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Jingyu Wang

wangjingyu98860163.com

Beijing Institute of Electronic System Engineering

Yiqun Zhang

Beijing Institute of Electronic System Engineering

Peifeng Chen

Beijing Institute of Electronic System Engineering

Junmeng Guan

Beijing Institute of Electronic System Engineering

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Reachable-Set-Based Strategies for Attacker-Defender-Target Games with a Non-Dominant Attacker

Jingyu Wang[1*], Yiqun Zhang[1†], Peifeng Chen[1†], Junmeng Guan[1†]

[1*]Beijing Institute of Electronic System Engineering, Beijing, 100854, China.

*Corresponding author(s). E-mail(s): wangjingyu9886@163.com;
Contributing authors: yiqunzhang19@163.com; cpf62023@126.com;
guanjunmeng1996@163.com;
†These authors contributed equally to this work.

Abstract

The Attacker-Defender-Target (ADT) differential game provides an important framework for modeling modern aerial combat and missile defense. This paper aims to address the limitations in existing research, which typically assumes scenarios where the attacker possesses overwhelming superiority, fails to account for coordination between the defender and the target, and often neglects singular surfaces in the state space, leading to incomplete strategy solutions. To overcome these issues, the following contributions are made. First, this paper considers a relatively balanced scenario, characterized by the fact that the attacker's reachable set does not completely contain the target's reachable set. Second, based on a reachable-set analysis, the payoff function is reformulated as the distance between the boundaries of the players' reachable sets, enabling the derivation of strategies that achieve coordination between the defender and the target. Third, the singular surfaces arising in the game are thoroughly analyzed, allowing for the derivation of complete strategies. Finally, simulations under both linearized and nonlinear planar engagement kinematics are conducted. The nonlinear results show close agreement with the analytically predicted switching times and attacker-target miss distance, demonstrating the applicability of the proposed framework beyond the linearized terminal-projection model.

Keywords: Differential games, Attacker-Defender-Target games, Reachable sets, Singular surfaces, Cooperative strategies

1 Introduction

In aerial combat scenarios, preventing fighter jets or other high-value targets from being destroyed by incoming missiles has always been a crucial challenge. One solution to this problem is for the target to launch a defensive missile to intercept the incoming threat. Boyell (1976) first proposed the attacker-defense-target (ADT) system in 1976. The most intuitive approach for the defensive missile is to place itself on the line connecting the target and the incoming attacker missile, thereby preventing the attacker from hitting the target. References Yamasaki and Balakrishnan (2010b), Yamasaki and Balakrishnan (2010a), Yamasaki et al. (2013) have extensively investigated this guidance approach. This approach typically assumes that the attacker's strategy and the target's evasion strategy are known to the defender. It then focuses on analyzing the optimal maneuvering strategies for the defender. Another approach is based on optimal control theory, which assumes only the attacker's maneuvering strategy is known and analyzes how the target and defender should cooperate to reduce the required acceleration of the defensive missile. Shima and other researchers at the Technion-Israel Institute of Technology have made significant contributions in this field Shima (2011), Prokopov and Shima (2013), Weiss and Shima (2016), Sinha et al. (2022).

In 1965, Yu-Chi Ho et al. Ho et al. (1965) proposed the Linear Quadratic Differential Game (LQDG) method. Based on this framework, Rusnak et al Rusnak (2004) investigated the three-player game problem. LQDG-based approaches provide analytical cooperative strategies but generally rely on quadratic payoffs and may not explicitly enforce the attacker-defender capture constraint Perelman et al. (2011)Saurav et al. (2019). To address this issue, Rubinsky et al. proposed a Boundary Differential Game (BDG) guidance law Rubinsky and Gutman (2014). BDG-based approaches address this issue through terminal projection and explicit defender-avoidance constraints, allowing the attacker to first bypass the defender and subsequently pursue the target Rubinsky and Gutman (2015),Sun et al. (2018),Sun et al. (2019),Wei et al. (2025). Although this approach is easy to analyze, its essence lies in transforming the three-player game into two pursuit-evasion games. This simplification prevents coordination between the defender and the target.

Recent studies have further considered hybrid differential-game guidance, multi-players switching strategies, time delays, incomplete information, and learning-based approaches Liang et al. (2020),Tao et al. (2023),Gong et al. (2023), Gong et al. (2024),Manoharan and Baliyarasimhuni (2023),Pansera et al. (2026),Mozhegova and Petrov (2026),Zatezalo and Stipanović (2026).

However, several issues remain insufficiently addressed in existing BDG-based studies. First, most analyses consider cases in which the attacker possesses sufficient reachable-set superiority, in which case the attacker's strategy becomes relatively simple (this will be elaborated in Section 2. Second, the conventional terminal miss-distance payoff does not explicitly incorporate the defender's state, which makes it difficult to derive coordinated equilibrium strategies for the defender and the target within a unified framework. Third, the singular structures separating different strategy regions have not been systematically characterized, which may lead to incomplete descriptions of the equilibrium strategy.

This paper aims to fill the three research gaps in BDG guidance methods identified above. First, this paper investigates ADT differential games involving a non-dominant attacker, for which the optimal strategies are more intricate. Secondly, the optimal strategies for players are proposed, which enables the derivation of cooperation strategies for defender and target. Finally, a comprehensive analysis of singular curves is conducted, leading to a complete and rigorous solution for the ADT game.

The remainder of this paper is organized as follows. Section 2 introduces the linearized motion models and the problem description is provided. Section 3 analyzes boundaries of the reachable sets for the players. Section 4 constructs the payoff function, derives the equilibrium strategies, and analyzes the singular curves. Finally, the Simulation section validates the proposed strategies under both linear and nonlinear engagement kinematics in section 5.

2 Problem statement

The specific game scenario is illustrated in Fig. 1, where V_A, a_A, V_D, a_D, V_T and a_T represent the velocities and accelerations of the attacker, defender, and target, respectively. The distance between the attacker and defender and between the attacker and target are denoted by r_{AD} and r_{AT}, respectively. Here, u_i denotes the projection of the acceleration vector $\mathbf{a}_i$ onto the direction perpendicular to the line of sight, for $i \in \{A, D, T\}$, which is $u_i = a_i \cos\theta_i$.

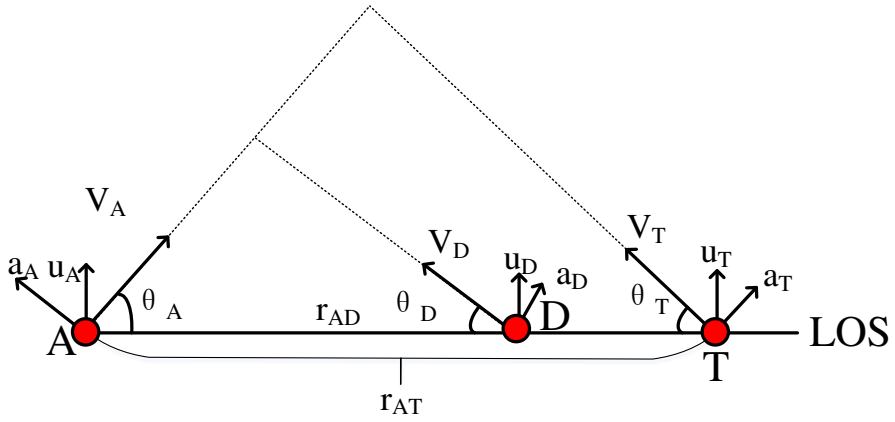


Fig. 1 The collision triangle between the attacker, the defender, and the target.

If the game takes place in the vicinity of the collision triangle, the following assumptions can be made:

Assumption 1. At the initial moment, the velocity components of the players perpendicular to the line of sight are approximately equal.

Assumption 2. The relative velocities between the attacker and the defender, as well as between the attacker and the target, are high enough that the maneuvers along

the line-of-sight direction can be neglected, so that the control acceleration is applied only in the direction perpendicular to the line of sight, i.e., $\dot{r}_{Ai} = const, i = D, T$.

Based on assumption 2, the interception time between the attacker and the defender t_1, and the interception time between the attacker and the target t_2 are also constants. Given initial distances r_{AD}^0 and r_{AT}^0, t_1 and t_2 can be written as

$$t_1 = \frac{r_{AD}^0}{\dot{r}_{AD}},\ t_2 = \frac{r_{AT}^0}{\dot{r}_{AT}} \tag{1}$$

Define the relative position perpendicular to the line of sight between two players as the miss distance at time t, denoted as $x_A(t), x_D(t), x_T(t)$, and for brevity, it will be abbreviated as x_A, x_D, x_T in the following text. The state variables are defined as$X = [x_A, x_D, x_T, \dot{x}_A, \dot{x}_D, \dot{x}_T]^T$, and the differential equation for the state variables can be derived

$$\dot{X} = AX + B\left[u_A, u_D, u_T\right]^T \tag{2}$$

where

$$A = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, B = \begin{bmatrix} 0_{3\times3} \\ I_{3\times3} \end{bmatrix} \tag{3}$$

Denote the maximum lateral acceleration being u_i^m. The admissible control set for player i is defined as $u_i \in \mathbf{U_i} = \{u_i | u_i \in [-u_i^m, u_i^m]\},\quad i = A, D, T$.

The geometric relationship among players during the game can be illustrated as shown in Fig. 2. $V_{A,r}$,$V_{D,r}$ and $V_{T,r}$ denotes the projection of the players' velocities onto the line-of-sight (LOS) direction, as illustrated in Fig. 1. Based on assumption 2, $V_{A,r}$,$V_{D,r}$ and $V_{T,r}$ remain unchanged during the game.

The set of all points a player can reach at t_1, t_2 is referred to as its reachable set at that time. In Fig. 2, the line segment D_1D_2 represents the defender's reachable set at time t_1. Similarly, the line segment T_1T_2 represents the target's reachable set at time t_2. The attacker's reachable set at t_1 is the line segment A_1A_2. However, its reachable set at t_2 comprises two distinct segments, A_3A_4 and A_5A_6. These two parts correspond to the two situations in which the attacker bypasses the defender: from the left side or from the right side.

