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Chaos is not an invariant of fractional-order systems: the clock shapes orbits, attractors and bifurcations

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Research Article

Keywords: Fractional Calculus, Fractional Differential Equations, Nonlinear Dynamics, Chaos Theory, Dynamical Systems, Bifurcation Theory

Posted Date: October 29th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11533019/v1

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Chaos is not an invariant of fractional-order systems: the clock shapes orbits, attractors and bifurcations

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Abstract

The qualitative theory of differential equations rests on a fact so elementary that it is rarely stated: multiplying a vector field by a positive function of the state, a *clock*, changes the speed at which the orbits are traversed but not the orbits. We show that for fractional-order systems in the Caputo sense this fact is false. The discrepancy between the system with a clock and the reference system read in the transformed time is governed, at short times, by two universal functions of the order α, which we obtain in closed form and which vanish only for $\alpha = 1$: $c(\alpha)$ measures the change in the time law, $d(\alpha)$ the change in the geometry of the orbits. Local stability and Matignon's critical order, on the contrary, are exactly invariant, so the effect is nonlinear and global. Numerically, a clock deforms the limit cycle of the fractional van der Pol oscillator by up to 16% and, in the fractional Chen system, turns a periodic regime into a chaotic one at the same order and from the same initial datum, something impossible in integer order. A general rule emerges: the local bifurcations of equilibria do not move, while the global ones shift linearly with the strength of the clock. The consequence reaches every field in which fractional models carry a state-dependent kinetic factor, from viscoelastic and glassy materials to electrochemical, biological and reacting systems: there the time in which memory accumulates is a physical hypothesis with observable consequences, and a fractional attractor is a property of the field together with its clock.

Every student of dynamical systems learns that speeding a system up or slowing it down at different points of its state space, by multiplying its equations by a positive function, changes how fast the trajectories are travelled but not their shape: the picture of the dynamics belongs to the equations, not to the clock that runs them. Fractional-order models, which describe systems that remember their past with a power-law weight, are widely used in physics, chemistry, engineering and biology, and are routinely analysed with the same pictures. We show that for them this basic fact fails. A clock that depends on the state changes the shape of the trajectories, by an amount fixed by universal functions of the order, and it can even turn periodic motion into chaos. Whenever a fractional model contains a factor such as a temperature-dependent rate, the time in which memory accumulates becomes a physical choice with visible consequences.

1 Introduction

Since Poincaré, the qualitative study of ordinary differential equations has been founded on the separation between the geometry of the orbits and the time law with which they are traversed. If $\dot{x} = f(x)$ and g is a positive function of the state, the system $\dot{x} = g(x)f(x)$ has exactly the same orbits as the first: every solution of one is a solution of the other run with a different clock. The phase portrait, the equilibria, the limit cycles, the attractors, the bifurcations and the signs

of the Lyapunov exponents are all invariant under this operation, and the theory treats them as geometric objects. This freedom is used all the time, often without being named: when time is rescaled to make a model dimensionless, when a kinetic factor is absorbed into a new unit of time, when a field is regularized near a singularity by multiplying it by a function that vanishes there, when models that differ by a state-dependent speed factor are compared.

Fractional-order systems, in which the first derivative is replaced by a Caputo derivative of order $\alpha \in (0,1)$, have been studied for thirty years with the tools of that theory: their phase portraits are drawn, their strange attractors are plotted, their chaos thresholds are computed as the order varies, their Lyapunov exponents are defined [1, 3, 4, 5, 6, 7]. It is known that temporal nonlocality makes these systems different from a classical dynamical system: the solutions do not form a flow, because the semigroup property fails, and trajectories issued from different points at different times can intersect in state space [3, 8]. But this difference concerns the initial time, and it can be sidestepped by fixing the origin of time at the preparation of the system. The question we ask is a different one, and as far as we know it has not been asked: with the origin of time fixed, are the orbits of a fractional system invariant under a change of clock?

The answer is negative, and it is quantitative. We prove that the difference between the solution of the system with clock g and the solution of the original system reparametrized with the same clock is, at short times, $c(\alpha)\,(\nabla g \cdot f)\, f\, t^{2\alpha}$, with $\nabla g \cdot f$ and f evaluated at the initial point, where $c(\alpha)$ depends only on the order: it vanishes for $\alpha = 1$, where the classical theorem is recovered, and tends to one as the order tends to zero. The function $c(\alpha)$ measures how much a system of order α "feels" the clock it is run with. This first discrepancy is tangent to the orbit, and by itself it does not show that the orbits change; we prove that at the next order, $t^{3\alpha}$, the discrepancy acquires a transverse component governed by a second universal function, $d(\alpha)$, also zero for $\alpha = 1$ and nonzero otherwise, so that the orbits of a fractional system are different curves for different clocks. We then show what remains invariant: the local stability of equilibria and the critical order at which they lose it, because Matignon's criterion depends on the argument of the eigenvalues and not on their modulus. The non-invariance is therefore a nonlinear and global fact, invisible in the linearization but visible in limit cycles, attractors and bifurcation cascades. We verify it numerically on four systems of increasing difficulty, the last of which shows that the clock can change not only the shape but the nature of an attractor, and from that example we extract a general rule on when this happens; we then draw the consequences for the way fractional attractors are drawn, compared and identified.

The question is not only of mathematical interest. Fractional models are used wherever a system weights its past with a power law: in the viscoelasticity of polymers and biological tissues, at electrochemical interfaces, whose double layer behaves as a constant-phase element, in anomalous diffusion and transport, in the design of fractional controllers [1]. In almost all of them the field contains a factor that depends on the state, an Arrhenius rate, an enzymatic activity, a mobility, and in integer order it makes no difference whether that factor multiplies the field or is absorbed into an intrinsic time. Our result shows that in fractional order the two choices are different physical models, that their difference can reach the very nature of the dynamics, and that its size is predicted by a universal function of the order. In the physics of polymers and glasses the second choice, a memory that accumulates in an intrinsic material time, has been the standard hypothesis for decades [23, 24], whereas many fractional models make the first without saying so. For those who study fractional chaos the consequence is equally direct: minimal orders for chaos, bifurcation diagrams and drawn attractors are properties of the particular form given to the field, not invariants of the system, while the necessary condition for chaos based on Matignon's criterion survives every change of clock. The theorems of Section 3, the examples of Section 4 and the consequences of Section 5 make these statements precise.

2 Preliminaries and notation

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth vector field and $x_0 \in \mathbb{R}^d$. The Caputo derivative of order $\alpha \in (0, 1]$ of a function x is

$$(D^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}\, \dot{x}(\tau)\, \mathrm{d}\tau, \tag{1}$$

which for $\alpha = 1$ reduces to the ordinary derivative. The Cauchy problem $D^\alpha x = F(x)$, $x(0) = x_0$, is equivalent to the Volterra integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(x(\tau))\, \mathrm{d}\tau, \tag{2}$$

which we take as the definition of a solution [3]. For smooth fields the solution admits, at short times, an expansion in powers of t^α, obtained by inserting into (2) the Taylor expansion of F about x_0 and integrating term by term. The operator that appears in (2), $I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, \mathrm{d}\tau$, is the Riemann–Liouville fractional integral, and on powers it acts as $I^\alpha t^{k\alpha} = \Gamma(1+k\alpha)\, t^{(k+1)\alpha} / \Gamma(1+(k+1)\alpha)$. The first two terms are

$$x(t) = x_0 + \frac{F_0}{\Gamma(1+\alpha)}\, t^\alpha + \frac{DF_0\, F_0}{\Gamma(1+2\alpha)}\, t^{2\alpha} + O(t^{3\alpha}), \tag{3}$$

with $F_0 = F(x_0)$ and DF_0 the Jacobian matrix of F at x_0. For $\alpha = 1$ one recovers the Taylor expansion $x_0 + F_0 t + \frac{1}{2} DF_0 F_0 t^2$.

