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A Conceptual Framework for AI Supported Representational Coordination and Transfer in Calculus

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Perspective

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Abstract

Calculus learners may carry out routine procedures while struggling to connect symbolic, graphical, numerical, verbal and contextual representations, particularly in unfamiliar problems (García-García et al., 2025). Digital environments can display linked representations, and generative artificial intelligence (AI) adds a conversational interface; neither feature by itself establishes that learners coordinate the mathematical meanings across forms. This Perspective proposes a framework for designing and investigating AI-supported representational coordination in calculus. Rather than treating AI as an answer generator, it considers a bounded role for a conversational system: eliciting a learner's interpretation, responding to the learner's stated reasoning, and inviting the learner to test a claim across representations. The paper adapts established work on multiple representations, covariational reasoning and transfer to propose four iterative stages—notice, translate, coordinate and transfer. Its proposed contribution is an analytic chain linking (i) the learner evidence available to the system, (ii) a contingent AI move, (iii) the learner's subsequent mathematical action and (iv) an observable outcome that can be studied. A net-change task illustrates the framework, but it has not been implemented or evaluated. The paper makes no effectiveness claims; it offers a set of design propositions and testable hypotheses for future research.

1 Introduction

Learning calculus involves interpreting relationships across symbolic expressions, graphs, tables, verbal descriptions and real-world contexts. Research on derivatives and definite integrals documents difficulties in connecting symbolic, graphical and numerical forms and interpreting calculus ideas in applications (García-García et al., 2025). A learner may integrate correctly yet be unable to explain what the result means on a graph, or read a graph accurately yet be unable to express the relationship symbolically. Difficulties applying calculus and related function concepts when contexts change have also been reported (Ceuppens et al., 2019; Cui, 2006). In this paper, an accumulation task is proposed as one possible setting in which to examine these difficulties; its sensitivity as an assessment of coordination remains a conjecture rather than an established finding.

Digital environments can make changing quantities and relationships visible and manipulable. Dynamic visualization has been examined as a support for calculus understanding (Kossivi, 2020), while data-driven recommendation has been explored for calculus learning (Hsu & Hsu, 2025). However, a display that links representations does not guarantee that learners will interpret the links or explain whether the forms are consistent. Work on multiple external representations argues that their instructional value depends on design and learner support (Ainsworth, 2006; Rexigel et al., 2024; Schwonke et al., 2009). This distinction motivates the present focus: not simply whether several representations are available, but what learners do to interpret and validate their mathematical relationship.

Conversational generative AI introduces a possible interactional affordance: it can ask a follow-up question in response to a learner's own explanation and can offer a contrasting case when a learner's

reasoning suggests a particular misconception. These are design possibilities, not established advantages over teachers or non-AI digital tools. They also create risks: a model may misread a learner's thinking, provide incorrect feedback, or disclose a complete solution too early. Existing research reports both potential scaffolding and limitations in mathematical reasoning and tutoring (Liu et al., 2023; Huda et al., 2025; Steindl et al., 2025). The framework therefore treats responsiveness as a property to be specified and evaluated, not as an inherent benefit of AI.

The research gap is framed here as a question for investigation, not as a claim that no related framework exists. The DeFT framework explains functions of multiple representations (Ainsworth, 2006), and research on translation, coordination and transitions between representations offers established theoretical resources (Duval, 2006; Adu-Gyamfi et al., 2012; Bossé et al., 2014). Research on dynamic and computer-based mathematics environments is also relevant to any novelty claim, including work on transitions between related artifacts (Panorkou et al., 2022). Reviews of AI in mathematics education map a broad range of opportunities, risks and pedagogical safeguards (Awang et al., 2025; Fikriani et al., 2026). Against this background, this paper asks a narrower design question: how might learner-responsive conversational moves be mapped to specific representational actions and observable evidence of coordination in calculus? The present narrative review does not establish that this intersection is unprecedented; a systematic search and closer comparison with related work are needed before making a strong novelty claim.