Based on the players' current coordinates and lateral velocities, their zero effort coordinates at times t_1 and t_2 can be determined as

$$\begin{aligned} Z_A\left(t, t_1\right) &= x_A(t) + \dot{x}_A(t)\left(t_1 - t\right), \\ Z_A\left(t, t_2\right) &= x_A(t) + \dot{x}_A(t)\left(t_2 - t\right), \\ Z_D\left(t, t_1\right) &= x_D(t) + \dot{x}_D(t)\left(t_1 - t\right), \\ Z_T\left(t, t_2\right) &= x_T(t) + \dot{x}_T(t)\left(t_2 - t\right) \end{aligned} \tag{4}$$

The coordinates of the reachable set boundary are denoted as $\bar{z}_{D_1}, \bar{z}_{D_2}, \bar{z}_{T_1}, \bar{z}_{T_2}$ and $\bar{z}_{A_i}, i = 1 \sim 6$. These coordinates incorporate the players' maneuver strategies and are determined by their zero-effort coordinates and maximum lateral acceleration. Then the boundaries of the reachable sets for the defender and the target at time t_1

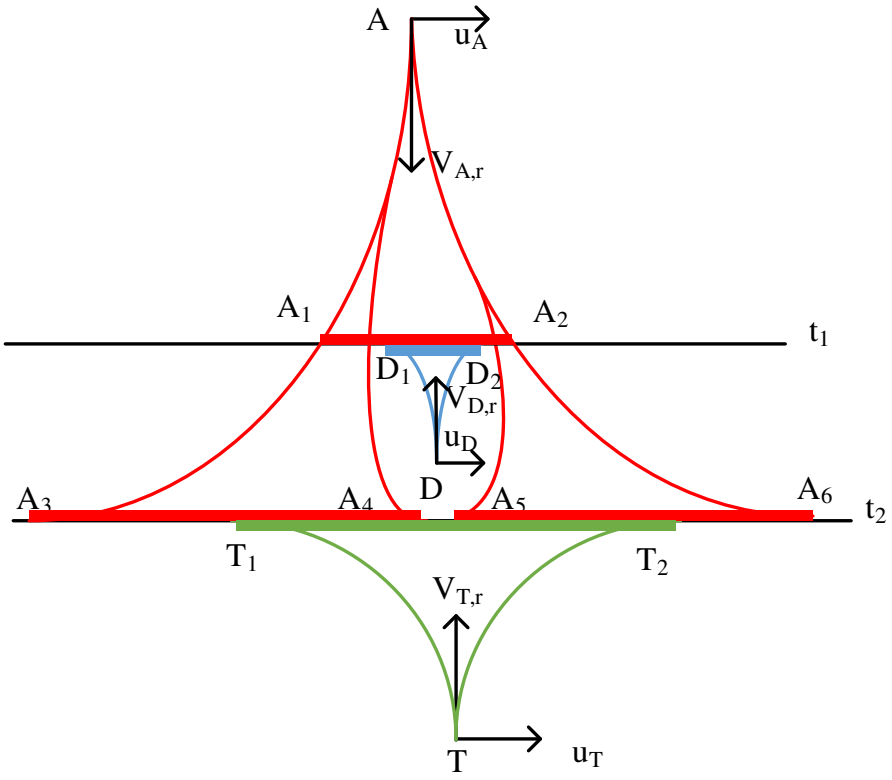


Fig. 2 The reachable set for players.

and t_2 are given by:

$$\begin{aligned}
\bar{z}_{D_1}(t,t_1) &= Z_D(t,t_1) - \frac{1}{2}u_D^m(t_1-t)^2, \\
\bar{z}_{D_2}(t,t_1) &= Z_D(t,t_1) + \frac{1}{2}u_D^m(t_1-t)^2, \\
\bar{z}_{T_1}(t,t_2) &= Z_T(t,t_2) - \frac{1}{2}u_T^m(t_2-t)^2, \\
\bar{z}_{T_2}(t,t_2) &= Z_T(t,t_2) + \frac{1}{2}u_T^m(t_2-t)^2
\end{aligned} \tag{5}$$

This paper investigates scenarios in which the attacker possesses superior maneuverability compared to both the defender and the target. Therefore,

$$u_A^m > u_D^m, u_A^m > u_T^m \tag{6}$$

holds. It is evident that the larger the u_A^m is, the larger the reachable set of attacker becomes. When u_A^m is sufficiently large, either situation $[T_1,T_2] \in [A_3,A_4]$ or $[T_1,T_2] \in [A_5,A_6]$ will occur. In such cases, regardless of the target's maneuvers, it cannot escape

the attacker's reachable set and will ultimately be captured by the attacker. These cases are relatively straightforward and have been thoroughly studied in Rubinsky and Gutman (2014), Sun et al. (2018) and Sun et al. (2019). Therefore, this paper is primarily based on the following assumption:

$$\begin{cases} [T_1, T_2] \nsubseteq [A_3, A_4] \\ [T_1, T_2] \nsubseteq [A_5, A_6] \end{cases} \tag{7}$$

Problem statement: The payoff function is defined as:

$$J = \|x_A(t_2) - x_T(t_2)\| \tag{8}$$

The attacker aims to minimize the terminal miss distance, while the defender and the target aim to maximize it. The equilibrium strategies can be written as

$$[u_A^*, u_D^*, u_T^*] = \underset{u_A \in \mathbf{U_A}}{argmin} \max_{u_D \in \mathbf{U_D}, u_T \in \mathbf{U_T}} J(u_A, u_D, u_T) \tag{9}$$

This inequality defines the saddle-point equilibrium for the three-player zero-sum differential game considered here. The equilibrium value J^* under u_A^*, u_D^*, u_T^* satisfy:

$$J(u_A, u_D^*, u_T^*) \geq J^*\left(u_A^*, u_D^*, u_T^*\right) \geq J(u_A^*, u_D, u_T) \tag{10}$$

Under Assumption (7), the target can always select a terminal position outside the attacker's reachable set. Therefore, a nonzero terminal miss distance can always be guaranteed.

3 Players' reachable sets

This section computes the boundary of the reachable sets of the players, serving as a foundation for solving the payoff function in the subsequent analysis.

Among them, the computation of $\bar{z}_{A_3}$ and $\bar{z}_{A_6}$ is similar to that in Eq. (5) and is straightforward; hence, it will not be elaborated here. However, $\bar{z}_{A_4}(t, t_2)$ and $\bar{z}_{A_5}(t, t_2)$ are relatively special, as the attacker must first maneuver to one side to bypass the defender's interception before heading toward the boundary. The coordinates of $\bar{z}_{A_4}(t, t_2)$ and $\bar{z}_{A_5}(t, t_2)$ will be solved in the following.

Before solving for the attacker's reachable set at time t_2, it is necessary to determine whether the attacker can bypass the defender's interception. To bypass the defender, the attacker must satisfy one of the following two inequalities:

$$x_A(t_1) - x_D(t_1) \leq -r, \;\; x_A(t_1) - x_D(t_1) \geq r \tag{11}$$

where r is the defender's capture radius.

Lemma 1. Define

$$\Delta^{\pm}(t) = \pm\left[Z_A\left(t,t_1\right) - Z_D\left(t,t_1\right)\right] + \frac{1}{2}\left(u_A^m - u_D^m\right)\left(t_1 - t\right)^2 - r \tag{12}$$

Assume that the defender always maneuvers with its maximum acceleration and that its acceleration is directed in the same direction as the attacker's. The attacker can bypass the defender from the left at t_1 if and only if $\Delta^-(t) \geq 0$, and from the right if and only if $\Delta^+(t) \geq 0$.

Proof First prove that the attacker can bypass the defender on the left if and only if $\Delta^-(t) \geq 0$. When the defender executes a maximum acceleration maneuver to the left, it will reach D_1 at t_1. Therefore, for the attacker to bypass the defender from the left without being captured, the condition $\bar{z}_{D_1}\left(t,t_1\right) - \bar{z}_{A_1}\left(t,t_1\right) - r \geq 0$ must hold, which is

$$\bar{z}_{D_1}\left(t,t_1\right) - \bar{z}_{A_1}\left(t,t_1\right) - r = \left[Z_D\left(t,t_1\right) - Z_A\left(t,t_1\right)\right] + \frac{1}{2}\left(u_A^m - u_D^m\right)\left(t_1 - t\right)^2 - r = \Delta^-(t) \geq 0 \tag{13}$$

Similarly, the attacker can bypass the defender from the right if and only if $\Delta^+(t) \geq 0$ holds. Then the lemma is proved. □

If both $\Delta^+(t) < 0, \Delta^-(t) < 0$, it means that the attacker cannot bypass the defender's interception.

Based on Lemma 1, when $\Delta^+(t) \geq 0$, the attacker can choose to bypass the defender from the right side and reach A_5. To reach A_5, the attacker first performs a maximum acceleration maneuver to the right until time t_+, and then switches to a maximum acceleration maneuver in the opposite direction. There exists an optimal $t_+ \in [t,t_1]$ that minimizes $\bar{z}_{A_5}$. The attacker's strategy can be expressed as:

$$u_A(\tau) = \begin{cases} u_A^m, \tau \in [t,t_+) \\ -u_A^m, \tau \in [t_+,t_2] \end{cases} \tag{14}$$

Lemma 2. For any $t \in [0,t_1]$ and $\Delta^+(t) \geq 0$, The minimum value of $\bar{z}_{A_5}(t)$ that the attacker can achieve without being intercepted by the defender is:

$$\bar{z}_{A_5}\left(t,t_2\right) = Z_A\left(t,t_2\right) + \frac{1}{2}u_A^m\left(t_2 - t\right)^2 - u_A^m\left(t_2 - t_+\right)^2 \tag{15}$$

where

$$t_+ = t_1 - \sqrt{\frac{\Delta^+(t)}{u_A^m}} \tag{16}$$

Proof Since the attacker must be positioned to the right of the defender and remain uncaptured at time t_1, the attacker must satisfy

$$x_A\left(t_1\right) - \bar{z}_{D_2}\left(t,t_1\right) \geq r \tag{17}$$

According to strategy (14), Eq. (17) can be rewritten as:

$$Z_A\left(t,t_1\right) + \frac{1}{2}u_A^m\left(t_1 - t\right)^2 - u_A^m\left(t_1 - t_+\right)^2 \geq \bar{z}_{D_2}\left(t,t_1\right) + r \tag{18}$$

