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Integral equations of contact problems of elasticity theory for stratified media

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Abstract

The formulation of an integral equation for the contact problem for a stratified multi-layered medium is considered using the example of a pure shear of a multi-layered (stratified) coating, bonded to a rigid base, subjected to the action of a rigid strip indenter. The thickness and shear modulus of each coating layer are known in advance. The solution to the contact problem is constructed using the Fourier integral transform, applied to the equilibrium differential equations for each layer and to the interface boundary conditions for layer couplings. Interface boundary conditions are written in a matrix form as recurrent relations. Taking into account the coating support conditions and mixed contact boundary conditions, the solution to the contact problem (contact stresses) is reduced to a Fredholm integral equation of the first kind of the Fourier convolution type. The transform of the integral equation kernel is expressed in a matrix form and contains all the mechanical and geometric parameters of the multilayered coating. An analytical solution to the integral equation is constructed in the form of a functional series in a small geometric parameter. Formulas to determine the stress-strain state in an arbitrary layer (in any of the layers) of a stratified medium are obtained for such cases as layered rock, underground structures, mines, etc., as well as in microelectronics, where the number of stratified material layers may exceed one hundred, and in biomechanics, where the layering of biomaterials ranges from ten or more layers of elements within the human or animal body.

Keywords: stratified coating, multilayered coating, contact problem, integral equation, matrix form

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Introduction

The widespread use of multilayered materials — which have enhanced both the strength and performance characteristics of products made from them, particularly in microelectronics — began in the middle of the last century. Since the 1980s, indentation methods have been used to evaluate the mechanical properties of multilayer materials. The fundamental basis of indentation methods lies in analytical solutions to the corresponding contact problems (based on the elasticity theory, dilatational elasticity theory, viscoelasticity theory, continuum mechanics, etc.) concerning the settling of an indenter into a deformable material. In this case, analytical solutions to contact problems for stratified materials can be obtained only by reducing them to corresponding integral equations with a kernel transform represented in an analytical form that incorporates the parameters of all the medium's layers. Furthermore, to perform a reverse analysis of stratified materials for the purpose of determining each layer parameters and their stress-strain state, it is necessary to have an equation for the free surface outside the contact area between the indenter and the stratified medium. The primary difficulty in deriving integral equations for contact problems involving multilayered materials lies in the need to analytically solve a system of linear algebraic equations — arising from the satisfaction of interface boundary conditions — whose order is proportional to the number of material layers. To overcome these difficulties, a group of researchers [1-9] developed algorithms for transforming the matrix functions of the system of linear algebraic equations — which describes interface couplings and boundary conditions at the contact interface of a stratified medium and at its support surface — in order to obtain the transforms of integral equation kernels in a compact analytical matrix form. This made it possible to obtain solutions to integral equations for problems with a small number of layers. Also, this group of researchers has made progress in solving the problem for continuously inhomogeneous media. Another group of researchers [10-19] reduced the solution of axisymmetric contact problems for multilayered materials involving circular and annular indenters to solving a system of dual integral equations; the compliance functions were expressed as the sum of solutions for the contacting layers, resulting in cumbersome formulas. After isolating the principal parts of the compliance functions, the solution of the dual integral equations has been reduced to a system of singular integral equations which, due to their complexity, were solved numerically. Next group of researchers [20-24] reduced the solution of the contact problem for a multilayered material to solving a dual integral equation with a kernel the asymptotic properties of which were determined numerically. A comparison of numerical and analytical asymptotic solutions suggested the possibility of approximating the numerical asymptotic solutions with analytical counterparts. For the past two decades, this group of researchers has primarily focused on problems involving continuously inhomogeneous media.

Since the end of the last century, a large group of researchers has solved a series of interesting applied problems some of which were based on previously developed theoretical and experimental solution methods. A significant number of articles and monographs [25–30] are devoted to solving problems concerning the indentation of multilayered materials including the cases accounting for contact adhesion as well as the deformation state of multilayered materials subjected to various types of mechanical and thermo-mechanical loading, sliding contacts of rough multilayer material surfaces, thermoelastic frictional heating at the contact interface, and many other relevant problems. However, the problems addressed in the above studies involved materials and coatings with a small number of layers, and their solutions did not require the development of a methodology for multilayered and stratified materials and coatings. The present review highlights key areas of research and does not claim to provide an exhaustive bibliography of studies related to the contact interaction of multilayered materials and stratified media.

This paper proposes an efficient method for deriving integral equations and systems of integral equations for contact problems involving an indentation of multilayered materials, utilizing matrix functions that describe the mechanical coupling of elastic layers within the multilayered material. The construction of layer matrix functions differs from that proposed by the above group of researchers [1–9] and offers a number of advantages. The derivation of integral equations for contact problems involving multilayered materials is demonstrated by solving the contact problem of pure shear applied by a strip indenter to an elastic N-layered coating bonded to a rigid base. The asymptotic properties of the matrix functions of the transforms of integral equations — represented in a matrix form — are studied in detail. An analytical solution is presented for the integral equation obtained for an N-layered coating, constructed as a power series in a characteristic geometric parameter. The developed solution method makes it possible to determine the stress-strain state of each layer using the derived formulas. As an example, the integral equations for the contact problem of shear of a three-layered coating are derived.

2. Pure shear of a stratified coating by an indenter

A stratified coating of thickness h, comprising N layers of elastic materials, is considered. The shear modulus of the k-th layer is μ_k, and its thickness is Δh_k, where $k = 1 - N$. The thickness of the N-layered coating is equal to the sum of the thicknesses of the layers $h = \sum_{k=1}^{N} \Delta h_k$.

Between coating layers at $y = y_k$, $k = 1 - N$, where $y_k = h - \sum_{i=1}^{k} \Delta h_i$, the layers are coupled in such a way that both displacements and stresses are equal

$$w_k(x, y_k) = w_{k+1}(x, y_k) \text{ at } -\infty < x < \infty,\ k = 1 - (N-1) \tag{2.1}$$

$$\sigma^k(x, y_k) = \sigma^{k+1}(x, y_k) \text{ at } -\infty < x < \infty,\ k = 1 - (N-1) \tag{2.2}$$

