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Bounding company concentration in mineral markets from limited production disclosures

Juan Ignacio Guzmán

jguzmanb@uc.cl

Pontificia Universidad Católica de Chile

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Bounding company concentration in mineral markets from limited production disclosures

Abstract

Mineral-market analysts often know the largest companies' output but lack the complete distribution needed for the Herfindahl–Hirschman index (HHI). We calculate the HHI range consistent with these disclosures and report its midpoint with a maximum possible error. Building on established concentration bounds, the method combines individual leading shares with a further concentration ratio, keeps shares linked to a common uncertain production total, and allows for reporting precision. The joint calculation respects the size boundary shared by the newly disclosed group and smaller producers. Historical observations for 24 mineral industries illustrate the information gained by retaining individual shares instead of their sum. In a 2024 copper example, requesting the combined share of the eight largest companies, rounded to a whole percentage, meets a 150-point accuracy objective under stated reporting allowances; it does not always meet a 100-point objective. An independent 2024 U.S. soda ash capacity census provides a complete company benchmark: the two largest shares plus a four-company ratio rounded to 91 percent locate HHI within 60 points, compared with 462 points from the two shares alone. This capacity exercise is separate from production concentration. The results guide disclosure requests and identify when tighter computation can avoid additional collection. The guarantees concern compatible company distributions, not sampling uncertainty. Applications to market power and merger analysis additionally require relevant-market and competitive-unit evidence.

Keywords: Company concentration; Herfindahl–Hirschman index; Market power; Mineral markets; Partial identification; Production disclosure

JEL classification: C43; D43; L13; L41; L72

1 Introduction

Mineral-market analysts can often document a few large producers without assembling a complete company census. Annual reports identify major operations, while private suppliers, joint ventures and minority interests are harder to reconcile. The four-company concentration ratio, C_4, is therefore easier to obtain than a complete Herfindahl–Hirschman index (HHI). The practical question is

whether the available information is accurate enough for the intended decision and, if not, what further disclosure would suffice.

The question matters because corporate concentration is one input into the assessment of supplier bargaining and market power. In a homogeneous-product Cournot model, the share-weighted Lerner index is proportional to HHI and inversely proportional to the absolute elasticity of market demand (Cowling & Waterson, 1976). Related studies examine company concentration in cobalt mining and refining (Ericsson et al., 2024), supplier relationships in iron and steelmaking (Florén et al., 2019), and governance concerns surrounding minority shareholdings in lithium production (Soberón Bravo, 2023). These applications require information about companies and their competitive relationships. Concentration by producing country answers a different question.

The 2024 copper example illustrates that decision. Four leading equity-production shares leave a midpoint HHI estimate with a maximum possible error of about 168 points. Requesting C_8, the combined share of the eight largest companies, to the nearest whole percentage meets a 150-point objective for every feasible response under the stated reporting allowances. It does not always meet a 100-point objective. A separate complete 2024 company-capacity census for U.S. soda ash checks the method against a directly calculated HHI. The historical iodine–iron ore comparison examines a different limitation: uncertainty in the production denominator can reverse an apparent concentration ordering.

Tilton and Guzmán (2016) report a nearly linear cross-industry relationship between company HHI and C_4. A fitted conversion is convenient, but the same aggregate can describe many company-size distributions. Sleuwaegen and Dehandschutter (1986) describe the classical feasible region as a horn. Its boundaries and the midpoint error principle are established results. Figure 1 applies that framework to mineral disclosures with explicit attribution.

There is a substantial literature on estimating company HHI from partial information. Michelini and Pickford (1985) study concentration-ratio data; Sleuwaegen et al. (1986, 1989) discuss leading-share observation and desired precision. McCloughan and Abounoori (2003) examine grouped firm data with size-class counts and totals. Naldi and Flamini (2014, 2017) analyze missing firms and packed-share bounds; Kvålseth (2018) gives midpoint estimates with error allowances; Tóth (2016) considers a known participant count. These studies concern firms or general index mathematics. Country concentration cannot substitute for the missing company distribution.

The contribution is an applied method for evaluating disclosures when the information is linked. First, a finite joint calculation makes the newly reported group and smaller producers respect the same ranking boundary. Second, a sharp denominator calculation keeps all leading shares tied to one uncertain world total. Third, a conservative reporting envelope combines individual and aggregate allowances and can be tightened by subdivision. Established HHI bounds, midpoint estimation and precision-guided collection provide the foundations. We quantify the additional gains from these linked calculations rather than claim priority for those foundations or a universally new HHI–C_4 inequality.

The evidence has three roles. The 24-industry historical panel illustrates breadth, and 22 same-source HHI chart readings provide a consistency check. Copper equity-production disclosures for 2023–2024 demonstrate a practical choice between further requests. A separate USGS soda ash census supplies an independently sourced complete company-capacity benchmark. Production and capacity are kept separate, and the source company definitions are explicit. None of these descriptive boundaries is assumed to define an antitrust market.

Section 2 defines the economic object and introduces the copper example. Section 3 develops the bounds and reporting treatments. Section 4 presents the historical evidence, denominator sensitivity, copper comparisons and independent soda ash benchmark. Section 5 evaluates disclosure requests and conditional market-power and merger applications. Section 6 explains implementation, and Section 7 concludes. Proofs, calculation procedures and the remaining numerical inputs appear in the appendices.

2 Market power and the object being measured

2.1 Why HHI can be economically informative

Let firm i supply quantity q_i, let total market quantity be Q, and let its market share be $s_i = q_i/Q$. With one homogeneous product and a common market price $P(Q)$, an independently profit-maximizing Cournot firm choosing an interior quantity satisfies $P + q_i P' = MC_i$, where MC_i is marginal cost. Define the positive absolute price elasticity of market demand as $\eta = -P/(QP')$. Its Lerner index and the industry share-weighted index are then

$$L_i = \frac{P - MC_i}{P} = \frac{s_i}{\eta}, \qquad \overline{L} = \sum_i s_i L_i = \frac{HHI}{10000\eta}. \tag{1}$$

The identity is standard (Cowling & Waterson, 1976; Tilton & Guzmán, 2016). It explains why incomplete concentration information can matter for a structural interpretation of margins. For example, an HHI interval of 400–600 points maps to a weighted Lerner interval of 0.04–0.06 if the assumed demand elasticity is one. This is an illustration of the model, not a markup estimate for a mineral industry.

The Cournot identity requires a common homogeneous product, independently chosen interior quantities and the same competitive units used in HHI. Mine-production shares can differ from relevant-market sales shares. Joint products, quality, inventories, capacity constraints, long-term contracts and shared control can change the economic interpretation. Precise concentration measurement alone does not identify conduct, marginal costs or welfare.

Proposition 1. (Conditional translation into market power bounds). Suppose the stated Cournot identity holds for the same market and shares used to calculate HHI. If $HHI \in [L, U]$ and $0 < \eta_- \leq \eta \leq \eta_+$, then

$$\frac{L}{10000\eta_+} \leq \overline{L} \leq \frac{U}{10000\eta_-}. \tag{2}$$

Proof. Substitute the identity and use the fact that its right side increases with HHI and decreases with positive elasticity. These are valid bounds; they are sharp only when the relevant endpoint combinations are jointly feasible under the economic model. The result is a direct application of the established Cournot identity. ■

The value of a concentration disclosure is consequently conditional. With elasticity fixed at one, reducing an HHI maximum error from 200 to 100 points reduces the corresponding conditional Lerner error from 0.02 to 0.01. Uncertainty about elasticity or the relevant competitive units may be more consequential than this arithmetic improvement.

2.2 Companies and production definitions

A company concentration measure requires a product and stage, geography, period, denominator and grouping rule. A mine, a listed producer and an independent corporate group are different units. Country HHI answers a different question. All empirical bounds below use company or producer units within an explicitly stated production or capacity boundary.

Two conventions are useful descriptive measures. Ownership-equivalent production allocates an operation's output proportionately to equity interests. Controlled production allocates output

according to a specified control convention. Either can inform mineral-market analysis, but a competition application must establish which entities decide output or marketing. The updated copper example uses the source's equity-production allocation consistently in both years.

A contemporary production table also needs a consistent treatment of acquisitions and divestitures. Actual-year ownership allocations differ from a year-end structure applied to the entire year's production. Neither should silently stand in for the other. Mine output, recovered metal, payable production, sales, capacity and refined products likewise require separate definitions. Converting all quantities to tonnes does not resolve those differences.

The grouping convention can dominate missing-producer uncertainty. Suppose a 100-unit operation is owned 60 percent by A and 40 percent by B; their other operations produce 20 and 10 units, while C produces 70. Equity quantities are 80, 50 and 70, yielding HHI of 3,450. A control convention assigning the whole operation to A gives 120, 10 and 70 and HHI of 4,850. The 1,400-point difference is definitional. This hypothetical illustration assumes the control rule; equity percentages alone do not establish it.

2.3 Information and reporting units

Throughout the paper, s_i denotes a fractional company share: a ten-percent share is 0.10. Companies are indexed from largest to smallest within the specified production or capacity boundary, so $s_1 \geq s_2 \geq \cdots$ and s_i is the share of the ith largest company. The index HHI is measured in points, with 10,000 for a monopoly. Then

$$C_m = \sum_{i=1}^{m} s_i\,, \qquad HHI = 10000 \sum_i s_i^2. \tag{3}$$

We use C4 and C8 for C_4 and C_8. Tables and graph labels display them as whole percentages. Calculations retain the original numerical precision. An illustrative interval displayed as 5–95 percent is rounded outward. Its endpoints are bounds, rather than measured point values. Where a new aggregate itself would be reported only to the nearest whole percentage point, the analysis explicitly allows for that reporting interval.

An aggregate C4 reveals the combined share of the four largest companies. Four individual leading shares reveal that total and its distribution among those firms. An additional C8 adds information if the individual shares are retained. A list of four documented companies with unverified global ranks is different: it establishes neither C4 nor a size limit for omitted producers. The distinction determines which bound is valid.

Company counts are positive integers. Shares are nonnegative and sum to one over the adopted market. We pad a finite distribution with zero shares when a concentration ratio refers to more positions than there are positive firms. A verified leading group is ordered $p_1 \geq \cdots \geq p_m > 0$, with $\sum p_i \leq 1$. The number of omitted positive firms is unrestricted unless stated otherwise. A sharp closed bound cannot be tightened using the specified information; an endpoint may be a limit approached by increasingly many small firms.

2.4 A worked copper example for 2024

The National Mining Association's table, sourced to S&P Global Market Intelligence, reports 2024 equity-production quantities of 1,497, 1,421, 1,253 and 1,009 thousand tonnes for BHP, Codelco, Freeport-McMoRan and Glencore. With the USGS world mine-production estimate of 23,000 thousand tonnes, their combined share is displayed as C4 of 23 percent. Calculations divide each quantity by the common world total without first rounding the resulting shares. These are company quantities and a world denominator, not country concentration measures. Appendix B3 supplies the inputs and their allocation convention.

Under the source's leading-company ranking, every omitted producer has a share no greater than Glencore's 1,009/23,000. Dividing the unreported production among increasingly many small companies makes its squared-share contribution approach zero. Packing it into as many 1,009-thousand-tonne companies as possible, plus one remainder, gives the opposite endpoint. The compatible HHI interval is 129.45–465.04 points. Its midpoint is 297.25 and its maximum absolute error is at most 167.80 points, treating the reported quantities and world estimate as fixed numerical inputs.

The first eight companies together account for 8,185 thousand tonnes, displayed as C8 of 36 percent. The next four jointly contribute 3,005 thousand tonnes. Their smallest member also limits every smaller company. Respecting that shared restriction gives the HHI interval 172.13–380.38, with an error bound of 104.13 points. An upper-endpoint allocation has twenty-three additional companies of 751.25 thousand tonnes each and one of 541.25 thousand tonnes. Four of the twenty-three form the next group. Equal next-group quantities and an increasingly diffuse smaller fringe approach the lower endpoint.

These endpoints condition on the source figures. If the additional C8 is disclosed only as 36 percent, with a nearest-whole-percentage reporting interval, the error guarantee becomes 113.5 points. A

separate disclosure sensitivity allows each retained leading share to vary by ±0.05 percentage point. Its conservative guarantee is 119.1. That allowance describes a reporting scenario, not all uncertainty in the source quantities or world total. Section 5.1 distinguishes these response-specific guarantees from the largest error over responses not yet known.