The paper proposes that AI may be useful when it acts as a representational mediator without taking over the learner's mathematical work. The proposed contribution is a design-and-analysis schema connecting a learner's expressed reasoning to a contingent AI move, a learner action, and evidence that can be inspected by a researcher or teacher. This is more specific than a list of prompts: the same prompt should not be prescribed regardless of what a learner says. For example, if a learner claims that the reservoir is fullest when the flow rate is largest, the system would ask the learner to compare the rate with the accumulated quantity and test the claim using a table or graph, rather than simply supply the correct answer. This is a proposed interaction pattern, not a tested result. The paper is conceptual and makes no claims about learning effectiveness.

Scope of the literature. This is a narrative conceptual review, not a systematic review. Sources were selected for their relevance to multiple representations, covariational reasoning in calculus, transfer, digital mathematics environments and generative AI in mathematics education. No exhaustive search protocol was applied, and the reference list should not be taken as comprehensive. In particular, the account of related dynamic and computer algebra environments requires further systematic development. Accordingly, the paper offers a provisional design framework and does not claim priority over all existing approaches.

Section 2 sets out the theoretical foundations. Section 3 specifies the hypothesised functions and risks of AI as a representational mediator. Section 4 presents the framework and Section 5 illustrates it with

an accumulation task. Section 6 derives design principles, and Section 7 discusses contributions, limitations and a research agenda.

2 Theoretical foundations

2.1 Learning with multiple representations

The DeFT framework (Design, Functions, Tasks) distinguishes three functions that multiple representations can serve. They can *complement* one another by offering different information or supporting different processes, *constrain* one another's interpretation, and help learners *construct* deeper understanding through abstraction, extension and relation (Ainsworth, 2006). The value of a set of representations therefore depends on how learners use it, not on its mere presence.

Mathematics education distinguishes transformations within a representation (often termed treatments) from conversions between representations. Duval (2006) argues that conversion between registers is a major source of difficulty in mathematical comprehension and is central to the development of coordination; coordination should therefore not be presented as an alternative to conversion. Related work examines translation errors and the ways learners coordinate representations (Adu-Gyamfi et al., 2012; Bossé et al., 2014). Building on this literature, the present paper treats translation as one activity through which coordination may develop, while distinguishing a mere change of form from a reasoned comparison and validation of meaning across forms. The proposed AI design question is how a conversational system might elicit and respond to these activities, not whether conversion can be bypassed.

The paper uses the following working definition, adapted to AI-mediated learning.

Definition 1

(Representational coordination). Representational coordination is the learner's activity of interpreting, translating and validating meaning across two or more representations of the same mathematical relationship. The paper additionally proposes, as a conjectured component for AI-mediated settings, that the learner justify why a given representation is appropriate for the question at hand.

Coordination differs from switching, in which a learner moves from one representation to another (for example, plots a function after being given a formula) without checking that the two agree or explaining what each reveals. It also differs from display, in which a tool presents several forms at once. Figure 1 summarises the three situations. A meta-analysis of more than two external representations in STEM indicates that their benefits depend on the provision of appropriate support (Rexigel et al., 2024). Any account of AI support must therefore address representational overload: adding more displays, or letting an AI add them on request, risks fragmenting attention rather than integrating understanding.

2.2 Calculus as covariational reasoning

Calculus ideas can be characterised through reasoning about quantities that change together. Covariational reasoning was developed as a lens on how learners model dynamic functions and rates of change (Carlson et al., 2002) and has been connected to the derivative (Martínez-Miraval et al., 2023) and to the definite integral through Riemann sums (Martínez-Miraval & García-Rodríguez, 2022). A recent study of second-order covariation with digital tools reports features of those tools that supported students' covariational reasoning (Bagossi, 2024). Extending this lens to the definite integral as accumulation is the interpretive step taken here. Net change, the reconstruction of a total change from its rate (Greefrath et al., 2016), is the contextual meaning foregrounded in this paper; area under a curve remains a valid and useful geometric interpretation in Cartesian settings (White Brahmia & Thompson, 2025). If r(t) is the rate of change of a quantity A(t), then:

$A(x) = A(0) + \int_0^{\square} r(t)\,dt$, $A'(x) = r(x)$

A learner can read this relationship graphically (the sign and size of r determine the slope of A), numerically (accumulating products of rate and time interval), verbally (“the amount has increased by…”) and contextually (what “net” means in the situation). The paper proposes this multi-way reading as an instructional goal and a candidate indicator of conceptual understanding, not as an established necessary condition.

