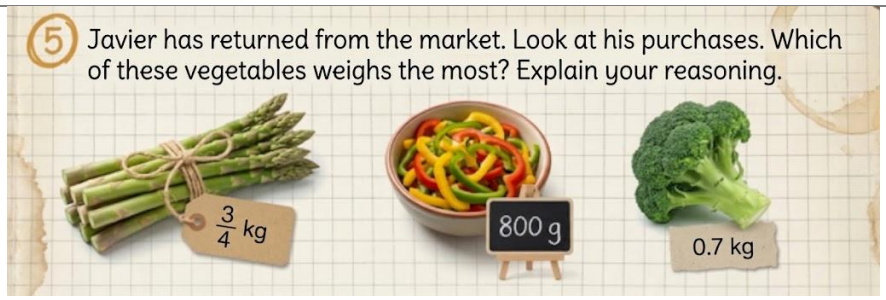
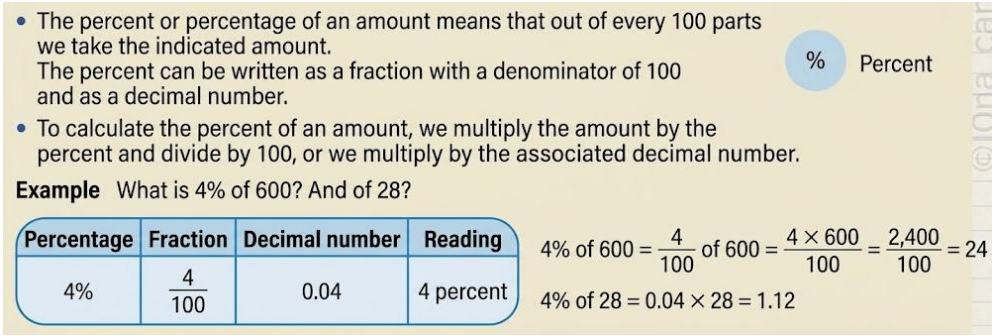
Magnitude	
Equivalence	Verify that the ratios between distance and fuel consumed are equivalent, forming a proportion: $\frac{1}{2.5} = \frac{2}{5} = \frac{4}{10} = 0.4$. The numerical value resulting from relating two directly proportional magnitudes is called the constant of proportionality.
Estimation	The car's fuel gauge shows that the gas tank is at 48% capacity. If you pour in a jerrycan containing exactly $\frac{1}{2}$ of a tank, will the gas overflow? Use one-half (50%) as a benchmark to explain your answer without performing mathematical calculations.
Density	Write two decimal numbers that lie between each of the following pairs of fractions: $\frac{8}{5}$ and $\frac{9}{5}$; $\frac{11}{4}$ and $\frac{22}{8}$; $\frac{36}{5}$ and $\frac{37}{5}$
Transcoding	Find the decimal value corresponding to each of these fractions: $a.\ \frac{2}{4} =$ $b.\ \frac{6}{10} =$ $c.\ \frac{1}{4} =$ $d.\ \frac{3}{5} =$ $e.\ \frac{9}{12} =$
Arithmetic	 **d.** How much does 1 and a quarter kilos of this cheese cost?
Mixed	

Fig. 1 Examples of Multiple-Notation Items by Dimension

Note. Magnitude, Authors' own elaboration based on SM, Grade 5, Book 2, p. 83. Equivalence, Authors' own elaboration based on Edelvives, Grade 6, p. 88. Estimation, authors' own elaboration. Density, Authors' own elaboration based on Santillana, Grade 6, Book 2, p. 123. Transcoding, Authors' own elaboration based on Edelvives, Grade 5, p. 88. Arithmetic, Authors' own elaboration based on Edelvives, Grade 6, p. 81; Mixed (equivalence–transcoding), Authors' own elaboration based on Santillana, Grade 6, Book 2, p. 138.

2.3. Reliability Analysis

The initial coding phase consisted of filtering items for the explicit presence or absence of symbolic notations (fractions, decimals, percentages, or their equivalent verbal terms). Because identifying single notations entails minimal interpretive variance, the inter-rater reliability protocol was reserved for the classification of multiple-notation items across cognitive dimensions (including the *coexistence* screening category). Two coders independently analyzed a 10% subsample of the multiple-notation items, selected via proportional stratified random sampling across the five publishers, achieving an inter-coder agreement of 91.1%. Cohen's kappa yielded an index of $\kappa = .86$ ($p < .001$), indicating "almost perfect" agreement (Landis & Koch, 1977). Discrepancies regarding theoretical boundaries (specifically between *arithmetic* and *coexistence*, and between *equivalence* and *transcoding*) were resolved by consensus to unify the coding criteria, after which a single researcher coded the remainder of the corpus.

2.4. Statistical Data Analysis

Given the categorical nature of the variables, non-parametric tests for nominal data were used, assuming a significance level of $\alpha = .05$.

Deviations of observed frequencies from an expected uniform distribution across categories were evaluated using one-way chi-square (χ^2) goodness-of-fit tests, estimating effect sizes with Cohen's w (reference thresholds: .10 small, .30 medium, .50 large; Cohen, 1988). Differences across textbook publishers and grade levels were tested using χ^2 tests of independence (applying Monte Carlo simulations with 2,000 replicates when expected cell frequencies were < 5). For these bivariate contrasts, following Cohen (1988), effect sizes were quantified using Cramér's V, adjusting the effect size reference thresholds for each contingency table by dividing the baseline values (.10, .30, .50) by the square root of the smallest dimension's degrees of freedom (df^* = min(rows − 1, columns − 1)).

Significant overall effects were followed up using two post-hoc procedures: (1) pairwise χ^2 comparisons (or Fisher's exact tests when expected cell frequencies were minimal) with Bonferroni correction to control for Type I error inflation, quantifying effect sizes via Cohen's *w* for one-dimensional goodness-of-fit comparisons and Cohen's *h* for two-sample proportion contrasts (.20 small, .50 medium, .80 large); and (2) standardized residual analyses (*z*), setting a critical threshold of $|z| > 1.96$ ($p < .05$) to determine significant overrepresentation or underrepresentation across categories and publishers. All data processing was executed in R using the rstatix and dplyr packages.

3. Results

3.1. Distribution of Items Across Notations

3.1.1. Global Results

Analysis of the total sample ($N = 8,304$) revealed an imbalance in the distribution of notations, $\chi^2(3) = 4,789.45$, $p < .001$, with a large effect size ($w = 0.76$). Task distribution exhibited a bias toward fractions ($n = 3,873$; 46.6%) and decimals ($n = 3,415$; 41.1%), whereas the presence of percentages ($n = 550$; 6.6%) and multiple-notation items ($n = 466$; 5.6%) was almost residual.