3 Bounds and maximum estimation errors

3.1 Verified leading companies

Suppose the first m companies really are the largest, with observed individual shares $p_1 \geq \cdots \geq p_m > 0$. Here $p_i = s_i$ for $i = 1, \ldots, m$. The notation p_i identifies the retained observations. The variables needed for the leading-share calculation are

$$c = \sum_{i=1}^{m} p_i\,, \quad h = \sum_{i=1}^{m} p_i^2\,, \quad d = 1 - c, \quad t = p_m. \tag{4}$$

Here c is the share already accounted for; d is the unreported production share. The quantity $10000h$ is the observed companies' contribution to HHI, not the complete index. The smallest observed share, t, is also an upper bound on every omitted share because the leading ranks have been verified. The total number of producers is left unknown.

Proposition 2. (Established bounds from leading company shares). For nonnegative remaining shares summing to d, each at most t, the smallest closed interval containing all compatible finite-company HHIs is

$$L = 10000h, \qquad U = 10000\{h + kt^2 + v^2\}, \quad k = \lfloor d/t \rfloor, \quad v = d - kt. \tag{5}$$

The integer k counts how many additional companies could each have share t if production were concentrated as much as the ranking permits. The remainder v, between zero and t, is the share of one final company in that construction. These are established bounds from the leading-company literature, especially Naldi and Flamini (2017). We restate them to make their assumptions and implementation explicit.

Proof. The sum of squared omitted shares is nonnegative. Dividing d equally among sufficiently many producers, each no larger than t, makes that sum approach zero. For the upper bound, transfer share from a smaller to a larger omitted firm while preserving the cap t. If $x \geq y$, the change from replacing them by $x + \delta, y - \delta$ is $2\delta(x - y) + 2\delta^2 \geq 0$. Repeating transfers leaves k firms of size t, possibly one firm of size v, and no other positive omitted share. This construction attains the upper

endpoint. Thus no narrower closed interval works. If $d > 0$, the lower endpoint is an infimum approached by increasingly many firms; no finite collection of positive omitted shares attains it.■

Corollary 2.1. (Established midpoint error guarantee). For any valid interval $[L, U]$, the midpoint estimate and its guaranteed maximum absolute error over the reported interval are

$$\widehat{HHI} = \frac{L+U}{2}, \qquad E_{\max} = \frac{U-L}{2}. \tag{6}$$

Proof. An estimate a has worst possible absolute error $\max\{|a - L|, |U - a|\}$ over the closed interval. Its minimum is attained by equalizing the endpoint distances. Endpoint limits give the same supremum when an endpoint is unattained. For Proposition 2, every omitted square is at most t times the corresponding share, so $E_{\max} \leq 5000dt$.■

The midpoint and half-width representation is explicit in Kvålseth (2018, equations 29–30), and precision-based collection is discussed by Sleuwaegen et al. (1986, 1989). We use $E = (U - L)/2$ as an error bound in HHI points. With sharp closed endpoints it is the minimax absolute error over compatible distributions. For a conservative outer interval it remains a guarantee, but need not be the smallest attainable guarantee for the original evidence. It is not a standard error, confidence interval or expected prediction error. If the midpoint is rounded for use, add that rounding displacement to its guarantee.

When only the concentration ratio $C_m = c$ is known, the packing-corrected sharp endpoints are broader. With $a = c/m$ and $j = \lfloor 1/a \rfloor$, their sharp closed endpoints are

$$L^{C_m} = 10000c^2/m, \qquad U^{C_m} = 10000\max\{c^2, ja^2 + (1 - ja)^2\}. \tag{7}$$

The integer remainder in the packing calculation is retained. Appendix A2 provides a proof of the aggregate-only bound and its relationship to the established concentration-ratio literature. Its role here is to supply a valid comparison with the richer individual-share information.

3.2 Retaining individual shares and adding C8

Suppose the first m individual shares are retained and the additional aggregate $C_{m+q} = z$ becomes available. The next q companies jointly account for $b = z - c$, and the smaller residual firms account for $e = 1 - z$. The variables m and q are company counts, not production shares. For four retained shares plus C8, $m = q = 4$. With a positive residual and finitely many positive firms, the feasible aggregate satisfies $c < z \leq \min\{1, c + qt\}$. The case $z = c = 1$ is degenerate and already determines HHI.

Proposition 3. (Bounds with an additional concentration ratio). Under verified leading ranks and a common production definition, the sharp lower closed endpoint is $10000(h + b^2/q)$. The sharp upper endpoint is obtained by maximizing the sum of the packed next-group and residual-fringe contributions over their common boundary share, as specified and proved in Appendix A. For feasible z, both endpoints are nondecreasing in z. Adding the aggregate to retained information cannot increase $E_{\max}$.

For four retained shares and C_8, the smallest share among firms ranked fifth to eighth also caps every smaller firm. Proposition 3 optimizes over that common boundary, with a finite candidate calculation in Appendix A3. Separate group maxima can imply incompatible boundary sizes and yield a wider interval. The calculation is sharp for exact inputs; reporting envelopes introduced later are conservative and are not claimed sharp.

Corollary 3.1. (Whole-percentage disclosure). If the additional concentration ratio in Proposition 3 is reported as an integer N percent under nearest-percentage-point rounding, intersect its feasible range with $[(N - 0.5)/100, (N + 0.5)/100]$. A valid HHI interval uses the lower endpoint at the lowest feasible aggregate and the upper endpoint at the highest feasible aggregate.

Proof. Every true aggregate compatible with that report lies in this closed reporting interval. Proposition 3 makes the lower and upper bounds nondecreasing in the aggregate, so their respective outer endpoints bound every such value. The construction treats rounding ties conservatively. ■

Corollary 3.2. (Comparison with separate packing). Let $b = z - c > 0$ be the next q firms' total share and $e = 1 - z$ the remaining share. Write $P_v(u)$ for the largest sum of squares of nonnegative shares with total v and cap u, as defined in Appendix A1. A valid separate upper bound is

$$U_{\text{sep}} = 10000\{h + P_b(t) + P_e(b/q)\}, \qquad U(z) \le U_{\text{sep}}. \tag{8}$$

Proof. Every next-group share is at most t. Its smallest member is no greater than b/q, so every smaller firm is capped by b/q. Bounding the two contributions separately gives the displayed expression. The jointly feasible maximum in Proposition 3 cannot exceed this bound. The inequality can be strict because the two separate maxima need not respect a common boundary. ■

For copper in 2024, the separate upper endpoint is approximately 394.61, whereas the joint upper endpoint is approximately 380.38. The lower bound is unchanged. The respective midpoint guarantees are 111.24 and 104.13 points. Thus the joint calculation satisfies a 110-point objective

when the separate calculation does not, using the same exact-input evidence. Section 4.3 also reports the 2023 comparison under the same equity-allocation convention.

C8 alone is not an addition to four retained individual shares. Replacing C4 by C8 discards information about how the first group is composed, and neither aggregate generally determines the other. The C8-only maximum error can therefore exceed the C4-only error. The guarantee cannot worsen when the new ratio is added to the same retained individual shares.

3.3 A common uncertain production denominator

Shares often use the same estimated world-production total. Independent intervals for each share ignore this dependence. Suppose the leading quantities $q_1 \geq \cdots \geq q_m > 0$ are fixed, their total is $S = \sum_i q_i$, their squared total is $A = \sum_i q_i^2$, and the world total satisfies $Q \in [Q_-, Q_+]$ with $Q_- \geq S$ and $Q_+ > 0$. The leading ranking and production definition are maintained for every admissible total.

Proposition 4. (Bounds with a common uncertain denominator). With an unknown finite number of omitted firms, the sharp closed HHI bounds under these assumptions are

$$L_Q = 10000\,\frac{A}{Q_+^2}, \qquad U_Q = 10000\,\frac{A+P_{Q_- - S}(q_m)}{Q_-^2}. \tag{9}$$

Proof. For fixed Q, apply Proposition 2 in quantity units: the residual is $Q - S$ and its cap is q_m. The lower bound $10000A/Q^2$ decreases in Q. On a packing segment write $Q - S = kq_m + v$, with $0 \leq v \leq q_m$. Apart from the factor 10000, the derivative of the upper bound has the sign of $v(S + kq_m) - (A + kq_m^2)$. Since $A \geq q_m S$, this expression is nonpositive. The upper bound is continuous at packing transitions and therefore is nonincreasing throughout the interval. Its maximum is at Q_- and the lower minimum at Q_+. Packed residual quantities attain the upper endpoint. A diffuse residual at Q_+ approaches the lower endpoint.■

This measurement extension of the established packing bound retains the dependence created by a common denominator. Its sharpness is conditional on fixed company quantities, verified ranks and consistent units. Section 4.2 examines denominator sensitivity; uncertain company quantities require additional evidence.

3.4 Ranking and reporting limitations

When leading ranks are not established, documented shares p_i with $h = \sum p_i^2$ and residual $d = 1 - \sum p_i$ give the sharp closed interval $[10000h, 10000(h + d^2)]$. A supported cap u on every omitted

corporate group replaces d^2 by $P_d(u)$. A cap on individual mines is insufficient when companies own several mines. Propositions 5 and 6 in Appendices A4 and A6 provide these results, their proofs, and associated concentration-ratio bounds.

If both retained shares and a new aggregate have reporting intervals, Appendix A7 gives a conservative outer interval that includes all their permitted combinations. It also covers the uncertain increment in a hypothetical merger of retained firms. Because some linked quantities are bounded separately, the interval can be wider than necessary. Corollary 7.1 shows how subdividing the share intervals yields another valid bound without a new disclosure. These refinements should be considered before interpreting an unresolved result as a need to collect more data.

A reporting tolerance represents only the error it is specified to cover. Printing precision does not bound every source error. Appendix C distinguishes these deterministic identification bounds from expectations, variances and probability statements, which require a separately supported stochastic model.

4 Mineral evidence and an independent capacity benchmark

4.1 Company observations and their provenance

Table 7.1 of Tilton and Guzmán (2016, pp. 163–165) reports four leading production shares for 24 mineral industries, mostly for 2013, with diamonds and niobium explicitly dated 2012. Chrome is included in the company panel. We interpret the book's "State of Kazakhstan" entry as the corporate producer Kazchrome (ENRC), with state participation. Papp (2016, p. 17.5) identifies Kazchrome as an ENRC subsidiary and reports the government's 40-percent stake in ERG after the December 2013 acquisition of ENRC. The historical 17.3-percent share and ranking remain inputs from the book. The chrome calculation is conditional on interpreting that entry as Kazchrome; the ownership evidence does not independently establish this assignment.

Appendix B2 reports the historical producer identities. The source allocates jointly owned production proportionately to equity interests and uses world-production denominators. Rankings and allocations are maintained conditionally; the underlying operation-level quantities and ownership links have not been independently reconstructed. This panel illustrates the bounds under the source definitions.

The baseline treats the printed individual shares as exact and obtains C4 by summing them, retaining their precision internally. Eleven separately printed source C4 totals differ from that sum by 0.1

percentage point; Appendix B1 records the differences. A separate nearest-tenth printing sensitivity allows each leading share to vary by ±0.05 percentage point. It is an assumed reporting model, including for integer-looking entries, and is not an issuer-certified error allowance.

Table 1. Historical company HHI bounds and maximum estimation errors.

Mineral	C4 %	HHI interval	Four-share error bound	C4-only error bound
Bauxite	51	735.2–1,086.9	175.8	967.8
Chrome	33	388.9–571.2	91.2	396.1
Cobalt	38	554.4–732.1	88.9	544.4
Copper	35	307.0–710.2	201.6	446.4
Diamonds	60	1,308.6–1,348.7	20.1	1,345.6
Gold	20	115.0–336.3	110.7	199.6
Iodine	56	1,222.2–1,438.2	108.0	1,192.9
Iron Ore	55	942.5–1,165.1	111.3	1,151.0
Lead	16	66.9–284.3	108.7	164.7
Lithium	87	2,297.0–2,406.0	54.5	2,838.4
Manganese	42	572.6–888.9	158.1	674.2
Molybdenum	39	462.2–803.2	170.5	570.4
Nickel	42	459.4–848.8	194.7	658.4
Niobium	99	6,518.3–6,519.8	0.8	3,660.6
Palladium	69	1,783.2–1,874.1	45.5	1,790.6
Platinum	91	2,668.0–2,742.0	37.0	3,132.8
Potash	67	1,137.0–1,554.0	208.5	1,683.4
Rhodium	88	2,586.8–2,693.5	53.3	2,917.3
Silver	19	88.0–419.7	165.9	186.6
Tin	44	617.3–1,004.0	193.3	739.3
Titanium	44	823.1–984.2	80.5	719.5
Uranium	60	893.5–1,435.4	270.9	1,336.6

Mineral	C4 %	HHI interval	Four-share error bound	C4-only error bound
Zinc	26	187.5–503.6	158.1	245.8
Zirconium	65	1,269.0–1,474.0	102.5	1,584.4

Note. Calculations use Appendix B1's unrounded share sums; C4 is shown as a whole percentage. HHI bounds are rounded outward and errors upward.