Substituting Eq. (12) into Eq. (18) and simplifying yields:

$$\Delta^{+}(t) - u_A^m \left(t_1 - t_+\right)^2 \geq 0 \tag{19}$$

Since $\Delta^{+}(t) \geq 0$ and $t_+ \in [t, t_1]$, there must exist a solution to Eq. (19), which is

$$t_+ \geq t_1 - \sqrt{\frac{\Delta^{+}(t)}{u_A^m}} \tag{20}$$

At time t, the attacker's coordinates are $x_A(t)$. Following strategy (14), it reaches A_5 at time t_2, therefore, $\bar{z}_{A_5}$ can be expressed as

$$\begin{aligned} \bar{z}_{A_5}(t, t_2) &= x_A(t) + \dot{x}_A(t)(t_2 - t) + \int_t^{t_2} u_A(\tau)(t_2 - \tau)d\tau \\ &= Z_A(t, t_2) + \frac{1}{2}u_A^m \left(t_2 - t\right)^2 - u_A^m \left(t_2 - t_+\right)^2 \end{aligned} \tag{21}$$

From Eq. (21), it can be seen that when $t_+ \in [t, t_1]$, the larger the t_+ is, the larger $\bar{z}_{A_5}(t, t_2)$ becomes. Therefore, when the inequality in Eq. (20) takes equality, $\bar{z}_{A_5}(t, t_2)$ attains its minimum value, leading to the Eq. (16). Then the lemma is proved. □

Similarly, when $\Delta^{-}(t) \geq 0$, the boundary coordinates of the reachable set. can be expressed as:

$$\bar{z}_{A_4}(t, t_2) = Z_A(t, t_2) - \frac{1}{2}u_A^m \left(t_2 - t\right)^2 + u_A^m \left(t_2 - t_-\right)^2 \tag{22}$$

where

$$t_- = t_1 - \sqrt{\frac{\Delta^{-}(t)}{u_A^m}} \tag{23}$$

The attacker's strategy for reaching A_4 can be expressed as:

$$u_A(\tau) = \begin{cases} -u_A^m, \tau \in [t, t_-) \\ u_A^m, \tau \in [t_-, t_2] \end{cases} \tag{24}$$

4 The optimal strategies and the singular curves

First, the changes in the boundary of reachable sets of the players under different strategies are examined to facilitate subsequent analysis.

Differentiating both sides of Eq. (4) yields:

$$\begin{matrix} \dot{Z}_A(t, t_2) = u_A(t)\left(t_2 - t\right), & \dot{Z}_A(t, t_1) = u_A(t)\left(t_1 - t\right) \\ \dot{Z}_D(t, t_1) = u_D(t)\left(t_1 - t\right), & \dot{Z}_T(t, t_2) = u_T(t)\left(t_2 - t\right) \end{matrix}, t \in [0, t_1] \tag{25}$$

Differentiating Eq. (5) and then substituting the Eq. (25) into the result yields:

$$\begin{matrix} \dot{\bar{z}}_{T_1}(t, t_2) = \left(u_T(t) + u_T^m\right)\left(t_2 - t\right), & \dot{\bar{z}}_{T_2}(t, t_2) = \left(u_T(t) - u_T^m\right)\left(t_2 - t\right) \\ \dot{\bar{z}}_{D_1}(t, t_1) = \left(u_D(t) + u_D^m\right)\left(t_1 - t\right), & \dot{\bar{z}}_{D_2}(t, t_1) = \left(u_D(t) - u_D^m\right)\left(t_1 - t\right) \end{matrix}, t \in [0, t_1] \tag{26}$$

Similarly, it follows that

$$\begin{aligned}\dot{\bar{z}}_{A_3}(t,t_2) &= \left(u_A(t)+u_A^m\right)(t_2-t)\\ \dot{\bar{z}}_{A_6}(t,t_2) &= \left(u_A(t)-u_A^m\right)(t_2-t)\end{aligned}, t\in[0,t_1] \tag{27}$$

Substituting Eq. (16) in Eq. (21), and differencing the result yields:

$$\dot{\bar{z}}_{A_5}(t,t_2) = \dot{\bar{z}}_{A_6}(t,t_2) - \frac{\sqrt{u_A^m}\,(t_2-t_+)}{\sqrt{\Delta^+(t)}}\dot{\Delta}^+(t), t\in[0,t_+] \tag{28}$$

where

$$\dot{\Delta}^+(t) = \left(u_A(t)-u_A^m-u_D(t)+u_D^m\right)(t_1-t), t\in[0,t_+] \tag{29}$$

Similarly

$$\dot{\bar{z}}_{A_4}(t,t_2) = \dot{\bar{z}}_{A_3}(t,t_2) + \frac{\sqrt{u_A^m}\,(t_2-t_-)}{\sqrt{\Delta^-(t)}}\dot{\Delta}^-(t), t\in[0,t_-] \tag{30}$$

where

$$\dot{\Delta}^-(t) = \left(-u_A(t)-u_A^m+u_D(t)+u_D^m\right)(t_1-t), t\in[0,t_-] \tag{31}$$

According to Eq. (28) and (29), when $u_A(t)=u_A^m$ and $u_D(t)=u_D^m$, $\dot{\bar{Z}}_{A_5}(t)=0$ holds. For any initial time t_0, define:

$$\begin{aligned}J_{T_2A_6}(t_0) &= \bar{z}_{T_2}(t_0,t_2)-\bar{z}_{A_6}(t_0,t_2)\\ J_{A_5T_1}(t_0) &= \bar{z}_{A_5}(t_0,t_2)-\bar{z}_{T_1}(t_0,t_2)\\ J_{T_2A_4}(t_0) &= \bar{z}_{T_2}(t_0,t_2)-\bar{z}_{A_4}(t_0,t_2)\\ J_{A_3T_1}(t_0) &= \bar{z}_{A_3}(t_0,t_2)-\bar{z}_{T_1}(t_0,t_2)\end{aligned} \tag{32}$$

and

$$J_m(t_0) = \max\left\{J_{T_2A_6}(t_0), J_{A_3T_1}(t_0), \min\left\{J_{A_5T_1}(t_0), J_{T_2A_4}(t_0)\right\}\right\} \tag{33}$$

Based on assumption (7), at least one of the pairs $J_{T_2A_6}$ and $J_{A_5T_1}$, as well as $J_{T_2A_4}$ and $J_{A_3T_1}$ has a positive value, then $J_m(t)$ is always positive. In fact, under the open-loop strategy, $J_m$corresponds to the value of the game that shown in Eq. (8). This will be proved in Lemma 4 in the subsequent analysis.

Lemma 3. Depending on the position of $Z_T(t_0,t_2)$ relative to the attacker's reachable set, the value of $J_m(t_0)$ is given by:

$$J_m(t_0)=\begin{cases} J_{T_2A_6}(t_0), if\ Z_T(t_0,t_2)\in\left[\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2},+\infty\right) \\ J_{A_5T_1}(t_0), if\ Z_T(t_0,t_2)\in\left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2},\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2}\right) \\ J_{T_2A_4}(t_0), if\ Z_T(t_0,t_2)\in\left(\frac{\bar{z}_{A_3}+\bar{z}_{A_4}}{2},\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}\right) \\ J_{A_3T_1}(t_0), if\ Z_T(t_0,t_2)\in\left(-\infty,\frac{\bar{z}_{A_3}+\bar{z}_{A_4}}{2}\right] \end{cases} \tag{34}$$

For notational simplicity, $\bar{z}_{A_i}(t_0,t_2), i=3\sim 6$ is abbreviated as $\bar{z}_{A_i}$ here.

Proof When $Z_T(t_0,t_2)\in\left[\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2},+\infty\right)$, it can be seen in Fig. 3, Substituting $\bar{z}_{T_1},\bar{z}_{T_2}$ from Eq. (5) into $J_{T_2A_6}(t_0), J_{A_5T_1}(t_0)$ in Eq. (32) yields:

$$\begin{aligned} J_{T_2A_6}(t_0)&=Z_T(t_0,t_2)+\frac{1}{2}u_T^m(t_2-t_0)^2-\bar{z}_{A_6}(t_0,t_2) \\ J_{A_5T_1}(t_0)&=\bar{z}_{A_5}(t_0,t_2)-Z_T(t_0,t_2)+\frac{1}{2}u_T^m(t_2-t_0)^2 \end{aligned} \tag{35}$$

Substituting $Z_T(t_0,t_2)\geq\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2}$ into Eq. (35) yields:

$$\begin{aligned} J_{T_2A_6}(t_0)&\geq\frac{\bar{z}_{A_5}-\bar{z}_{A_6}}{2}+\frac{1}{2}u_T^m(t_2-t_0)^2 \\ J_{A_5T_1}(t_0)&\leq\frac{\bar{z}_{A_5}-\bar{z}_{A_6}}{2}+\frac{1}{2}u_T^m(t_2-t_0)^2 \end{aligned} \tag{36}$$

Therefore, $J_{A_5T_1}(t_0)\leq J_{T_2A_6}(t_0)$. As previously mentioned, at least one of $J_{A_5T_1}(t_0)$ and $J_{T_2A_6}(t_0)$ must be greater than zero. Therefore, it follows that $J_{T_2A_6}(t_0)>0$. Since the target has inferior maneuverability compared to the attacker, when $J_{T_2A_6}(t_0)>0$, it necessarily follows that $J_{A_3T_1}(t)<0$, then $\min\{J_{A_3T_1}(t_0),J_{T_2A_4}(t_0)\}<0$ can be deduced. Therefore, it follows that $J_m(t_0)=J_{T_2A_6}(t_0)$. The other three cases are similar and are not elaborated here. The lemma is proved. □

Lemma 4. For any t_0 satisfying $\Delta^{\pm}(t_0)>0$, the equilibrium open-loop strategies of the players is

$$u_A(t)\begin{cases} u_A^m, t\in(t_0,t_2), if\ Z_T(t_0,t_2)\in\left[\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2},+\infty\right) \\ \begin{cases} u_A^m, t\in(t_0,t_+) \\ -u_A^m, t\in[t_+,t_2] \end{cases}, if\ Z_T(t_0,t_2)\in\left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2},\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2}\right) \\ \begin{cases} -u_A^m, t\in(t_0,t_-) \\ u_A^m, t\in[t_-,t_2] \end{cases}, if\ Z_T(t_0,t_2)\in\left(\frac{\bar{z}_{A_3}+\bar{z}_{A_4}}{2},\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}\right) \\ -u_A^m, t\in(t_0,t_2), if\ Z_T(t_0,t_2)\in\left(-\infty,\frac{\bar{z}_{A_3}+\bar{z}_{A_4}}{2}\right] \end{cases} \tag{37}$$