2.3 Transfer beyond familiar tasks

Transfer is defined here as adapting conceptual relationships and strategies to structurally unfamiliar problems. Near transfer occurs between tasks of similar format, for example, from one rate-of-change graph to another. Far transfer occurs across contexts and representations (Barnett & Ceci, 2002), for example, from a water-volume scenario to a battery-charge scenario in which the same covariational structure appears as a table (an illustrative example of the author's own, not drawn from the cited taxonomy). Barnett and Ceci offer a taxonomy of dimensions of transfer; they do not establish that learners must justify representational choices for far transfer to occur.

The paper therefore proposes justification of representational choices as an *observable indicator*, not as a defining requirement of far transfer. The proposal is motivated by work on mindful abstraction and schema formation, in which comparing contrasting cases supports the abstraction of transferable structure (Alfieri et al., 2013; Salomon & Perkins, 1989). In this paper, transfer is operationalised as solving unfamiliar calculus problems through coordinated representational reasoning, with justification of representational choices as a conjectured observable component.

3 AI as a representational mediator

3.1 Hypothesised functions

The framework positions AI as a possible mediator whose role is to elicit and respond to learners’ representational reasoning, rather than to perform the reasoning on their behalf. The four functions

below are design conjectures, not proven affordances. To distinguish a contingent conversational move from a fixed worksheet prompt, the system must use evidence from the learner's preceding response to select or adapt its next move. Whether current systems can do this accurately, consistently and educationally productively is an empirical question. Evidence on prompt-moderated mathematics tutoring is mixed, so the design should be compared with non-AI alternatives rather than assumed superior (Steindl et al., 2025).

- **Prompting comparison and translation.** Prompts ask students to compare two representations of the same situation, or to translate from one form to another and explain what changed and what stayed the same.
- **Contingent questioning. A follow-up question should be selected in relation to the learner's stated reasoning, not merely delivered as the next item in a fixed script. For example, if a learner says, "The reservoir is fullest at t = 0 because the rate is largest then," a contingent response could ask, "What does the rate tell you about whether the volume is increasing or decreasing between t = 0 and t = 3? How could you check the volume itself?" If the learner instead correctly identifies that the volume rises until the rate reaches zero but cannot justify why, the next move could ask the learner to compare the sign of the rate with the slope of the accumulated-volume graph. These examples illustrate the intended interaction logic; they do not demonstrate that an AI system can diagnose the response reliably. A study would need to record the learner response, the system's next move and whether that move appropriately addresses the mathematical issue.**
- **Counterexamples and contrasting cases.** AI can in principle generate cases where a tempting generalisation fails, such as a rate that is positive but decreasing, prompting learners to refine their reasoning.
- **Error diagnosis for reflection. A system may tentatively identify a possible mismatch between representations, but it should present that diagnosis as a hypothesis for the learner to test, not as an authoritative judgement. For example, it could ask whether a claimed maximum refers to the rate or to the accumulated quantity and invite a graph or numerical check. Evaluation must examine both the mathematical correctness of the diagnosis and whether it supports learner reasoning; those outcomes cannot be inferred from a fluent response alone.**

3.2 Risks and design boundaries

AI-generated explanations may be fluent but incorrect. Three design boundaries follow.

- Every AI-generated claim must be open to verification through at least one independent route: a graph, a numerical check, a symbolic derivation or a contextual interpretation.
- Students should be prompted to critique and correct AI explanations. This is a design recommendation motivated by broader evidence of learner dependence on generative AI and the value of metacognitive evaluation (Fan et al., 2025). That evidence comes from university students

completing a writing task; it does not test whether critiquing AI explanations improves calculus learning, which remains to be examined.