Post-hoc comparisons confirmed that the presence of fractions was greater than that of percentages, $\chi^2(1) = 2,496.57$, $p < .001$, $w = 0.75$, and multiple-notation tasks, $\chi^2(1) = 2,675.19$, $p < .001$, $w = 0.78$. Likewise, decimals were more frequent than both percentages, $\chi^2(1) = 2,070.17$, $p < .001$, $w = 0.72$, and multiple-notation tasks, $\chi^2(1) = 2,240.81$, $p < .001$, $w = 0.76$. Although the prevalence of fractions over decimals was statistically significant, $\chi^2(1) = 28.78$, $p < .001$, its effect size was negligible ($w = 0.06$). No significant differences were observed between percentages and multiple-notation tasks.

3.1.2. Results by Grade Level

Cross-grade analysis across grade levels (Grade 2: $n = 116$; Grade 3: $n = 518$; Grade 4: $n = 1,380$; Grade 5: $n = 3,067$; Grade 6: $n = 3,223$) revealed a main effect of grade on notation distribution, $\chi^2(12) = 958.10$, $p < .001$, with a medium effect size ($V = .20$) (see Figure 2).

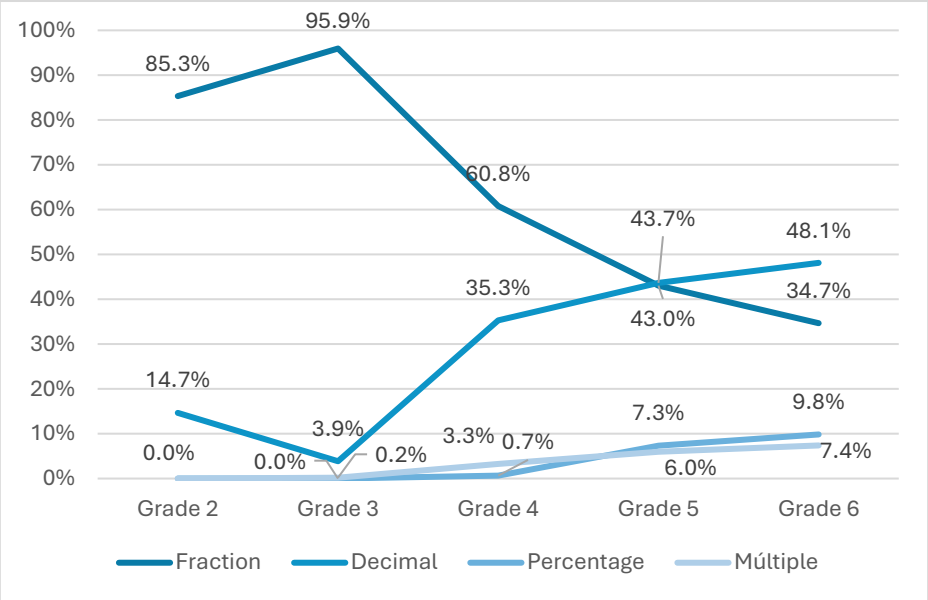


Fig. 2 Relative Evolution of Rational Number Notations by Grade Level

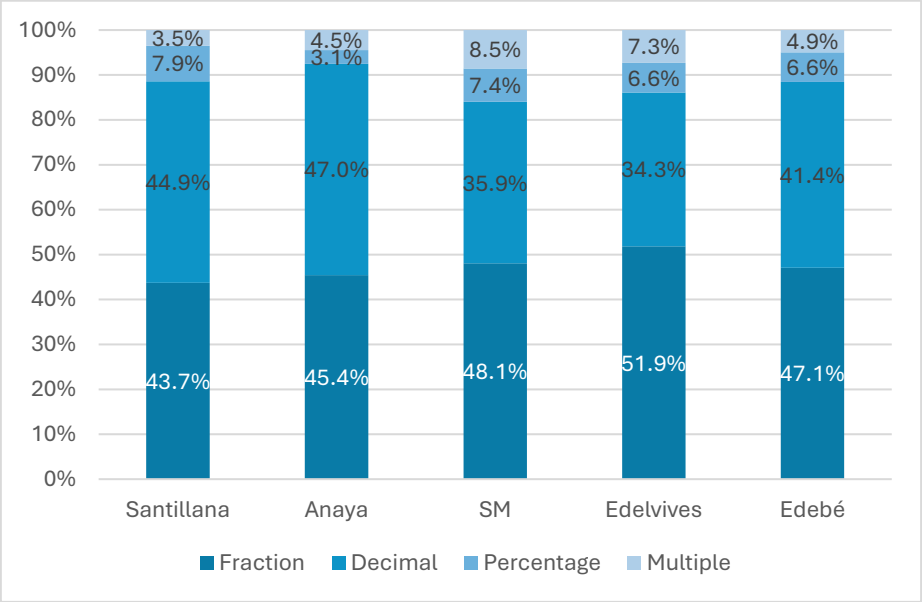
Post-hoc comparisons characterized this evolution in relative proportions across grades. First, fractions were the most prevalent notation in Grades 2 and 3 (85.3% and 95.9% of rational number items, respectively). However, between Grades 3 and 4, the proportion of fractions experienced a drastic drop (from 95.9% to 60.8%; $\chi^2(1) = 223.25$, $p < .001$, $h = 0.95$), continuing to decline to its lowest frequency in Grade 6 (34.7%). Simultaneously, decimals—which were residual in Grade 3 (3.9%)—increased substantially in Grade 4 (35.3%; $\chi^2(1) = 189.99$, $p < .001$, $h = 0.88$) and continued to increase until becoming the most frequent notation by Grade 6 (48.1%).

In contrast, percentages and multiple-notation tasks appeared almost exclusively in the upper grades. Percentages did not reach a meaningful presence until Grade 5 (7.3%), approaching 10% (9.8%) in Grade 6. In parallel, multiple-notation tasks experienced an increase between Grades 4 and 5 (from 3.3% to 6.0%; $\chi^2(1) = 14.33$, $p < .001$, $h = 0.13$), reaching their maximum in Grade 6 (7.4%), although this latter increase was not statistically significant.