A *clock* is a smooth positive function $g : \mathbb{R}^d \to (0, \infty)$. Given the reference system $D^\alpha X = f(X)$, $X(0) = x_0$, we call system with clock g the system $D^\alpha x = g(x) f(x)$ with the same initial datum, and transformed time the function $s(t) = \int_0^t g(x(\tau))\, \mathrm{d}\tau$, which is increasing. The classical theorem, which we state for comparison, says that in integer order the system with clock is the reference system read in the transformed time.

Theorem 1 (Clock covariance, $\alpha = 1$). *If $\dot{x} = g(x) f(x)$ and $\dot{X} = f(X)$ with $x(0) = X(0) = x_0$, then $x(t) = X(s(t))$ for every t, with $\dot{s} = g(x(t))$, $s(0) = 0$. In particular the two solutions describe the same orbit.*

Proof. Let σ be the solution of the scalar problem $\dot{\sigma} = g(X(\sigma))$, $\sigma(0) = 0$, which exists and is increasing because $g > 0$. With $y(t) = X(\sigma(t))$, the chain rule gives $\dot{y} = \dot{X}(\sigma)\, \dot{\sigma} = f(y)\, g(y)$ with $y(0) = x_0$, so y and x solve the same Cauchy problem and coincide by uniqueness; hence $\dot{\sigma} = g(x(t))$, that is $\sigma = s$. □

The proof uses the chain rule, $\frac{\mathrm{d}}{\mathrm{d}t} X(s(t)) = X'(s)\dot{s}$. The Caputo derivative obeys no chain rule [10], and this is the point from which everything follows.

3 What remains invariant and what does not

We begin with what fractional order preserves: constant clocks.

Proposition 2 (Constant clocks). *If $D^\alpha X = f(X)$, $X(0) = x_0$, and $\rho > 0$ is a constant, then $x(t) = X(\rho^{1/\alpha} t)$ solves $D^\alpha x = \rho f(x)$, $x(0) = x_0$. The orbits of the two systems coincide.*

Proof. From (1), with the substitution $\tau = \tau'/r$, one gets $D^\alpha[X(r\,\cdot)](t) = r^\alpha (D^\alpha X)(rt)$ for every constant $r > 0$; with $r = \rho^{1/\alpha}$ one has $D^\alpha x(t) = \rho\, f(X(rt)) = \rho f(x(t))$. □

The unit of time, then, does not matter: changing it amounts to multiplying the field by a constant, and the orbits do not notice. What matters is the clock that depends on the state.

Theorem 3 (Clock non-covariance, $0 < \alpha < 1$). *Let x and X be the solutions of $D^\alpha x = g(x)f(x)$ and $D^\alpha X = f(X)$ with the same initial datum x_0, and let $g(x_0) = 1$, an assumption that is not restrictive by Proposition 2. Let $s(t) = \int_0^t g(x(\tau))\,\mathrm{d}\tau$. Then, as $t \to 0$,*

$$x(t) - X(s(t)) = c(\alpha)\,\big(\nabla g(x_0)\cdot f(x_0)\big)\,f(x_0)\,t^{2\alpha} + o(t^{2\alpha}),$$
$$c(\alpha) = \frac{1}{\Gamma(1+2\alpha)} - \frac{\alpha}{\Gamma(1+\alpha)\,\Gamma(2+\alpha)}. \tag{4}$$

Proof. We apply (3) to the field $F = gf$. Since $g(x_0) = 1$ we have $F_0 = f_0$ and $DF_0 = Df_0 + f_0\,\nabla g_0^{\top}$, where $f_0 = f(x_0)$, Df_0 is the Jacobian of f and $\nabla g_0 = \nabla g(x_0)$; with $\gamma = \nabla g_0 \cdot f_0$,

$$x(t) = x_0 + \frac{f_0}{\Gamma(1+\alpha)}t^\alpha + \frac{Df_0 f_0 + \gamma f_0}{\Gamma(1+2\alpha)}t^{2\alpha} + O(t^{3\alpha}).$$

For the transformed time, $g(x(\tau)) = 1 + \nabla g_0 \cdot (x(\tau) - x_0) + O(\tau^{2\alpha}) = 1 + \gamma\,\tau^\alpha/\Gamma(1+\alpha) + O(\tau^{2\alpha})$, and integrating

$$s(t) = t + \frac{\gamma}{\Gamma(2+\alpha)}t^{1+\alpha} + O(t^{1+2\alpha}), \qquad s(t)^\alpha = t^\alpha + \frac{\alpha\gamma}{\Gamma(2+\alpha)}t^{2\alpha} + O(t^{3\alpha}),$$

where we used $(1+\alpha)\Gamma(1+\alpha) = \Gamma(2+\alpha)$. Substituting into the expansion of X, which is (3) with $F = f$,

$$X(s(t)) = x_0 + \frac{f_0}{\Gamma(1+\alpha)}t^\alpha + \Big[\frac{Df_0 f_0}{\Gamma(1+2\alpha)} + \frac{\alpha\gamma f_0}{\Gamma(1+\alpha)\Gamma(2+\alpha)}\Big]t^{2\alpha} + O(t^{3\alpha}),$$

and the difference from $x(t)$ is $\gamma f_0 t^{2\alpha}$ times the bracket in (4). □

The theorem says that the discrepancy at the first non-vanishing order points along f_0, that is along the tangent to the orbit, and is proportional to the variation of the clock along the motion, $\nabla g \cdot f$, and to a universal function of the order. The properties of $c(\alpha)$ can be read off its expression. For $\alpha = 1$, $c = 1/2 - 1/2 = 0$, and Theorem 1 is recovered. As $\alpha \to 0$, $c \to 1$: a system of order close to zero feels the clock maximally. For $\alpha = 1/2$ one has the closed form $c(1/2) = 1 - 4/(3\pi) \simeq 0.576$. Near $\alpha = 1$, $c(\alpha) = \frac{3}{4}(1-\alpha) + O((1-\alpha)^2)$, as one checks by differentiating. The function is plotted in Figure 1; it decreases from one to zero, with a slight maximum of 1.002 around $\alpha \simeq 0.03$ due to the term $2\gamma_E\alpha$, where $\gamma_E \simeq 0.577$ is Euler's constant, in the expansion of $1/\Gamma(1+2\alpha)$.

Corollary 4 (Reparametrization group). *For $\alpha = 1$ the set of orbits of $\dot{x} = f(x)$ is invariant under multiplication of the field by any positive function of the state. For $0 < \alpha < 1$ it is invariant under multiplication by positive constants (Proposition 2), while for every clock g with $\nabla g \cdot f \neq 0$ at x_0 the time law along the orbit is altered already at order $t^{2\alpha}$ (Theorem 3) and the geometry of the orbit at order $t^{3\alpha}$ (Theorem 5 below).*

Theorem 3 concerns the time law, because the first-order discrepancy is tangent to the orbit and could in principle be reabsorbed by a reparametrization. To decide whether the orbits change one has to push the expansion one order further and ask whether there exists a reparametrization $\sigma(t)$ such that $x(t) = X(\sigma(t))$. The answer is negative, and the size of the discrepancy is again a universal function of the order.