- Student agency is essential to the proposed framework. Learners should make and explain representational choices and perform the core translation and validation. AI may offer a question, counterexample or tentative diagnosis, but should not automatically select every representation, silently repair all mismatches or complete the coordination task. The degree of assistance should be logged and treated as a variable in evaluation, because a system that supplies the reasoning may improve immediate answer accuracy without supporting independent transfer.

4 A conceptual framework: notice, translate, coordinate, transfer

Figure 2 presents four iterative stages rather than a mandatory linear sequence. The stages describe learner activity, while AI moves are selected in response to the evidence available at each point. A failed validation may prompt a return to noticing a feature, revising a translation or reconsidering an interpretation. In an implemented system, the response-to-prompt mapping would need to be specified in advance or transparently logged; otherwise, researchers could not determine whether the interaction was genuinely contingent or simply followed a fixed script.

- **Notice.** The learner attends to key features within a single representation, such as intercepts, sign changes, steepness, units and direction of change.
- **Translate.** The learner expresses the relationship in a second representation, for example moving from a verbal description to a graph, or from a graph to a table.
- **Coordinate.** The learner compares representations, resolves mismatches and validates meaning across them, using at least two mutually validating representations to justify a claim.
- **Transfer.** The learner applies and adapts the coordinated understanding to a task whose context changes while its covariational structure is preserved.

Table 1 operationalises the proposed framework for future analysis. Its design proposition is the mapping from the learner's current evidence to an AI move, then to a learner action and a conjectured observable outcome. The prompt examples are intended to be selected or adapted in response to the learner's preceding explanation, rather than delivered as a fixed sequence regardless of what the learner says. These outcomes are proposed indicators for study, not validated measures or established criteria.

Table 1
Stages, AI prompt types, learner actions and conjectured observable outcomes

Stage	AI prompt type	Learner action	Conjectured observable outcome
Notice	Attention-directing question selected in response to the learner's initial account (e.g., "What changes as t increases?")	Identifies relevant features within one representation and explains their meaning	Learner names relevant features and units; initial reasoning is recorded
Translate	If the learner's account is unclear, ask for a learner-produced second representation (e.g., "Can you show that change in a table?")	Produces a second representation and explains what was preserved; the AI does not generate it for the learner	Second representation is consistent with the first, or a mismatch is identified
Coordinate	Select a comparison question that targets the mismatch or claim in the learner's response (e.g., "Do your graph and numbers agree? How do you know?")	Compares representations, reconciles mismatches and validates meaning	Learner justifies a claim using at least two mutually validating representations
Transfer	Offer a changed-context task that preserves the underlying covariational structure; follow-up questions depend on the learner's explanation	Adapts the strategy to an unfamiliar context and explains the relationship used	Correct reasoning and a rationale for representational choices; proposed indicator, not a defining requirement of far transfer

5 Illustrative calculus task design

The following task illustrates how the framework could be used to design and analyse an AI-supported learning interaction. It has not been implemented, and the sample dialogue is illustrative rather than evidence of effectiveness. To test the framework, a study would need to retain learners' initial predictions, subsequent responses, AI messages and revisions so that the contingency and mathematical appropriateness of the interaction could be evaluated.

A reservoir has water entering and leaving. The net flow rate r(t), in thousands of litres per hour, is given graphically and in Table 2 for $0 \le t \le 8$ hours; the underlying function is $r(t) = 3 - t$, which learners do not need to be told. The rate is positive for $0 \le t < 3$ and negative for $3 < t \le 8$. The volume at $t = 0$ is not given, so learners work with changes in volume. Learners are asked: *At what time is the reservoir fullest, and is it fuller at t = 8 than at t = 0?*