Table 1 shows that, across the 24 industries, the median maximum possible error is approximately 853.50 HHI points from C4 alone and 109.63 points from the four individual shares. The reduction between these two medians is 87.2 percent. It is not the median of industry-specific percentage reductions and does not measure a change in actual concentration. Four, nine and twenty-one industries respectively meet 50-, 100- and 200-point tolerances. Under the printing sensitivity the median outer error is 115.43 points and the corresponding counts are three, nine and twenty (Appendix B4).

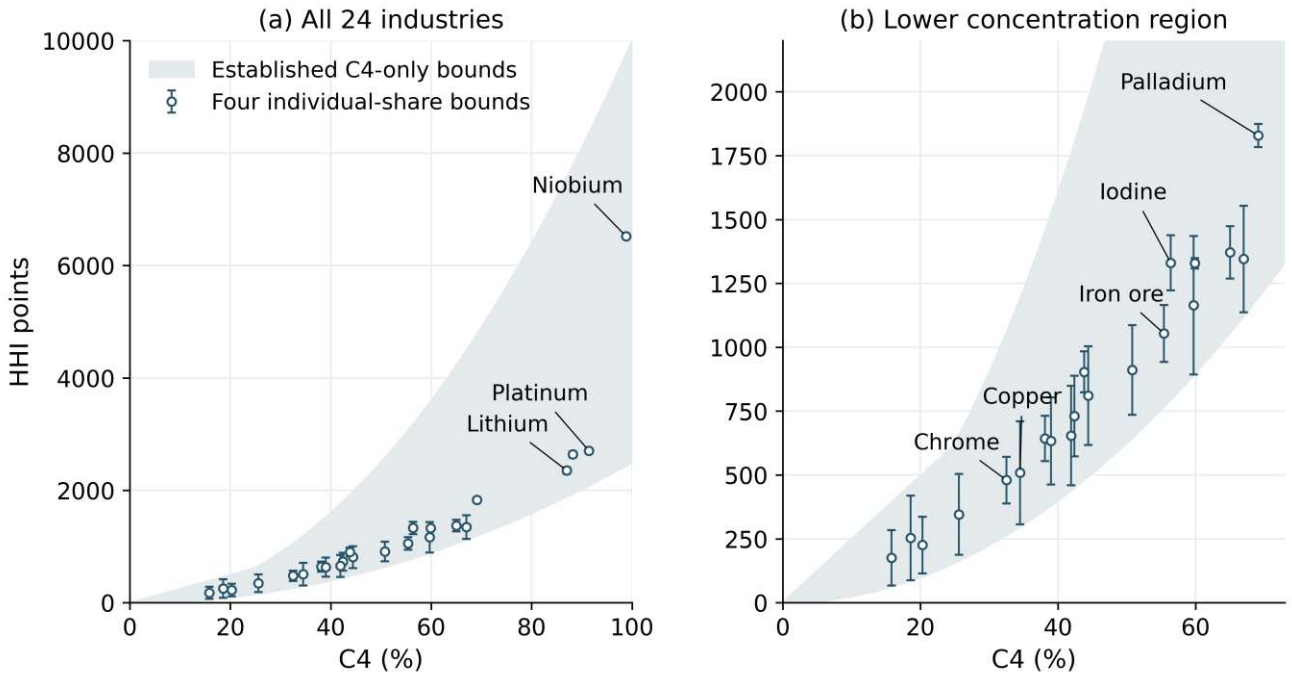


Figure 1. Historical company concentration within the established HHI–C_4 horn.

Note. Author's intervals and midpoints are based on Tilton and Guzmán (2016), Table 7.1; horn: Sleuwaegen and Dehandschutter (1986).

The size of the fourth company matters because it limits every omitted producer under the ranking assumption. Neither C4 nor the number of disclosed companies alone determines the remaining uncertainty. Rounding the HHI bounds outward to whole points gives iodine's interval of 1,222–

1,439, entirely above iron ore's 942–1,166, even though the displayed C4 values are 56 and 55 percent. C4 alone does not identify this ordering. The stated printing sensitivity preserves it, but that result does not establish robustness to all production-data revisions.

4.2 Sensitivity to a shared world total

To examine a different uncertainty, hold the reported leading quantities fixed and allow each industry's world total to lie within $Q_0(1 \pm \rho)$ of its baseline. Normalize $Q_0 = 1$ so that the printed fractional shares are the quantity inputs. Apply Proposition 4 to each industry. The adverse comparison uses iodine's largest admissible denominator and iron ore's smallest. The same ranking is assumed throughout. These analyst-selected denominator scenarios are separate from the printing sensitivity and are not estimates of source error; equality leaves the ordering unresolved. Table 2 reports these scenarios.

Table 2. World-production denominator sensitivity of the iodine and iron ore ordering.

Denominator allowance ±%	Iodine lower HHI bound	Iron ore upper HHI bound	Ordering identified
0.0	1,222.2	1,165.1	Yes
0.5	1,210.1	1,173.3	Yes
1.0	1,198.1	1,182.1	Yes
1.5	1,186.4	1,191.5	No
2.0	1,174.8	1,201.5	No

Note. Only the common production denominators vary; these are sensitivity scenarios. Bounds are rounded outward, and equality is unresolved.

The ordering survives a ±1 percent allowance but not ±1.5 percent. The crossover is approximately ±1.38 percent under this scenario. This does not estimate the reliability of either world total. It identifies how accurate the denominator would have to be, within the maintained specification, for the ranking to remain guaranteed. More precise information about small producers cannot substitute for checking the common denominator.

4.3 Copper in 2023 and 2024 and the gain from grouped information

The copper company quantities come from the National Mining Association's S&P Global table for 2020–2024 (National Mining Association, n.d.). It applies each mine's December 31 equity shares to calendar-year production, so its allocations can differ from company annual reports after ownership

changes. We use that convention consistently in 2023 and 2024. Appendix B3 reports all company quantities and the USGS world totals of 22,600 and 23,000 thousand tonnes (U.S. Geological Survey, 2025a, pp. 64–65). The 2024 total is an estimate. The world-production boundary includes copper in concentrates and recoverable electrowon cathode; the published totals are rounded.

Both years display C_4 of 23 percent and C_8 of 36 percent, although the retained quantities and producer composition differ. First Quantum is seventh in 2023 and absent from the 2024 top ten; Nornickel appears in 2024. Each year's quantities and denominator are used separately. The resulting HHI describes source-defined equity-production units, pending any consolidation required for a competitive-group application.

Table 3. Copper HHI bounds use company equity-production information for 2023 and 2024.

Information retained	**2023 HHI interval**	**Error bound**	**2024 HHI interval**	**Error bound**
C4 alone	135.9–578.6	221.4	126.8–557.3	215.3
C8 alone	165.5–1,324.2	579.4	158.3–1,266.5	554.1
Four individual shares	138.1–496.6	179.3	129.4–465.1	167.8
Four shares + C8 separate bound	180.8–401.1	110.2	172.1–394.7	111.3
Four shares + C8 joint bound	180.8–386.1	102.7	172.1–380.4	104.2
Eight individual shares	182.0–348.1	83.1	174.3–324.3	75.0
Ten individual shares	193.2–322.1	64.4	183.8–304.6	60.4

Note. Exact inputs are from Appendix B3; bounds are rounded outward and errors upward. Separate and joint bounds use Corollary 3.2 and Proposition 3.

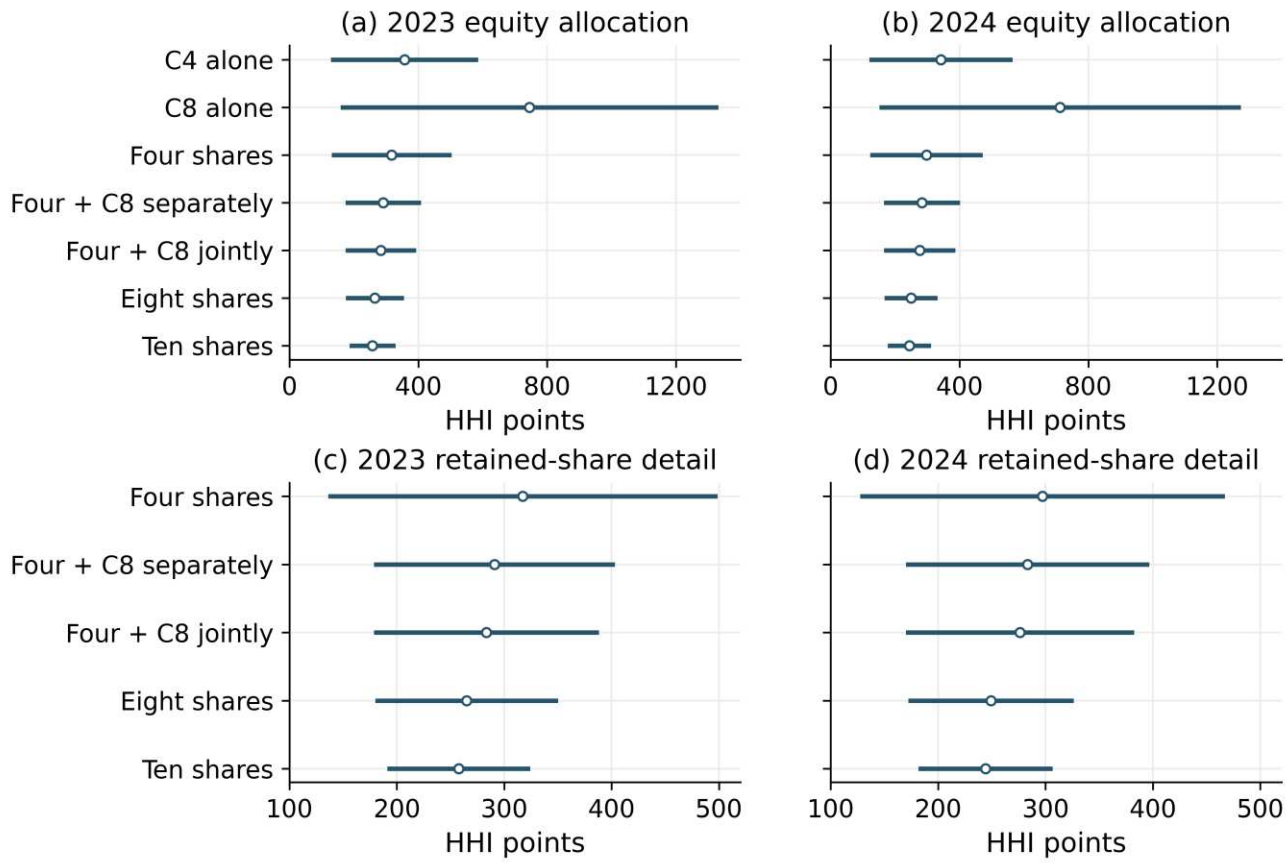


Figure 2. Copper HHI intervals show the value of retaining individual company shares.

Note. Hollow points are midpoints; lower panels magnify retained-share cases. Both years use equity production from Appendix B3.

Table 3 and Figure 2 show that the joint calculation narrows the 2024 copper interval by about 14.23 points relative to separate packing, reducing its half-width by 7.12 points. The corresponding 2023 reductions are 15.05 and 7.53. These incremental gains come from respecting the common group boundary. The larger improvement over C4 alone also uses the additional individual information.

The ten-share intervals, 193.25–322.03 for 2023 and 183.82–304.59 for 2024, overlap. They therefore do not establish a strict year-to-year ordering of complete company HHI. Neither interval is an observed complete HHI. Appendix B3 shows possible signed errors relative to each year's richer ten-share interval. The aggregate-only midpoint exceeds every compatible ten-share HHI in both cases.

4.4 Comparing historical HHI values with the bounds

We compare our calculated HHI bounds with HHI readings based on Tilton and Guzmán (2016). The HHI inputs are readings from the bars in its Figure 7.3 (p. 169), calibrated to the zero and 7,000-point

grid lines, rounded to the nearest ten HHI points and assigned a ±20-point reading allowance. This allowance describes chart reading and rounding; it does not quantify the underlying production-data error. Figure 3 compares these readings with our bounds and the source's reported HHI–C4 conversion.

The comparison uses 22 historical industries. Diamonds and niobium are excluded because the share table dates their observations 2012 whereas the HHI figure is labeled 2013. Appendix B7 reports every reading interval, including chrome's 380–420 points. All 22 overlap the corresponding leading-share HHI bounds. This is a consistency comparison based on Tilton and Guzmán (2016), not an independent validation against complete company distributions.