3.1.3. Results by Publisher

Analysis by textbook revealed significant differences in the distribution of notations, $\chi^2(12) = 150.42$, $p < .001$, albeit with a small global effect size ($V = .08$). Standardized residual analyses (Table 1) identified divergent instructional profiles in the treatment of rational numbers (see Figure 3).

Fig. 3 Distribution of Notations by Publisher



Standardized residual analysis revealed two distinct patterns. First, SM and Edelvives significantly overrepresented multiple-notation tasks and underrepresented decimals, with Edelvives also showing a significant overrepresentation of fractions. Conversely, Santillana and Anaya exhibited the opposite pattern, underrepresenting multiple-notation tasks while overrepresenting decimals. Within this latter group, Santillana overrepresented percentages and underrepresented fractions, whereas Anaya showed a significant underrepresentation of percentages. Finally, Edebé showed no statistically significant deviations from the overall sample distribution.

Table 1

Standardized Residuals (z) for Notation Distribution by Publisher

Notation	SM (n = 2,095)	Santillana (n = 2,461)	Anaya (n = 1,340)	Edebé (n = 1,436)	Edelvives (n = 972)
Fractions	1.51	-3.46*	-0.96	0.36	3.53*
Decimals	-5.57*	4.49*	4.78*	0.20	-4.56*
Percentages	1.75	3.09*	-5.73*	-0.01	-0.19
Multiple notation	6.74*	-5.44*	-1.97*	-1.21	2.29*

Note. * $p < .05$ (indicates a statistically significant standardized residual, $|z| > 1.96$).

3.1.4. Curricular Trajectories by Publisher

Cross-grade analysis across publishers revealed that, although all textbook series converged toward a similar notation distribution by Grade 6, their **instructional sequences** differed markedly. As shown in Figure 4, three distinct progression patterns emerged in the evolution of results across grade levels.

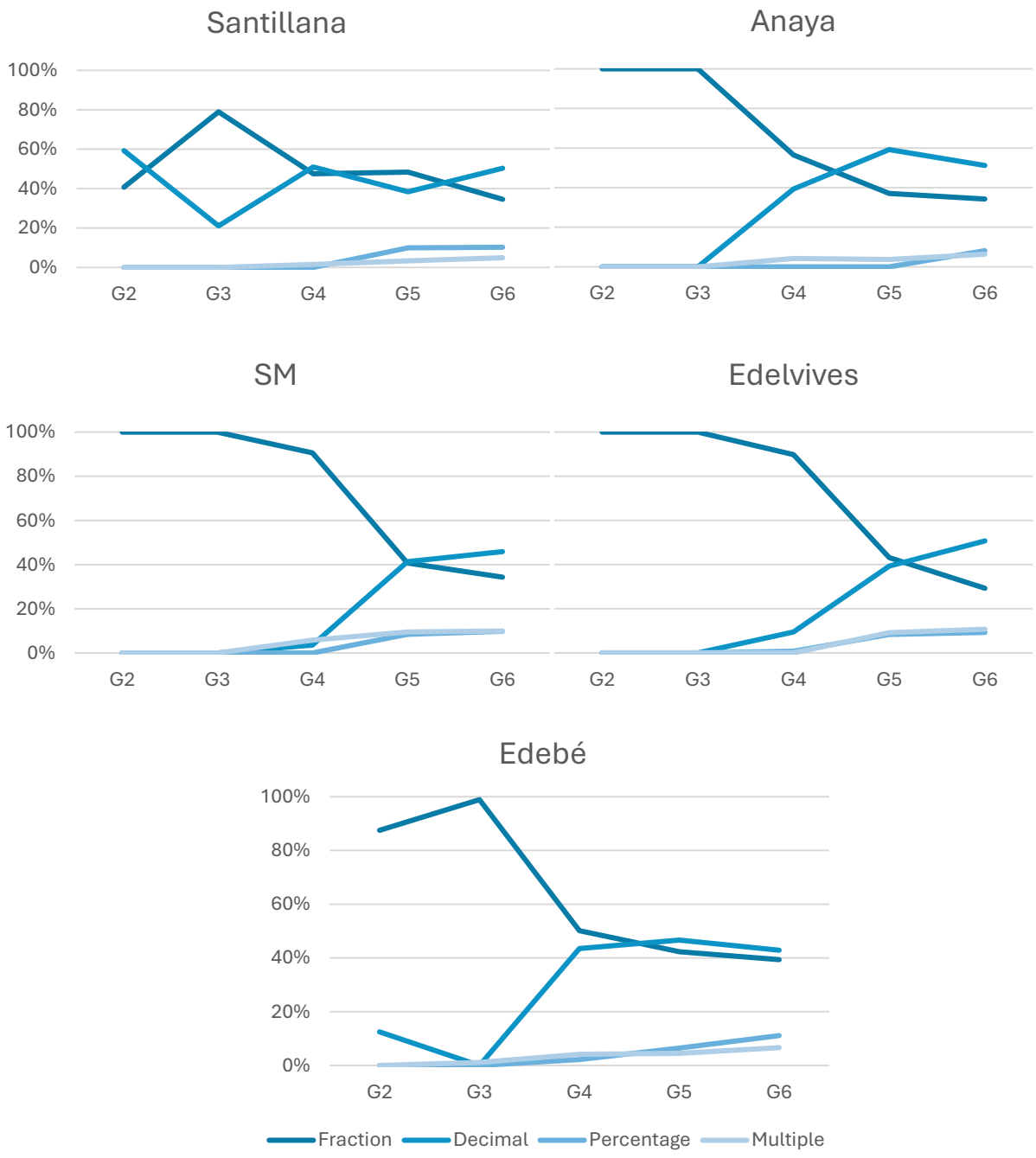


Fig. 4 Relative Evolution of Notations by Publisher and Grade Level

The first trend, concerning the progression of decimal notation, followed three differentiated patterns. First, Santillana introduced decimals early, accounting for 59.3% of its rational number items in Grade 2,

contextualizing these (still scarce) items in everyday situations involving coins. The second pattern reflected a substantial increase in the proportion of decimals in Grade 4 across Santillana, Edebé, and Anaya (reaching between 39.3% and 51.0% of items). This represented the first appearance of this notation in Anaya and its consolidation in Edebé after a residual presence in Grade 2, triggering a corresponding drop in the proportion of fractions in both series. This year-over-year jump between Grades 3 and 4 was statistically significant across all three publishers: Santillana ($\chi^2(1) = 27.84$, $p < .001$, $h = 0.64$), Anaya ($\chi^2(1) = 54.46$, $p < .001$, $h = 1.36$), and Edebé ($\chi^2(1) = 60.64$, $p < .001$, $h = 1.44$). Finally, SM and Edelvives delayed the introduction of decimals until Grade 5, maintaining an almost exclusive predominance of fractions in Grade 4 (≈ 90%). Their subsequent increases in Grade 5 were equally significant: SM ($\chi^2(1) = 113.41$, $p < .001$, $h = 1.02$) and Edelvives, $\chi^2(1) = 39.06$, $p < .001$, $h = 0.73$.