Table 2
Net flow rate supplied to learners (thousand litres per hour)

t (hours)	0	1	2	3	4	5	6	7	8
r(t)	3	2	1	0	−1	−2	−3	−4	−5

- **Predict.** Before consulting the AI, learners record a prediction and a reason in their own words. This preserves agency and gives a baseline against which later reasoning can be compared.
- **Notice and translate. The system first asks the learner to interpret the sign of r(t), then asks the learner—not the AI—to sketch the change in volume and construct a table of approximate net change over successive hours. If the learner says that negative flow means the volume is negative, the system could ask the learner to distinguish the sign of a rate from the sign of a quantity and check the distinction against the starting volume. If the learner correctly says that volume is decreasing after t = 3 but cannot explain why, the system could ask the learner to connect the negative rate to the slope of the accumulated-volume graph. The system's next question is thus contingent on the response, in principle; whether an actual model follows this pattern reliably would have to be tested. With the values in** Table 2, **the hourly net changes are 2.5, 1.5, 0.5, − 0.5, − 1.5, − 2.5, − 3.5 and − 4.5 thousand litres, so cumulative change is 2.5, 4.0, 4.5, 4.0, 2.5, 0, − 3.5 and − 8.0. The trapezoidal rule gives these values exactly because r is linear.**
- **Coordinate.** The AI asks whether the sketch and the table agree on when the volume is greatest and on whether V(8) > V(0). A common misconception is to confuse features of a function with those of its rate graph, for example assuming that a maximum of one occurs at the same x-value as a maximum of the other. A contingent prompt ("What would have to be true of r at the moment the volume stops increasing?") invites learners to relate the sign change of r to the extremum of the volume through the relationship in Section 2.2. Learners must defend their final answer using at least two representations that validate each other, such as the graph of r and the table of accumulated change. The expected result (Fig. 3) is that the volume is greatest at t = 3, returns to its starting level at t = 6, and is lower at t = 8 than at t = 0.

- **Verify and critique. To avoid implying that a model spontaneously produced a known error, the illustrative activity uses a deliberately prepared incorrect explanation as a comparison stimulus. For example: "The reservoir is fullest when the rate is largest, at t = 0." Learners must locate and correct the error using a graph or numerical check. In a separate AI condition, a model-generated explanation could be used only if its actual output is preserved and its correctness independently checked. This distinction matters: a pre-authored error can test learners' critique of an explanation, but it does not establish the AI's ability to generate useful misconceptions or diagnose learner thinking.**
- **Transfer.** A final task changes the context while preserving the covariational structure (Fig. 4). A battery's net charging current i(t), in amperes, is given only as a table: 4, 3, 2, 1, 0, − 1, −2 at t = 0, 1, …, 6 hours. The changing quantity is charge, in ampere-hours, which is the time integral of current; a power profile would instead determine stored energy and would require assumptions about voltage,

so the task is posed in terms of current. The simplified model ignores capacity limits and charging efficiency. Learners decide which representation best supports the question of when the charge is greatest and whether it is higher at t = 6 than at t = 0, and justify the choice. The expected result is a maximum net gain of 8 Ah at t = 4 and a net gain of 6 Ah at t = 6.

6 Design principles for AI-supported calculus learning

The framework yields five design principles, stated as conjectures to be tested.

- **Reasoning before procedure.** Prompts should request reasoning, comparison and verification before offering procedural suggestions.
- **Purposeful sequencing.** Representations may be sequenced around conceptual purpose, with simultaneous presentation requiring an instructional rationale to limit representational overload (Ainsworth, 2006; Seufert, 2003). Which sequences work, and for whom, remains an open question: existing evidence on structured or prompt-moderated AI tutoring is mixed (Bastani et al., 2025; Steindl et al., 2025), so sequencing should not be treated as an established lever.
- **Visible uncertainty.** Students should be prompted to critique and correct AI explanations, so that fallibility is part of the learning activity. This recommendation rests on a broader concern about uncritical reliance (Bastani et al., 2025), as noted in Section 3.2.
- **Teacher orchestration.** Teachers should lead shared discussion of competing representations and solution pathways, so that AI support complements rather than replaces classroom mathematical discourse.
- **Evidence beyond accuracy.** Learning should be evaluated through transfer tasks, explanations and representational justifications alongside answer accuracy.