Theorem 5 (Non-invariance of the orbits). *Under the hypotheses of Theorem 3, let $u = f_0/|f_0|$ and, for a vector v, let $v_\perp = v - (v\cdot u)u$ be its component orthogonal to f_0. The distance between the point $x(t)$ and the orbit of X is, as $t \to 0$,*

$$\operatorname{dist}\big(x(t), \{X(s)\}_{s\geq 0}\big) = |d(\alpha)|\,|\nabla g_0 \cdot f_0|\,\big|(Df_0 f_0)_\perp\big|\,t^{3\alpha} + o(t^{3\alpha}),$$
$$d(\alpha) = \frac{1 + \Gamma(1+2\alpha)/\Gamma(1+\alpha)^2}{\Gamma(1+3\alpha)} - \frac{2\,\Gamma(1+\alpha)}{\Gamma(1+2\alpha)^2}, \tag{5}$$

and the transverse discrepancy points along $d(\alpha)\,(\nabla g_0 \cdot f_0)\,(Df_0 f_0)_\perp$.

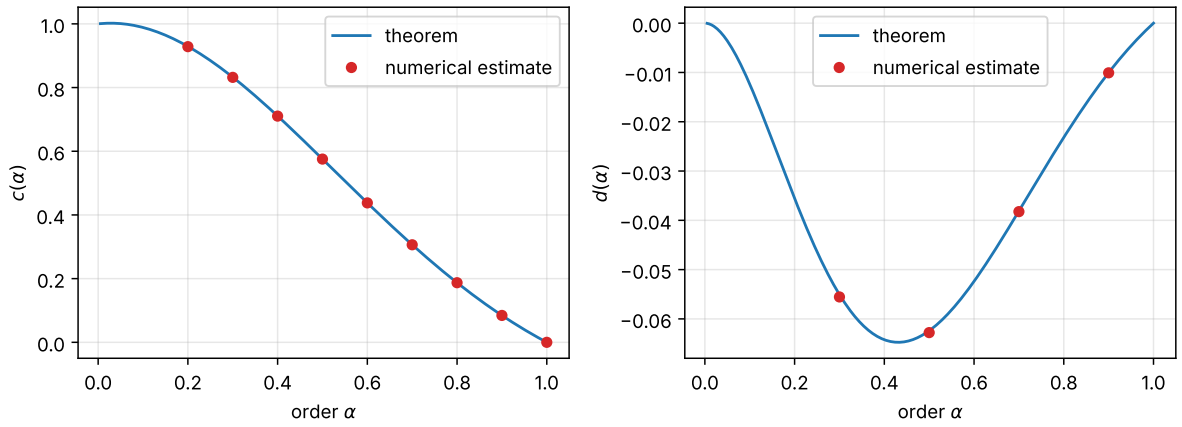

Figure 1: The two universal functions of the order: left, $c(\alpha)$ of Theorem 3; right, $d(\alpha)$ of Theorem 5 (lines), with the numerical estimates (dots) obtained from system (6) as described in Section 4; the relative error is below 0.1% for c and below 1% for d.

Proof. Set $\tau = t^\alpha$, a symbol that in this proof denotes this variable and not an integration variable, and write the two solutions as series, $X(s) = x_0 + a_1\varsigma + a_2\varsigma^2 + a_3\varsigma^3 + O(\varsigma^4)$ with $\varsigma = s^\alpha$, and $x(t) = x_0 + \tilde{a}_1\tau + \tilde{a}_2\tau^2 + \tilde{a}_3\tau^3 + O(\tau^4)$. Iterating (2) with the second-order Taylor expansion of F, the term following (3) is

$$a_3 = \frac{DF_0^2F_0 + \theta\, D^2F_0[F_0, F_0]}{\Gamma(1+3\alpha)}, \qquad \theta = \frac{\Gamma(1+2\alpha)}{2\,\Gamma(1+\alpha)^2},$$

where $D^2F_0[v,v]$ is the quadratic form of the second derivatives of F at x_0; for $\alpha = 1$, $\theta = 1$ and the Taylor term $\frac{1}{6}(DF^2F + D^2F[F,F])$ is recovered. For $F = f$ the coefficients are $a_1 = f_0/\Gamma(1+\alpha)$, $a_2 = Df_0f_0/\Gamma(1+2\alpha)$ and a_3 as above. For $F = gf$ with $g(x_0) = 1$ one has $DF_0 = Df_0 + f_0\nabla g_0^\top$ and $D^2F_0[v,v] = D^2f_0[v,v] + 2(\nabla g_0 \cdot v)\, Df_0v + D^2g_0[v,v]\, f_0$, whence, with $\gamma = \nabla g_0 \cdot f_0$,

$$\tilde{a}_1 = a_1, \qquad \tilde{a}_2 = a_2 + \frac{\gamma f_0}{\Gamma(1+2\alpha)}, \qquad \tilde{a}_3 = a_3 + \frac{(1+2\theta)\,\gamma\, Df_0f_0}{\Gamma(1+3\alpha)} + (\text{multiple of } f_0).$$

We look for a reparametrization $\varsigma = \tau(1 + b_1\tau + b_2\tau^2 + \dots)$ that makes $X(s) = x(t)$. Substituting, $X = x_0 + a_1\tau + (a_2 + a_1b_1)\tau^2 + (a_3 + 2a_2b_1 + a_1b_2)\tau^3 + O(\tau^4)$. At order τ^2 equality requires $a_1b_1 = \gamma f_0/\Gamma(1+2\alpha)$, which has the solution $b_1 = \gamma\,\Gamma(1+\alpha)/\Gamma(1+2\alpha)$ because a_1 is parallel to f_0: the tangential discrepancy of Theorem 3 can thus be reabsorbed, at the price of a reparametrization that differs from the transformed time $s(t)$. At order τ^3 the unknown b_2 acts only along f_0, and the component orthogonal to f_0 of the residual $\tilde{a}_3 - a_3 - 2a_2b_1$ cannot be cancelled; it equals

$$\Big[\frac{1+2\theta}{\Gamma(1+3\alpha)} - \frac{2\,\Gamma(1+\alpha)}{\Gamma(1+2\alpha)^2}\Big]\gamma\,(Df_0f_0)_\perp = d(\alpha)\,\gamma\,(Df_0f_0)_\perp.$$

The distance from the orbit is the modulus of the component of the residual orthogonal to the local tangent of the curve, which differs from u by $O(\tau)$; since the residual is $O(\tau^3)$, the correction is $O(\tau^4)$. □

The function $d(\alpha)$ is plotted in Figure 1, right panel. For $\alpha = 1$ one has $d = (1+2)/6 - 2/4 = 0$, and the orbits coincide at this order as at every other. For $0 < \alpha < 1$ the function is negative, with minimum -0.065 at $\alpha \simeq 0.43$; near one, $d(\alpha) = -(1-\alpha)/12 + O((1-\alpha)^2)$, and near zero, $d(\alpha) = -(\pi^2/6)\,\alpha^2 + O(\alpha^3)$. The negative sign on the whole interval is equivalent to the inequality $\binom{2\alpha}{\alpha}\big(1 + \binom{2\alpha}{\alpha}\big) < 2\binom{3\alpha}{\alpha}$ between generalized binomial coefficients, with equality at the

endpoints; we know no elementary proof of it and have verified it numerically on a grid of two thousand points with thirty-digit arithmetic. The theorem says that the orbits differ as soon as the clock varies along the motion, $\nabla g \cdot f \neq 0$, and the orbit is curved at x_0, $(Df_0 f_0)_\perp \neq 0$; since x_0 is an arbitrary point, the orbits of the two systems are different curves unless g is constant along every orbit of f, in which case they coincide exactly by Proposition 2. The reparametrization group of a fractional system is therefore the group of clocks that are first integrals of the field, and no larger.