The book's fitted conversion as printed, expressed for fractional $c = C_4$, is $\widehat{H}_{\text{line}} = -720 + 3400c$. Using the unrounded totals from the four retained shares, its point prediction lies outside the identified interval in 13 of the 22 comparisons. Figure 3 shows why a high cross-industry fit does not guarantee a feasible company HHI for a particular industry. The original fitted relationship remains a descriptive summary of its source sample; no new predictive fit is estimated from these approximate readings.

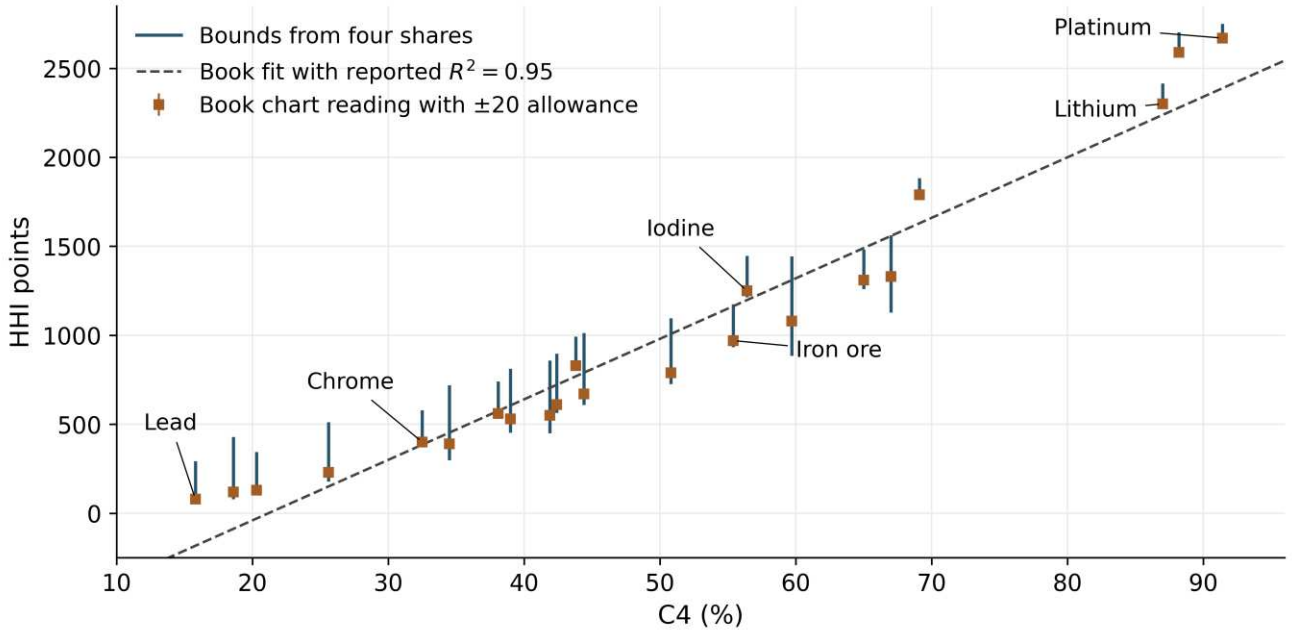


Figure 3. Historical company HHI values illustrate the limits of a fitted C_4 conversion.

Note. Author's calculations based on Tilton and Guzmán (2016), Figures 7.3–7.4; chart readings allow ±20 HHI points. Inputs are in Appendices B1 and B7.

4.5 An independent company capacity benchmark

A complete distribution permits a direct check against HHI rather than against another interval. The USGS 2024 soda ash census lists six U.S. plants operated by five producer companies (U.S. Geological Survey, 2025b, Table 3). We aggregate Genesis Alkali's Granger and Green River plants and retain the other four listed producer units. Table 4 supplies every capacity input. These are company shares of reported U.S. nameplate capacity, not country shares, output shares, or a finding about the relevant sales market.

The five reported capacities sum to 15.30 million short tons. On these fixed published inputs, full company-capacity HHI is 2,252.55 points; C_2 and C_4 are displayed as 53 and 91 percent. For the information comparison we retain the exact underlying quantities and initially withhold both the remaining company details and the total company count. This masking exercise uses a real complete census; it is a check of the bounds and information requirements, not an out-of-sample forecast.

Table 4. Reported U.S. soda ash capacities identify five producer companies in 2024.

Producer company	Listed plants	Capacity	Capacity share %
Genesis Alkali Wyoming	Granger; Green River	1.30 + 3.55 = 4.85	31.70
Searles Valley Minerals	Trona	1.45	9.48
Sisecam Chemicals Wyoming	Green River	3.25	21.24
Solvay Chemicals	Green River	2.95	19.28
Tata Chemicals North America	Green River	2.80	18.30
Total	Six plants	15.30	100.00

Note. Capacities are million short tons; shares use the 15.30 total. Source: U.S. Geological Survey (2025b), Table 3.

With only C_4 retained, the compatible interval is 2,048.59–8,194.40 points. Two individual leading shares give 1,456.06–2,379.43. Adding exact C_4 to those two shares, using Proposition 3 with two companies in the additional group, gives 2,162.25–2,264.09. The maximum midpoint error falls from 461.69 to 50.92 points, and its actual error against the complete census is 39.39 points when rounded upward. If C_4 is instead disclosed only as the whole-percentage report 91, the rounding rule in Corollary 3.1, applied to the four-company ratio, gives the interval 2,161.39–2,279.41, with a

maximum error of 59.01 points and an actual midpoint error of 32.16 points. Thus an illustrative 60-point objective is met even with aggregate rounding. Figure 4 compares these intervals with the benchmark.

Four retained individual shares instead give 2,162.73–2,252.56, with maximum error 44.91 points. The complete-census HHI lies at the exact upper endpoint because the omitted capacity belongs to one producer. If that single-company residual count is also disclosed, Proposition 6 identifies HHI exactly. In the two-share-plus-C_4 comparison, separate and joint packing coincide: this case verifies coverage but does not demonstrate a strict gain from enforcing the common boundary. The copper example supplies that strict comparison.

The benchmark is deliberately limited to the USGS-listed producer units and reported capacities. The Sisecam operation remains a single producer unit despite its disclosed partnership interests; we do not split its capacity into owners' equity shares. No cross-company consolidation or independent marketing structure is inferred from the census. Capacity utilization, imports, exports and sales arrangements would need separate treatment for a production or competition application.

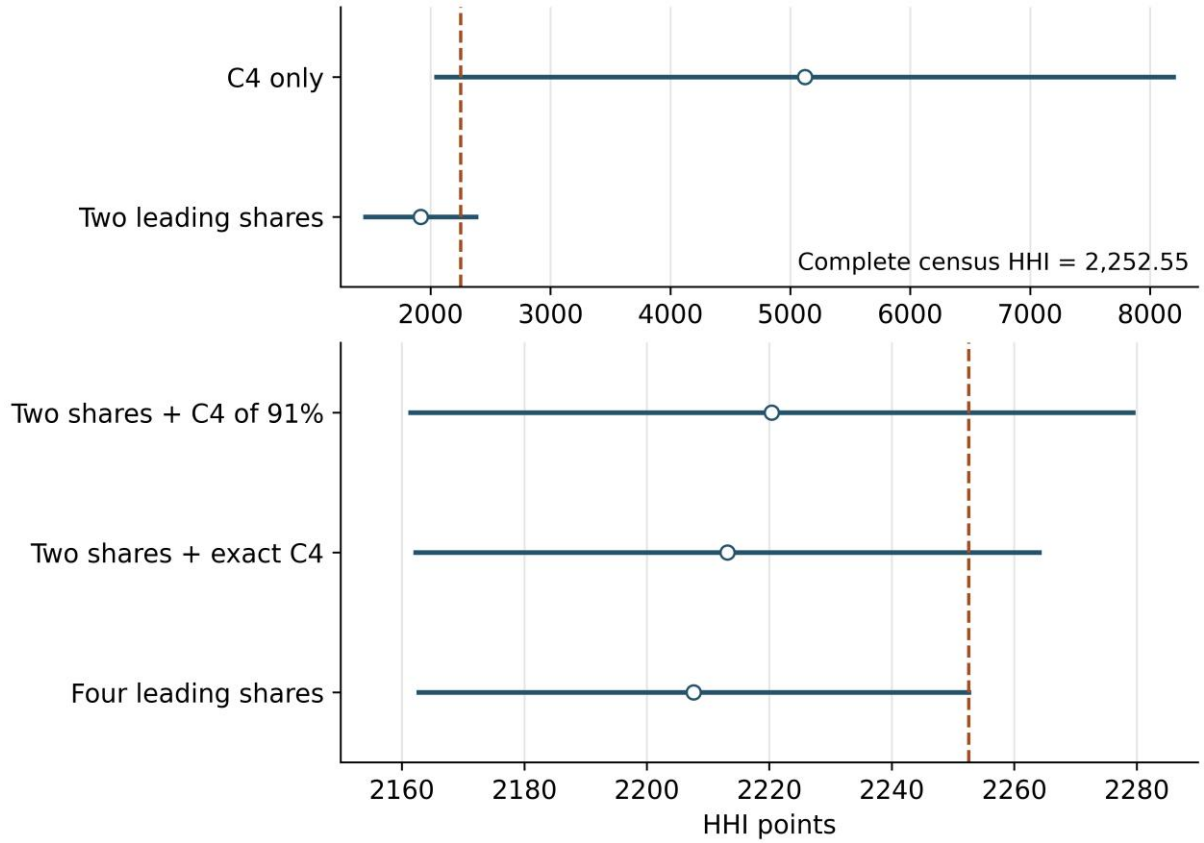


Figure 4. A complete soda ash capacity census checks the partial-information HHI intervals.

Note. The vertical line is the complete-census HHI; intervals initially omit the total company count. Inputs are in Table 4.

5 Disclosure choices and economic uses

5.1 Accuracy before and after a response

A request can be evaluated before its answer is known by calculating the error bound for every feasible response and taking the largest. Afterwards, the actual response determines a narrower conditional interval. Table 5 separates these questions and reporting assumptions in the 2024 copper example. The largest exact-share error occurs at a hypothetical 40-percent C8 report; with the retained-share allowance the largest outer error occurs at 39 percent. The world denominator is fixed in this reporting exercise.

Table 5. Copper error guarantees depend on the information and reporting assumptions.

Stage	Additional information	Retained shares	Error bound
Baseline	None	Exact	167.8
After response	C8 taken exactly from company quantities	Exact	104.2
After response	C8 reported as 36% with ±0.5 pp	Exact	113.5
After response	Same whole-percentage C8 report	Each ±0.05 pp	119.1
Before response	Every feasible whole-percentage C8 report	Exact	131.8
Before response	Every feasible whole-percentage C8 report	Each ±0.05 pp	137.2

Note. Error bounds are HHI points, rounded upward; uncertain-share entries are conservative outer bounds (Proposition 7).

Compare requesting the exact fifth leading share, C_8 to the nearest whole percentage, or an exact supported cap on every omitted company. Only C_8 includes response rounding in this benchmark; operational fifth-share or cap requests would need their own allowances. A fifth share adds both a squared-share contribution and an omitted-company cap. A cap supplies only the latter. Collection costs are not measured. Table 6 compares these requests.

Table 6. Additional disclosure requests provide different guaranteed precision.

Additional request	Largest error bound across responses	2024 response	Error bound at that response
Exact fifth leading share	158.2	4.29%	156.3

Additional request	Largest error bound across responses	2024 response	Error bound at that response
C8 to nearest whole percentage	131.8	36%	113.5
Exact supported cap on all omitted companies	167.8	4.29%	165.5

Note. Fifth-share and cap responses use 986/23,000, displayed as 4.29%; retained shares are exact and error bounds are rounded upward.

For a 150-point objective, only the C8 request among these three guarantees sufficient precision before its answer is known. At 160 points, both C8 and the fifth share suffice. At 170 points, the baseline already suffices. With exact retained shares, the largest error guarantees, rounded upward, are 131.8 points for the rounded C8 request and 158.2 for the fifth share. The analyst compares these guarantees with the required tolerance and chooses among qualifying requests using their collection costs. The number of statistics requested is not itself a cost estimate.

The C_8 calculation enumerates feasible whole-percentage response bins. For an exact fifth share, the largest error is found at the packing-segment endpoints and the zero limit; in the 2024 example it is below 158.2 points. A cap request can return the existing fourth-share cap, leaving the baseline guarantee unchanged. Whole-percentage C_8 reports range from 23 to 40 with exact retained shares and from 22 to 40 under their reporting allowance. No probabilities are assigned to these responses. Figure 5 displays the rounded aggregate response guarantees.