7 Discussion

7.1 Contributions

The paper adapts established distinctions among treatment, conversion and coordination to a design question for conversational AI. Its proposed contribution is not a new theory of representational learning, nor evidence that AI improves transfer. Instead, it offers a provisional analytic schema that links learner evidence, a contingent AI move, a learner action and a conjectured observable outcome. The illustrative reservoir-to-battery task shows how this schema might guide task design and data collection. Its novelty remains provisional: related work on linked representations, dynamic environments and transitions between artifacts must be examined more comprehensively before the framework can be claimed to extend that literature. The schema’s value will depend on whether it enables researchers to describe interactions more precisely and test mechanisms that would otherwise remain implicit.

7.2 Limitations

The framework is conceptual and has not been empirically validated. The stages may not fit all learners or calculus topics, and the illustrative tasks focus on accumulation with linear rate functions. The proposed contingency depends on a model correctly interpreting learner responses, which cannot be assumed. The literature review is narrative rather than systematic and does not yet provide a comprehensive comparison with work on linked representations in dynamic mathematics and computer algebra environments. The paper therefore cannot establish originality across the full literature or claim effectiveness. Future work should first map the related literature and refine the constructs, then pilot the interaction protocol and evaluate mathematical accuracy, contingency, learner agency and transfer. Existing evidence on AI chatbots is qualified; for example, a meta-analysis reported stronger effects in shorter than longer interventions, raising the possibility of novelty effects (Wu & Yu, 2024).

7.3 Research agenda

- **Effectiveness.** Test whether AI-mediated coordination improves transfer relative to non-AI multi-representational instruction, using non-routine transfer tasks and process data on representation use.
- **Prompt design and sequencing.** Test the prompt types in Table 1 and their ordering against unstructured AI use, given the mixed evidence on prompt moderation.
- **Prior knowledge.** Examine which prompt sequences support learners with different levels of prior calculus knowledge, including whether novices need more structure to avoid overload.
- **Teachers.** Investigate how teachers interpret, adapt and regulate AI-supported representational activities.
- **Equity.** Study how differences in AI literacy, access, language and prior digital experience shape who benefits.

8 Conclusion

This paper has proposed a conceptual framework for investigating how conversational AI might support representational coordination in calculus. Its four stages—notice, translate, coordinate and transfer—are intended to organise learner activity and connect it to contingent AI moves and observable evidence. The framework does not establish that AI improves learning, nor that its contribution is distinct from existing digital tools. Those claims require a fuller review of related work and empirical comparison with non-AI instruction. The next steps are to strengthen the literature mapping, specify and pilot the interaction protocol, and test whether the proposed indicators are reliable and whether learners demonstrate transfer on unfamiliar tasks.

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- **Use of AI tools:** Generative artificial intelligence tools were used to assist with language revision and checking arithmetic in the illustrative calculus task. The author reviewed and revised the manuscript and takes responsibility for the final content. The figures were created by the author without the use of generative AI.