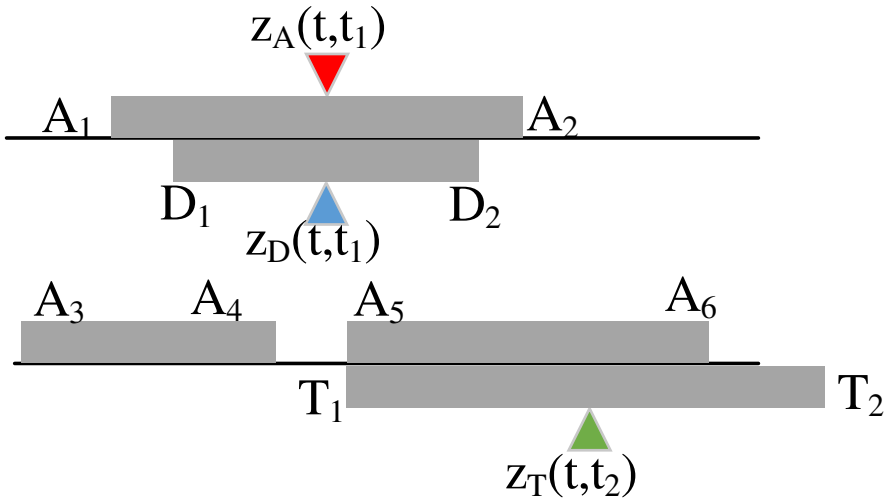


Fig. 3 The reachable set for $Z_T(t_0,t_2) \in \left[\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2},+\infty\right)$.

$$u_D(t)\begin{cases} u_D^m, t\in(t_0,t_1), if\ Z_T(t_0,t_2)\in\left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2},+\infty\right) \\ -u_D^m, t\in(t_0,t_1), if\ Z_T(t_0,t_2)\in\left(-\infty,\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}\right) \end{cases} \tag{38}$$

$$u_T(t)\begin{cases} u_T^m, t\in(t_0,t_2), if\ Z_T(t_0,t_2)\in\left[\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2},+\infty\right) \\ -u_T^m, t\in(t_0,t_2), if\ Z_T(t_0,t_2)\in\left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2},\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2}\right) \\ u_T^m, t\in(t_0,t_2), if\ Z_T(t_0,t_2)\in\left(\frac{\bar{z}_{A_3}+\bar{z}_{A_4}}{2},\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}\right) \\ -u_T^m, t\in(t_0,t_2), if\ Z_T(t_0,t_2)\in\left(-\infty,\frac{\bar{z}_{A_3}+\bar{z}_{A_4}}{2}\right) \end{cases} \tag{39}$$

And the value of the game is $J_m(t_0)$, that is $J^* = J_m(t_0)$. For notational simplicity, $\bar{z}_{A_i}(t_0,t_2), i=3\sim 6$ is abbreviated as $\bar{z}_{A_i}$ here.

Proof When $Z_T(t_0,t_2) \in \left[\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2},+\infty\right)$, which can be seen in Fig. 3. According to Lemma 3, the corresponding value is $J_m(t_0) = J_{T_2A_6}(t_0)$. Substituting the proposed controls into Eq. (27) and (26) gives $\dot{J}_{T2A6}(t) = 0$, any admissible unilateral deviation of the attacker increases the corresponding terminal separation, whereas any unilateral deviation of the maximizing team decreases it. Hence, the saddle-point condition in Eq. (10) is satisfied.

If $Z_T(t_0,t_2) \in \left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2},\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2}\right)$, which can be seen in Fig. 4. According to Lemma 3, the corresponding value is $J_m(t_0) = J_{A5T1}(t_0)$. Based on Eq. (26) and (28), the defender and target controls given above maximize this separation, while the attacker's strategy minimizes it. Therefore, the saddle-point condition is satisfied.

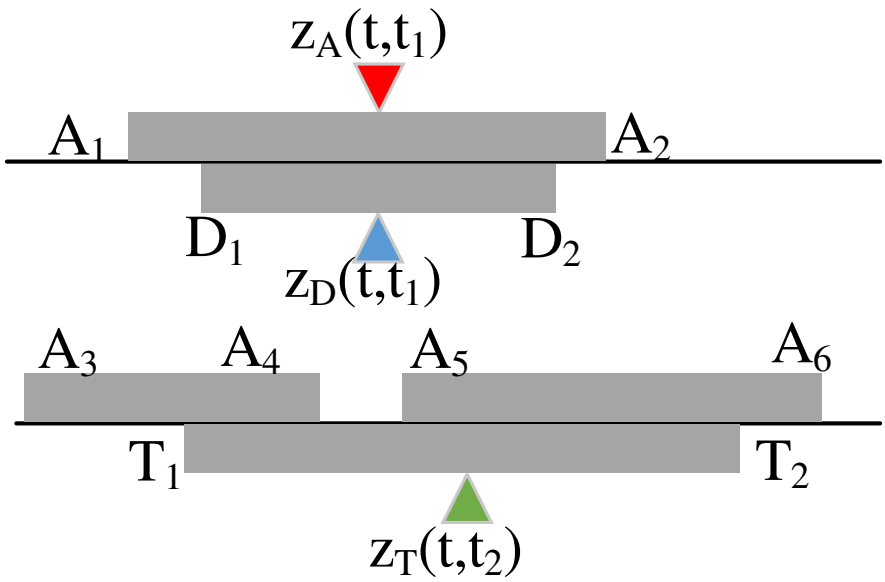


Fig. 4 The reachable set for $Z_T(t_0,t_2) \in \left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}, \frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2}\right)$.

The remaining cases are similar to the two scenarios described above and are not elaborated here. The theorem is thus proved. □

The following lemma states that the state converge to the interface between the two adjacent strategy regions.

Lemma 5. Assuming at the initial time, $\Delta^-(0) > 0, \Delta^+(0) > 0$, and $Z_T(0,t_2) \in \left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2} - \varepsilon_-, \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2} + \varepsilon_+\right]$, where

$$\varepsilon_\pm = \pm\frac{(u_T^m + u_A^m)}{2}\left(2t_{\Delta^\mp}t_2 - {t_{\Delta^\mp}}^2\right) \mp \frac{\Delta^\pm(0)}{2} \mp \sqrt{u_A^m \Delta^\pm(0)}\,(t_2 - t_1) \tag{40}$$

$$t_{\Delta^\pm} = t_1 - \sqrt{t_1^2 - \frac{\Delta^\pm(0)}{u_A^m - u_D^m}} \tag{41}$$

And the strategies for players are

$$\begin{cases} u_A(t) = u_A^m sgn(Z_T(0,t_2) - \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}) \\ u_D(t) = u_D^m sgn(Z_T(0,t_2) - \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}) \\ u_T(t) = -u_T^m sgn(Z_T(0,t_2) - \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}) \end{cases}, t \in [0,t_1] \tag{42}$$

then $\exists t_c \in [0, \max\{t_{\Delta^\pm}\}]$ makes $Z_T(t_c,t_2) = \frac{\bar{z}_{A_4}(t_c,t_2)+\bar{z}_{A_5}(t_c,t_2)}{2}$.

Proof This proof first consider the case in which $Z_T(0,t_2) \in \left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}, \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}+\varepsilon_+\right]$, where the players' strategies would be

$$\begin{cases} u_A(t) = u_A^m \\ u_D(t) = u_D^m \\ u_T(t) = -u_T^m \end{cases}, t \in [0, t_+] \tag{43}$$

Substituting ε_+ in Eq. (40) into $Z_T(0,t_2) < \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}+\varepsilon_+$, the initial condition satisfied

$$Z_T(0,t_2) - \frac{(u_T^m+u_A^m)}{2}\left(2t_{\Delta^-}t_2 - {t_{\Delta^-}}^2\right) - Z_A(0,t_2) + \frac{\Delta^+(0)}{2} + \sqrt{u_A^m\Delta^+(0)}\,(t_2-t_1) \le 0 \tag{44}$$

Integrating Eq. (25) under strategy (43) gives:

$$\begin{aligned} Z_A(t,t_2) &= Z_A(0,t_2) + \frac{1}{2}u_A^m\left({t_2}^2 - (t_2-t)^2\right) \\ Z_A(t,t_1) &= Z_A(0,t_1) + \frac{1}{2}u_A^m\left({t_1}^2 - (t_1-t)^2\right) \\ Z_D(t,t_1) &= Z_D(0,t_1) + \frac{1}{2}u_D^m\left({t_1}^2 - (t_1-t)^2\right) \\ Z_T(t,t_2) &= Z_T(0,t_2) - \frac{1}{2}u_T^m\left({t_2}^2 - (t_2-t)^2\right) \end{aligned}, t \in [0, t_+] \tag{45}$$

Substituting the two intermediate equations from Eq. (45) into Eq. (12), we get

$$\Delta^-(t) = \Delta^-(0) + \left(u_A^m - u_D^m\right)\left(t^2 - 2t_1 t\right),\ \Delta^+(t) = \Delta^+(0) \tag{46}$$

Substituting $t = t_{\Delta^-}$ into the first equation of Eq. (46), where t_{Δ^-} can be seen in Eq. (41), it follows that:$\Delta^-(t_{\Delta^-}) = 0$.

Substituting $t = t_+ = t_1 - \sqrt{\frac{\Delta^+(0)}{u_A^m}}$ into Eq. (46) yields:

$$\begin{aligned} \Delta^-(t_+) &= -Z_A(0,t_1) + Z_D(0,t_1) - \frac{1}{2}\left(u_A^m - u_D^m\right)\left(t_1^2 + \frac{2\Delta^+(0)}{u_A^m}\right) - r \\ \Delta^+(t_+) &= Z_A(0,t_1) - Z_D(0,t_1) + \frac{1}{2}\left(u_A^m - u_D^m\right)t_1^2 - r \end{aligned} \tag{47}$$

From Eq. (47), it follows that

$$\Delta^-(t_+) = -\Delta^+(t_+) - \left(u_A^m - u_D^m\right)\frac{2\Delta^+(0)}{u_A^m} \tag{48}$$

Since $\Delta^+(t_+) = \Delta^+(0) > 0$, it follows that $\Delta^-(t_+) < 0$. According to the first equation in Eq. (46), $\Delta^-(t), t \in [0, t_1]$ decreases over time. Since $\Delta^-(0) > 0$, $\Delta^-(t_{\Delta^-}) = 0$, and $\Delta^-(t_+) < 0$, it follows that $t_{\Delta^-} < t_+$.