An infinite rigid strip indenter ($-a \le x \le a$, $-\infty < z < \infty$) of width $2a$, bonded to the surface of the upper coating layer at $y = y_0 = h$, is displaced by a distance δ along the axis Oz under the action of a force T applied to the indenter and directed along its axis of symmetry, which coincides with the coordinate axis Oz (Fig. 1)

$$w_1(x, y_0) = -\delta \text{ at } -a \le x \le a,\ \delta > 0 \tag{2.3}$$

The bottom N-th layer of the multilayer coating is rigidly bonded along the bottom surface of the N-th layer at $y = y_N = 0$ to a non-deformable base

$$w_N(x, y_N) = 0 \text{ at } -\infty < x < \infty,\ y_N = 0 \tag{2.4}$$

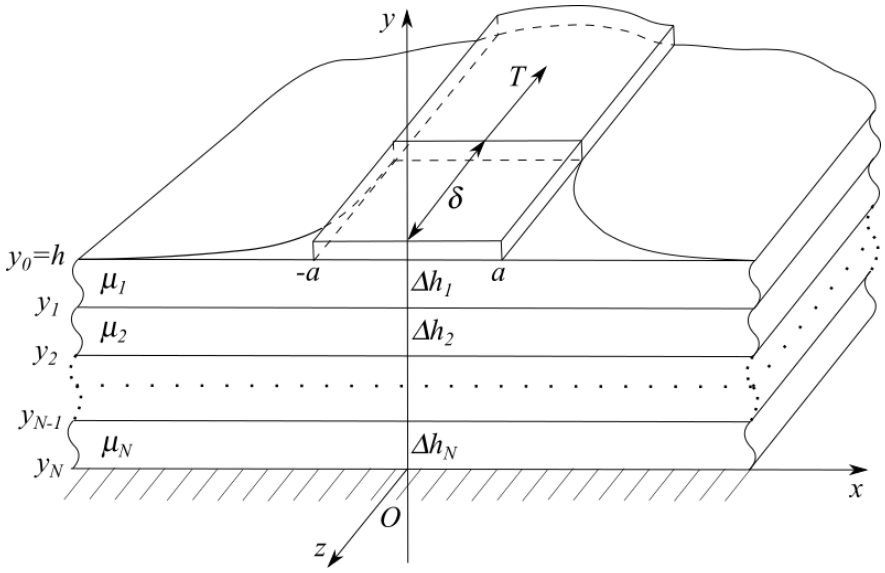


Fig. 1. Scheme of the contact problem regarding shearing of a N-layered coating by a rigid strip indenter

When solving the contact problem, the contact stresses arising at the base of the indenter during shearing of the multilayered foundation (2.3) are to be determined

$$\sigma_y^1(x, y_0) = \begin{cases} -\varphi(x), & -a \le x \le a \\ 0, & -\infty < x < -a, \quad a < x < \infty \end{cases} \tag{2.5}$$

The formulation of the contact problem is completed by the differential equations describing the behavior of the shear displacements $w_k(x, y)$ in each of the N layers of the stratified coating

$$\Delta w_k(x, y) = 0 \text{ at } -\infty < x < \infty,\ y_k \le y \le y_{k-1},\ k = 1 - N \tag{2.6}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is Laplace operator [31].

3. Formulation of the auxiliary problem

The indentation process of stratified coatings is modeled by formulating and analytically solving the corresponding contact problems. When deriving analytical solutions for indentation problems, contact problems are reduced to integral equations. To simplify and schematize the derivation of integral equations for contact problems involving multilayered materials, the corresponding non-mixed problems (often referred to as auxiliary problems) are formulated and solved [31]. As a result of solving auxiliary problems, all the fundamental formulas required to derive the integral equations for the corresponding contact problems are prepared. The differential equations and interface boundary conditions for the auxiliary problem concerning shear of an N -layered coating are presented layer-by-layer in Table 1.

Table 1

0	$y_0 = h$	$\sigma_y^1(x, y_0) = -\varphi(x)$ at $-\infty < x < \infty$	(3.1)
1	$y_1 = h - \Delta h_1$	$\Delta w_1(x, y) = 0$ at $-\infty < x < \infty,\ y_1 \le y < y_0$ $w_2(x, y_1) = w_1(x, y_1),\ \sigma_y^2(x, y_1) = \sigma_y^1(x, y_1)$	(3.2) (3.3)
2	$y_2 = h - \sum_{i=1}^{2} \Delta h_i$	$\Delta w_2(x, y) = 0$ at $-\infty < x < \infty,\ y_2 \le y < y_1$ $w_3(x, y_2) = w_2(x, y_2),\ \sigma_y^3(x, y_2) = \sigma_y^2(x, y_2)$	(3.4) (3.5)
……...			
k	$y_k = h - \sum_{i=1}^{k} \Delta h_i$	$\Delta w_k(x, y) = 0$ at $-\infty < x < \infty,\ y_k \le y < y_{k-1}$ $w_{k+1}(x, y_k) = w_k(x, y_k),\ \sigma_y^{k+1}(x, y_k) = \sigma_y^k(x, y_k)$	(3.6) (3.7)
……...			
$N-1$	$y_{N-1} = h - \sum_{i=1}^{N-1} \Delta h_i$	$\Delta w_{N-1}(x, y) = 0$ at $-\infty < x < \infty,\ y_{N-1} \le y < y_{N-2}$ $w_N(x, y_{N-1}) = w_{N-1}(x, y_{N-1}), \sigma_y^N(x, y_{N-1}) = \sigma_y^{N-1}(x, y_{N-1})$	(3.8) (3.9)
N	$y_N = h - \sum_{i=1}^{N} \Delta h_i$	$\Delta w_N(x, y) = 0$ at $-\infty < x < \infty,\ y_N \le y < y_{N-1}$ $w_N(x, y_N) = 0$	(3.10) (3.11)

In the presented boundary conditions, the shear stresses $\sigma_y^k(x, y)$ are calculated using the formula

$$\sigma_y^k(x, y) = \mu_k w_{k,y}(x, y) \text{ at } -\infty < x < \infty,\ y_k \le y \le y_{k-1} \tag{3.12}$$

where the subscript y in $w_{k,y}$ denotes the partial derivative of $w_k(x, y)$ with respect to y, and the function $\varphi(x)$ in (3.1) is assumed to be known.

Thus, the auxiliary problem is described by conditions (3.1)–(3.11) of Table 1.

4. Solution to the auxiliary problem

The solution to the auxiliary problem formulated in Section 3, presented in the form of equations and boundary conditions (3.1)-(3.12), is sought using the Fourier integral transform [32]

$$w^F(\alpha, y) = \int_{-\infty}^{\infty} w(x, y) \exp(i\alpha x) dx, \quad w(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w^F(\alpha, y) \exp(-i\alpha x) d\alpha \tag{4.1}$$

where $w^F(\alpha, y)$ is the Fourier transform (image) of the original function $w(x, y)$.