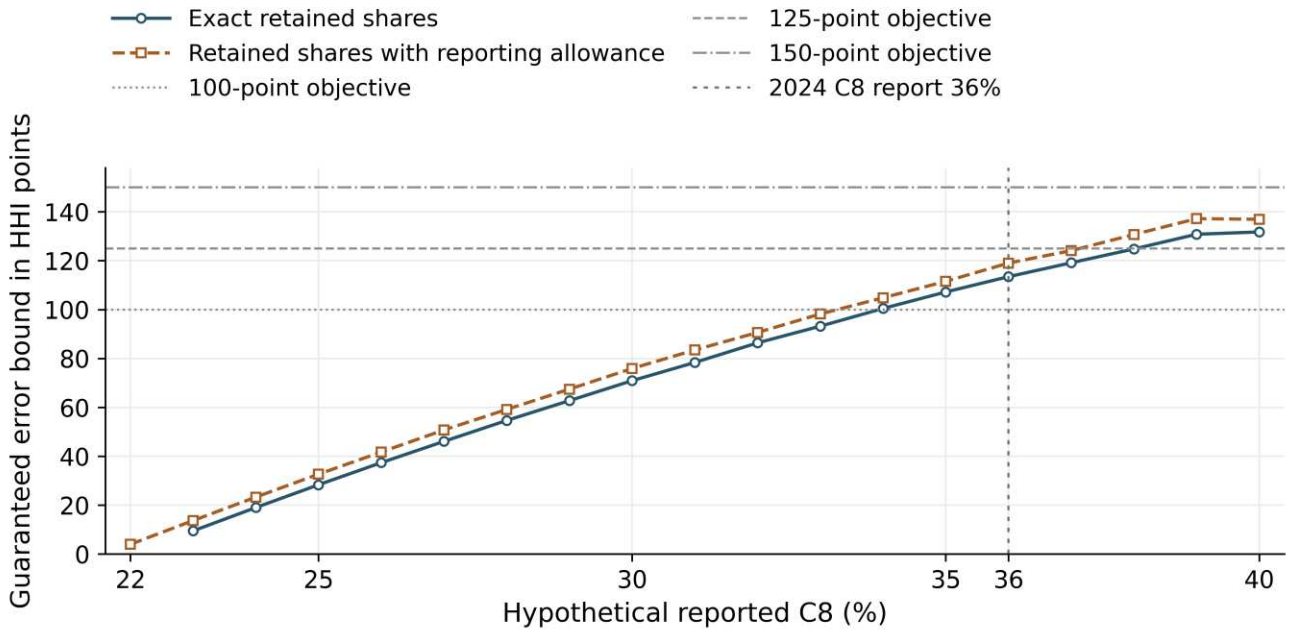


Figure 5. A rounded C_8 request can meet an accuracy target before its value is known.

Note. C8 is rounded to the nearest 1%; the second curve also allows ±0.05 percentage point per retained share. Inputs are in Appendix B3.

With uncertain retained shares included, the largest response-specific outer error is at most 137.2 points. Thus the 150-point objective still qualifies. A 100-point objective does not qualify for every response. At the 2024 report of 36 percent the outer error is 119.1. Eight individually retained shares with the same ±0.05-percentage-point disclosure allowance instead give 81.2. The appropriate next request changes with the required precision. These are conditional reporting guarantees; an uncertain world total must additionally be treated through Proposition 4 or a joint quantity calculation.

5.2 When concentration accuracy improves a market power inference

The Cournot translation in Proposition 1 is conditional on independently chosen quantities in the relevant homogeneous-product market. With demand elasticity fixed at one, the 2024 four-share interval gives a share-weighted Lerner index of about 1.3–4.7 percent. Adding exact C8 narrows it to about 1.7–3.8 percent. These are model illustrations, not estimates of copper margins, and require a compatible competitive-group definition.

If elasticity remains anywhere in [0.5,2], the corresponding ranges are about 0.6–9.3 and 0.9–7.6 percent. The midpoint error bound falls from about 4.3 to 3.4 percentage points, but substantial economic uncertainty remains. Even if HHI were known exactly at an illustrative 250 points, the Lerner interval would still be 1.25–5 percent, with a half-width of 1.875 percentage points, if both

elasticity endpoints remain admissible. An objective of one percentage point therefore cannot be met by further HHI precision alone in that scenario.

The elasticity is market demand over the selected product boundary and time horizon, not a single producer's demand. If the economic objective concerns margins, research on demand or competitive decision rights can have greater value than another small producer's output. A highly precise ownership-equivalent concentration statistic does not establish independent output-setting units.

5.3 Hypothetical merger comparisons

For a comparison with a threshold T, an interval entirely below or above T resolves that comparison; an interval crossing it does not. A chosen maximum error is a separate requirement. A wide interval can resolve a distant threshold, while a narrow one can still straddle a nearby threshold. These rules can be applied before requesting more information, so that the requested disclosure has a clear purpose.

The same idea extends to a hypothetical merger of two independently measured firms. Holding all other shares and the denominator fixed, the established accounting identity is

$$\Delta HHI = 20000 p_i p_j, \qquad HHI_{post} \in [L + \Delta HHI, U + \Delta HHI]. \tag{10}$$

The maximum estimation error is unchanged by this fixed translation. The identity does not predict subsequent output, prices or efficiencies. Nocke and Whinston (2022) explain why the interpretation of concentration screens depends on the competitive environment. Global equity-production shares also require reconciliation to the relevant competitors, product and geography before they can support an actual competition proceeding.

For bauxite, titanium and uranium, consider a hypothetical combination of the first two source companies and an illustrative postmerger comparison value of 1,500 HHI points. The requested ratio is explicitly premerger C8. The leading shares vary within ±0.05 percentage point, and the C8 report within ±0.5 percentage point. Proposition 7 propagates these allowances together, including the uncertain merger increment. Table 7 reports sufficient classifications from the initial conservative envelope.

Table 7. Sufficient premerger C_8 reports classify the illustrative historical mergers.

Historical mineral	Feasible N	Entire interval below 1,500	Unresolved	Entire interval above 1,500
Bauxite	51–81	51–62	63–77	78–81
Titanium	44–56	44–46	47–56	None
Uranium	60–100	60–69	70–83	84–100

Note. C8 reports are whole percentages; leading shares are from Appendix B1. These are sufficient outer-envelope rules, with equality unresolved.

An unresolved interval from this envelope can reflect a loose calculation as well as a lack of information. We therefore divide each of the four retained-share intervals into eight equal parts, evaluate the same valid envelope over all 4,096 sub-boxes, and take their combined outer interval. Intersecting it with the original bound preserves validity. Corollary 7.1 proves the refinement rule, and Appendix A8 gives the calculation procedure. Table 8 reports the refinement.

Table 8. Refining the same reporting evidence can resolve an illustrative merger comparison.

Mineral and C8 report %	Initial postmerger interval	Refined postmerger interval	Refined comparison
Bauxite 63	1,364.27–1,505.40	1,366.32–1,499.95	Below
Bauxite 77	1,493.77–1,670.32	1,498.27–1,664.04	Unresolved
Titanium 47	1,433.36–1,502.98	1,433.83–1,499.20	Below
Uranium 70	1,394.13–1,510.28	1,395.84–1,502.70	Unresolved
Uranium 83	1,498.78–1,683.40	1,502.77–1,676.35	Above

Note. The same evidence is subdivided into 4,096 boxes; entries are conservative outer bounds rounded outward to two decimals.

Bauxite at a 63-percent report and titanium at 47 percent become entirely below the illustrative comparison value. Uranium at 83 percent becomes entirely above it. The other displayed cases remain unresolved. This demonstrates a practical distinction: some additional data requests can be avoided by tightening the calculation. An unresolved refined interval still need not be the information-theoretic limit of the original evidence.

The parties' exact shares already identify the fixed-share increments: 604.54, 616.80 and 486.64 points respectively. C8 informs the concentration level, not these increments. For another objective, Nocke and Whinston's (2022, Section II.A) homogeneous-goods Cournot model gives a consumer-surplus-neutral proportional marginal-cost reduction of $r = 2ab/\{\eta(a+b) - a^2 - b^2\}$ for party

shares $a, b,$ relative to their output-weighted premerger marginal cost. It depends on those shares and demand elasticity, not outsiders' concentration conditional on them. At elasticity one, the bauxite example gives approximately 21.0 percent for every compatible outsider distribution. C8 therefore adds no information for that model's criterion. The illustration requires its constant-cost and equilibrium assumptions; it does not extend to every theory of merger harm.

The level comparison and the unilateral-harm calculation answer different questions. The 1,500-point value is an analytical example, not a current legal threshold or a merger clearance rule. A real assessment also requires relevant-market definition, competitive units, entry, capacity and other case evidence. The purpose is to match information requests to a specified concentration or economic objective.

6 Applying the method to mineral production evidence

Start with a written measurement specification: product and stage, geography, period, denominator, ownership date and company grouping. Copper here means worldwide mine production allocated by year-end equity interests. Soda ash in Section 4.5 means reported U.S. producer nameplate capacity. A legal application additionally requires a relevant-market definition and evidence identifying independent output or marketing decisions.

A usable C_8 request must use the same specification, identify the combined share of the eight largest groups, verify the ranking boundary, and state the denominator and rounding rule. If leading ranks cannot be established, use the unranked-subset bound or a supported omitted-company cap. Reconcile inconsistent definitions before calculation.

Before seeking smaller producers' quantities, check whether ownership and marketing arrangements establish the intended units. Joint-venture agreements or operating rights may matter more than numerical precision. A separate request for demand evidence may likewise be more useful when the objective concerns margins rather than the concentration index itself.

Appendix A8 gives the calculation procedure. The manuscript tables supply all empirical numerical inputs, including the complete soda ash capacity census. Supporting calculation files are available from the corresponding author on reasonable request. Apply the stated bounds first, and refine a conservative reporting calculation before commissioning further collection.

The independent capacity census checks the bounds against one complete recent producer distribution; it does not validate global production HHIs across all minerals. The historical observations retain their original dates, while copper uses 2023–2024 disclosures. Unsupported source errors, changed product boundaries and missing subsidiaries within a reported company total remain outside the guarantees.

7 Conclusions

Company disclosures can provide useful HHI measurements without a complete distribution when their units and ranks are credible. The joint calculation, common-denominator bounds and reporting refinements turn partial information into explicit accuracy guarantees. In copper in 2024, a whole-percentage C_8 request meets a 150-point objective under the specified allowances, although it does not always meet a 100-point objective.

The independent 2024 U.S. soda ash benchmark checks the method against a complete reported company-capacity distribution. Two retained shares plus C_4 rounded to 91 percent reduce the maximum possible error from about 462 to 60 points; the actual midpoint error is about 32 points. Four individual shares reduce the guarantee below 45 points. This is a capacity result, separate from production concentration and from any market-power inference.

The economic lesson is to match the next information request to the decision. Better computation can resolve some comparisons without more disclosure, while denominator, demand or corporate-control evidence can matter more than another small producer's quantity. Costs are needed to choose among requests that meet the objective. The contribution complements established concentration bounds with linked calculations and documented mineral applications.

Data availability

All empirical numerical inputs, chart-reading intervals, reporting allowances and disclosure-response values used in the analysis are reported in the manuscript tables. Hypothetical allocations, denominator scenarios and elasticity assumptions are stated with the calculations. Supporting calculation files are available from the corresponding author on reasonable request.

Use of generative artificial intelligence

Generative artificial intelligence assisted drafting and review of the manuscript, including language editing. The author takes responsibility for the final text, calculations and source interpretations.

Appendix A Proofs and implementation

A1 The packing function

For unallocated production share $v \geq 0$ and a maximum individual share $t > 0$, define

$$P_v(t) = kt^2 + (v - kt)^2, \qquad k = \lfloor v/t \rfloor. \tag{11}$$

Proposition 2 proves that this is the largest squared-share contribution of a residual of size v whose members are each at most t. Set $P_0(t) = 0$. Where a zero cap is used as a limit, $P_v(0)$ denotes the limiting value zero as t decreases to zero. Positive v cannot be allocated to a finite collection of companies with exactly zero shares. The function is nondecreasing in both mass and cap: more mass can be added to a feasible allocation, and a larger cap enlarges its feasible set. Also $P_v(t) \leq vt$, implying the zero limit.

A2 Classical concentration ratio bounds

We prove the formula based only on C_m in Section 3.1, using normalized squared shares and multiplying the final endpoints by 10,000. The Cauchy–Schwarz inequality gives a leading-group contribution of at least c^2/m. Set its shares equal to $a = c/m$ and divide the residual $d = 1 - c$ among sufficiently many firms to approach this lower endpoint.