The second trend corresponded to percentages. All publishers introduced percentages in Grade 5 except Anaya, which presented them in Grade 6. Given their absence or marginal presence in Grade 4, Fisher's exact test was employed, confirming significant increases between Grades 4 and 5 in Santillana ($p < .001$, $h = 0.64$), SM ($p < .001$, $h = 0.59$), Edelvives ($p = .001$, $h = 0.41$), and Edebé ($p = .004$, $h = 0.22$).

Finally, substantial variation emerged in the presentation of multiple-notation items in the upper grades. Edelvives exhibited the largest increase, rising from a complete absence in Grade 4 to 9.2% in Grade 5 (Fisher's exact test, $p < .001$, $h = 0.62$). SM increased the presence of these items progressively, reaching 9.9% in Grade 6 without a statistically significant year-over-year increase. In contrast, Santillana presented the lowest proportion of multiple-notation items: by the end of elementary school, these items represented only 4.9% of its total, a significantly lower proportion than in SM ($p < .001$, $h = 0.19$) and Edelvives ($p < .001$, $h = 0.22$, Fisher's exact test).

3.2. Combinations within Multiple-Notation Items

3.2.1. Global Results

Analysis of the subsample of multiple-notation items ($n = 466$) revealed a highly uneven distribution, $\chi^2(3) = 332.39$, $p < .001$, with a very large effect size ($w = 0.84$). Standardized residual analysis showed that this imbalance was driven by a marked overrepresentation of the fraction–decimal combination (61.4%; $z = 15.71$). Conversely, all remaining combinations were significantly underrepresented:

fraction–percentage (16.3%; $z = -3.75$), decimal–percentage (12.0%; $z = -5.61$), and the simultaneous combination of all three notations (10.3%; $z = -6.35$).

3.2.2. Results by Publisher

Comparing this distribution via a χ^2 test of independence (with Monte Carlo simulation) revealed significantly different combination patterns across publishing programs, $\chi^2(12) = 46.71$, $p < .001$, with a moderate effect size ($V = .18$). As shown in Figure 5 and the standardized residual analysis (see Table 2), two main tendencies emerged across the textbooks.

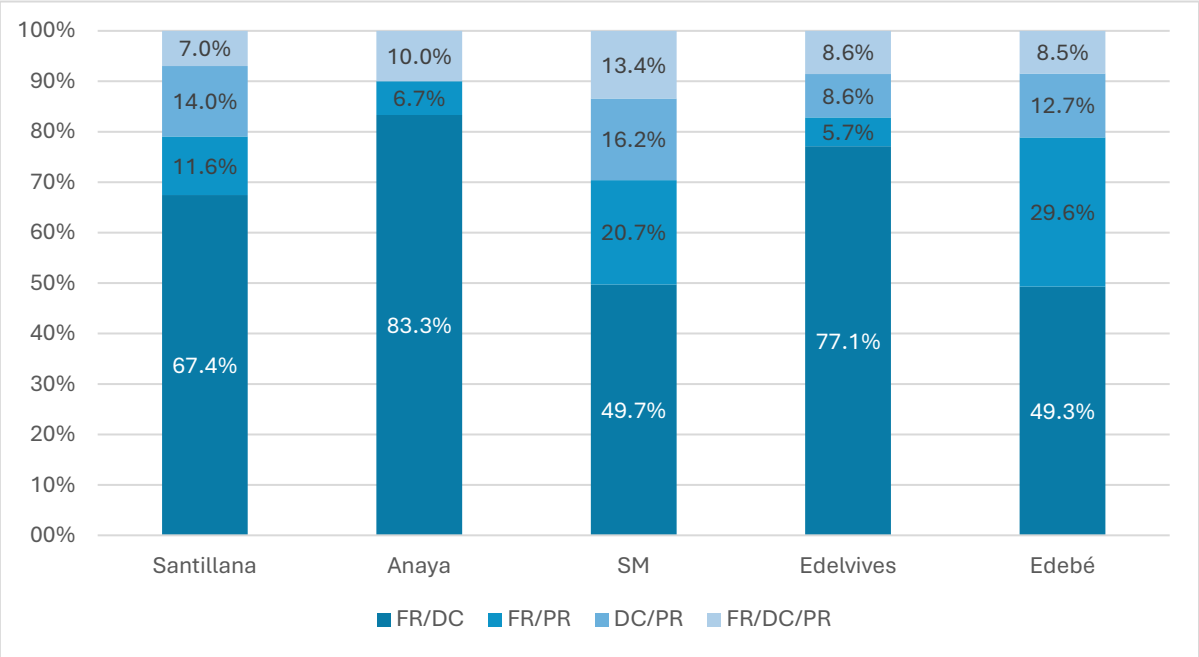


Fig. 5 Distribution of Multiple-Notation Items by Publisher. Note. FR/DC (Fraction–Decimal), FR/PR (Fraction–Percentage), DC/PR (Decimal–Percentage), FR/DC/PR (Fraction–Decimal–Percentage)

The first tendency consisted of a restricted integration of the different notations. Anaya and Edelvives concentrated their tasks almost exclusively on the fraction–decimal interaction. In the case of Anaya, this combination accounted for 83.3% of its multiple-notation items, significantly limiting fraction–percentage tasks and completely excluding the decimal–percentage interaction. Edelvives similarly presented very few items combining fractions and percentages (5.7%).

The second tendency, represented by SM and Edebé, reflected greater diversification. Although both devoted roughly half of their multiple-notation tasks to the fraction–decimal interaction (49.7% and 49.3%, respectively), this proportion represented a significant underrepresentation relative to the overall

sample mean. In SM, both the fraction–percentage (20.7%) and decimal–percentage (16.2%) combinations were significantly overrepresented. Meanwhile, Edebé presented the highest proportion of tasks linking fractions and percentages (29.6%).

Finally, Santillana was the only textbook series lacking significant residuals, mirroring the global trend of the study.