Substituting the first two equation of Eq. (45) into Eq. (22) and (21) yields

$$\begin{aligned} \bar{z}_{A_4}(t,t_2) &= Z_A(0,t_2) + \frac{1}{2}u_A^m\left({t_2}^2 - (t_2-t)^2\right) - \frac{1}{2}u_A^m(t_2-t)^2 + u_A^m(t_2-t_-)^2 \\ \bar{z}_{A_5}(t,t_2) &= \bar{z}_{A_5}(0,t_2) = Z_A(0,t_2) + \frac{1}{2}u_A^m{t_2}^2 - u_A^m\left(t_2 - t_1 + \sqrt{\frac{\Delta^+(0)}{u_A^m}}\right)^2 \end{aligned} \tag{49}$$

Substituting $\Delta^-(t_{\Delta^-}) = 0$ into (23) yields $t_- = t_1$ when $t = t_{\Delta^-}$. Then

$$\bar{z}_{A_4}(t_{\Delta^-},t_2) = Z_A(0,t_2) + \frac{1}{2}u_A^m t_2^2 - u_A^m(t_2-t_{\Delta^-})^2 + u_A^m(t_2-t_1)^2 \tag{50}$$

holds. From Eq. (49), (50) and the last equation in Eq. (45), it follows that:

$$\begin{aligned} Z_T\left(t_{\Delta^-}, t_2\right) - & \frac{\bar{z}_{A_4}\left(t_{\Delta^-}, t_2\right) + \bar{z}_{A_5}\left(t_{\Delta^-}, t_2\right)}{2} \\ & = Z_T\left(0, t_2\right) - \frac{\left(u_T^m + u_A^m\right)}{2}\left(2t_{\Delta^-}t_2 - {t_{\Delta^-}}^2\right) - Z_A\left(0, t_2\right) \\ & + \frac{\Delta^+(0)}{2} + \sqrt{u_A^m \Delta^+(0)}\left(t_2 - t_1\right) \end{aligned} \tag{51}$$

The right-hand side of Eq. (51) is exactly the left-hand side of Eq. (44). Then $Z_T\left(t_{\Delta^-}, t_2\right) \leq \frac{\bar{z}_{A_4}\left(t_{\Delta^-}, t_2\right) + \bar{z}_{A_5}\left(t_{\Delta^-}, t_2\right)}{2}$ holds. Together with the initial inequality and continuity of the three terms, it necessarily follows that $\exists t_c \in [0, t_{\Delta^-}]$ makes $Z_T\left(t_c, t_2\right) = \frac{\bar{z}_{A_4}(t_c, t_2) + \bar{z}_{A_5}(t_c, t_2)}{2}$ holds.

For the case $Z_T\left(0, t_2\right) \in \left[\frac{\bar{z}_{A_4} + \bar{z}_{A_5}}{2} - \varepsilon_-, \frac{\bar{z}_{A_4} + \bar{z}_{A_5}}{2}\right]$, the proof procedure is identical to this and will therefore not be repeated. Then the lemma is proved. □

After time t_c, Taking any time t as the initial time, and executing strategy (42), the closed-loop strategy can be obtained as:

$$\begin{cases} u_A(t) = u_A^m sgn(Z_T(t, t_2) - \frac{\bar{z}_{A_4}(t, t_2) + \bar{z}_{A_5}(t, t_2)}{2}) \\ u_D(t) = u_D^m sgn(Z_T(t, t_2) - \frac{\bar{z}_{A_4}(t, t_2) + \bar{z}_{A_5}(t, t_2)}{2}) \\ u_T(t) = -u_T^m sgn(Z_T(t, t_2) - \frac{\bar{z}_{A_4}(t, t_2) + \bar{z}_{A_5}(t, t_2)}{2}) \end{cases}, t \in [t_c, t_1] \tag{52}$$

Lemma 6. For $t > t_c$, if $\Delta^{\pm} > 0$ and condition

$$Z_T(t, t_2) = \lim_{\delta \to 0} \frac{\bar{z}_{A_4}(t, t_2) + \bar{z}_{A_5}(t, t_2)}{2} + \delta \tag{53}$$

are satisfied, and the players execute strategy (52), the state will converge again to $Z_T(t, t_2) = \frac{\bar{z}_{A_4}(t, t_2) + \bar{z}_{A_5}(t, t_2)}{2}$ within finite time.

Proof Substituting $\Delta^{\pm} > 0$ into Eq. (40) and (41) yields a finite value for $\varepsilon_{\pm}(t)$. Since δ is an infinitesimal quantity, it follows that $Z_T\left(t, t_2\right) \in \left[\frac{\bar{z}_{A_4}(t, t_2) + \bar{z}_{A_5}(t, t_2)}{2} - \varepsilon_-(t), \frac{\bar{z}_{A_4}(t, t_2) + \bar{z}_{A_5}(t, t_2)}{2} + \varepsilon_+(t)\right]$. Following the same argument as in Lemma 5, it can be proved that the state converges to $Z_T(t, t_2) = \frac{\bar{z}_{A_4}(t, t_2) + \bar{z}_{A_5}(t, t_2)}{2}$ within finite time; the detailed proof is omitted. □

The switching strategy in Eq. (52) is understood in the Filippov sense when the state lies on the switching manifold. The corresponding equivalent control and its admissibility will be analyzed in the following lemma.

Combined Lemma 5 and Lemma 6, the time evolution curves of $\bar{z}_{A_4}(t, t_2)$, $\bar{z}_{A_5}(t, t_2)$ and $Z_T(t, t_2)$ are shown in Fig. 5 and 6. Specifically, Fig. 5 corresponds to case

$Z_T(0,t_2) > \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}$. At the initial time, the players' strategy is given by Eq. (43). Based on Eq. (45), and (49), the evolution curves of $\bar{z}_{A_4}(t,t_2), \bar{z}_{A_5}(t,t_2)$ and $Z_T(t,t_2)$ can be plotted. Under this strategy, $\bar{z}_{A_5}(t,t_2)$ remains constant, $\bar{z}_{A_4}(t,t_2)$ increases over time, and $Z_T(t,t_2)$ decreases over time, until at timet_c, $Z_T(t,t_2) = \frac{\bar{z}_{A_4}(t,t_2)+\bar{z}_{A_5}(t,t_2)}{2}$ holds. Subsequently, the players switch to strategy (52). Under this strategy, $\bar{z}_{A_5}(t,t_2)$ decreases, $\bar{z}_{A_4}(t,t_2)$ increases, and $Z_T(t,t_2)$ remains near curve $Z_T(t,t_2) = \frac{\bar{z}_{A_4}(t,t_2)+\bar{z}_{A_5}(t,t_2)}{2}$. The symmetric case $Z_T(0,t_2) < \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}$ is illustrated in Fig. 6.

It can be observed that on the two sides of curve $Z_T(t,t_2) = \frac{\bar{z}_{A_4}(t,t_2)+\bar{z}_{A_5}(t,t_2)}{2}$, the sign function in strategy (52) takes opposite signs. Moreover, under this strategy, the state trajectories evolve along this curve. In differential game theory, it is often referred to as the "switch envelope curve," which is a type of singular curve analyzed in this paper. Although this set is referred to as a singular surface in the general state space, it appears as a singular curve in the reduced two-dimensional state space considered here.

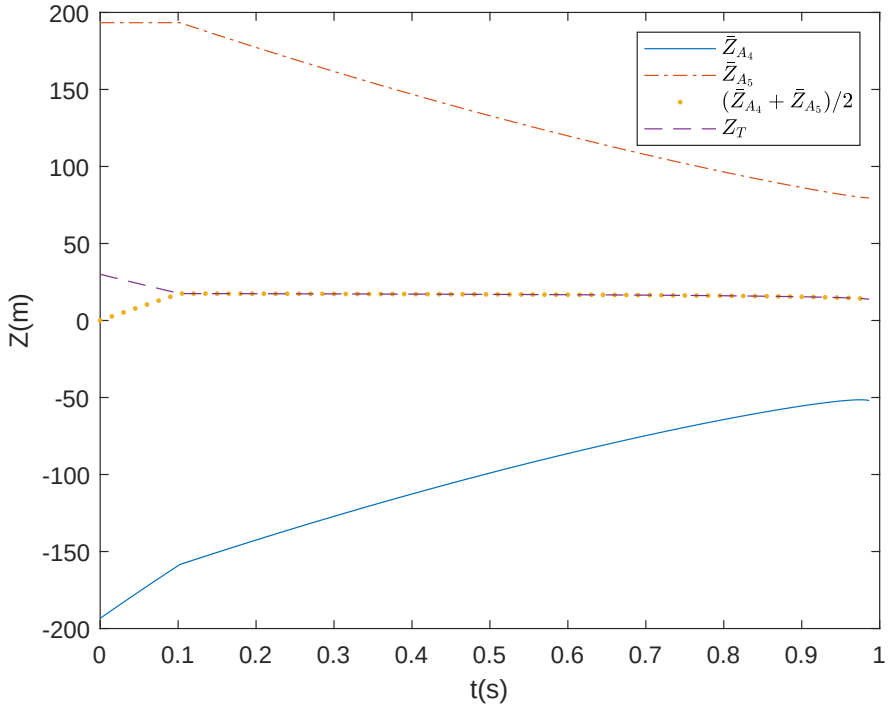


Fig. 5 Singular curve for $Z_T(0,t_2) \geq \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}$.