For $m = 1$, the packing argument gives upper endpoint $P_1(c)$, which is at least c^2. For $c = 1$, the endpoints are $1/m$ and one, with a limiting endpoint when all m shares must be positive. Consider $m > 1$ and $0 < c < 1$. Conditional on the last leading share $t \in (0, a]$, transfers maximize the leading group's contribution at $(c - (m-1)t, t, \ldots, t)$. The exact conditional upper objective is

$$f(t) = [c - (m-1)t]^2 + (m-1)t^2 + P_d(t). \tag{12}$$

Between consecutive packing points $t = d/k$, this function is a convex quadratic, so a maximum occurs at a packing point, at a, or at the zero limit c^2. At a packing point its value is

$$G(t) = c^2 + [1 - (2m-1)c]t + m(m-1)t^2. \tag{13}$$

This is convex in t. Among packing points no greater than a, only the largest or the zero limit can maximize it. Let $K = \lceil d/a \rceil$, let $b = d/K \leq a$ be that largest point, and let $r = a/b \geq 1$. Direct expansion on the last packing segment gives

$$f(a) - f(b) = b^2[K(K-1) - m(m-1)](r-1)^2. \tag{14}$$

If $K \geq m$, the difference is nonnegative. If $K < m$, expansion instead gives

$$f(b) - c^2 = b^2\{K - (m-1)(2mr - m)\} \le b^2\{K - m(m-1)\} \le 0. \tag{15}$$

Consequently every candidate is bounded above by $\max\{c^2, f(a)\}$. Since $f(a) = ma^2 + P_d(a) = P_1(a)$, this is the asserted upper endpoint. Equal leading shares and a packed fringe attain $P_1(a)$. One leading share approaching c, the other $m-1$ shares tending to zero, and an increasingly diffuse fringe below them approach c^2. Both constructions preserve C_m. This establishes the sharp closed interval, including the integer remainder correction. ■

This proof concerns the classical concentration-ratio problem. Sleuwaegen and Dehandschutter (1985, Appendix) and Sleuwaegen et al. (1986, 1989) discuss packing and integer corrections. Kvålseth (2018) presents a convenient upper envelope. We retain the exact packed expression without attributing priority for its present rearrangement.

A3 Proof and finite calculation for Proposition 3

Let the next q company shares be $y_1 \ge \cdots \ge y_q = x$, with sum $b = z - c$. The quantity x is the smallest share in this next group, which also caps the smaller residual firms. These next-group shares lie in $[x, t]$ and include x, so feasibility requires $qx \le b \le (q-1)t + x$. Therefore

$$\ell = \max\{0, b - (q-1)t\} \le x \le u = b/q. \tag{16}$$

Also $0 \le b \le qt$ and $e = 1 - z \ge 0$. When $b = 0$, a finite ranked market is feasible only if $e = 0$, in which case the index is already known. We henceforth take $b > 0$.

For a fixed $x < t$, put $r_x = (b - qx)/(t - x)$ and define an integer packing count and a remaining intermediate share by

$$a_x = \min\{q - 1, \lfloor r_x \rfloor\}, \qquad v_x = b - a_x t - (q - a_x - 1)x. \tag{17}$$

The transfer argument in Proposition 2 puts a_x next-group shares at t, one at v_x, and the remaining $q - a_x - 1$ at x. Their maximum squared contribution is

$$B(x) = a_x t^2 + v_x^2 + (q - a_x - 1)x^2. \tag{18}$$

If $x = t$, feasibility requires $b = qt$ and $B(t) = qt^2$. The smaller residual has total share e and cap x, giving maximum contribution $P_e(x)$. Thus the point upper endpoint is

$$U(z) = 10000\left\{h + \max_{x \in [\ell, u]}[B(x) + P_e(x)]\right\}. \tag{19}$$

At zero, use the right limit when $e > 0$. The next-group and fringe maximizing constructions respect the same boundary x and are jointly feasible. For the lower endpoint, Cauchy–Schwarz gives $\sum y_i^2 \geq b^2/q$. Equal next-group shares $b/q \leq t$ attain that contribution, while increasingly many smaller firms make the remaining contribution tend to zero. This proves the claimed lower endpoint and sharpness.

The upper calculation is finite. Start with ℓ, u and all next-group breakpoints $(b - at)/(q - a)$ for integers $a = 0, \ldots, q - 1$ that fall in the interval. Within each consecutive segment $[v, w]$, the packing count a_x is fixed and $B(x)$ is a convex quadratic. Fringe breakpoints have the form $x = e/j$. When $e > 0$ and $w > 0$, their integer indices satisfy

$$j_{\min} = \max\{1, \lceil e/w \rceil\}, \qquad j_{\max} = \lfloor e/v \rfloor \quad (v > 0). \tag{20}$$

If the range is nonempty, retain only $e/j_{\min}$ and $e/j_{\max}$. When $v = 0$, retain $e/j_{\min}$ and the existing zero limit. Between consecutive breakpoints the complete objective is convex, so its maximum is at an endpoint. At a fringe breakpoint $P_e(x) = ex$; these values lie on the convex function $B(x) + ex$ within the next-group segment, and only its extreme fringe breakpoints can maximize them. The number of candidates is proportional to q, even though the number of omitted firms is unrestricted.

For completeness, the closure contains every intermediate HHI value. For a sufficiently large fixed total company count, ordered share constraints with the prescribed leading shares and next-group sum define a convex connected set. HHI is continuous, so its image is an interval. Padding with zero shares nests the sets as count grows. The closure of their union therefore fills the interval between the proved endpoints.

The lower endpoint is nondecreasing in z because $b = z - c \geq 0$. To establish the upper endpoint's monotonicity, increase the next group's mass by transferring output from the smaller residual firms. Fill the largest next-group company below t before proceeding to the next, preserving ranking. Every recipient is at least as large as a donor, so the squared sum weakly increases. Transfers remain possible until $z = \min\{1, c + qt\}$. Taking suprema proves that $U(z)$ is nondecreasing. Finally, adding an observed aggregate restricts the original feasible set; its lower endpoint cannot fall and its upper endpoint cannot rise relative to the same retained shares without that aggregate. This proves Proposition 3 and the monotonicity used in Corollary 3.1.■

A4 Bounds for unranked documented companies

Documented producers whose leading ranks have not been established require a different bound. Let $p_1, \dots, p_K$ be exact full-group shares for an arbitrary documented subset. Write $c = \sum p_i$, $h = \sum p_i^2$, $d = 1 - c$, and let A_j be the sum of its largest $\min(j, K)$ shares, with $A_0 = 0$.

Proposition 5. (Bounds without verified leading ranks). With no further restriction on omitted companies and an unknown total count,

$$10000h \le HHI \le 10000(h + d^2), \qquad A_N \le C_N \le \max\{A_N, d + A_{N-1}\}. \tag{21}$$

These are sharp closed bounds. In particular, the maximum estimation error for the HHI midpoint is $5000d^2$.

Proof. Dividing the residual among arbitrarily many sufficiently small firms gives the HHI lower limit and makes C_N approach A_N. Allocating the entire residual to one independent omitted company maximizes its sum of squares, giving the HHI upper endpoint. Any set of N largest firms either contains only documented firms, whose sum cannot exceed A_N, or includes omitted firms. In the second case their combined mass is at most d and the documented component is at most A_{N-1}. Allocating all d to one firm attains the larger of these two candidate sums after sorting. The HHI error follows by taking half the width.■

This elementary rank-robust formulation is proved for transparency. No claim of mathematical priority is made. Its practical implication is substantial. With exact unranked shares, an error tolerance of 200, 100 or 50 HHI points requires production coverage of at least approximately 80.0, 85.9 or 90.0 percent, respectively, absent any supported cap on omitted firms. By contrast, verified leading shares of 10, 8, 7 and 5 percent give the sharp interval 238–588 points and $E_{\max} = 175$. The same four shares without verified ranks give 238–5,138 points and $E_{\max} = 2{,}450$. These are mathematical illustrations, not new mineral observations.

Suppose the documented groups' shares satisfy $a_i \le p_i \le b_i$ and the nonempty feasible set is

$$\mathcal{S} = \{x: a_i \le x_i \le b_i,\ \textstyle\sum_i x_i \le 1\}. \tag{22}$$

The sharp closed lower HHI endpoint for this box model is $10000\sum a_i^2$. The upper endpoint is the maximum of $10000[\sum x_i^2 + (1 - \sum x_i)^2]$ over its vertices. To see why the vertex list matters, a point on $\sum x_i = 1$ with two free coordinates can be perturbed in opposite directions, and a point below that plane with one free coordinate can be perturbed in that coordinate. Such points are not vertices. The

vertices therefore consist of feasible original box corners and points on the plane where all but at most one nonfixed coordinate are at a bound. Convexity of the upper objective proves the enumeration; Proposition 5 establishes its interpretation at each vertex. The vertex construction avoids an arbitrary truncation of the omitted-company count.

For C_N, let $a_i^{\downarrow}$ denote the lower share bounds sorted from largest to smallest. Its lower endpoint is $\sum_{i=1}^{\min(N,K)} a_i^{\downarrow}$. For an all-documented selection S of $\min(N,K)$ groups, the largest attainable sum is $\min\{\sum_{i\in S} b_i\,, 1-\sum_{i\notin S} a_i\}$. Maximize this expression over such selections. The other upper candidate, permitting an omitted firm in the leading group, is

$$1-\sum_i a_i+\sum_{i=1}^{\min(N-1,K)} a_i^{\downarrow}. \tag{23}$$

The larger of these two candidates is the upper endpoint. The first is attainable by choosing the selected coordinates within their intervals and minimizing the other coordinates. The clipped sum enforces feasibility. For the second, use lower-bound documented shares and put all residual mass into one omitted company. Proposition 5 bounds every possible selection. This proves the interval concentration-ratio construction.

The bounds are sharp for the stated share box. Correlations induced by a common denominator or ownership restriction can make the box wider than the original jointly feasible evidence. In that situation the box bounds are conservative for the evidence. Neither statement covers an unrecorded source error or an omitted subsidiary within a supposed full-group total.■

A5 Sensitivity to printed leading shares

Corollary A1. (Conservative reporting precision allowance). Suppose the verified leading shares are printed as p_i and each true value is within fractional tolerance δ. Define $a_i=\max\{0,p_i-\delta\}$, $b_i=\min\{1,p_i+\delta\}$ and $d_+=1-\sum a_i$. Every feasible true ranked head within these intervals satisfies

$$L_{round}=10000\sum a_i^2, \qquad U_{round}=10000\min\{1,\sum b_i^2+P_{d_+}(b_m)\}. \tag{24}$$

Proof. Its observed squared contribution lies between $\sum a_i^2$ and $\sum b_i^2$. Its residual mass is no larger than d_+ and its omitted-company cap is no larger than b_m. Monotonicity of the packing function in both arguments and the universal 10,000-point cap give the upper bound. Nonnegativity gives the lower bound. Maximizing linked quantities separately can widen the interval, so joint sharpness is not claimed.■

The historical sensitivity uses $\delta = 0.0005$, corresponding to assumed nearest-0.1-percentage-point reporting for each company share, including integer-looking table entries. This is a specified sensitivity model, not an issuer-certified error allowance. Appendix B reports its resulting maximum errors. Corollary 3.1 addresses aggregate rounding conditional on exact retained shares. Proposition 7 combines both reporting allowances for the operational rules in Table 7.

A6 Bounds with an omitted company cap and count

Some mineral associations can establish a maximum size for every omitted supplier without disclosing their identities or certifying a complete rank order. That evidence is valuable even for an arbitrary documented company subset.

Proposition 6. (Bounds with an omitted company cap and count). Retain the exact full-group shares in Proposition 5 and suppose every omitted company's share is at most a supported value $u > 0$. The sharp closed HHI bounds are

$$10000h \leq HHI \leq 10000\{h + P_d(u)\}. \tag{25}$$

Here $P_d(u)$ is the packed contribution defined in Appendix A1. If there are at most $R \geq 1$ omitted companies and $Ru \geq d$, replace the lower endpoint by $10000(h + d^2/R)$. The upper endpoint is unchanged. If $d = 0$, HHI is known. The following proof also gives the associated bounds on C_N.

This is a direct extension of the established packing and Cauchy–Schwarz arguments, with no priority claim. Without count information the maximum error is at most $5000du$. Thus a sufficient cap for an error tolerance ε is $u \leq \varepsilon/(5000d)$. A supported supplier-size cap can be more informative than adding several small companies to an unranked list. The cap must apply to complete corporate groups; a limit on mine or subsidiary size is insufficient.

With omitted mass d and cap u, Proposition 2's transfer argument gives maximum squared contribution $P_d(u)$. An unbounded number of sufficiently small companies gives lower limit zero. With at most R omitted companies, padding with zeros and applying Cauchy–Schwarz gives $\sum r_j^2 \geq d^2/R$. Equal shares $d/R \leq u$ attain this lower endpoint. The packing maximum uses $\lceil d/u \rceil \leq R$ companies, so it remains feasible. This proves Proposition 6, including sharpness for the specified information. These are extensions of established arguments, not claimed new inequalities in their underlying mathematical form.