There is, however, something that remains exactly invariant, and it is important to say so in order not to overstate the result.

Proposition 6 (Invariance of local stability)**.** *The systems $D^\alpha x = f(x)$ and $D^\alpha x = g(x)f(x)$ have the same equilibria, and at every equilibrium x^* the asymptotic stability, the instability and the critical order $\alpha^* = \frac{2}{\pi}\min_k |\arg \lambda_k|$ coincide, where λ_k are the eigenvalues of $Df(x^*)$.*

Proof. Since $g > 0$, $gf = 0$ if and only if $f = 0$. At an equilibrium the Jacobian of gf is $g(x^*)Df(x^*) + f(x^*)\nabla g^\top = g(x^*)Df(x^*)$, whose eigenvalues are $g(x^*)\lambda_k$. Matignon's criterion [11] states that the equilibrium is asymptotically stable if $|\arg \lambda_k| > \alpha\pi/2$ for every k and unstable if $|\arg \lambda_k| < \alpha\pi/2$ for some k; multiplication by a positive constant does not change the arguments. □

It follows that the threshold in α at which an equilibrium loses stability, the fractional analogue of the Hopf bifurcation [12], and the necessary condition for the existence of chaotic attractors based on the same criterion [5], are invariant under a change of clock. The non-covariance of Theorem 3 is not visible in the linearization: it is an effect of the nonlinearity, and it shows up in the global objects, limit cycles, attractors, bifurcation cascades, as we now show.

Remark 7. Theorem 3 is a quantitative form of the known fact that the Caputo derivative does not obey the chain rule [10]: the quantity $c(\alpha)$ measures, to first order and at the level of solutions, the effect of that failure when the composition is with a clock that varies along the motion. The result is also complementary to that of Cong and Tuan [8], according to which Caputo equations do not generate a two-parameter dynamical system: there, invariance under translation of the initial time is missing; here, invariance under reparametrization of time is missing; and the two failures have the same origin, memory. Finally, the result should not be confused with the theory of fractional derivatives with respect to another function, the ψ-Caputo derivatives [9], which describe a memory that accumulates in a deformed time $\psi(t)$ fixed in advance, and with which covariance under that deformation holds by construction. Here the deformation $s(t) = \int_0^t g(x(\tau))\,\mathrm{d}\tau$ depends on the solution itself, and the question is not how to differentiate in a deformed time, but which of two different models one is writing when a kinetic factor multiplies the field.

4 Numerical verification

Unless otherwise stated, the integrations use the fractional Adams–Bashforth–Moulton predictor-corrector scheme [13], which for $\alpha = 1$ reduces to Heun's method and which introduces no truncation of the memory; the localization of the edges of the periodic window of the Chen system uses a second, independent integrator based on the diffusive representation, described there. The steps are given case by case, and every result was repeated with the step halved.

The coefficient $c(\alpha)$. Consider the nonlinear oscillator

$$f(x) = \big(x_2,\ -x_1 - \tfrac{1}{2}x_2 + \tfrac{3}{10}x_1^3\big), \qquad x_0 = (1, \tfrac{1}{2}), \qquad g(x) = 1 + \tfrac{4}{5}(x_1 - 1), \tag{6}$$

for which $g(x_0) = 1$ and $\gamma = \nabla g_0 \cdot f_0 = \frac{4}{5}\cdot\frac{1}{2} = \frac{2}{5}$. For every order we integrate x and X up to a time T chosen so that T^α is small, compute $s(T)$ by the trapezoidal rule, interpolate X at $s(T)$

Table 1: Transverse coefficient: distance of $x(t)$ from the orbit of X divided by $t^{3\alpha}$, extrapolated to $t \to 0$ from the series solutions, and prediction of Theorem 5.

α	$d(\alpha)$	measured	predicted	cosine
0.3	-0.0551	0.01428	0.01416	0.9987
0.5	-0.0624	0.01614	0.01604	0.9990
0.7	-0.0381	0.00983	0.00980	0.9993
0.9	-0.0100	0.00259	0.00258	0.9996

Table 2: Coefficient $c(\alpha)$: value from Theorem 3 and numerical estimate from system (6).

α	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
theorem	0.9294	0.8327	0.7107	0.5756	0.4379	0.3063	0.1871	0.0844
estimate	0.9284	0.8320	0.7103	0.5754	0.4378	0.3063	0.1871	0.0844

and take the scalar product of the difference $x(T) - X(s(T))$ with f_0; by Theorem 3 the ratio between this quantity and $\gamma|f_0|^2 T^{2\alpha}$ tends to $c(\alpha)$ as $T \to 0$ with a remainder proportional to T^α, which we remove with two values of T ($T^\alpha = 0.02$ and 0.04) and linear extrapolation, using 800 steps per window; doubling the number of steps leaves the estimates unchanged beyond the fourth digit. Table 2 reports the comparison: the agreement is within 0.1% at every order from 0.2 to 0.9, and for $\alpha = 1$ the estimate is zero within 10^{-6}.

The coefficient $d(\alpha)$. The transverse term is smaller than the tangential one by a factor of order $|d/c|\, t^\alpha$, and to measure it we avoided discretization altogether: for polynomial fields such as (6) the solution of (2) can be built as a series in $\tau = t^\alpha$, coefficient by coefficient, through the recursion $a_{k+1} = \Gamma(1+k\alpha)[F(x)]_k / \Gamma(1+(k+1)\alpha)$, where $[F(x)]_k$ is the coefficient of τ^k in the series of F along the solution, computed with truncated-series arithmetic. With fourteen terms the two solutions are exact, for the values of τ used, far beyond the precision required, and the same construction reproduces $c(\alpha)$ to six digits. For each order we computed the distance of the point $x(t)$ from the curve $X(s)$, minimizing over s, for $\tau = 0.04$ and 0.08, and extrapolated to $\tau \to 0$ the ratio between this distance and $\tau^3 = t^{3\alpha}$. The comparison with the prediction $|d(\alpha)|\,\gamma\,|(Df_0 f_0)_\perp|$, where in the example $\gamma = 2/5$ and $|(Df_0 f_0)_\perp| = 0.643$, is in Table 1: the agreement is within one percent, and the cosine between the measured transverse discrepancy and the predicted direction $d(\alpha)\gamma(Df_0 f_0)_\perp$ exceeds 0.998, which also confirms the sign of d.

The limit cycle of the fractional van der Pol oscillator. The van der Pol oscillator $f(x) = (x_2,\ \mu(1-x_1^2)x_2 - x_1)$ with $\mu = 1$ has at the origin the eigenvalues $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, with argument $\pi/3$; by Proposition 6 the origin is unstable for $\alpha > 2/3$ with any clock, and in that regime the fractional system possesses a limit cycle [14]. Strictly speaking a Caputo system has no exact non-constant periodic solutions [15, 16]; here and below, by limit cycle, or simply cycle, we mean the attractor of an asymptotically periodic solution, which is what one observes once the transient has died out. We compare the cycle of the reference system with that of the system with clock $g(x) = e^{0.6x_1}$, which varies along the cycle by up to a factor $e^{2.4} \simeq 11$, integrating from $x_0 = (2, 0)$ up to $T = 200$ and keeping the last 40 time units, when the amplitude is stationary to within 10^{-4}. As a measure of the geometric difference we use the maximum distance between the two closed curves, that is the maximum, over the points of one, of the distance from the polyline of the other, in both directions.