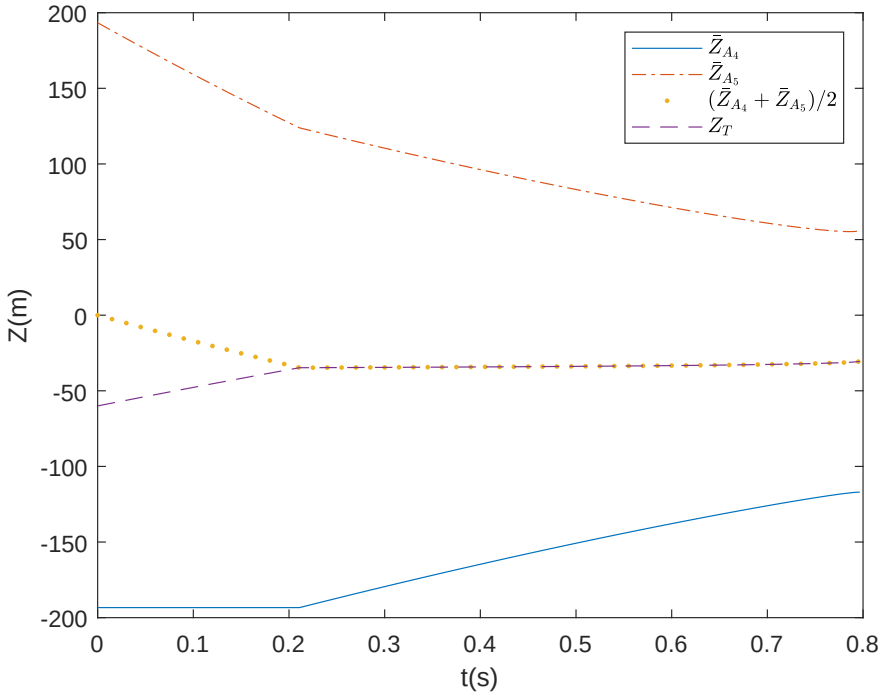


Fig. 6 Singular curve for $Z_T(0,t_2) < \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}$.

When $Z_T(t,t_2) = \frac{\bar{z}_{A_4}(t)+\bar{z}_{A_5}(t)}{2}$, it has $J_{A_5T_1}(t) = J_{T_2A_4}(t)$, and

$$J_m(t) = J_{A_5T_1}(t) = J_{T_2A_4}(t) = \frac{\bar{z}_{T_2}(t,t_2) - \bar{z}_{T_1}(t,t_2)}{2} + \frac{\bar{z}_{A_5}(t,t_2) - \bar{z}_{A_4}(t,t_2)}{2} \tag{54}$$

By substituting Eq. (5) into Eq. (54), it follows that

$$J_m(t) = J_{A_5T_1}(t) = J_{T_2A_4}(t) = \frac{u_T^m\left(t_2-t\right)^2}{2} + \frac{\bar{z}_{A_5}(t,t_2) - \bar{z}_{A_4}(t,t_2)}{2} \tag{55}$$

Lemmas 5 and 6 respectively prove that under certain conditions, the game will enter the switch envelope and propagate along it. The following Lemma 7 will prove that on the switch envelope, $J_m(t)$ decreases over time until it reaches its minimum at a certain instant, at which point the game departs from the singular curve. Define the switching function as

$$g(t) = Z_T(t,t_2) - \frac{\bar{z}_{A4}(t,t_2) + \bar{z}_{A5}(t,t_2)}{2} \tag{56}$$

The switch envelope is characterized by $g(t) = 0$.

On the switch envelope, the players' strategy can be seen in (52). Therefore, the equivalent control can be written as

$$u_A(t) = u_A^m \sigma(t),\ u_D(t) = u_D^m \sigma(t),\ u_T(t) = -u_T^m \sigma(t),\ \sigma(t) \in [-1, 1] \tag{57}$$

Lemma 7. Define

$$F_{\pm}(t) = (u_A^m + u_T^m)(t_2 - t_1) + (u_T^m + u_D^m)(t_1 - t) - \frac{(t_2 - t_1)\sqrt{u_A^m}(u_A^m - u_D^m)(t_1 - t)}{\sqrt{\Delta^{\pm}(t)}} \tag{58}$$

and that the switch envelope is locally attractive at t_c, i.e.,$F_+(t_c) > 0,\ F_-(t_c) > 0$. Then there exists a finite time $t^* > t_c$ such that

$$F_+(t^*)F_-(t^*) = 0 \tag{59}$$

Moreover, $J_m(t)$ is strictly decreasing for $t \in [t_c, t^*)$, and $\dot{J}_m(t^*) = 0$ holds.

Proof According to Eq. (28) - (31), under the equivalent control (57), the derivatives of $\bar{z}_{A_4}$ and $\bar{z}_{A_5}$ can be written as

$$\begin{aligned} \dot{\bar{z}}_{A_5}(t, t_2) &= [\sigma(t) - 1]Q_+(t), \\ \dot{\bar{z}}_{A_4}(t, t_2) &= [\sigma(t) + 1]Q_-(t), \end{aligned} \tag{60}$$

where

$$Q_{\pm}(t) = u_A^m(t_2 - t) - \frac{\sqrt{u_A^m}(u_A^m - u_D^m)(t_1 - t)(t_2 - t_{\pm})}{\sqrt{\Delta^{\pm}(t)}} \tag{61}$$

From Eq. (16) and (23),

$$t_2 - t_{\pm} = t_2 - t_1 + \sqrt{\frac{\Delta^{\pm}(t)}{u_A^m}} \tag{62}$$

Substituting (62) into (61) gives

$$Q_{\pm}(t) = u_A^m(t_2 - t_1) + u_D^m(t_1 - t) - \frac{(t_2 - t_1)\sqrt{u_A^m}(u_A^m - u_D^m)(t_1 - t)}{\sqrt{\Delta^{\pm}(t)}} \tag{63}$$

Therefore,

$$F_{\pm}(t) = u_T^m(t_2 - t) + Q_{\pm}(t) \tag{64}$$

Differentiating the switching function (56) yields

$$\dot{g}(t) = \dot{Z}_T(t, t_2) - \frac{\dot{\bar{z}}_{A4}(t, t_2) + \dot{\bar{z}}_{A5}(t, t_2)}{2} \tag{65}$$

Since

$$\dot{Z}_T(t, t_2) = -u_T^m \sigma(t)(t_2 - t) \tag{66}$$

substituting Eq. (60) and (64) into (65) gives

$$\dot{g}(t) = -\frac{1}{2}\left[(F_+(t) + F_-(t))\sigma(t) + F_-(t) - F_+(t)\right] \tag{67}$$

When $g > 0$, strategy (52) gives $\sigma = 1$, then $\dot{g}(t) = -F_-(t)$ holds, When $g < 0$, $\sigma = -1$, and $\dot{g}(t) = F_+(t)$ holds. Consequently, the switch envelope is locally attractive if

$$F_+(t) > 0, F_-(t) > 0 \tag{68}$$

On the switch envelope, $\dot{g} = 0$. It follows from (67) that the equivalent switching variable is

$$\sigma_{\text{eq}}(t) = \frac{F_+(t) - F_-(t)}{F_+(t) + F_-(t)} \tag{69}$$

Under condition (68), $|\sigma_{\text{eq}}| < 1$, and hence the state can remain on the switch envelope. If $F_- = 0$, then $\sigma_{\text{eq}} = 1$, whereas if $F_+ = 0$, then $\sigma_{\text{eq}} = -1$. Thus, the switch-envelope motion loses its interior equivalent control when $F_+F_- = 0$.

It remains to prove that such a time necessarily exists. From Eq. (12),

$$\Delta^+(t) + \Delta^-(t) = (u_A^m - u_D^m)(t_1 - t)^2 - 2r \tag{70}$$

Since $u_A^m > u_D^m$, the right-hand side decreases continuously with time and becomes zero at

$$t_b = t_1 - \sqrt{\frac{2r}{u_A^m - u_D^m}} \tag{71}$$

Therefore, starting from $\Delta^+(t_c) > 0$ and $\Delta^-(t_c) > 0$, at least one of Δ^+ and Δ^- must approach zero at a finite time before or at t_b.

Without loss of generality, suppose that $\Delta^-(t) \to 0^+$ as $t \to t_\Delta^-$. From Eq. (58),

$$\lim_{t \to t_\Delta^-} F_-(t) = -\infty \tag{72}$$

Since $F_-(t_c) > 0$ and $F_-(t)$ is continuous for $\Delta^-(t) > 0$, the intermediate value theorem guarantees that there exists at least one $t^* \in (t_c, t_\Delta)$ such that $F_-(t^*) = 0$. The case in which Δ^+ approaches zero first is identical. Hence, a finite t^* satisfying (59) necessarily exists.

Finally, differentiating Eq. (55) yields

$$\dot{J}_m = -u_T^m(t_2 - t) + \frac{1}{2}\left(\dot{\bar{z}}_{A5} - \dot{\bar{z}}_{A4}\right). \tag{73}$$

Substituting (60) and (69) into (73) gives

$$\dot{J}_m(t) = -\frac{2F_+(t)F_-(t)}{F_+(t) + F_-(t)} \tag{74}$$

Therefore, $\dot{J}_m(t) < 0$ for $t \in [t_c, t^*)$, whereas $\dot{J}_m(t^*) = 0$ in the nondegenerate case. Thus, $J_m(t)$ reaches its minimum value along the admissible switch-envelope segment at $t = t^*$. The lemma is proved. □

From Eq. (59), the departure time t^* can be characterized implicitly. If $F_\pm(t^*) = 0$, then

$$\sqrt{\Delta^\pm(t^*)} = \frac{(t_2 - t_1)\sqrt{u_A^m}(u_A^m - u_D^m)(t_1 - t^*)}{(u_A^m + u_T^m)(t_2 - t_1) + (u_D^m + u_T^m)(t_1 - t^*)} \tag{75}$$

Equation (75) provides an implicit characterization of the optimal departure time t^*. Hence, t^* is the optimal departure time from the switch envelope for the attacker.