For the concentration ratio with an unknown omitted count, the associated sharp closed endpoints are

$$A_N \le C_N \le \max_{0\le j\le N}\{A_{N-j} + \min(d, ju)\}. \tag{26}$$

Any selection of N firms contains j omitted firms. They contribute at most $\min(d, ju)$, and the documented firms at most A_{N-j}. Conversely, assign this much omitted mass to at most j firms below the cap and divide any remaining mass among arbitrarily many small firms. The largest N shares have a sum at least that candidate; the preceding upper bound makes the maximal candidate sharp in closure. The lower limit A_N follows by diffusing the residual. If a finite omitted count is imposed, the HHI formula above remains sharp, but this concentration-ratio lower bound is not generally sharp. ■

A7 Joint reporting uncertainty and refinement

An operational rule must include all reporting allowances it purports to cover. Let verified leading shares satisfy $a_i \le p_i \le b_i$, and suppose an additional ratio satisfies $z_- \le C_{m+q} \le z_+$. A whole-percentage report supplies one such interval. Keep the dependence between premerger HHI and the merger increment instead of adding an independently optimized increment to an already optimized HHI.

Proposition 7. (A conservative joint reporting envelope). Define $c_- = \sum a_i$, $c_+ = \sum b_i$, $u = b_m$ and $\bar{z} = \min\{z_+, c_- + qu, 1\}$. Let $T_q(b, e, u)$ be the next-group and fringe maximum in Appendix A3, excluding the retained head. Define $G(p) = \sum p_i^2$ before a merger, and $G(p) = \sum p_i^2 + 2p_i p_j$ if retained companies i and j are hypothetically combined. For nonempty feasible evidence with $\bar{z} > c_-$, valid HHI bounds are

$$L_J = 10000\left\{G(a) + \frac{\max(0, z_- - c_+)^2}{q}\right\}, \tag{27}$$

$$U_J = 10000\{G(b) + T_q(\bar{z} - c_-, 1 - \bar{z}, u)\}. \tag{28}$$

The endpoints may be intersected with zero and 10,000 and any other valid interval for the same evidence. The already complete case is handled directly. Appendix A7 proves validity. Joint sharpness is not claimed: the construction deliberately maximizes some linked quantities separately. Its midpoint therefore has a valid maximum error over this outer interval, although a sharper calculation might improve it. We use this proposition for the combined reporting rules below.

Define $T_q(b, e, u)$ by the maximization of $B(x) + P_e(x)$ in Appendix A3, using next-group total b, fringe total e and next-group cap u. It excludes the retained head. This maximum is nondecreasing in b at fixed e, u when $b \le qu$: add mass to the largest nonsaturated next-group share, preserving

ranking. It is also nondecreasing along $(b, e) = (z - c_-, 1 - z)$ for $c_- < z \leq \min(1, c_- + qu)$, by transferring fringe output to larger next-group firms. Both transfers weakly increase the squared sum. The zero-next-group case is understood as a closure limit if a positive fringe remains.

For any feasible true head p and aggregate z, let $c = \sum p_i$. Its next-group total is $b = z - c \geq 0$, and its fringe total is $e = 1 - z$. Since $c \leq c_+$ and $z \geq z_-$, Cauchy–Schwarz gives a next-group contribution of at least $\max(0, z_- - c_+)^2/q$. The fringe is nonnegative and G is increasing in each coordinate, proving the lower bound.

For the upper bound, its retained contribution is at most $G(b_1, \ldots, b_m)$. Every next-group share is at most $u = b_m$. If $z \leq c_- + qu$, first enlarge the next-group total from $z - c$ to $z - c_-$ at fixed fringe. Then use the transfer monotonicity in z up to $\bar{z}$. This gives a contribution at most $T_q(\bar{z} - c_-, 1 - \bar{z}, u)$. If $z > c_- + qu$, bound the next-group contribution by qu^2 and the fringe by $P_{1-z}(u)$. Enlarging its mass to $1 - c_- - qu$ gives the same stated expression at the saturated next group $\bar{z} = c_- + qu$. This proves U_J. The merger version is valid because $G(p) = \sum p_i^2 + 2p_i p_j$ is increasing in every nonnegative coordinate; its increment and the original head are bounded jointly by the same two coordinate vectors. ■

All share boxes have nonnegative bounds and a nonempty admissible ranked information set. In a merger application the ratio is premerger, and the parties belong to the retained group. The construction bounds their squared contribution and increment through the same coordinate endpoints. Its validity does not imply joint sharpness.

Corollary 7.1. (Refinement without additional evidence). Suppose the admissible evidence set $\mathcal{S}$ is covered by finitely many sub-boxes $\mathcal{S}_j$. Let $[L_j, U_j]$ be valid outer bounds on each nonempty part, and $[L_0, U_0]$ any initial bound for the original evidence. Then another valid interval is

$$[\max\{L_0, \min_j L_j\},\ \min\{U_0, \max_j U_j\}]. \tag{29}$$

Proof. Every compatible distribution belongs to at least one covered part and hence lies in the union of its valid intervals, which is contained in $[\min_j L_j, \max_j U_j]$. Intersecting two valid outer intervals preserves validity and cannot widen the initial interval. Sub-boxes known to be empty may be discarded; retaining them is conservative. No claim that finite subdivision yields sharp bounds is needed. ■

The numerical refinement in Table 8 applies Proposition 7 to eight equal subdivisions of each of the four original leading-share intervals. The premerger C8 reporting bin remains unchanged. All 4,096 boxes are evaluated, and endpoints are combined using Corollary 7.1. In these examples the reported aggregate is strictly above each retained-share total; the complete-market and zero-mass limits are handled directly in general.

A8 Calculation procedure and verification

The following procedure makes the finite calculation reproducible without imposing an arbitrary maximum number of omitted firms.

1. Establish the company units and input domains. Require integer counts, ordered nonnegative leading shares, a common denominator and a nonempty information set. For interval evidence, require an ordered head and an aggregate compatible with $c \le z \le \min(1, c + qt)$. When positive smaller production remains, equality $z = c$ is a closure limit rather than a finite positive-share allocation.

2. For a packing calculation with mass v and cap $u > 0$, set $k = \lfloor v/u \rfloor$ and return $ku^2 + (v - ku)^2$. Return zero for zero mass. A zero cap with positive mass is used only as an explicitly stated limiting value.

3. For exact leading shares and aggregate, form $b = z - c$, $e = 1 - z$, lower boundary $\ell = \max(0, b - (q - 1)t)$ and upper boundary b/q. Start the candidate list with these endpoints and the next-group breakpoints $(b - at)/(q - a)$ for integer $a = 0, \ldots, q - 1$ that lie inside.

4. Sort and deduplicate the next-group breakpoints. On each consecutive interval $[v, w]$, add only the first and last feasible fringe breakpoints e/j, as specified in Appendix A3. If the left endpoint is zero, retain the zero limit and first finite breakpoint. Evaluate $B(x) + P_e(x)$ on this finite list and take its largest value.

5. For a whole-percentage response N, intersect $[(N - 0.5)/100, (N + 0.5)/100]$ with the feasible aggregate range. Use its lower endpoint for the lower HHI bound and its upper endpoint for the upper bound. With uncertain leading shares, use Proposition 7, retaining all covered combinations.

6. To compare requests before a response, enumerate the feasible response bins and take the largest resulting half-width. A request qualifies if that largest bound does not exceed the declared target.

Compare collection costs only among qualifying requests. For a level comparison, classify only when the entire valid interval is strictly on one side of the value.

7. If a conservative comparison is unresolved, subdivide the original reporting boxes and combine valid intervals using Corollary 7.1. Discard only boxes certified empty. Retain uncertain boxes conservatively. Keep rational arithmetic at packing points; round displayed lower endpoints down and upper endpoints and error guarantees up.

The implementation uses rational arithmetic for packing candidates, reporting bins and the subdivision examples. A separately constructed rational check compared reduced candidates with a more extensive enumeration in 300 cases and tested 1,000 generated finite markets against the joint reporting envelope, including retained-company mergers. Additional checks for the common-denominator result use generated finite markets and monotonicity across packing transitions. These checks support implementation consistency; they supplement the proofs and do not establish mathematical priority or validate historical corporate definitions.

Appendix B Historical inputs and numerical comparisons

B1 Reported production shares and included observations

Table 9 reports the historical company shares and separately printed source totals.

Table 9. Historical source shares cover 24 mineral industries.

Mineral	Year	p1 %	p2 %	p3 %	p4 %	C4 %	Source C4 minus share sum (pp)
Bauxite	2013	18.1	16.7	8.6	7.4	51	0.1
Chrome	2013	17.3	7.7	4.8	2.7	33	0.0
Cobalt	2013	20.8	9.4	5.0	2.9	38	0.1
Copper	2013	10.2	9.8	8.2	6.3	35	0.0
Diamonds	2012	26.9	22	10	1	60	0.0
Gold	2013	7.5	5.7	4.3	2.8	20	-0.1
Iodine	2013	31.5	11.9	7.9	5.1	56	0.0
Iron Ore	2013	22.8	16.8	10.7	5.1	55	0.1
Lead	2013	5.5	4.2	3.5	2.6	16	-0.1
Lithium	2013	36	26	15	10	87	0.0
Manganese	2013	20	8.7	8.1	5.6	42	0.0
Molybdenum	2013	17.4	8.5	7.4	5.7	39	0.0
Nickel	2013	12.8	12	10.2	6.9	42	0.0

Mineral	Year	p1 %	p2 %	p3 %	p4 %	C4 %	Source C4 minus share sum (pp)
Niobium	2012	80	6.7	6.4	5.7	99	0.1
Palladium	2013	36	18.6	11.5	3.0	69	0.0
Platinum	2013	40.4	27.6	12.1	11.3	91	0.0
Potash	2013	19	18	16	14	67	0.0
Rhodium	2013	41.6	24.9	11.5	10.2	88	-0.1
Silver	2013	5.7	4.5	4.3	4.1	19	-0.1
Tin	2013	20.6	9.6	7.2	7	44	-0.1
Titanium	2013	25.7	12	3.2	2.9	44	0.0
Uranium	2013	15.8	15.4	14.8	13.7	60	0.1
Zinc	2013	10.4	6.3	4.6	4.3	26	-0.1
Zirconium	2013	26	19	14	6	65	0.0

Note. Shares are percentages; C4 is displayed without decimals. Source: Tilton and Guzmán (2016), Table 7.1.

B2 Historical company identities and measurement conventions

Table 10 identifies the historical producer units, including the conditional chrome interpretation.

Table 10. Historical producer names identify the four leading companies.

Mineral	First	Second	Third	Fourth
Bauxite	Alcoa	Rio Tinto (Alcan)	US Rusal	BHP Billiton
Chrome	Kazchrome (ENRC; assumed interpretation)	Tata Iron & Steel	Glencore Xstrata	International Minerals Resources
Cobalt	Glencore Xstrata	Freeport-McMoRan	Gecamines	Vale
Copper	Freeport-McMoRan	Codelco	Glencore Xstrata	BHP Billiton
Diamonds	Alrosa	Anglo American (De Beers)	Rio Tinto	BHP Billiton
Gold	Barrick	Newmont	Anglo American	Goldcorp
Iodine	SQM	Cosayach	ACF Minera	Bullmine
Iron Ore	Vale	Rio Tinto	BHP Billiton	Fortescue Metals Group
Lead	Glencore Xstrata	BHP Billiton	Doe Run Company	Volcán Compañía Minera
Lithium	Talison	SQM	Rockwood Lithium	FMC Lithium

Mineral	First	Second	Third	Fourth
Manganese	BHP Billiton	Eramet	Consmin	Vale
Molybdenum	Freeport-McMoRan	Codelco	Grupo México	China Molybdenum
Nickel	Norilsk Nickel	Vale	Glencore Xstrata	BHP Billiton
Niobium	CBMM	Anglo American	Iamgold	Grupo Parana-Panema
Palladium	Norilsk Nickel	Anglo American	Impala Platinum	Stillwater Mining Company
Platinum	Anglo American	Impala Platinum	Lonmin	Norilsk Nickel
Potash	PotashCorp	Uralkali	Mosaic	Belaruskali
Rhodium	Anglo American	Impala Platinum	Lonmin	Norilsk Nickel
Silver	Peñoles	KGHM	Glencore Xstrata	BHP Billiton
Tin	Yunnan Tin Group	Glencore Xstrata	PT Timah	Minsur
Titanium	Rio Tinto	Tronox	Sierra Rutile	Iluka Resources
Uranium	KazAtomProm	Cameco	Areva	ARMZ–Uranium One
Zinc	Glencore Xstrata	Vedanta Resources	Teck Cominco	Minmetals
Zirconium	Iluka Resources	Rio Tinto	Tronox	Cristal

Note. Source: Tilton and Guzmán (2016), Table 7.1; chrome identity is conditional (Section 4.1), and Stillwater's name is standardized.