Table 2

Standardized Residuals (z) for the Distribution of Multiple-Notation Items by Publisher

Multi-Notation Interaction	Santillana (n = 86)	Anaya (n = 60)	SM (n = 179)	Edelvives (n = 70)	Edebé (n = 71)
Fraction–Decimal	1.06	3.74*	−4.08*	2.94*	−2.27*
Fraction–Percentage	−1.29	−2.17*	2.01*	−2.60*	3.29*
Decimal–Percentage	1.36	−3.06*	2.19*	−1.00	−0.60
Fraction–Decimal–Percentage	−1.57	−0.13	1.62	−0.61	0.23

Note. * $p < .05$ (indicates a statistically significant standardized residual, $|z| > 1.96$).

3.3. Analysis of the Cognitive Demand of Multiple-Notation Items

3.3.1. Global Results

From the initial multiple-notation sample ($n = 466$), 34 items (7.3%) were categorized as mere coexistence. Because they did not require cognitive interaction across notations, they were excluded from subsequent analyses.

Analysis of the remaining corpus ($n = 432$) revealed a heavily skewed distribution of cognitive demand (see Table 3), $\chi^2(2) = 416.35$, $p < .001$, with a very large effect size ($w = 0.98$). The instructional approach was overwhelmingly dominated by procedural items (78.9%, $z = 16.42$), whose overrepresentation was explained almost entirely by transcoding items ($\chi^2(1) = 140.65$, $p < .001$, $w = 0.64$). As a consequence, conceptual items showed a marked underrepresentation (17.4%, $z = −5.75$) and were restricted almost entirely to equivalence, to the detriment of magnitude and density, with a complete absence of estimation items ($\chi^2(2) = 79.76$, $p < .001$, $w = 1.03$). Finally, mixed conceptual–procedural items were virtually residual (3.7%, $z = −10.67$).

Table 3

Frequencies and Standardized Residuals (z) for Cognitive Demand Domains and Dimensions

Domain	Dimension	n	% (Within Domain)	% (Total n = 432)	Residuals (z)
Procedural	Transcoding	280	82.1	64.8	8.39*
	Arithmetic	61	17.9	14.1	−8.39*
	Total Procedural	341	100.0	78.9	16.42*
Conceptual	Equivalence	61	81.3	14.1	7.20*
	Magnitude	12	16.0	2.8	−2.60*
	Density	2	2.7	0.5	−4.60*
	Estimation	0	0.0	0.0	—
	Total Conceptual	75	100.0	17.4	−5.75*
Mixed Domain	Total Mixed	16	100.0	3.7	−10.67*

Note. * $p < .05$ (indicates a statistically significant standardized residual, $|z| > 1.96$). Within-domain residuals were computed via one-way goodness-of-fit tests for the Procedural domain ($df = 1$) and for the three observed categories of the Conceptual domain ($df = 2$, excluding Estimation due to zero observed frequency; if Estimation is included assuming a 4-category uniform distribution, $\chi^2(3) = 131.35$, $p < .001$, with residuals of $z = 9.76$* for Equivalence, $z = -1.56$ for Magnitude, $z = -3.87$* for Density, and $z = -4.33$* for Estimation).

3.3.2. Results by Publisher

At the global level, no statistically significant differences were found across textbook publishers in the overall distribution of procedural, conceptual, and mixed items, $\chi^2(8) = 9.04$, $p = .339$, $V = .10$. As shown in Figure 6, the procedural domain predominated across all textbook series (ranging from 69.6% in Edelvives to 82.1% in SM), to the detriment of the conceptual domain and, most notably, the mixed domain.

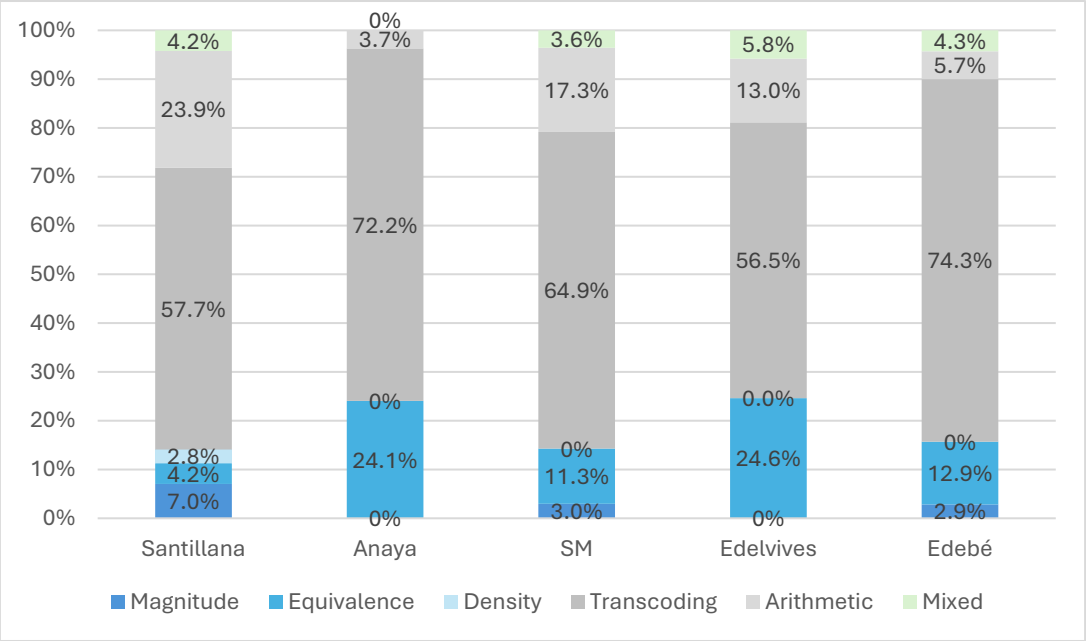


Fig. 6 Percentage Distribution of Cognitive Demand Dimensions by Publisher

However, within-domain analyses confirmed the existence of significant differences among textbook publishers in the internal distribution of both domains (see Table 4). Within the procedural domain, $\chi^2(4) = 15.22$, $p = .004$, $V = .21$, Anaya and Edebé concentrated their tasks almost exclusively on transcoding. In contrast, Santillana showed a more balanced distribution, with a significant overrepresentation of cross-notation arithmetic. Within the conceptual domain, $\chi^2(8) = 29.95$, $p < .001$ (Monte Carlo simulation), $V = .45$, Edelvives and Anaya devoted all of their conceptual items (100%) to equivalence. Conversely, Santillana was the only publisher to significantly underrepresent equivalence in order to include more magnitude and density items.