To solve the auxiliary problem, the Fourier transform is successively applied to the Laplace differential equations (3.2), (3.4), (3.6), (3.8), (3.10); to the interface boundary conditions (3.3), (3.5), (3.7), (3.9), (3.11); to the condition at the coating surface (3.1); and to the condition regarding the support of the N -layered coating on a rigid base (3.11). Assuming that the displacements $w_k(x,y)$, $k=1-N$ in each layer vanish at infinity as $|x|\to\infty$, $y_k \le y \le y_{k-1}$, along with their partial derivatives with respect to x, and applying the Fourier transform to Laplace equation (3.6) for the k -th layer reduces it to a second-order ordinary differential equation

$$w^F_{k,yy}(\alpha,y)-\alpha^2 w^F_k(\alpha,y)=0 \text{ at } y_k \le y \le y_{k-1},\ k=1-N \tag{4.2}$$

where $w^F_k(\alpha,y)$ is the Fourier transform of the function $w_k(x,y)$, yy in the subscript means the second derivative with respect to y. It should also be noted here that the general solution to such a differential equation is the transform

$$w^F_k(\alpha,y)=A_k\,\mathrm{sh}(\alpha y)+B_k\,\mathrm{ch}(\alpha y) \text{ at } y_k \le y \le y_{k-1},\ k=1-N \tag{4.3}$$

where A_k, B_k, $k=1-N$ are unknown constants depending on the Fourier parameter α, determined from the $2N$ boundary conditions (3.1)–(3.11).

Applying the Fourier transform to boundary condition (3.1) on the surface of the N -layered coating yields the following expression in terms of Fourier transforms

$$\sigma^{1,F}_y(\alpha,y_0)=-\varphi^F(\alpha),\ \ y_0=h \tag{4.4}$$

where $\sigma^{1,F}_y(\alpha,y_0)$ is a shear stress transform, $\varphi^F(\alpha)$ is a transform of the function $\varphi(x)$. Applying the Fourier transform to the interface boundary conditions (3.7) for the k -th layer leads to conditions in terms of the transforms

$$w^F_{k+1}(\alpha,y_k)=w^F_k(\alpha,y_k),\ k=1-(N-1) \tag{4.5}$$

$$\sigma^{k+1,F}_y(\alpha,y_k)=\sigma^{k,F}_y(\alpha,y_k),\ k=1-(N-1) \tag{4.6}$$

The boundary condition (3.11) for the support of an N -layered coating takes the following form in terms of Fourier transforms

$$w^F_N(\alpha,y_N)=0,\ y_N=0 \tag{4.7}$$

Note that the formulas for calculating stresses (3.12) in terms of Fourier transforms have the form

$$\sigma^{k,F}_y(\alpha,y)=\mu_k w^F_{k,y}(\alpha,y),\ y_k \le y \le y_{k-1},\ k=1-N \tag{4.8}$$

To determine the transform $w^F_k(\alpha,y)$, $k=1-N$ from (4.3) — and, consequently, the unknown constants A_k, B_k, $k=1-N$ — it is necessary to substitute the expression for the transform $w^F_k(\alpha,y)$ in the form of (4.3) and the expression for the transform $\sigma^{k,F}_y(\alpha,y)$ in the form of (4.8) into the boundary conditions (4.4)-(4.7) expressed in terms of Fourier transforms. The boundary condition (4.4) then takes the form

$$\mu_1\alpha\left(A_1\,\mathrm{ch}(\alpha y_0)+B_1\,\mathrm{sh}(\alpha y_0)\right)=-\varphi^F(\alpha),\quad y_0=h \tag{4.9}$$

The interface boundary conditions (4.5) and (4.6) take the form

$$A_{k+1}\,\mathrm{sh}(\alpha y_k)+B_{k+1}\,\mathrm{ch}(\alpha y_k)=A_k\,\mathrm{sh}(\alpha y_k)+B_k\,\mathrm{ch}(\alpha y_k) \tag{4.10}$$

$$\mu_{k+1}\alpha\left(A_{k+1}\,\mathrm{ch}(\alpha y_k)+B_{k+1}\,\mathrm{sh}(\alpha y_k)\right)=\mu_k\alpha\left(A_k\,\mathrm{ch}(\alpha y_k)+B_k\,\mathrm{sh}(\alpha y_k)\right) \tag{4.11}$$

where $k=1-(N-1)$.

Boundary condition (4.7) at $y=y_N$ takes the form

$$A_N\,\mathrm{sh}(\alpha y_N)+B_N\,\mathrm{ch}(\alpha y_N)=0,\quad y_N=0 \tag{4.12}$$

To determine the unknown constants A_k, B_k, $k=1-N$ in the total amount $2N$, we have a non-homogeneous system of linear algebraic equations, consisting of equations (4.9)-(4.12), yielding a total of $2N$

equations. Thus, the number of unknowns in the problem coincides with the number of equations in the non-homogeneous system of linear algebraic equations (4.9)-(4.12) for determining the $2N$ unknowns A_k, B_k, $k=1-N$. To facilitate the solution of the specified non-homogeneous system of linear algebraic equations, the following matrix functions are introduced [7].

$$C_m^k(\alpha)=\begin{pmatrix} \mathrm{sh}(\alpha y_k) & \mathrm{ch}(\alpha y_k) \\ \mu_m\alpha\,\mathrm{ch}(\alpha y_k) & \mu_m\alpha\,\mathrm{sh}(\alpha y_k)\end{pmatrix} \quad X_k=\begin{pmatrix} A_k \\ B_k\end{pmatrix} \quad m,k=1-N \tag{4.13}$$

As a result, the internal part of the system of linear algebraic equations of the problem in the form (4.10), (4.11) is written in the following matrix form

$$C_{k+1}^k(\alpha)X_{k+1}=C_k^k(\alpha)X_k \quad k=1-(N-1) \tag{4.14}$$

The matrices $C_m^k(\alpha)$ from (4.13) are non-singular, since their determinants are non-zero

$$\det\left(C_m^k(\alpha)\right)\neq 0\,,$$

and the matrices

$$C_m^{k,-1}(\alpha)=\begin{pmatrix} -\mathrm{sh}(\alpha y_k) & (\mu_m\alpha)^{-1}\mathrm{ch}(\alpha y_k) \\ \mathrm{ch}(\alpha y_k) & -(\mu_m\alpha)^{-1}\mathrm{sh}(\alpha y_k)\end{pmatrix}, \quad m,k=1-N \tag{4.15}$$

are inverses of the matrices $C_m^k(\alpha)$ and the following equalities hold

$$C_m^{k,-1}(\alpha)\cdot C_m^k(\alpha)=E\,, \quad m,k=1-N \tag{4.16}$$

where E is the second-order identity matrix.

First, we obtain the relationship between X_k and X_{k+1} by multiplying equality (4.14) from the left by the inverse matrix $C_m^{k,-1}(\alpha)$, and taking (4.16) into account, we have

$$C_k^{k,-1}(\alpha)\cdot C_{k+1}^k(\alpha)\cdot X_{k+1}=X_k\,, \quad k=1-(N-1) \tag{4.17}$$

where preceding X_k arises from the following X_{k+1}.

Subsequently, the relations (4.17) between X_k $k=1-(N-1)$, derived from (4.14), are used. The result is a chain of matrix equations for the problem, combined with the boundary conditions (4.9) (at the coating surface) and (4.12) (at the interface with the substrate), as listed in the second column of Table 2.