Figure 2 and Table 3 summarize the result. For $\alpha = 1$ the distance between the cycles is 0.037 with step 0.02 and 0.0085 with step 0.01: it falls by a factor of four when the step is halved, as a second-order discretization error should, and it is bound for zero. For every $\alpha < 1$ the distance

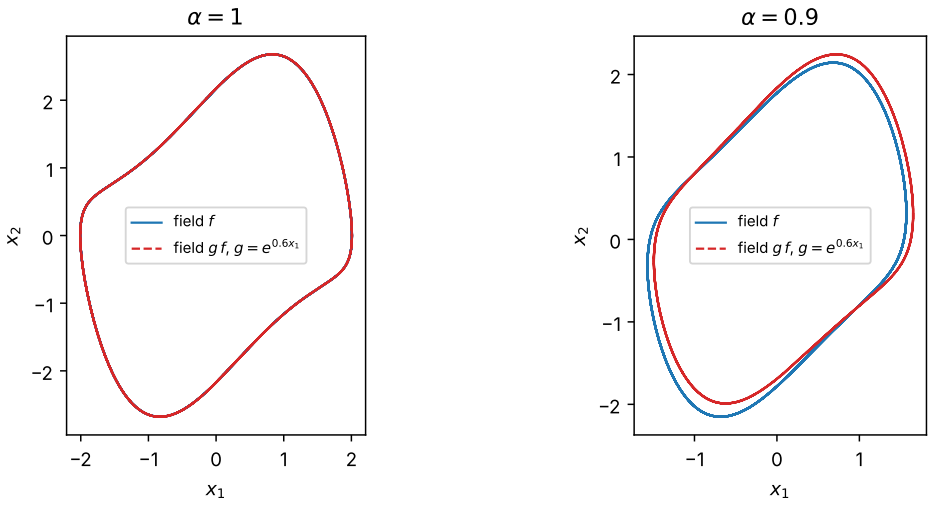

Figure 2: Limit cycle of the fractional van der Pol oscillator with $\mu = 1$ for the field f (solid line) and for the field $g\,f$ with $g = e^{0.6x_1}$ (dashed). For $\alpha = 1$ the two curves coincide, as Theorem 1 prescribes; for $\alpha = 0.9$ they are two different cycles, of amplitude 1.570 and 1.652.

Table 3: Fractional van der Pol, $\mu = 1$: maximum distance between the limit cycles of the two clocks for two integration steps, and amplitude in x_1 of the two cycles ($h = 0.01$).

α	distance, $h = 0.02$	distance, $h = 0.01$	amplitude f	amplitude gf	difference
1	0.0370	0.0085	2.009	2.009	$< 0.1\%$
0.95	0.1085	0.1065	1.795	1.846	2.8%
0.9	0.1715	0.1712	1.570	1.652	5.2%
0.85	0.1954	0.1945	1.335	1.431	7.1%
0.8	0.1797	0.1787	1.089	1.180	8.4%
0.75	0.1327	0.1323	0.821	0.890	8.5%

is instead stable under the change of step, to within one percent, and amounts to between 6% and 16% of the amplitude of the cycle; the amplitude of the oscillation differs between the two clocks by about 3% at $\alpha = 0.95$ and by more than 8% for $\alpha \le 0.8$. The two curves in the right panel of Figure 2 are two different limit cycles of the same vector field, read with two clocks: in integer order this would be a contradiction, in fractional order it is what Theorem 5 prescribes, here on a global object.

The fractional Rössler system. The Rössler system $f(x) = (-x_2 - x_3,\ x_1 + a x_2,\ b + x_3(x_1 - c))$ with $a = b = 0.2$ and $c = 5.7$ [17] is the canonical example of a period-doubling cascade, and in fractional form it has been studied by many authors to determine the minimal order at which chaos survives [4, 5]. The inner equilibrium has eigenvalues $0.097 \pm 0.995\,i$ and -5.69, with minimum argument 1.474 rad, so that by Proposition 6 it is stable for $\alpha < 0.938$ with any clock; above this threshold, which the two clocks share exactly, the cascade opens. We computed the bifurcation diagram, the local maxima of x_1 in the last 150 time units out of 400, for α from 0.93 to 1 in steps of 0.005, with the clock $g = 1$ and with the moderate clock $g = e^{0.1x_1}$, which varies by a factor of about seven along the attractor. Figure 3 shows the two diagrams: the qualitative sequence is the same, birth of the cycle at 0.94, period two at 0.97, period four at 0.98, chaos from 0.985, and the thresholds coincide within the resolution of the step in α; the values of the maxima differ between the clocks by about one percent, as one expects from $c(\alpha) \le 0.04$ in this range. With the moderate clock, then, the cascade is robust and the non-invariance is visible

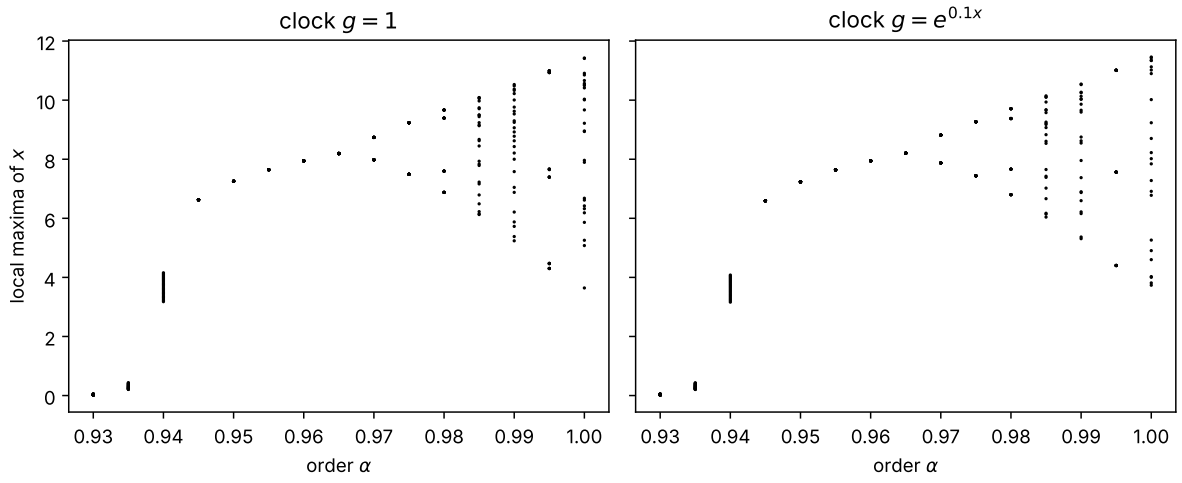

Figure 3: Bifurcation diagram of the fractional Rössler system ($a = b = 0.2$, $c = 5.7$) as a function of the order, for the clock $g = 1$ and for the moderate clock $g = e^{0.1x_1}$. The instability threshold of the equilibrium, $\alpha = 0.938$, is the same by construction; the sequence of period doublings coincides within the resolution in α, and the maxima differ by about one percent.

only in the details.

A stronger clock, $g = e^{0.25x_1}$, which varies by a factor of about one hundred along the attractor, does not move the period doublings either: for orders between 0.96 and 0.98 the sequence of periods is the same for the two clocks and the doubling occurs between 0.965 and 0.97, but the cycles change measurably, and the separation between the two maxima of the period-two cycle at $\alpha = 0.97$ goes from 0.76 to 0.53, stably under the halving of the step. Up to order 0.99 we observed no change of regime that survives the halving of the step: on this example the clock changes the cycles, which in integer order it would leave untouched, but not their nature.