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Figures

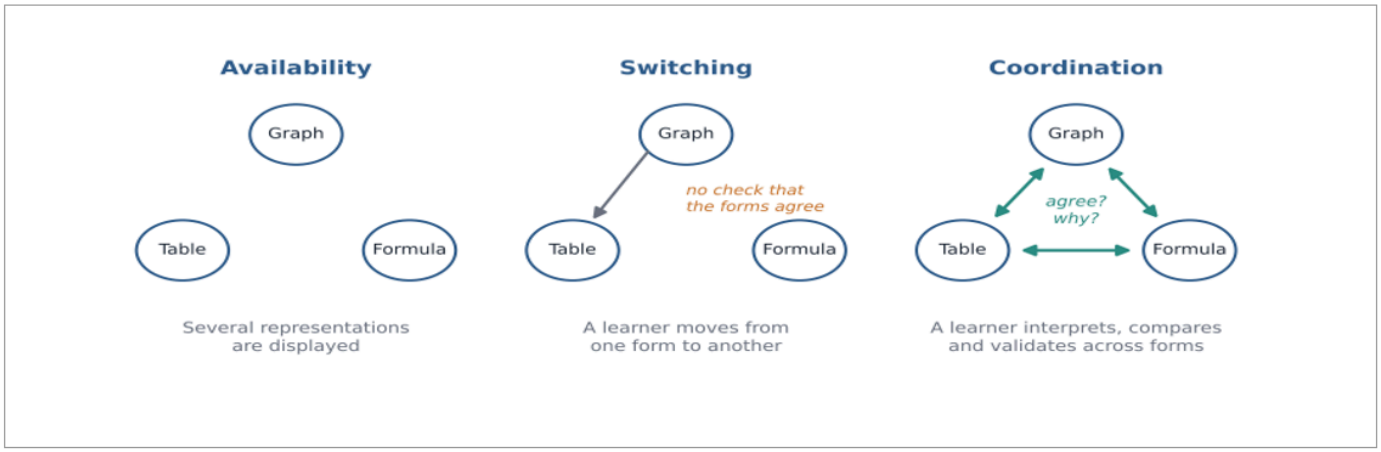


Figure 1

Availability, switching and coordination of representations. Only coordination involves checking whether the forms agree and explaining why.

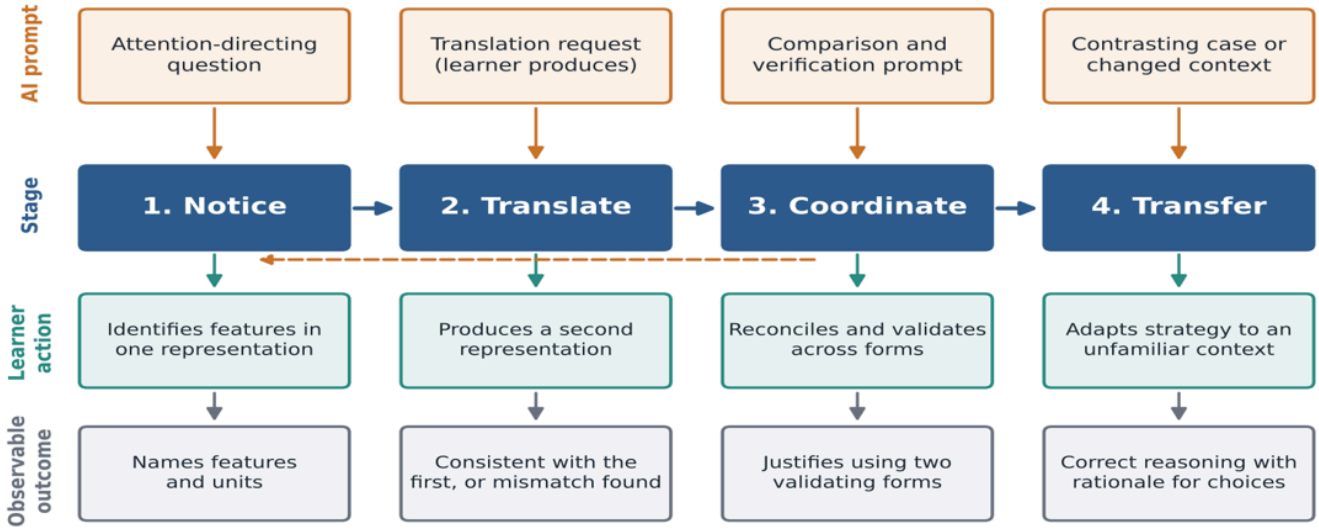


Figure 2

The notice–translate–coordinate–transfer framework. Each stage is linked to an AI prompt type, a learner action and an observable outcome. The dashed arrow indicates that a failed validation at the coordinate stage returns the learner to notice.

Supplementary Files

This is a list of supplementary files associated with this preprint. Click to download.

- supplementaryframeworktable.docx