Starting from the bottom row of the second column of Table 2, we successively substitute X_N into X_{N-1}, X_{N-1} into X_{N-2}, ..., X_3 into X_2, X_2 into X_1, recording the results in the third column of formulas using matrices $C_m^{N-1}(\alpha)$ $m=1-N$, starting with (4.24) and ending with (4.20). As a result, in the third column of (4.20), we obtain a relation expressing X_1 in terms of X_N in the form

$$X_1=C_1^{N-1}(\alpha)\cdot X_N \text{ where } C_1^{N-1}(\alpha)=\prod_{k=1}^{N-1}C_k^{k,-1}(\alpha)\cdot C_{k+1}^k(\alpha), \quad C_1^0=E \tag{4.26}$$

The elements of the matrix function $C_1^{N-1}(\alpha)$ are functions of α and are written in the form $c_{ij}^1(\alpha)$ $i,j=1,2$. In the general case, the matrix $C_m^{N-1}(\alpha)$ $m=1-N$ has the form

$$\begin{gathered} C_m^{N-1}(\alpha)=\prod_{k=m}^{N-1}C_k^{k,-1}(\alpha)\cdot C_{k+1}^k(\alpha) \text{ for } m=1-N \\ C_m^{N-1}(\alpha)=E \text{ for } m>N-1 \end{gathered} \tag{4.27}$$

elements $c_{ij}^{m}(\alpha)$ $i,j=1,2$, $m=1-N$ of which are functions of α.

Table 2

0	$\hat{C}(\alpha,y_0)\cdot X_1=-\varphi^F(\alpha)$	$\hat{C}(\alpha,y_0)\cdot C_1^{N-1}(\alpha)\cdot X_N=-\varphi^F(\alpha)$	(4.19)
1	$X_1=C_1^{1,-1}(\alpha)\cdot C_2^1(\alpha)\cdot X_2$	$X_1=C_1^{N-1}(\alpha)\cdot X_N$	(4.20)
2	$X_2=C_2^{2,-1}(\alpha)\cdot C_3^2(\alpha)\cdot X_3$	$X_2=C_2^{N-1}(\alpha)\cdot X_N$	(4.21)
……...			
k	$X_k=C_k^{k,-1}(\alpha)\cdot C_{k+1}^k(\alpha)\cdot X_{k+1}$	$X_k=C_k^{N-1}(\alpha)\cdot X_N$	(4.22)
……...			
$N-2$	$X_{N-2}=C_{N-2}^{N-2,-1}(\alpha)\cdot C_{N-1}^{N-2}(\alpha)\cdot X_{N-1}$	$X_{N-2}=C_{N-2}^{N-1}(\alpha)\cdot X_N$	(4.23)
$N-1$	$X_{N-1}=C_{N-1}^{N-1,-1}(\alpha)\cdot C_N^{N-1}(\alpha)\cdot X_N$	$X_{N-1}=C_{N-1}^{N-1}(\alpha)\cdot X_N$	(4.24)
N	$C(\alpha,y_N)\cdot X_N=0 \quad X_N=\begin{pmatrix}A_N\\0\end{pmatrix}$	$X_N=A_NC_0^1 \quad C_0^1=\begin{pmatrix}1\\0\end{pmatrix}$	(4.25)

Relations (4.20)-(4.24) from the second column made it possible to establish a matrix relationship between the unknown matrices X_1 and X_N (the third column), the elements of which are determined using the remaining unused boundary conditions (4.9) and (4.12) — written in a matrix form in (4.19) and (4.25), respectively — where the adopted notations are

$$\hat{C}(\alpha,y_0)=\begin{pmatrix}\alpha\,\mathrm{ch}(\alpha y_0) & \alpha\,\mathrm{sh}(\alpha y_0)\end{pmatrix} \text{ for } y_0=h \tag{4.28}$$

$$C(\alpha,y_N)=\begin{pmatrix}\mathrm{sh}(\alpha y_N) & \mathrm{ch}(\alpha y_N)\end{pmatrix} \text{ for } y_N=0 \tag{4.29}$$

after substituting $y_N=0$ into which we get that

$$C(\alpha,y_N)=\begin{pmatrix}0 & 1\end{pmatrix}$$

Thus, solving the system of linear algebraic equations (4.19)-(4.25) located in the second column for the $2N$ unknowns reduced to solving two matrix equations (4.19) and (4.25) and the matrix equality (4.26)

$$\hat{C}(\alpha,y_0)\cdot X_1=-\varphi^F(\alpha) \text{ for } y_0=h \tag{4.30}$$

$$C(\alpha,y_N)\cdot X_N=0 \text{ for } y_N=0 \tag{4.31}$$

$$C_1^{N-1}(\alpha)\cdot X_N=X_1 \tag{4.32}$$

After substituting (4.29) into (4.25) we obtain that

$$B_N=0,\ X_N=\begin{pmatrix}A_N\\0\end{pmatrix}=A_N\begin{pmatrix}1\\0\end{pmatrix},\ C_0^1=\begin{pmatrix}1\\0\end{pmatrix} \tag{4.33}$$

Now, to determine the elements (or, more precisely, the element A_N) it is sufficient to substitute X_1 from (4.32) into (4.30), and from the resulting equation

$$\hat{C}(\alpha,y_0)\cdot C_1^{N-1}(\alpha)\cdot X_N=-\varphi^F(\alpha) \tag{4.34}$$

we obtain that

$$A_N=-\frac{\varphi^F(\alpha)}{\hat{C}(\alpha,y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1} \tag{4.35}$$

To determine X_1, it is now sufficient to use relation (4.32) by substituting the previously determined values X_N (or A_N) from (4.35). As a result, in matrix form we have

$$X_1 = C_1^{N-1}(\alpha)\cdot X_N = C_1^{N-1}(\alpha)\cdot A_N C_0^1 \tag{4.36}$$

After substituting A_N from (4.35) into (4.36), we obtain

$$X_1 = -\varphi^F(\alpha)\frac{C_1^{N-1}(\alpha)\cdot C_0^1}{\hat{C}(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1} \tag{4.37}$$

To derive the integral equation for the contact problem and to determine the deformed state in the k-th layer ($k = 1-N$), it is necessary to determine the displacement function $w_k(x,y)$ at $-\infty < x < \infty$, $y_k \le y \le y_{k-1}$, $k = 1-N$. For this purpose, the transform $w_k^F(\alpha, y)$ is determined according to (4.3)

$$w_k^F(\alpha, y) = A_k \operatorname{sh}(\alpha y) + B_k \operatorname{ch}(\alpha y) \text{ at } y_k \le y \le y_{k-1},\ k = 1-N \tag{4.38}$$

which, in matrix form, takes the form

$$w_k^F(\alpha, y) = C(\alpha, y)\cdot X_k,\ C(\alpha, y) = \begin{pmatrix}\operatorname{sh}(\alpha y) & \operatorname{ch}(\alpha y)\end{pmatrix} \tag{4.39}$$
$$\text{for } y_k \le y \le y_{k-1},\ k = 1-N$$