Finally, given its scarcity ($n = 16$), the mixed domain was analyzed descriptively. Its presence was anecdotal across all textbooks, being largely restricted to the equivalence–transcoding dyad: SM contributed 6 tasks, Edelvives 4, and Edebé and Santillana contributed 3 each (with 0 in Anaya).

Table 4

Standardized Residuals (z) for Cognitive Demand Dimensions by Publisher

Domain	Dimension	Santillana (n = 71)	Anaya (n = 54)	SM	Edelvives (n = 69)	Edebé (n = 70)

				(n = 168)		
Conceptual	Magnitude	3.15*	-1.73	0.78	-2.05*	0.21
	Equivalence	-4.48*	1.90	-0.33	2.25*	0.04
	Density	3.65*	-0.66	-0.98	-0.78	-0.59
Procedural	Transcoding	-2.49*	2.32*	-1.24	-0.17	2.30*
	Arithmetic	2.49*	-2.32*	1.24	0.17	-2.30*

Note. * $p < .05$ (indicates a statistically significant standardized residual, $|z| > 1.96$).

4. Discussion

Deep understanding of rational numbers requires integrating the learning of fractions, decimals, and percentages into a unified cognitive structure. According to the Integrated Theory of Numerical Development (Braithwaite et al., 2022; Siegler et al., 2011), consolidating this cross-notation number sense entails conceptually and procedurally interconnecting six core dimensions: magnitude, equivalence, estimation, transcoding, density, and arithmetic (Lamon, 2020; Tian et al., 2020). Because international evidence suggests that instructional materials often present a fragmented view of rational numbers (e.g., Braithwaite & Siegler, 2018; Siegler & Oppenzato, 2021; Siegler & Tian, 2022), this study evaluated the instructional design of elementary school textbooks in Spain to determine whether the implemented textbook curriculum genuinely fosters integrated learning or perpetuates representational segregation. Our findings clearly point to the latter scenario. The persistent segregation of fractions, decimals, and percentages in the Spanish textbooks analyzed in this study validates the "biased input hypothesis" (Siegler et al., 2020; Siegler & Oppenzato, 2021). By isolating notations into independent instructional units across 94.4% of all tasks, textbooks bypass the simultaneous co-occurrence required to develop an integrated number sense. This makes it difficult for students to unify notations into a continuous system, potentially obstructing the final developmental stage posited by the Integrated Theory of Numerical Development (Braithwaite et al., 2022; Siegler et al., 2011). Moreover, this disconnection across notations is compounded by a curricular trajectory across school grades that imposes sequences of blocked practice rather than interleaved learning (Rohrer & Taylor, 2007). Under this rigid sequencing, fractions dominate early instruction but drop sharply when decimals are introduced midway through elementary school, relegating percentages and multiple-notation tasks to the final grades. While this sequential introduction partially reflects official Spanish curricular guidelines (which formally schedule percentages in Grades 5 and 6), even in the upper grades (once all three notations are active) multiple-

notation tasks remain restricted to just 6% to 7.4% of the instructional corpus. This progression illustrates the traditional linear instructional approach criticized in the literature, which weakens cognitive flexibility and conceptual transfer by forcing students to work predominantly in isolation with one notation at a time (Schiller et al., 2025a; Siegler & Oppenzato, 2021). Thus, textbook design helps explain why students often end up developing a precarious sense of cross-notation equivalence after years of formal schooling (Lemonidis & Pilianidis, 2020).

Furthermore, the marginalization of percentages (under 7% of all items) and their delayed introduction in elementary school constitute a major pedagogical limitation. Percentages serve as a "conceptual bridge" between whole and rational numbers due to their standardized scale based on a constant denominator of 100, making them the most cognitively intuitive rational number notation (Lemonidis & Pilianidis, 2020; Powell & Nelson, 2021; Tian et al., 2020). Although Spanish curriculum standards delay formal percentage instruction until the final elementary cycle, relegating percentages entirely to a late appearance prevents Spanish school textbooks from capitalizing earlier on the informal intuitions and everyday experiences that students already possess. As Lamon (2020) warns, this instructional delay risks reducing percentage learning to the rote memorization of mechanical calculation rules and rigid algorithms detached from spatial meaning. Consequently, students are deprived of early opportunities to use the intuition of the base-100 scale to estimate magnitudes and reflectively guide their translations between fractions and decimals.

The results also uncover an "illusion of integration" within multiple-notation tasks. Nearly 80% of these valid items posed purely procedural demands (with almost 65% involving isolated transcoding), whereas the mixed conceptual–procedural approach was anecdotal (3.7%). This marginalization runs counter to the cognitive principle of iterative development, which posits that conceptual and procedural knowledge should mutually reinforce each other (Rittle-Johnson, 2021; Xu et al., 2026). By depriving students of tasks in which transcoding serves a conceptual purpose (e.g., estimating magnitudes to reject inconsistent solutions), textbooks hinder the consolidation of a unified representational system on the mental number line (Hiebert & Carpenter, 1992).

This algorithmic bias, which prioritizes routine over cross-notation magnitude comprehension, aligns with the deficient instructional input documented in the United States (Siegler & Oppenzato, 2021). Just as co-occurrence is minimal there, it is equally scarce in the Spanish market: cross-notation arithmetic and

magnitude comparison were merely residual (both under 1% of the corpus), while cross-notation density and estimation were virtually nonexistent. This striking disproportion validates Lamon's (2020) warnings regarding traditional instruction: by jumping prematurely to calculation rules devoid of spatial and quantitative meaning, textbook design offers limited support for articulating the underlying network of rational number dimensions across notations.

A direct risk of such instruction without conceptual grounding is impulsive calculation (Schiller & Siegler, 2023): operating immediately on individual digits without comprehending overall numerical magnitude. Lacking cross-notation conceptual anchors, students tend to process fractions and decimals as they would whole numbers, developing persistent misconceptions such as transcoding via digit concatenation (e.g., translating ¼ as 0.14 or 1.4; Nesher & Peled, 1986; Schiller & Siegler, 2023). As with percentages, this mechanical approach weakens the scaffolding across notations and helps explain why these equivalence biases become entrenched over time.

At the same time, our results demonstrate that despite sharing structural deficiencies—namely, a global procedural bias—textbooks exhibit substantial design differences across publishers regarding their instructional sequencing and specific cross-notation combinations. This divergence challenges the assumed homogeneity of the textbook market (Siegler & Oppenzato, 2021; Siegler & Tian, 2022) and provides an empirical basis for understanding why textbook choice can differentially impact student achievement (Agodini et al., 2009). When analyzing the alignment of each publishing program with evidence-based suitability criteria, an instructional trade-off emerges between the quantity of cross-notation exposure and its cognitive quality.