The fractional Chen system: chaos is not an invariant. The Rössler system above 0.95 is the wrong place to look for a qualitative change, because there $c(\alpha) \leq 0.04$. One needs a system that stays chaotic at lower orders, and the Chen system, which for the state (x, y, z) reads $f(x, y, z) = (a(y - x),\ (c - a)x - xz + cy,\ xy - bz)$, with $a = 35$, $b = 3$, $c = 28$ [18] is the natural candidate: its nontrivial equilibria have eigenvalues -18.4 and $4.21 \pm 14.9\,i$, with minimum argument 1.295 rad, so that the critical order is $\alpha^* = 0.824$, the same for every clock, and the system is chaotic on almost the whole interval from 0.83 to 1 [19]. As clock we chose an Arrhenius-type factor, $g = e^{\beta(z-25)/20}$, in which the variable z, which in Lorenz-type systems is related to temperature, speeds up or slows down all rates; $\beta = 0$ is the trivial clock, and for $\beta = \pm 1$, since z ranges roughly between 5 and 45 on the attractor, g varies between e^{-1} and e. To decide the regime we used the largest Lyapunov exponent $\lambda_{\max}$, defined, as in [20], through the Caputo variational equation $D^\alpha \delta = D(gf)(x)\,\delta$ for a small perturbation δ of the solution, integrated together with the system by the same scheme, from the point $(-3, 2, 20)$, with step 0.004 up to $T = 150$. We did not renormalize δ at intervals, as Benettin's algorithm does, because in a system with memory rescaling the present value without its stored history would alter the variational equation: with $|\delta(0)| = 10^{-150}$ the growth stays within double precision, and $\lambda_{\max}$ is the slope of $\ln|\delta|$ over the second half of the integration. In integer order the exponents with and without clock are related by $\lambda_g = \langle g \rangle \lambda$, where $\langle g \rangle$ is the time average of g along the orbit (the reason is recalled in Section 5), and this provides a check: for $\alpha = 1$ the three clocks give $\lambda = 2.04$, 1.93 and 1.92, with $\langle g \rangle = 1$, 0.90 and 1.01, that is the same regime for all, and values consistent with the relation within the precision of a finite-time estimate.

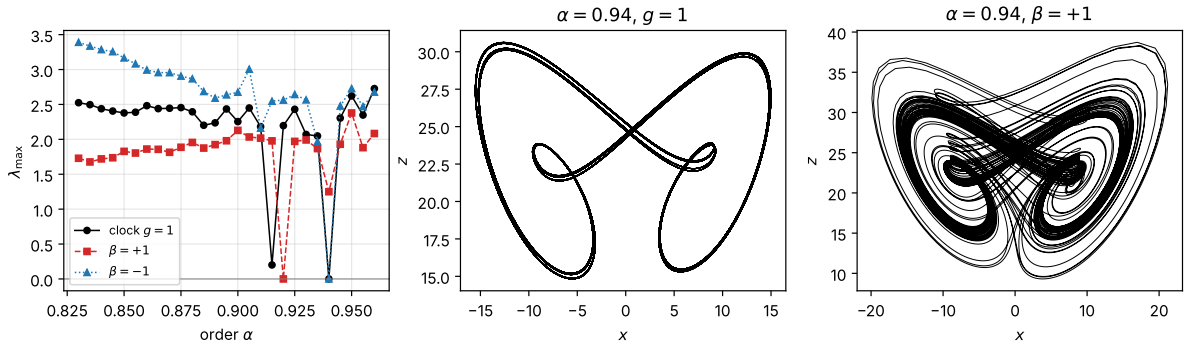

Figure 4: Fractional Chen system with the Arrhenius clock $g = e^{\beta(z-25)/20}$. Left: largest Lyapunov exponent as a function of the order for $\beta = 0$, $+1$ and -1 (integration step 0.004, $T = 150$, order step 0.005); the periodic windows, where $\lambda_{\max}$ vanishes, sit at different positions for different clocks. Middle and right: x–z projection of the solution for $\alpha = 0.94$ with the trivial clock, a cycle, and with $\beta = +1$, a chaotic attractor; same field, same order, same initial datum.

Figure 4, left, shows $\lambda_{\max}$ as a function of the order for the three clocks, from 0.83 to 0.96 in steps of 0.005. The system is chaotic almost everywhere, with $\lambda_{\max}$ between 1.7 and 3.4 depending on the clock, but the periodic windows, in which the exponent vanishes, are not in the same places: for $\alpha = 0.94$ the trivial clock and the clock with $\beta = -1$ give a cycle, with $\lambda_{\max} = 0.000$ and few distinct maxima of z, while the clock with $\beta = +1$ gives a chaotic attractor with $\lambda_{\max} = 1.25$. We subjected this case to two checks: with the step halved to 0.002 the exponents are 0.000, 0.92 and 0.000 for $\beta = 0$, $+1$ and -1, and with the time doubled to 300 they are 0.000, 1.21 and 0.000; the middle and right panels of the figure show the two solutions. One may object that at $\alpha = 0.94$ the system sits in a thin periodic window and that the clock acts as a small perturbation that moves it; this is true, and it is exactly the point: in integer order no clock, however strong, can move a window, because Theorem 1 forbids it, and the check at $\alpha = 1$ confirms it. In fractional order a moderate Arrhenius factor is enough for the same field, at the same order and from the same point, to be periodic with one clock and chaotic with another. The chaos of a fractional system is not a property of the vector field.

The shift of the window and the general rule. The Chen system at $\alpha = 0.94$ is nothing special as a system; what is special is the point where it sits, inside a narrow periodic window embedded in chaos, hence a short distance from two global bifurcations, the saddle-node that opens the window and the crisis that closes it. If the change of clock acts as a perturbation that moves those edges, the shift must have a direction fixed by the sign of β and be regular in β. To check this we located the two edges of the window for five values of β between -1 and $+1$, sweeping α from 0.934 to 0.947 in steps of 0.0005. For this sweep we used a second integrator, independent of the first: the diffusive representation of the system [2, 7]. It rests on the identity $t^{\alpha-1}/\Gamma(\alpha) = \frac{\sin\alpha\pi}{\pi}\int_0^\infty w^{-\alpha}e^{-wt}\,\mathrm{d}w$, which turns (2) into a superposition of exponential modes, $x(t) = x_0 + \frac{\sin\alpha\pi}{\pi}\int_0^\infty w^{-\alpha}\varphi_w(t)\,\mathrm{d}w$ with $\dot\varphi_w = -w\varphi_w + g(x)f(x)$ and $\varphi_w(0) = 0$: each mode relaxes at rate w and is driven by the field. We discretized the integral with two hundred modes logarithmically spaced between 10^{-8} and 10^4 plus a zero-rate mode for the tail, which reproduces the kernel $t^{\alpha-1}/\Gamma(\alpha)$ with a relative error of a few 10^{-6}, integrated with a second-order exponential predictor-corrector at step 0.005, from the point $(-9, -5, 14)$, for a time 400 of which the first 200 were discarded. The regime was classified by the number of distinct maxima of z in the second half: at most a dozen for periodic motion, more than one hundred and fifty for chaos, with in between an intermittent band that appears always and only on the left side of the window, as is proper for a window born by saddle-node and killed by crisis. Table 4 reports the result. The same integrator, with the variational equation on the augmented system and

Table 4: Edges of the periodic window of the fractional Chen system near $\alpha = 0.94$ as a function of the parameter β of the Arrhenius clock (resolution 0.0005 in α).