Theorem 1. Suppose that at the beginning of the game, $\Delta^-(0) > 0$ and $\Delta^+(0) > 0$ holds. Then based on the following strategies, the value of the game is as follows

1. If $Z_T(0, t_2) \in \left[\frac{\bar{z}_{A_5} + \bar{z}_{A_6}}{2}, +\infty\right)$, then the equilibrium strategy is then given by

$$u_A^*(t) = u_A^m, u_T^*(t) = u_T^m, t \in [0, t_2] \tag{76}$$

and the game value in this case is $J^* = J_{T_2A_6}(0)$

2. If $Z_T(0,t_2) \in \left(\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}+\varepsilon_+, \frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2}\right)$, then the equilibrium strategy is then given by

$$u_A^*(t) = \begin{cases} u_A^m, t \in (0,t_+) \\ -u_A^m, t \in [t_+,t_2] \end{cases}, u_T^*(t) = -u_T^m, t \in [0,t_2], u_D^*(t) = u_D^m, t \in [0,t_1] \tag{77}$$

and the game value in this case is $J^* = J_{A_5T_1}(0)$

3. If $Z_T(0,t_2) \in \left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}-\varepsilon_-, \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}+\varepsilon_+\right]$ then the equilibrium strategy is then given by

$$u_A^*(t) = \begin{cases} u_A^m sgn\left(Z_T(0,t_2) - \frac{\bar{z}_{A_4}(0,t_2)+\bar{z}_{A_5}(0,t_2)}{2}\right), t \in (0,t_c] \\ u_A^m sgn\left(Z_T(t,t_2) - \frac{\bar{z}_{A_4}(t,t_2)+\bar{z}_{A_5}(t,t_2)}{2}\right), t \in (t_c,t^*] \\ \pm u_A^m, t \in (t^*,t_\pm] \\ \mp u_A^m, t \in (t_\pm,t_2] \end{cases}$$

$$u_D^*(t) = \begin{cases} u_D^m sgn\left(Z_T(0,t_2) - \frac{\bar{z}_{A_4}(0,t_2)+\bar{z}_{A_5}(0,t_2)}{2}\right), t \in (0,t_c] \\ u_D^m sgn\left(Z_T(t,t_2) - \frac{\bar{z}_{A_4}(t,t_2)+\bar{z}_{A_5}(t,t_2)}{2}\right), t \in (t_c,t^*] \\ \pm u_D^m, t \in (t^*,t_1] \end{cases} \tag{78}$$

$$u_T^*(t) = \begin{cases} -u_T^m sgn\left(Z_T(0,t_2) - \frac{\bar{z}_{A_4}(0,t_2)+\bar{z}_{A_5}(0,t_2)}{2}\right), t \in (0,t_c] \\ -u_T^m sgn\left(Z_T(t,t_2) - \frac{\bar{z}_{A_4}(t,t_2)+\bar{z}_{A_5}(t,t_2)}{2}\right), t \in (t_c,t^*] \\ \mp u_T^m, t \in (t^*,t_2] \end{cases}$$

where t^* is determined by Lemma 7, and the corresponding game value is $J_m(t^*)$

4. If $Z_T(0,t_2) \in \left(\frac{\bar{z}_{A_3}+\bar{z}_{A_4}}{2}, \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}-\varepsilon_-\right)$, then the equilibrium strategy is then given by

$$u_A^*(t) = \begin{cases} -u_A^m, t \in (0,t_-) \\ u_A^m, t \in [t_-,t_2] \end{cases}, u_T^*(t) = u_T^m, t \in [0,t_2], u_D^*(t) = -u_D^m, t \in [0,t_1] \tag{79}$$

and the game value in this case is $J^* = J_{T_2A_4}(0)$

5. If $Z_T(0,t_2) \in \left(-\infty, \frac{\bar{z}_{A_3}+\bar{z}_{A_4}}{2}\right]$, then the equilibrium strategy is then given by

$$u_A^*(t) = -u_A^m, u_T^*(t) = -u_T^m, t \in [0,t_2] \tag{80}$$

and the game value in this case is $J^* = J_{A_3T_1}(0)$

Proof At the initial time, if $Z_T(0,t_2) \in \left[\frac{\bar{z}_{A_5}+\bar{z}_{A_6}}{2}, +\infty\right)$, as shown in Fig. 3. The strategy at this point is the open-loop strategy from Lemma 4, resulting in a final miss distance of $J_{T_2A_6}(0)$. Similarly, Cases 2, 4, and 5 are identical to the open-loop strategy in Lemma 4 and will not be elaborated here.

For Case 3, similar to the Case 2 and Case 4, the players' strategies are the same as (42). Moreover, $\Delta^+(0) > 0, \Delta^-(0) > 0$. According to Lemma 5, $\exists t_c \in (0, t_{\Delta^+})$, makes $Z_T(t_c,t_2) = \frac{\bar{z}_{A_4}(t_c,t_2)+\bar{z}_{A_5}(t_c,t_2)}{2}$.

Under the strategies in Eq. (78) for $t \in (t_c, t^*]$, According to Lemma 6, the state evolves along $Z_T(t,t_2) = \frac{\bar{z}_{A_4}(t,t_2)+\bar{z}_{A_5}(t,t_2)}{2}$. The state follows the switch envelope on $t \in [t_c, t^*]$, at $t = t^*$, the equivalent control reaches the boundary of the admissible control set, and the attacker departs from the switch envelope. For the attacker, this moment is the optimal time to initiate maneuvering. The corresponding strategy is given by the part $[t^*, t_2]$ in Eq. (78), which is an open-loop strategy, regardless of the maneuvers of the defender and the target, the attacker's strategy remains unchanged. The optimality of this interval strategy is established in Lemma 4. Thus, $J_m(t^*)$ is the value of the game, the Theorem is thus proved. □

It is important to note that under the aforementioned strategies, when the moment t^* arrives, the state trajectory reaches the dispersal surface, from which two symmetric equilibrium continuations emanate. These two continuations correspond to bypassing the defender from opposite sides and yield the same game value $J_m(t^*)$.

5 Simulation

5.1 Validation under linear Engagement Kinematics

Consider the scenario illustrated in Fig. 2, $V_{A,r} = 3000m/s, V_{D,r} = 2500m/s, V_{T,r} = 2000m/s$, and at the initial moment, $y_A = 12500, y_D = 1500, y_T = 0$, consider the initial condition as

$$\begin{cases} u_A^m = 100\ m/s^2, u_D^m = 80\ m/s^2, u_T^m = 50\ m/s^2 \\ x_A^0 = x_D^0 = 0\ m, x_T^0 = 30\ m, r = 5\ m \end{cases} \tag{81}$$

Since the longitudinal distance between the players is sufficiently large compared to the lateral distance, it follows that $r_{AD}^0 \approx y_A - y_D, r_{AT}^0 \approx y_A - y_T$. Based on (1), the intersection time can be computed as $t_1 = 2s, t_2 = 2.5s$.

From the initial state

$$Z_T(0,t_2) \in \left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2} - \varepsilon_-, \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2} + \varepsilon_+\right] \tag{82}$$

With the strategies (78), Z_T decreases and $\bar{z}_{A_4}$ increases over time until $t = t_c \approx 0.1023s$, at which point $Z_T(t_c,t_2) = \frac{\bar{z}_{A_4}(t_c)+\bar{z}_{A_5}(t_c)}{2}$, as shown in Fig. 7. Thereafter,

$\bar{z}_{A_5}$ decreases, $\bar{z}_{A_4}$ increases, while Z_T remains equal to $\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}$. When $t = t^*$, the payoff function reaches its minimum value, which is computed using the event condition derived in Lemma 7,$F_+(t^*)F_-(t^*) = 0$.. For the parameters considered here, the resulting value is $t^* \approx 0.9866$ s, which coincides with the minimum of $J_m(t)$ observed in Fig. 6. After t^*, the game then enters the dispersal curve, the players will choose to maneuver in one direction.

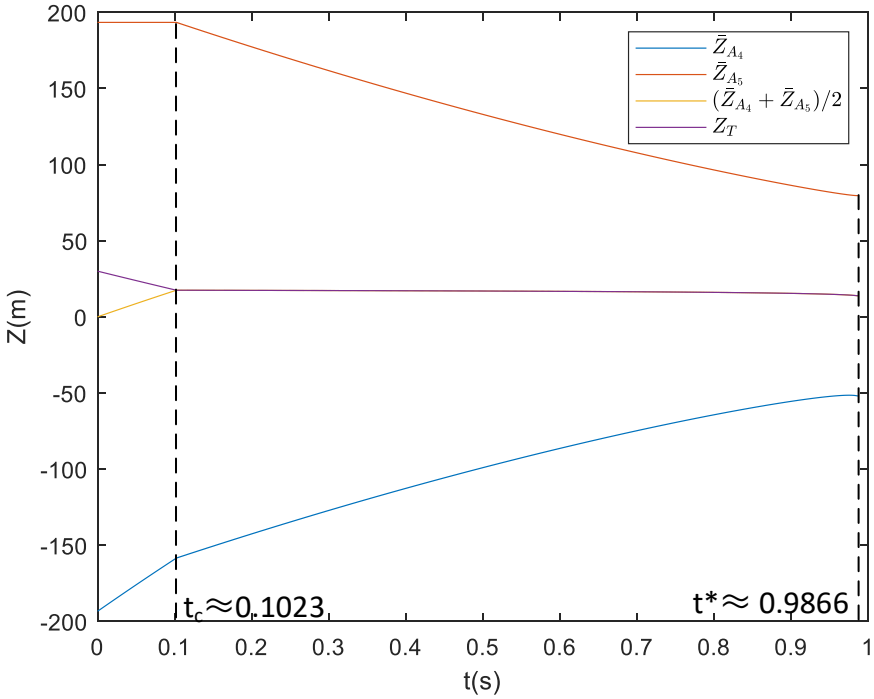


Fig. 7 The change of the Z_T, $\bar{z}_{A_4}$ and $\bar{z}_{A_5}$.

The movement trajectories of the players are shown in the Fig. 8. At the initial time of the game, $Z_T\,(0, t_2) \in \left[\frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}, \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2} + \varepsilon_+\right]$. The attacker and defender maneuver to the right, while the target maneuvers to the left. When $t = t_c$, $Z_T = \frac{\bar{z}_{A_4}+\bar{z}_{A_5}}{2}$, and the state trajectory enters the "switch envelope curve." This continues until $t = t^*$, and the miss distance reaches its minimum. At this point, the attacker initiates further maneuvers, choosing to bypass the defender from one side. The "switch envelope curve" transitions into the "dispersal curve." In the figure, it is assumed that the attacker maneuvers to the left. When $t = t_+$, the defender no longer poses a threat to the attacker, and the attacker switches control once again to begin pursuing the target.