On the one hand, Santillana displayed the most internally balanced distribution of cognitive demand within its cross-notation tasks: it offered greater internal conceptual and procedural diversity (including cross-notation magnitude, density, and arithmetic) and introduced decimal notation early in Grade 2 through everyday coin contexts (which can provide an informal anchor if subsequently connected to continuous positional value rather than processed as two separate whole numbers for euros and cents), although it was the publisher that presented the lowest overall frequency of multiple-notation items (3.5%). In contrast, SM was notable for including the highest proportion of multiple-notation items and establishing stronger links with percentages, but delayed the introduction of both decimals and percentages until Grade 5 and its conceptual focus remained unbalanced. Edebé and Edelvives exhibited

profiles of superficial notation integration: they posed high proportions of mechanical transcoding items and reduced the conceptual domain entirely or almost entirely to static equivalence. Finally, Anaya displayed the most restrictive profile overall by segregating notations, delaying their integration and the introduction of percentages until Grade 6, excluding decimal–percentage interactions, and relying disproportionately on purely procedural tasks.

Given these publisher differences, textbook selection is key to student learning, making it necessary to equip teachers with evidence-based evaluation criteria that enable them to identify publishing programs with greater cognitive potential (Hiebert & Carpenter, 1992). This selection should prioritize textbooks that minimize notation segregation (especially from the intermediate elementary grades onward) and anchor equivalence in continuous spatial models, such as the number line, rather than in the mechanical instruction of transcoding rules. Nevertheless, because all analyzed textbook programs remain far from achieving full curricular suitability (Lamon, 2020), the responsibility for actively connecting these notations in the classroom ultimately falls on the teacher.

5. Limitations and Future Directions

Several methodological and conceptual limitations in this study outline avenues for future work. First, cognitive demand and mathematical dimensions were coded exclusively within multiple-notation items to evaluate cross-notation integration. Future studies should extend this dimensional coding to single-notation items to determine whether Spanish textbooks address magnitude, estimation, and density within isolated notations while restricting cross-notation tasks to algorithmic translation.

Second, items were analyzed individually, omitting their didactic sequencing or spatial contiguity within the textbook page. Future research should evaluate whether the conceptual or mixed tasks identified function as localized strategic scaffolding that, acting as a "cognitive hook," prepares and facilitates the reasoning students need before tackling procedural transcoding tasks, as suggested by the iterative development model (Rittle-Johnson, 2021).

Third, basing the analysis on absolute item frequencies assumes that each item carries equal instructional weight, overlooking the fact that solving a complex conceptual item demands substantially more classroom time than executing multiple mechanical transcoding exercises. Future research should

incorporate classroom observations to evaluate the actual time-on-task and instructional weight teachers assign to these different types of activities.

Fourth, the sample was restricted to elementary education in Spain. It would be valuable to extend this analysis both to compulsory secondary education (Grades 7–10, where representational unification of rational numbers becomes indispensable for success in algebra; Siegler et al., 2012) and to other international educational systems.

Finally, future work should move beyond cognitive demand coding to analyze the semantic alignment of textbook tasks (Qiu & Wang, 2021). Doing so would make it possible to determine whether Spanish textbooks incur implicit biases by systematically associating fractions with discrete contexts (counting/partitioning) and decimals with continuous ones (measurement).

6. Statements and Declarations

Ethics approval

Ethics declaration: not applicable.

Consent to participate

Consent to Participate declaration: not applicable.

Consent to publish

Consent to publish declaration: not applicable.

Funding

This study was funded by the Spanish Ministry of Science and Innovation under Grant PID2022-139703NB-I00.

Competing Interests

Financial interests: Santiago Vicente previously held a research transfer contract with Grupo Editorial SM to design instructional models for primary mathematics textbooks. This contract has expired and has in no way influenced the design, analysis, or interpretation of the results of this study. The remaining authors declare they have no relevant financial or non-financial interests to disclose.

Data availability

The datasets generated and/or analysed during the current study are available from the corresponding author on reasonable request.

Authors' contributions

Conceptualization: Santiago Vicente, Paula Muñoz, Josetxu Orrantia.

Methodology: Santiago Vicente, Natalia Calvo, Paula Muñoz.

Formal analysis and investigation: Santiago Vicente.

Validation: Paula Muñoz.

Resources: Natalia Calvo.

Writing - original draft preparation: Santiago Vicente.

Writing - review and editing: Santiago Vicente, Paula Muñoz, Natalia Calvo, Josetxu Orrantia.

7. References

Agodini, R., Harris, B., Atkins-Burnett, S., Heaviside, S., Novak, T., & Murphy, R. (2009). *Achievement effects of four early elementary school math curricula: Findings from first graders in 39 schools—Executive summary* (NCEE 2009-4053). National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education.

Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology*, *113*(3), 447–455. https://doi.org/10.1016/j.jecp.2012.06.004

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–126). Academic Press. https://doi.org/10.4324/9781003726777-6

Braithwaite, D. W., McMullen, J., & Hurst, M. A. (2022). Cross-notation knowledge of fractions and decimals. *Journal of Experimental Child Psychology*, *213*, 105210. https://doi.org/10.1016/j.jecp.2021.105210

Braithwaite, D. W., & Siegler, R. S. (2018). Children learn spurious associations in their math textbooks: Examples from fraction arithmetic. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *44*(11), 1765–1777. https://doi.org/10.1037/xlm0000546

Ching, B. H.-H., Li, X. Y., & Chen, T. T. (2024). Cross-notation knowledge of rational numbers predicts fraction arithmetic. *British Journal of Educational Psychology*, *94*(3), 717–737. https://doi.org/10.1111/bjep.12674

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Lawrence Erlbaum Associates.

González-Peñaloza, C., Castro-Rodríguez, E., & Piñeiro, J. L. (2023). Los números racionales en los currículos de Educación General Básica de Chile (1974–2012). *Revista Electrónica Educare*, *27*(3), 1–19. https://doi.org/10.15359/ree.27-3.17247

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 65–97). Macmillan Publishing Co, Inc.

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *The Journal of Mathematical Behavior*, *14*(4), 365–378. https://doi.org/10.1016/0732-3123(95)90035-7

Kieren, T. E. (1980). Recent research in number learning. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125–149). Department of Elementary Education, University of Alberta.