For the included source rows, the common specification is worldwide ownership-equivalent mineral production, mostly 2013. Diamonds and niobium are 2012. Box 7.1 defines equity-proportional allocation of jointly owned output. The source does not supply an operation-level reconciliation or the underlying world quantities in this table. Mineral-specific physical stages, ownership dates and parent consolidation are therefore inherited conventions rather than independently certified metadata. The calculations maintain the reported ranking conditionally. Source producer names alone do not certify those assumptions for an enforcement case.

B3 Copper company inputs and signed error intervals

Table 11 reports the copper quantities and common production denominators.

Table 11. Leading copper company quantities use equity allocation in 2023 and 2024.

Rank	2023 company	Output (kt)	2024 company	Output (kt)
1	BHP	1,506	BHP	1,497
2	Codelco	1,424	Codelco	1,421
3	Freeport-McMoRan	1,272	Freeport-McMoRan	1,253
4	Glencore	1,068	Glencore	1,009
5	Southern Copper Corp.	902	Southern Copper Corp.	986
6	Zijin	810	Zijin	829
7	First Quantum	648	Rio Tinto	650
8	Rio Tinto	594	Anglo American	540
9	Anglo American	571	KGHM	534
10	KGHM	498	Nornickel	466
Q	World total	22,600	World total	23,000

Note. Equity output is in thousand tonnes. Sources: National Mining Association (n.d.) and U.S. Geological Survey (2025a); see Section 4.3.

Table 12 and Figure 6 give possible signed errors relative to the richer ten-share interval.

Table 12. Possible signed errors use the richer ten-company copper interval as a reference.

Information retained	2023 signed error	2024 signed error
C4 alone	-164.0–-35.2	-158.3–-37.4
C8 alone	-551.6–-422.8	-528.6–-407.7
Four individual shares	-124.1–4.7	-113.5–7.4
Four shares + C8 joint bound	-90.2–38.6	-92.5–28.4
Eight individual shares	-71.8–57.0	-65.5–55.4
Ten individual shares	-64.4–64.4	-60.4–60.4

Note. Signed errors equal feasible HHI minus each estimate, using the ten-share interval as reference; endpoints are rounded outward.

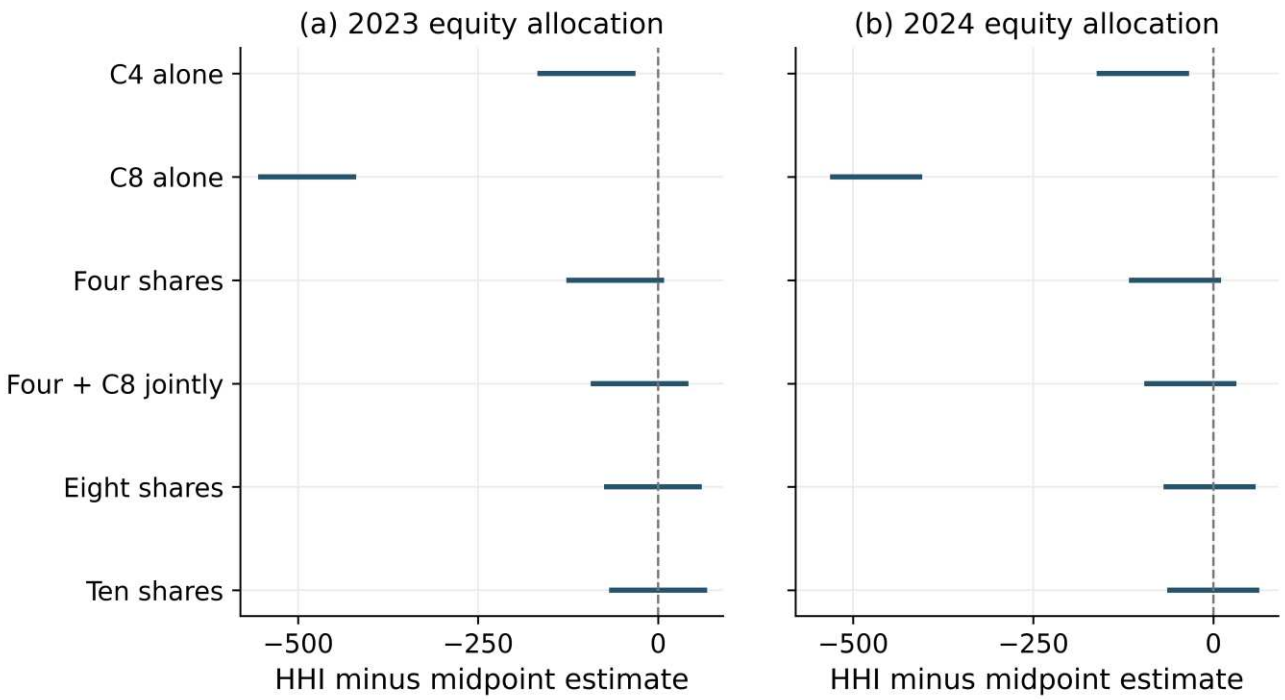


Figure 6. Possible copper estimation errors depend on the retained company information.

Note. Possible error ranges use each year's ten-share interval; Table 12 gives the endpoints. Source inputs are in Table 11.

B4 Printing sensitivity for the included company observations

Table 13 applies the stated printing allowance to each historical observation.

Table 13. Printing sensitivity widens the historical company HHI intervals.

Mineral	Outer HHI interval	Exact-input error bound	Printing outer error bound
Bauxite	730.1–1,095.5	175.8	182.7
Chrome	385.6–576.6	91.2	95.5
Cobalt	550.6–741.1	88.9	95.2
Copper	303.5–718.6	201.6	207.5
Diamonds	1,302.6–1,356.7	20.1	27.1
Gold	113.0–344.6	110.7	115.8
Iodine	1,216.6–1,446.9	108.0	115.2
Iron Ore	937.0–1,173.3	111.3	118.2
Lead	65.3–291.3	108.7	113.0

Mineral	Outer HHI interval	Exact-input error bound	Printing outer error bound
Lithium	2,288.3–2,416.7	54.5	64.2
Manganese	568.4–897.9	158.1	164.7
Molybdenum	458.3–810.5	170.5	176.1
Nickel	455.3–857.4	194.7	201.1
Niobium	6,508.4–6,530.2	0.8	10.9
Palladium	1,776.3–1,883.6	45.5	53.7
Platinum	2,658.8–2,754.7	37.0	47.9
Potash	1,130.3–1,564.6	208.5	217.2
Rhodium	2,578.0–2,703.9	53.3	62.9
Silver	86.1–424.7	165.9	169.3
Tin	612.9–1,011.4	193.3	199.2
Titanium	818.7–993.0	80.5	87.2
Uranium	887.5–1,446.7	270.9	279.6
Zinc	184.9–512.2	158.1	163.7
Zirconium	1,262.5–1,483.1	102.5	110.3

Note. Each leading share has an assumed ±0.05-percentage-point allowance; bounds are conservative (Corollary A1).

For the 24 industries the median error rises from approximately 109.63 to 115.43 points. Counts meeting 50-, 100- and 200-point tolerances change from 4, 9 and 21 to 3, 9 and 20. These are guarantees under the stated reporting model, not estimated frequencies of prediction success.

B5 Initial merger boundary intervals

Table 14 supplies the initial reporting envelopes behind the historical merger classifications.

Table 14. Initial reporting envelopes give the merger boundaries summarized in Table 7.

Mineral	Reported C8 %	Postmerger HHI interval	Comparison
Bauxite	62	1,358.7–1,493.2	Below
Bauxite	63	1,364.2–1,505.4	Unresolved
Bauxite	77	1,493.7–1,670.4	Unresolved

Mineral	Reported C8 %	Postmerger HHI interval	Comparison
Bauxite	78	1,506.7–1,689.3	Above
Titanium	46	1,432.3–1,489.0	Below
Titanium	47	1,433.3–1,503.0	Unresolved
Uranium	69	1,389.5–1,489.6	Below
Uranium	70	1,394.1–1,510.3	Unresolved
Uranium	83	1,498.7–1,683.4	Unresolved
Uranium	84	1,510.3–1,704.9	Above

Note. The first two firms merge; retained shares allow ±0.05 percentage point and C8 allows ±0.5 percentage point (Proposition 7).

B6 Complete copper response grid

Table 15 lists every feasible whole-percentage aggregate response in the copper exercise.

Table 15. Copper error bounds cover all feasible whole-percentage C_8 responses.

Reported C8 %	Error with exact retained shares	Error with joint reporting allowance
22	—	4.0
23	9.5	13.7
24	19.1	23.3
25	28.4	32.7
26	37.4	41.8
27	46.1	50.8
28	54.7	59.2
29	62.8	67.5
30	71.0	76.0
31	78.4	83.5
32	86.4	90.7
33	93.3	98.3
34	100.5	104.9
35	107.2	111.6
36	113.5	119.1
37	119.2	124.2
38	124.8	130.8
39	130.9	137.2
40	131.8	137.0

Note. Allowances are ±0.5 percentage point for C8 and ±0.05 per retained share; a dash denotes infeasibility with exact retained shares.

B7 HHI readings based on the historical source

Table 16 reports the chart-reading intervals used for the historical source-consistency comparison.

Table 16. Historical HHI readings provide a source-consistency comparison.

Mineral	C4 %	Chart-reading interval	Four-share HHI interval	Fitted line outside bound
Bauxite	51	770–810	735.2–1,086.9	No
Chrome	33	380–420	388.9–571.2	Yes
Cobalt	38	540–580	554.4–732.1	No
Copper	35	370–410	307.0–710.2	No
Gold	20	110–150	115.0–336.3	Yes
Iodine	56	1230–1270	1,222.2–1,438.2	Yes
Iron Ore	55	950–990	942.5–1,165.1	No
Lead	16	60–100	66.9–284.3	Yes
Lithium	87	2280–2320	2,297.0–2,406.0	Yes
Manganese	42	590–630	572.6–888.9	No
Molybdenum	39	510–550	462.2–803.2	No
Nickel	42	530–570	459.4–848.8	No
Palladium	69	1770–1810	1,783.2–1,874.1	Yes
Platinum	91	2650–2690	2,668.0–2,742.0	Yes
Potash	67	1310–1350	1,137.0–1,554.0	Yes
Rhodium	88	2570–2610	2,586.8–2,693.5	Yes
Silver	19	100–140	88.0–419.7	Yes
Tin	44	650–690	617.3–1,004.0	No
Titanium	44	810–850	823.1–984.2	Yes
Uranium	60	1060–1100	893.5–1,435.4	No
Zinc	26	210–250	187.5–503.6	Yes
Zirconium	65	1290–1330	1,269.0–1,474.0	Yes

Note. Chart readings allow ±20 HHI points around the nearest ten. Source: Tilton and Guzmán (2016), Figure 7.3, p. 169.

Appendix C Approximation and probabilistic interpretation

A mathematical explanation for local near-linearity is available under stable composition. Let the leading m shares be $c\alpha_i$, where $\sum\alpha_i = 1$, and the remaining shares be $(1-c)\beta_j$, where $\sum\beta_j = 1$. If these within-group profiles remain fixed and the ranking is preserved, define $A = \sum\alpha_i^2$ and $B = \sum\beta_j^2$. Then

$$HHI(c) = 10000\{Ac^2 + B(1-c)^2\}. \tag{30}$$

This is an identity under the stated fixed-composition assumptions. On $c \in [c_0 - w, c_0 + w]$, the deviation from its tangent at c_0 is exactly $10000(A+B)(c-c_0)^2$, and is therefore at most $10000(A+B)w^2$. The result follows by expanding the quadratic. It provides a transparent condition for a locally accurate straight-line approximation. It supplies no evidence that A and B are stable across minerals or through time. The sharp information bounds remain the appropriate guarantee when composition is unknown.

The deterministic interval also limits moments under any separately specified probability distribution supported on $[L, U]$. With midpoint $M = (L+U)/2$ and $E = (U-L)/2$, one has $|\mathbb{E}[HHI] - M| \leq E$ and $\mathrm{Var}(HHI) \leq \mathbb{E}[(HHI - M)^2] \leq E^2$. These follow directly from $|HHI - M| \leq E$. The data provide no probability distribution within the interval, so the midpoint is not an identified expected HHI and neither a sampling variance nor a confidence level is estimated.

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Table 3, U.S. producers of soda ash in 2024 [Data tables released August 6, 2025].

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