To solve this problem, we use Table 2 (or, more precisely, its third column). To construct $w_k^F(\alpha, y)$, $k = 1-N$, we begin with the bottom layer, where the transform $w_N^F(\alpha, y)$ is constructed using formula (4.39) and the now-known X_N from (4.25) of Table 2. By analogy with the construction of the third column of Table 2, the value $X_{N-1} = C_{N-1}^{N-1}(\alpha)\cdot X_N$ from the third column is substituted into the formula for $w_{N-1}^F(\alpha, y) = C(\alpha, y)\cdot X_{N-1}$, where $C_{N-1}^{N-1}(\alpha)$ is constructed using formula (4.27), and so on. The matrix X_N (or, more precisely, its non-zero part A_N) is determined by formula (4.35), and therefore

$$w_k^F(\alpha, y) = -\varphi^F(\alpha)\frac{C(\alpha, y)\cdot C_k^{N-1}(\alpha)\cdot C_0^1}{\hat{C}(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1} \text{ for } y_k \le y \le y_{k-1},\ k = 1-N \tag{4.40}$$

In (4.40) $C(\alpha, y)$ is from (4.39), $C_k^{N-1}(\alpha) = \{c_{ij}^k\}$ $i, j = 1,2$ is from (4.27), $C_1^{N-1}(\alpha) = \{c_{ij}^1\}$ $i, j = 1,2$ is from (4.26), $\hat{C}(\alpha, y_0)$ is from (4.28), C_0^1 is from (4.33).

The shear stress transform $\sigma_y^{k,F}(\alpha, y)$ for the k-th layer, defined in (4.8) in terms of the derivative of the displacement transform $w_k^F(\alpha, y)$, has the following form in matrix notation

$$\sigma_y^{k,F}(\alpha, y) = -\mu_k \varphi^F(\alpha)\frac{C'(\alpha, y)\cdot C_k^{N-1}(\alpha)\cdot C_0^1}{\hat{C}(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1} \text{ for } y_k \le y \le y_{k-1},\ k = 1-N \tag{4.41}$$

where the prime over $C(\alpha, y)$ denotes the derivative with respect to y.

The presence of displacement transforms $w_k^F(\alpha, y)$ and stress transforms $\sigma_y^{k,F}(\alpha, y)$ makes it possible, using the inverse Fourier transform according to the second formula in (4.1), to determine $w_k(x,y)$ and $\sigma_y^k(x,y)$ in the form

$$\begin{Bmatrix} w_k(x,y) \\ \sigma_y^k(x,y) \end{Bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{Bmatrix} w_k^F(\alpha, y) \\ \sigma_y^{k,F}(\alpha, y) \end{Bmatrix} \exp(-i\alpha x) d\alpha \tag{4.42}$$

$$\text{for } -\infty < x < \infty,\ y_k \le y \le y_{k-1},\ k = 1 - N$$

Taking into account formulas (4.40) and changing the order of integration in the resulting double integral (4.42) for $w_k(x,y)$, we write the expression as follows

$$w_k(x,y) = -\frac{1}{2\pi\mu_1}\int_{-a}^{a}\varphi(\xi)d\xi\int_{-\infty}^{\infty}W_k(\alpha,y)\exp(i\alpha(\xi-x))d\alpha \tag{4.43}$$

$$\text{for } -\infty < x < \infty,\ y_k \le y \le y_{k-1},\ k = 1 - N$$

To determine $W_k(\alpha,y)$, we use the formulas (4.40) for transforms $w_k^F(\alpha,y)\quad k = 1-N$, from which follows in matrix form

$$W_k(\alpha,y) = \frac{C(\alpha,y)\cdot C_k^{N-1}(\alpha)\cdot C_0^1}{\hat{C}(\alpha,y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1} \quad \text{for} \quad y_k \le y \le y_{k-1},\ k = 1-N \tag{4.44}$$

or in partially matrix form

$$W_k(\alpha,y) = \frac{c_{11}^k(\alpha)\mathrm{sh}(\alpha y)+c_{21}^k(\alpha)\mathrm{ch}(\alpha y)}{\alpha\left(c_{11}^1(\alpha)\mathrm{ch}(\alpha y_0)+c_{21}^1(\alpha)\mathrm{sh}(\alpha y_0)\right)} \quad \text{for} \quad y_k \le y \le y_{k-1},\ k = 1-N \tag{4.45}$$

where $c_{ij}^k(\alpha)$ are elements of the matrix $C_k^{N-1}(\alpha)$ from (4.27), y_k for $k = 1-N$ are the ordinates of the interfaces.

Remark 1. The function $w_1(x,y_0)$ from (4.43) for $y = y_0$ defines the mixed boundary condition of the contact problem on the upper surface of the N-layered coating, in accordance with condition (2.3).

The shear stresses $\sigma_y^k(x,y)$ in the coating layers are determined by the second formula in (4.42) and, after changing the order of integration, takes the form

$$\sigma_y^k(x,y) = -\frac{1}{2\pi}\frac{\mu_k}{\mu_1}\int_{-a}^{a}\varphi(\xi)d\xi\int_{-\infty}^{\infty}V_k(\alpha,y)\exp(i\alpha(\xi-x))d\alpha \tag{4.46}$$

$$\text{for } -\infty < x < \infty,\ y_k \le y \le y_{k-1},\ k = 1 - N$$

and the use of the transform (4.41) leads to the formula

$$V_k(\alpha,y) = \frac{C'(\alpha,y)\cdot C_k^{N-1}(\alpha)\cdot C_0^1}{\hat{C}(\alpha,y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1} \quad \text{for} \quad y_k \le y \le y_{k-1},\ k = 1-N \tag{4.47}$$

where $C'(\alpha,y) = \begin{pmatrix}\alpha\,\mathrm{ch}(\alpha y) & \alpha\,\mathrm{sh}(\alpha y)\end{pmatrix}$

or in partially matrix form

$$V_k(\alpha,y) = \frac{c_{11}^k(\alpha)\mathrm{ch}(\alpha y)+c_{21}^k(\alpha)\mathrm{sh}(\alpha y)}{c_{11}^1(\alpha)\mathrm{ch}(\alpha y_0)+c_{21}^1(\alpha)\mathrm{sh}(\alpha y_0)} \text{ for } y_k \le y \le y_{k-1},\ k = 1-N \tag{4.48}$$

Remark 2. For $k = 1$ and $y = y_0 = h$, from (4.46), we obtain the stress on the upper surface of the N-layered coating at $y = y_0$ (that is, we arrive at the boundary condition (2.5) for the contact problem concerning shear of an N-layered coating).