From Fig. 8, the attacker and the defender intersect at $y = 6500$ m, and the miss distance at t_1 is approximately 5.01 m (Refer to the small window indicated by the

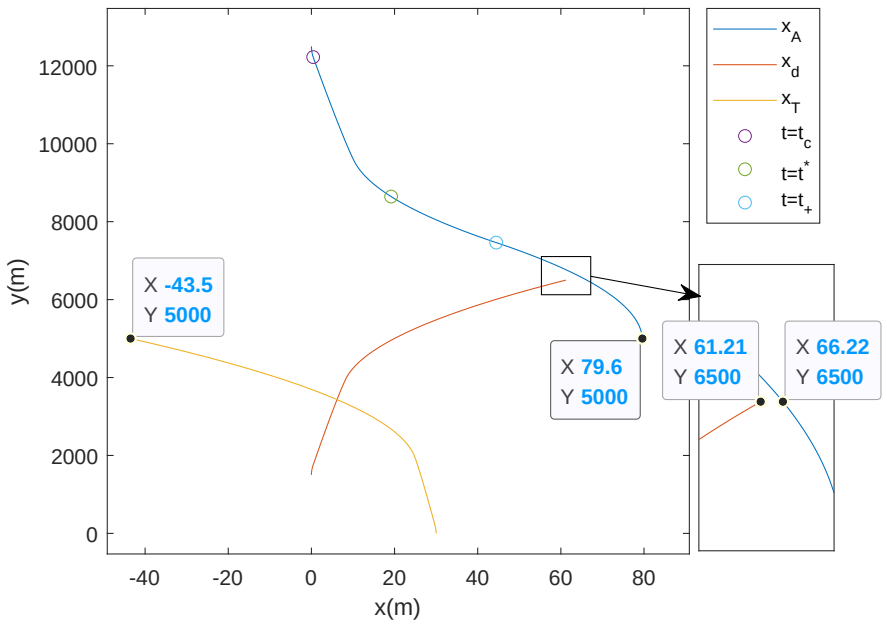


Fig. 8 Trajectories in linear engagement.

arrow in Fig. 8, which corresponds to the lethal radius of the defender $r = 5\ m$). The attacker and the target intersect at $y = 5000\ m$, with a final miss distance of 123.1 m.

5.2 Validation under Nonlinear Engagement Kinematics

To further investigate the applicability of the proposed strategy beyond the linearized terminal-projection model, an additional simulation is conducted using nonlinear planar engagement dynamics. The x-axis is selected as the nominal line-of-sight direction, whereas the y-axis denotes the transverse direction.

The initial positions of the attacker, defender, and target are $(x_A(0), y_A(0)) = (0, 0)$ m, $(x_D(0), y_D(0)) = (11000, 1000)$ m, $(x_T(0), y_T(0)) = (12500, 2530)$ m. The velocity magnitudes are $V_A = 3000\sqrt{2}$ m/s, $V_D = 2500\sqrt{2}$ m/s, $V_T = 2000\sqrt{2}$ m/s and the corresponding initial heading angles are $\theta_i(0) = \frac{\pi}{4}, i = A, D, T$.

Each player obeys the nonlinear constant-speed planar kinematics

$$\begin{aligned} \dot{x}_A &= V_A \cos\theta_A, \dot{y}_A = V_A \sin\theta_A, \\ \dot{x}_i &= -V_i \cos\theta_i, \dot{y}_i = V_i \sin\theta_i, i = D, T, \end{aligned} \tag{83}$$

and

$$\dot{\theta}_j = \frac{a_j}{V_j}, j = A, D, T, \tag{84}$$

where a_j is the signed normal acceleration. The acceleration bounds are $a_A^m = 100\sqrt{2}, a_D^m = 80\sqrt{2}, a_T^m = 50\sqrt{2}$ m/s^2.

At the initial headings, the velocity components along the x-axis are $3000, -2500, -2000m/s$, and the transverse components of the maximum normal accelerations are $u_A^m = 100, u_D^m = 80, u_T^m = 50$ m/s^2. Hence, linearization of the nonlinear model about the initial engagement geometry recovers exactly the parameters used in the preceding linearized simulation. Moreover, $Z_A(0,t_1) = Z_D(0,t_1) = 6000$ m, and$Z_A(0,t_2) = 7500$ m, $Z_T(0,t_2) = 7530$ m, so that $Z_T(0,t_2) - Z_A(0,t_2) = 30$ m. Thus, the nonlinear and linearized simulations have identical initial zero-effort geometry.

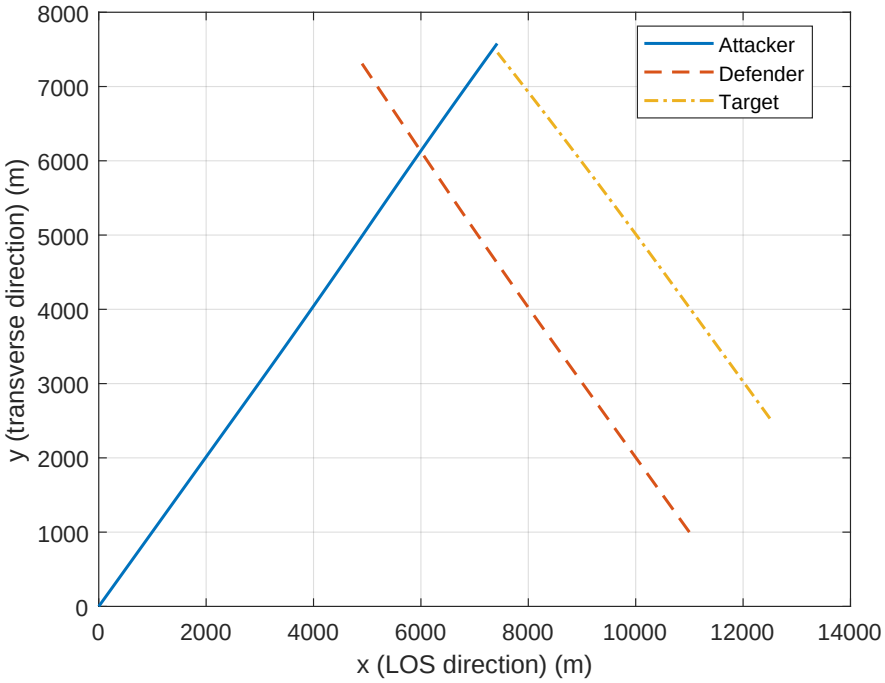


Fig. 9 Maneuvering trajectories of the players.

The nonlinear maneuvering trajectories and the corresponding inter-player distances are shown in Fig. 9 and 10. The nonlinear simulation gives $t_c = 0.102331$ s, $t^* = 0.987426$ s, $t_+ = 1.679594$ s. The corresponding values of t_c and t^* predicted by the linearized model are approximately 0.1023 s and 0.9866 s, respectively. Their relative differences are both below 0.1%, indicating that the switching structure predicted by the reachable-set analysis is preserved under the nonlinear engagement kinematics.

Because the heading angles vary during the engagement, the actual closest approach occurs slightly before the nominal terminal time. The minimum Euclidean attacker-target distance is $d_{AT}^{\min} = 122.075$ m, $t = 2.4965$ s.

For the attacker-defender engagement, the actual closest approach occurs at approximately $t = 2.02375$ s, and $d_{AD}^{\min} = 16.058$ m $> r$.

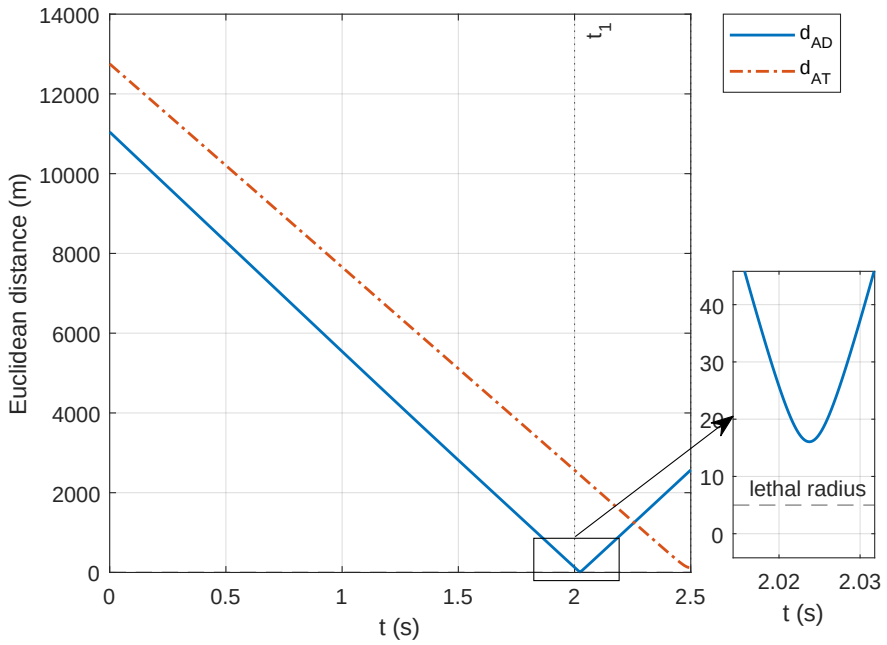


Fig. 10 Evolution of the Euclidean distances.

Therefore, the attacker successfully bypasses the defender while remaining outside its lethal radius. The nonlinear results confirm that the reachable-set-based switching structure remains valid even when the coupling between heading variation and longitudinal motion is retained.

6 Conclusions

This paper presented a reachable-set-based framework for analyzing the ADT differential game with a non-dominant attacker. Unlike previous studies that typically assume an overwhelmingly superior attacker, this paper considers a non-dominant attacker whose reachable set does not completely contain the target's reachable set.

First, the attacker's reachable set is computed. The results show that the reachable set at the interception time is influenced by the defender's state and strategy. Second, the initial relative positional relationship is classified into five cases, and for each case, the equilibrium strategies are derived. It is proved that the game value is given by the distance between the corresponding boundaries of the attacker's reachable set and a boundary of the target's reachable set. Third, it is shown that when the target's zero-effort coordinates at the initial time fall within a certain interval, the state trajectory will enter the interface between two cases, which corresponds to the first type of singular curve studied in this paper—namely, the switch envelope. Subsequently, the state remains on this singular curve for a certain period until the miss distance reaches its minimum, at which point the attacker begins to maneuver. Fourth, it is demonstrated that the game then enters the second type of singular curve, referred

to as the dispersal curve, where all the players have two optimal strategy options. Finally, simulations are conducted, which validate that the proposed strategies enable coordination between the defender and the target, while the attacker can successfully bypass the defender and pursue the target.

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