β	left edge (end of chaos)	right edge (crisis)	width
-1	0.9358	0.9418	0.0060
-0.5	0.9363	0.9423	0.0060
0	0.9373	0.9428	0.0055
$+0.5$	0.9383	0.9438	0.0055
$+1$	0.9398	0.9448	0.0050

Benettin renormalization, which on the augmented space is legitimate because there the system is a flow [7] and the modes carry the whole history, so that rescaling them rescales the past as well, recovers the case $\alpha = 0.94$ from another initial point and with another scheme: exponent 0.000 and -0.003 for $\beta = -1$ and 0, 0.81 for $\beta = +1$. The window moves as a whole towards higher orders as β increases: the right edge, that of the crisis, is the sharpest and goes from 0.9418 to 0.9448, described by a straight line with residuals of at most 0.0003, that is within the resolution, with slope $\kappa \simeq 0.0015$ in α per unit of β; the left edge moves in the same direction with a similar slope, about 0.002, and a slight curvature at the extremes, where the clock is no longer a small perturbation; the width stays between 0.005 and 0.006. Repeating the localization with the Lyapunov exponent instead of the maxima, the stretch with vanishing exponent goes from 0.9375–0.9415 for $\beta = -1$ to 0.9405–0.9445 for $\beta = +1$, with the same slope 0.0015 on both edges and constant width 0.004; the difference from the table lies in the intermittent band, which the count of maxima assigns to the window and the exponent to chaos. The case $\alpha = 0.94$ becomes a consequence: with $\beta = +1$ the window moves by about 0.0025 to the right and the point 0.94, which with the trivial clock sat 0.0027 from the left edge, comes to lie on the edge itself, in the intermittent band, which is why its Lyapunov exponent, about 1, is below the values 2–3 of full chaos; with $\beta = -1$ the window moves the other way and the point stays inside.

A simple reading follows, valid at least for this window: to first order the Arrhenius clock acts on the global bifurcations as an effective shift of the order, in the sense that the system with clock β at order α behaves, as far as its regime is concerned, like the system with the trivial clock at order $\alpha - \kappa\beta$; a clock that speeds up the upper part of the attractor is equivalent to a little more memory. Together with Proposition 6, this observation suggests a general rule, proved in its first part and verified numerically in its second: the local bifurcations of equilibria, the birth of a cycle and Matignon's threshold, do not move under any clock, because they depend on the linearization; the global bifurcations, period doublings, births and deaths of windows, crises, depend on the orbits and do move, to first order linearly in the strength of the clock. The fragility of a system under a change of clock is then its distance from the nearest global bifurcation, measured in units of $\kappa\beta$. This explains which systems are fragile: those with non-robust chaos, like Rössler and Chen, in which periodic windows are dense in parameter space and a global bifurcation is always nearby, so that a strong enough clock changes the regime somewhere and the case at 0.94 is one among many; and which are robust: the regimes decided by the stability of equilibria and the chaotic attractors without windows, in which the clock deforms the attractor but does not change its nature as long as the shift stays small. The coefficient κ is not universal, since it depends on the system and on the window, and we measured this: a second, narrower window, which with the trivial clock sits at 0.9145–0.915, moves linearly in β from the centre 0.9108 for $\beta = -1$ to the centre 0.920 for $\beta = +1$, with periodic points that remain periodic when the step is halved, that is with $\kappa \simeq 0.0046$, three times the value of the window at 0.94, while $c(\alpha)$ between the two orders changes only by a factor 1.6. The dependence on $c(\alpha)$ that Theorem 3 leads one to expect is therefore only one of the factors, and the window-dependent factor, that is how the variation of the clock is distributed along the orbit involved, is not negligible: the rule

is universal in form, that is in the sign and linearity of the shift, not in numbers.

5 Consequences

The first consequence concerns the way fractional attractors are presented. A strange attractor of a fractional-order system, drawn in state space, is not a property of the vector field: it is a property of the field and of the clock it was run with, and another clock, just as legitimate, would have produced a different attractor, with another shape, another amplitude and, near a bifurcation, possibly another nature. Does the shape of the attractor depend on the unit of time? No, by Proposition 2; it depends on how the clock varies with the state, and how much it depends on it is measured by $c(\alpha)$ and by $\nabla g \cdot f$. In the works that report the minimal order for chaos of a given system [4, 5], the number obtained is to be read as a property of the particular form of the field adopted, and not as an invariant of the system: a form equivalent in integer order, obtained by multiplying by a positive function, would give another minimal order. Matignon's necessary condition, on the contrary, is an invariant, and this makes it more robust than it is usually taken to be.

The second consequence concerns modelling. In many physical, chemical and biological models the field contains a state-dependent kinetic factor: a rate constant that depends on temperature, and temperature on the state; an enzyme concentration that modulates all rates; an activity, viscosity or density factor. In integer order the choice between writing the model with that factor and absorbing it into an intrinsic time is a convention without consequences for the orbits. In fractional order it is not: the two writings are different models, with different predictions, and Theorem 3 says by how much they differ at short times. Whoever writes $D^\alpha x = k(x)f(x)$ asserts that the memory of the system accumulates in physical time, and whoever writes $D^\alpha X = f(X)$ in an intrinsic time s asserts that it accumulates in the intrinsic time; these are two distinct physical hypotheses, both sensible, both falsifiable, and the choice between them must be declared and justified. The hypothesis that memory accumulates in an intrinsic time is not new in physics. It is the reduced time of nonlinear viscoelasticity, introduced by Leaderman [21] and developed in the free-volume clock models of polymers [22], and it is the material time with which Narayanaswamy described the structural relaxation of glasses [23], a concept recently validated directly on the nonlinear ageing of a molecular glass [24]. A similar question, where to place a space-dependent rate or force relative to the fractional operator, has been discussed for subdiffusion [25]. Fractional models that multiply the Caputo field by an Arrhenius factor make, often without saying so, the opposite hypothesis, memory in physical time; where the material time is established, Theorem 3 gives, at short times, the size of the error that this choice introduces in the orbits.

The same observation makes precise the distinction between memory and heterogeneity that we discussed elsewhere [7]. By the diffusive representation a Caputo system is itself a population of exponential modes with distributed rates, so the distinction cannot lie in the presence of the modes. It lies in how the kinetic factor acts on them: if it speeds up the relaxation of the memory modes as well as the forcing, the augmented system is an ordinary differential equation multiplied by g, it transforms covariantly and the memory accumulates in the intrinsic time; if it acts only on the forcing, the modes relax at their own pace in physical time and the system is not covariant. Clock non-covariance is therefore an observable signature not of memory as such, but of a memory that does not share the clock of the dynamics.

The third consequence concerns Lyapunov exponents. In integer order, if $x = X \circ s$ with $\dot{s} = g(x)$, the exponents of the system with clock are those of the reference system multiplied by the time average of g along the orbit, $\lambda_g = \langle g \rangle \, \lambda$, because a time t of the system with clock corresponds to a time $s(t)$ of the reference system and $s(t)/t \to \langle g \rangle$: the values change, the signs do not, and chaos is an invariant. In fractional order no relation of this kind can hold, because the orbits are different, and the Chen example shows that the sign of the largest exponent does

change with the clock: chaos, which in integer order is a reparametrization invariant, in fractional order is not. The definition of Lyapunov exponents for fractional systems through the diffusive representation [6, 7] remains valid clock by clock, but what it measures is, once again, a property of the field–clock pair.

6 Discussion

The result has a simple origin. The Caputo derivative weights the past with the kernel $(t-\tau)^{-\alpha}$ in physical time; if time is deformed by a clock that depends on the state, the deformation is not uniform along the past, and the kernel in the transformed time is no longer a power kernel. For a constant clock the deformation is uniform, the kernel remains a power up to a factor, and Proposition 2 is recovered. For a variable clock the first effect is that of Theorem 3, proportional to the rate of variation of the clock along the motion, $\nabla g \cdot f$, and to the function $c(\alpha)$, which vanishes only when the kernel is a delta, that is when there is no memory.