5. Integral equation of the contact problem

The displacement formula for $w_k(x,y)$ for $k = 1$ and $y = y_0$ is given by formula (4.42), in which the transform $w_1^F(\alpha,y_0)$ is defined in (4.40) in matrix form

$$w_1^F(\alpha, y) = -\varphi^F(\alpha)\frac{C(\alpha, y)\cdot C_1^{N-1}(\alpha)\cdot C_0^1}{\hat{C}(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1} \text{ for } y_1 \le y \le y_0 \tag{5.1}$$

Transitioning from y to y_0 in formula (5.1), we obtain the transform of the displacements at the surface of the upper layer

$$w_1^F(\alpha, y_0) = -\varphi^F(\alpha)\frac{C(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1}{\hat{C}(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1} \tag{5.2}$$

Using formula (4.43), we obtain the original of the displacement function $w_1(x, y_0)$ in the form

$$w_1(x, y_0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi^F(\alpha)\frac{C(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1}{\hat{C}(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1}\exp(-i\alpha x)d\alpha \text{ for } -\infty < x < \infty \tag{5.3}$$

Taking into account the mixed boundary conditions (2.3) and (2.5) for $y = y_0$, and changing the order of integration in (5.3), we obtain the integral equation for the contact shear problem under consideration

$$\int_{-a}^{a}\varphi(\xi)k_{N-1}(\xi - x)d\xi = 2\pi\mu_1\delta \text{ for } -a \le x \le a \tag{5.4}$$

the kernel of which $k_{N-1}(\zeta)$ has the form

$$k_{N-1}(\zeta) = \int_{-\infty}^{\infty}K_{N-1}(\alpha)\exp(i\alpha\zeta)d\alpha \tag{5.5}$$

and the kernel transform - function $K_{N-1}(\alpha)$ - takes the form in matrix notation

$$K_{N-1}(\alpha) = \frac{C(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1}{\hat{C}(\alpha, y_0)\cdot C_1^{N-1}(\alpha)\cdot C_0^1} \tag{5.6}$$

where matrix $C(\alpha, y_0)$ is defined in (4.39), matrix $C_1^{N-1}(\alpha) = \{c_{ij}^1\}$ $i, j = 1,2$ is defined in (4.26), matrix $\hat{C}(\alpha, y_0)$ is defined in (4.28), and matrix C_0^1 is defined in (4.33). We carry out the matrix multiplication in the numerator and denominator of transform (5.6) and obtain its partially matrix form

$$K_{N-1}(\alpha) = \frac{c_{11}^1(\alpha)\mathrm{sh}(\alpha y_0) + c_{21}^1(\alpha)\mathrm{ch}(\alpha y_0)}{\alpha\left(c_{11}^1(\alpha)\mathrm{ch}(\alpha y_0) + c_{21}^1(\alpha)\mathrm{sh}(\alpha y_0)\right)},\ y_0 = h \tag{5.7}$$

where $c_{11}^1(\alpha)$, $c_{21}^1(\alpha)$ are elements of the matrix $C_1^{N-1}(\alpha)$ and are functions of α, dependent on the problem parameters: the shear moduli of the layer materials μ_k and the layer thicknesses Δh_k $k = 1 - N$.

The integral equation (5.4)-(5.7) for the pure shear contact problem of the N-layered coating (given for $\mu_1 = \mu_k$ $k = 2 - N$ or for $N = 1$) reduces to the integral equation for the pure shear of a homogeneous layer of thickness h with shear modulus μ_1, rigidly bonded to a non-deformable substrate. In this case, the matrix elements $c_{11}^1 = 1$, $c_{21}^1 = 0$, since in this case relation (4.32) for $N = 1$ becomes $C_1^0 \cdot X_1 = X_1$, whence $C_1^0 = E$. Substituting determined values c_{11}^1, c_{21}^1 into (5.7), we find that the kernel transform $K_{1-1}(\alpha)$ takes the form

$$K_{1-1}(\alpha) = \alpha^{-1}\,\mathrm{th}(\alpha y_0) \tag{5.8}$$

which corresponds to the transform of the kernel of the integral equation for the contact problem of shear of a homogeneous layer bonded to a rigid substrate [31]. On the other hand, in this case, all matrix products in (4.26) (including those involving terms that reduce to the identity matrix according to (4.16)) simplify accordingly; it follows that the relevant quantities take specific forms, and formula (5.7) reduces to (5.8).

On the other hand, in this case, based on the fact that $\mu_1 = \mu_k \quad k = 2 - N$, all the matrix products in (4.26), reduce to identity matrices in accordance with (4.16), including $C_1^0 = E$. Also from $c_{11}^1 = c_{22}^1 = 1$, $c_{12}^1 = c_{21}^1 = 0$, and for $N = 1$ formula (5.7) for $K_{N-1}(\alpha)$ simplifies and takes the form (5.8).

To facilitate further analysis, the integral equation (5.4) is transformed into the corresponding dimensionless integral equation using the substitutions $\xi = a\xi'$, $d\xi = ad\xi'$, $x = ax'$, $\alpha = h^{-1}u$, $d\alpha = h^{-1}du$

$$\int_{-1}^{1} \varphi(\xi) k_{N-1}\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi\mu_1 \frac{\delta}{a} \text{ for } -1 \le x \le 1,\ \lambda = \frac{h}{a} \tag{5.9}$$

in which the kernel $k_{N-1}(\zeta)$ has the form

$$k_{N-1}(\zeta) = \int_0^{\infty} K_{N-1}(u)\cos(u\zeta)du \tag{5.10}$$

The transform $K_{N-1}(u)$ of the kernel of the integral equation (5.9) is represented in matrix form by the formula

$$K_{N-1}(u) = \frac{C(u)\cdot C_1^{N-1}(u)\cdot C_0^1}{\hat{C}(u)\cdot C_1^{N-1}(u)\cdot C_0^1} \tag{5.11}$$

where the matrix $C(u) = (\mathrm{sh}(u) \quad \mathrm{ch}(u))$ is defined in (4.39) for $\alpha h = u$, the matrix $C_1^{N-1}(u)$ is defined in (4.26), the matrix $\hat{C}(u) = h^{-1}u(\mathrm{ch}(u) \quad \mathrm{sh}(u))$ is defined in (4.28) for $\alpha h = u$, and the matrix C_0^1 is defined in (4.33). By performing the multiplication of the matrix functions in (5.11), we obtain the transform in a partially matrix form.

$$K_{N-1}(u) = \frac{c_{11}^1(u)\mathrm{sh}(u) + c_{21}^1(u)\mathrm{ch}(u)}{u\left(c_{11}^1(u)\mathrm{ch}(u) + c_{21}^1(u)\mathrm{sh}(u)\right)} \tag{5.12}$$

where $c_{ij}^1(u) \quad i, j = 1,2$ are elements of the matrix function $C_1^{N-1}(u)$ from (4.26).