Lamon, S. J. (2020). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (4th ed.). Routledge. https://doi.org/10.4324/9780203803165

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. *Biometrics*, *33*(1), 159–174. https://doi.org/10.2307/2529310

Lemonidis, C., & Pilianidis, N. (2020). The 8th grade students' competencies in alternating different symbolic representations of rational numbers. *International Electronic Journal of Mathematics Education*, *15*(3), em0587. https://doi.org/10.29333/iejme/7865

Liu, Q., & Braithwaite, D. W. (2023). Affordances of fractions and decimals for arithmetic. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *49*(9), 1459–1470. https://doi.org/10.1037/xlm0001161

Liu, Q., Wang, Y., & Braithwaite, D. W. (2025). Affordances of fractions and decimals for arithmetic among middle school students in the United States and China. *Journal of Experimental Child Psychology*, *253*, 106187. https://doi.org/10.1016/j.jecp.2024.106187

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review*, *38*, 201–221. https://doi.org/10.1016/j.dr.2015.07.008

McMullen, J., & Van Hoof, J. (2020). The role of rational number density knowledge in mathematical development. *Learning and Instruction*, *65*, 101228. https://doi.org/10.1016/j.learninstruc.2019.101228

Nesher, P., & Peled, I. (1986). Shifts in reasoning: The case of extending number concepts. *Educational Studies in Mathematics*, *17*(1), 67–79. https://doi.org/10.1007/bf00302379

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27–52. https://doi.org/10.1207/s15326985ep4001_3

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, *28*, 64–72. https://doi.org/10.1016/j.learninstruc.2013.05.003

Powell, S. R., & Nelson, G. (2021). University students' misconceptions about rational numbers: Implications for developmental mathematics and instruction of younger students. *Psychology in the Schools*, *58*(2), 307–331. https://doi.org/10.1002/pits.22448

Qiu, K., & Wang, Y. (2021). Conceptual distinctions and preferential alignment across rational number representations. *European Journal of Psychology of Education*, *36*(3), 865–881. https://doi.org/10.1007/s10212-020-00502-4

Rittle-Johnson, B. (2021). Iterative development of conceptual and procedural knowledge in mathematics learning. *Cognitive Development*, *58*, 101034. https://doi.org/10.1016/j.cogdev.2021.101034

Rohrer, D., & Taylor, K. (2007). The shuffling of mathematics problems improves learning. *Instructional Science*, *35*, 481–498. https://doi.org/10.1007/s11251-007-9015-8

Schiller, L. K. (2020). *Recharging rational number understanding* [Doctoral dissertation, Columbia University]. PsyArXiv. https://doi.org/10.31234/osf.io/42u3a

Schiller, L. K., Abreu-Mendoza, R. A., Fitzsimmons, C. J., Siegler, R. S., Thompson, C. A., & Rosenberg-Lee, M. (2025a). Lack of integrated number sense among college students: Evidence from rational number cross-notation comparison. *Journal of Experimental Psychology: Human Perception and Performance*, *51*(1), 70–91. https://doi.org/10.1037/xhp0001268

Schiller, L. K., Abreu-Mendoza, R. A., Siegler, R. S., Rosenberg-Lee, M., & Thompson, C. A. (2024). Building integrated number sense in adults and children: Comparing fractions-only training with cross-notation number line training. *Journal of Experimental Child Psychology*, *246*, 106017. https://doi.org/10.1016/j.jecp.2024.106017

Schiller, L. K., & Siegler, R. S. (2023). Integrated knowledge of rational number notations predicts children's math achievement and understanding of numerical magnitudes. *Cognitive Development*, *68*, 101380. https://doi.org/10.1016/j.cogdev.2023.101380

Schiller, L. K., Siegler, R. S., Abreu-Mendoza, R. A., Rosenberg-Lee, M., & Thompson, C. A. (2025b). The percentages-are-larger bias: Evidence from rational number cross-notation comparison.

Journal of Educational Psychology. Advance online publication. https://doi.org/10.1037/edu0000941

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Magnuson, K., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691–697. https://doi.org/10.1177/0956797612440101

Siegler, R. S., Im, S., Schiller, L. K., Tian, J., & Braithwaite, D. W. (2020). The sleep of reason produces monsters: How and when biased input shapes mathematics learning. *Annual Review of Developmental Psychology*, *2*(1), 413–435. https://doi.org/10.1146/annurev-devpsych-041620-031544

Siegler, R. S., & Oppenzato, C. O. (2021). Missing input: How imbalanced distributions of textbook problems affect mathematics learning. *Child Development Perspectives*, *15*(2), 76–82. https://doi.org/10.1111/cdep.12402

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–296. https://doi.org/10.1016/j.cogpsych.2011.03.001

Siegler, R. S., & Tian, J. (2022). Why do we have three rational number notations? The importance of percentages. *Advances in Child Development and Behavior*, *63*, 1–33. https://doi.org/10.1016/bs.acdb.2022.05.001

Sievert, H., van den Ham, A.-K., & Heinze, A. (2021). Are first graders' arithmetic skills related to the quality of mathematics textbooks? A study on students' use of arithmetic principles. *Learning and Instruction*, *71*, 101401. https://doi.org/10.1016/j.learninstruc.2020.101401

Thompson, C. A., Mielicki, M. K., Rivera, F., Fitzsimmons, C. J., Scheibe, D. A., Sidney, P. G., Schiller, L. K., Taber, J. M., & Waters, E. A. (2023). Leveraging math cognition to combat health innumeracy. *Perspectives on Psychological Science*, *18*(1), 152–177. https://doi.org/10.1177/17456916221083277

Tian, J., Braithwaite, D. W., & Siegler, R. S. (2020). How do people choose among rational number notations? *Cognitive Psychology*, *123*, 101333. https://doi.org/10.1016/j.cogpsych.2020.101333

Törnroos, J. (2005). Mathematics textbooks, opportunity to learn and student achievement. *Studies in Educational Evaluation*, *31*(4), 315–327. https://doi.org/10.1016/j.stueduc.2005.11.005

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Kluwer Academic Publishers.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181–209. https://doi.org/10.1080/07370001003676603

Xu, C., Di Lonardo Burr, S., Rodgers, L. J., Wylie, J., Si, J., Huang, B., Guo, K., & Li, H. (2026). A longitudinal investigation of the codevelopment and bidirectional relations among whole number arithmetic and conceptual and procedural fraction knowledge. *Journal of Educational Psychology*, *118*(2), 224–239. https://doi.org/10.1037/edu0000942