The result does not say that existing studies of fractional attractors are wrong; it says that their objects, the attractors and the thresholds, are properties of the field–clock pair and not of the field alone, and that two authors who model the same phenomenon with different clocks, both legitimate in integer order, obtain different attractors in fractional order. It is also worth being precise about the size of the effect, because it is easy to misjudge it in either direction. On one side, near $\alpha = 1$ the effect is small, proportional to $\frac{3}{4}(1-\alpha)$, and in the Rössler system with a moderate clock it does not move the thresholds within the resolution of our computation; those who work with orders 0.95 or above and away from bifurcations can in practice keep drawing attractors, knowing that they are defined up to a small clock-dependent error. Near a global bifurcation, however, small is not enough to protect the nature of the regime, as the Chen system at 0.94 shows, and the rule of Section 4 says when: when the distance from the bifurcation is smaller than the shift $\kappa\beta$ that the clock produces. On the other side, the effect grows quickly away from one: for orders around 0.5, where $c \simeq 0.58$, it is of the same order as the nonlinearity itself, and there is no longer any sense in which the phase portrait is a property of the field.

The result has three limitations. The first is that Theorems 3 and 5 are short-time statements: they say that the orbits separate and with which law, not how far apart they are at long times, where all orders and, for attractors, the global dynamics come into play; the numerical examples cover that part. The second is that the sign of $d(\alpha)$ is established numerically and not by proof, although for the conclusion it suffices that d does not vanish. The third is that we considered only the Caputo derivative with initial datum at $t = 0$ and zero history; for other operators with memory, and for non-zero histories, we expect the same phenomenon with another function of the order.

7 Conclusions

For an integer-order system the clock is a detail and the phase portrait is geometry; for a fractional-order system the clock is part of the model and the phase portrait depends on it. We have proved that the dependence appears in the time law at order $t^{2\alpha}$ with the universal coefficient $c(\alpha) = 1/\Gamma(1+2\alpha) - \alpha/(\Gamma(1+\alpha)\Gamma(2+\alpha))$ and in the geometry of the orbits at order $t^{3\alpha}$ with the universal coefficient $d(\alpha)$, both vanishing only for $\alpha = 1$, so that the orbits are invariant only under clocks that are constant along the orbits; that the local stability of equilibria and Matignon's critical order are exempt from it, and that the global objects, limit cycles and attractors, feel it to a degree that ranges from one percent near $\alpha = 1$ to a sizeable fraction for intermediate orders, up to swapping, in the Chen system at order 0.94, a cycle for a chaotic attractor: the chaos of a fractional system is not a clock invariant. The resulting rule is simple: the local bifurcations of equilibria do not move, the global ones translate linearly with the strength of the clock, and a system is fragile under a change of clock to the extent that it is

close to a global bifurcation. The practical moral is short: whoever uses a fractional model must declare in which time the memory accumulates, because two choices indistinguishable in integer order are two different models in fractional order, and whoever draws a fractional attractor draws the pair formed by the field and the clock. This holds in every discipline in which a fractional model carries a kinetic factor that depends on the state.

Data and code availability

The Python scripts that verify the coefficients $c(\alpha)$ and $d(\alpha)$, compute the van der Pol limit cycles, the Rössler bifurcation diagrams and the Lyapunov exponents of the Chen system with the Arrhenius clock, both with the Adams–Bashforth–Moulton scheme and with the diffusive representation, locate the edges of the periodic windows, and generate the figures, are available from the author and will be deposited in a public repository upon publication.

References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[2] G. Montseny, Diffusive representation of pseudo-differential time-operators, ESAIM: Proc. 5 (1998) 159–175.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics 2004, Springer, Berlin, 2010.

[4] C. Li and G. Chen, Chaos and hyperchaos in the fractional-order Rössler equations, Physica A 341 (2004) 55–61.

[5] M. S. Tavazoei and M. Haeri, A necessary condition for double scroll attractor existence in fractional-order systems, Phys. Lett. A 367 (2007) 102–113.

[6] R. Caponetto and S. Fazzino, A semi-analytical method for the computation of the Lyapunov exponents of fractional-order systems, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 22–33.

[7] S. Fazzino, Lyapunov exponents of fractional-order systems: a definition from the diffusive representation, the ageing of memory, and what memory truncation measures, Research Square preprint (2026), doi:10.21203/rs.3.rs-11383967/v1.

[8] N. D. Cong and H. T. Tuan, Generation of nonlocal fractional dynamical systems by fractional differential equations, J. Integral Equations Appl. 29 (2017) 585–608.

[9] R. Almeida, A Caputo fractional derivative of a function with respect to another function, Commun. Nonlinear Sci. Numer. Simul. 44 (2017) 460–481.

[10] V. E. Tarasov, No violation of the Leibniz rule. No fractional derivative, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 2945–2948.

[11] D. Matignon, Stability results for fractional differential equations with applications to control processing, in: Computational Engineering in Systems Applications, vol. 2, IMACS, Lille, 1996, pp. 963–968.

[12] M.-S. Abdelouahab, N.-E. Hamri and J. Wang, Hopf bifurcation and chaos in fractional-order modified hybrid optical system, Nonlinear Dyn. 69 (2012) 275–284.

[13] K. Diethelm, N. J. Ford and A. D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dyn. 29 (2002) 3–22.

[14] R. S. Barbosa, J. A. Tenreiro Machado, B. M. Vinagre and A. J. Calderón, Analysis of the Van der Pol oscillator containing derivatives of fractional order, J. Vib. Control 13 (2007) 1291–1301.

[15] M. S. Tavazoei and M. Haeri, A proof for non existence of periodic solutions in time invariant fractional order systems, Automatica 45 (2009) 1886–1890.

[16] E. Kaslik and S. Sivasundaram, Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions, Nonlinear Anal. Real World Appl. 13 (2012) 1489–1497.

[17] O. E. Rössler, An equation for continuous chaos, Phys. Lett. A 57 (1976) 397–398.

[18] G. Chen and T. Ueta, Yet another chaotic attractor, Int. J. Bifurcation Chaos 9 (1999) 1465–1466.

[19] C. Li and G. Peng, Chaos in Chen's system with a fractional order, Chaos Solitons Fractals 22 (2004) 443–450.

[20] C. Li, Z. Gong, D. Qian and Y. Chen, On the bound of the Lyapunov exponents for the fractional differential systems, Chaos 20 (2010) 013127.

[21] H. Leaderman, *Elastic and Creep Properties of Filamentous Materials and Other High Polymers*, The Textile Foundation, Washington, 1943.

[22] W. G. Knauss and I. J. Emri, Non-linear viscoelasticity based on free volume consideration, Comput. Struct. 13 (1981) 123–128.

[23] O. S. Narayanaswamy, A model of structural relaxation in glass, J. Am. Ceram. Soc. 54 (1971) 491–498.

[24] B. Riechers, L. A. Roed, S. Mehri, T. S. Ingebrigtsen, T. Hecksher, J. C. Dyre and K. Niss, Predicting nonlinear physical aging of glasses from equilibrium relaxation via the material time, Sci. Adv. 8 (2022) eabl9809.

[25] B. I. Henry, T. A. M. Langlands and P. Straka, Fractional Fokker–Planck equations for subdiffusion with space- and time-dependent forces, Phys. Rev. Lett. 105 (2010) 170602.