6. Properties of the kernel of an integral equation and its transform

To construct the solution to the integral equation (5.9) for the anti-plane contact problem for a pure shear of the N-layered elastic coating of thickness h by a rigid strip indenter of width $2a$ — where the coating is bonded to a non-deformable base (it is necessary, as an initial step, to identify the fundamental properties of the kernel transform $K_{N-1}(u)$ given by formula (5.11)). The identified properties of the transform $K_{N-1}(u)$ will help to investigate the fundamental properties of the primary characteristics of the integral equation (its kernel $k_{N-1}(\zeta)$, represented by formula (5.11)).

Lemma 1. In the complex plane of the integration variable $u = \sigma + i\tau$, the Fourier transform $K_{N-1}(u)$ is a real-valued even function for $\tau = 0$, and meromorphic function in the complex plane of the integration variable $u = \sigma + i\tau$.

To prove the lemma, we use the matrix representation of the transform $K_{N-1}(u)$ given by (5.11). The matrix function $C_1^{N-1}(u)$ appearing in the numerator and denominator of the transform $K_{N-1}(u)$ in (5.11) is an important component thereof and consists of pairwise products of the matrix functions $C_k^{k,-1}(u)\cdot C_{k+1}^k(u)$ from (4.15) and (4.13), respectively, each of which is explicitly represented by the matrix function

$$C_k^{k,-1}(u)\cdot C_{k+1}^k(u) = \begin{pmatrix} -\mathrm{sh}^2(u\gamma_k) + \beta_k \mathrm{ch}^2(u\gamma_k) & -(1-\beta_k)\mathrm{sh}(u\gamma_k)\mathrm{ch}(u\gamma_k) \\ (1-\beta_k)\mathrm{sh}(u\gamma_k)\mathrm{ch}(u\gamma_k) & \mathrm{ch}^2(u\gamma_k) - \beta_k \mathrm{sh}^2(u\gamma_k) \end{pmatrix} \tag{6.1}$$

where $\gamma_k = 1 - \sum_{i=1}^{k} \frac{\Delta h_i}{h}$, $\beta_k = \frac{\mu_{k+1}}{\mu_k}$.

The elements on the main diagonal of the matrix function (6.1) are even functions with respect to u, while the elements on the secondary diagonal are odd functions with respect to u. Based on this property of elements of the matrix function $C_k^{k,-1}(u) \cdot C_{k+1}^{k}(u)$, it can be shown that the elements $c_{11}^{1}(u)$ and $c_{22}^{1}(u)$ located on the main diagonal of the matrix function $C_1^{N-1}(u)$ are even functions of u, whereas the elements $c_{12}^{1}(u)$ and $c_{21}^{1}(u)$ are odd functions of u. The identified properties of elements $c_{ij}^{1}(u) \quad i, j = 1,2$ of the matrix function $C_1^{N-1}(u)$, taking into account the properties of matrix functions $C(u)$, $\hat{C}(u)$ and C_0^1, allow us to conclude that the kernel transform $K_{N-1}(u)$ from (5.11) of the integral equation (5.9) is an even function. In the complex plane of the integration variable $u = \sigma + i\tau$, the transform $K_{N-1}(u)$ from (5.11) of the integral equation (5.9) is represented as the ratio of integer functions of exponential type, is a regular function on the real axis ($\mathrm{Im}(u) = 0$) and meromorphic function in the complex plane of the integration variable $u = \sigma + i\tau$.

Lemma 2. The Fourier transform $K_{N-1}(u)$ (5.11), (5.12) has the following asymptotic representations on the real axis of the integration variable

$$K_{N-1}(u) = \frac{1}{|u|} + O\left(\frac{e^{-2|u|}}{|u|^3}\right) \text{ for } |u| \to \infty \tag{6.2}$$

$$K_{N-1}(u) = 1 + O\left(u^2\right) \text{ for } |u| \to 0 \tag{6.3}$$

The proof of (6.2) follows from the matrix form of the transform $K_{N-1}(u)$ given by (5.12), in which functions $\mathrm{sh}(u)$, $\mathrm{ch}(u)$, $\mathrm{sh}(u\gamma_k)$, $\mathrm{ch}(u\gamma_k)$ are replaced by their asymptotic forms

$$\begin{Bmatrix} \mathrm{sh}(u) \\ \mathrm{ch}(u) \end{Bmatrix} \approx \frac{1}{2} e^{u}\left(1 \mp e^{-2u}\right), \begin{Bmatrix} \mathrm{sh}(u\gamma_k) \\ \mathrm{ch}(u\gamma_k) \end{Bmatrix} \approx \frac{1}{2} e^{\gamma_k u}\left(1 \mp e^{-2\gamma_k u}\right) \quad \text{for } |u| \to \infty$$

and operations are performed on matrices.

The proof of (6.3) employs equivalent infinitesimal quantities and matrix operations.

Lemma 3. In the strip $0 \le |\zeta| < \infty$ and $|\eta| < 2$ of the complex plane $\zeta + i\eta$, the representation holds

$$k_{N-1}(\zeta) = -\ln|\zeta| - \mathcal{F}_{N-1}(\zeta)$$

$$\mathcal{F}_{N-1}(\zeta) = \int_0^{\infty} u^{-1}\left[\left(1 - uK_{N-1}(u)\right)\cos(u\zeta) - e^{-u}\right]du \tag{6.4}$$

where $\mathcal{F}_{N-1}(\zeta)$ is even function with respect to ζ, is regular function in the specified strip and is represented by an absolutely convergent series

$$\mathcal{F}_{N-1}(\zeta) = \sum_{k=0}^{\infty} d_k \zeta^{2k} \tag{6.5}$$

in which the coefficients of the series $d_k \quad k = 0,1,2,\ldots$ are represented by formulas

$$d_0 = \int_0^{\infty} u^{-1}\left(1 - uK_{N-1}(u) - e^{-u}\right)du \tag{6.6}$$

$$d_k = \frac{(-1)^k}{(2k)!}\int_0^{\infty}\left(1 - uK_{N-1}(u)\right)u^{2k-1}du\,,\ k = 1,2,3,...$$

The proof of the lemma is based on the expansion of the function $\cos(u\zeta)$ into a power series [33].

7. Solution of the integral equation (5.9)

The solution of the integral equation (5.9) with the logarithmic singularity (6.4) in the kernel $k_{N-1}(\xi)$ is carried out in three steps. On the first step of solving integral equation (5.9), both its left- and right-hand sides are differentiated, resulting in an equivalent singular equation. On the second step, the singular integral equation is inverted, resulting in an equivalent integral equation of the second kind [31] of the form

$$\varphi(x) = \frac{1}{\pi\sqrt{1-x^2}}\left[T - \frac{1}{\pi\lambda}\int_{-1}^{1}\frac{\sqrt{1-\xi^2}}{\xi - x}\,d\xi\int_{-1}^{1}\varphi(\tau)F'_{N-1}\left(\frac{\tau-\xi}{\lambda}\right)d\tau\right] \text{ for } -1 \le x \le 1 \tag{7.1}$$

in which $T = \int_{-1}^{1}\varphi(x)dx$ is given by the formula

$$T = \frac{1}{\ln 2\lambda}\left[\mu_1\frac{\delta}{a}\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi^2}} - \frac{1}{\pi}\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi^2}}\int_{-1}^{1}\varphi(\tau)F_{N-1}\left(\frac{\tau-\xi}{\lambda}\right)d\tau\right] \tag{7.2}$$

On the third step, the integral equation (7.1), taking into account (6.4) and (6.5), is solved using the method of successive iterative approximations or the asymptotic method of expansion in terms of a small parameter λ^{-1}. For the indenter with a flat base, the solution constructed in [33] in the form of an asymptotic series up to λ^{-6} has the form

$$\varphi(x) = \frac{T}{\pi\sqrt{1-x^2}}\left[1 - \frac{2d_1}{\lambda^2}\left(\frac{1}{2} - x^2\right) - \frac{4d_2}{\lambda^4}\left(\frac{7}{8} - x^2 - x^4\right) + O\left(\lambda^{-6}\right)\right] \text{ for } -1 \le x \le 1 \tag{7.3}$$

in which the shear force

$$T = \int_{-1}^{1}\varphi(t)dt \tag{7.4}$$

is calculated using the formula

$$T = \pi\mu_1\frac{\delta}{a}\left[\ln 2\lambda - d_0 - \frac{d_1}{\lambda^2} - \frac{d_1^2 + 9d_2}{4\lambda^4} + O\left(\lambda^{-6}\right)\right]^{-1} \tag{7.5}$$

The coefficients d_k $k = 0,1,2,...$ are calculated using formulas (6.6).

Another scheme for solving the integral equation (5.9)-(5.12) for arbitrary values of the parameter λ was developed in [35] and frequently used, for example, in [20-24].

Summarizing the solution to integral equation (5.9), the final theorem is formulated: if the right-hand side of equation (5.9) is constant on the interval $-1 \le x \le 1$, then integral equation (5.9) (and consequently, equations (7.1) and (7.2)) is uniquely solvable in the space $L_p(-1,1)$ $(1 < p < 2)$ for $0 < \lambda < \infty$, and its solution can be represented in the form $\varphi(x) = \omega(x)\left(1-x^2\right)^{-\frac{1}{2}}$, where $\omega(x) \in C(-1,1)$.

8. Analysis of the stress-strain state of interfaces using as an example a three-layered coating

In Section 4, formulas (4.43)-(4.45) for the displacements $w_k(x,y)$, $|x|<\infty$, $y_k \le y \le y_{k-1}$, $k=1-N$ in each layer of an N-layered coating and formulas (4.46)-(4.48) for the shear stresses $\sigma_y^k(x,y)$, $|x|<\infty$, $y_k \le y \le y_{k-1}$, $k=1-N$ in each of the layers are derived.

To analyze the stress-strain state of a three-layer coating, it is necessary to derive formulas for displacements $w_k(x,y)$ and stresses $\sigma_y^k(x,y)$ for $y=y_0$ and $y=y_3$ (at the coating boundaries) and for displacements $w_k(x,y_k)$ and stresses $\sigma_y^k(x,y_k)$ for $k=1,2$ (at the interfaces), for the case where a three-layered coating of thickness h is subjected to a surface shear of distance δ applied to the top layer by a rigid strip indenter, while the bottom layer is rigidly bonded to a non-deformable base.

For convenience of calculations in formula (4.43) we will make the substitutions $\xi = a\xi'$, $d\xi = ad\xi'$, $x = ax'$, $\alpha = h^{-1}u$, $d\alpha = h^{-1}du$ and obtain formulas for $w_k(x,y)$ in dimensionless variables, immediately replacing $y=y_k$, $k=1-3$ with γ_k

$$w_k(x,y_k) = -\frac{a}{\pi\mu_1}\int_{-1}^{1}\varphi(\xi)d\xi\int_0^{\infty}W_k(u,\gamma_k)\cos\left(u\frac{\xi-x}{\lambda}\right)du \quad \text{for } -\infty<x<\infty,\ k=1-3,\ \lambda=\frac{h}{a} \tag{8.1}$$

$$W_k(u,\gamma_k) = \frac{c_{11}^k(u)\mathrm{sh}(u\gamma_k)+c_{21}^k(u)\mathrm{ch}(u\gamma_k)}{u\left(c_{11}^1(u)\mathrm{ch}(u)+c_{21}^1(u)\mathrm{sh}(u)\right)} \quad \text{for } k=1-3, \tag{8.2}$$

$$y_k = h-\sum_{i=1}^{k}\Delta h_i = h\gamma_k,\ \gamma_k = 1-\sum_{i=1}^{k}\frac{\Delta h_i}{h},\ \gamma_0=1 \tag{8.3}$$

For $y=y_0=h$ the formula $W_1(u,\gamma_0)$ in (8.2) takes the form

$$W_1(u,\gamma_0) = \frac{c_{11}^1(u)\mathrm{sh}(u)+c_{21}^1(u)\mathrm{ch}(u)}{u\left(c_{11}^1(u)\mathrm{ch}(u)+c_{21}^1(u)\mathrm{sh}(u)\right)} \tag{8.4}$$

because $\gamma_0=1$.

Comparing (8.4) with the transform of the kernel (5.12) of the integral equation (5.9) and taking Remark 1 into account, we obtain the function $W_1(u,\gamma_0)$ from (8.4) as the transform of the kernel $K_{3-1}(u)$ of the integral equation for a three-layered coating — that is

$$K_{3-1}(u) = W_1(u,\gamma_0) = \frac{c_{11}^1(u)\mathrm{sh}(u)+c_{21}^1(u)\mathrm{ch}(u)}{u\left(c_{11}^1(u)\mathrm{ch}(u)+c_{21}^1(u)\mathrm{sh}(u)\right)}$$

in which $c_{ij}^1(u)$ $i,j=1,2$ are elements of the matrix $C_1^{3-1}(u)$

$$C_1^{3-1}(u) = \begin{pmatrix} c_{11}^1(u) & c_{12}^1(u) \\ c_{21}^1(u) & c_{22}^1(u) \end{pmatrix} \tag{8.5}$$

The matrix $C_1^{3-1}(u)$ is defined in (4.27), and $c_{ij}^1(u)$ $i,j=1,2$ have the form

$$\begin{aligned}
c_{11}^1(u) &= r(u)\left[\left(-\mathrm{th}^2(u\gamma_1)+\beta_1\right)\left(-\mathrm{th}^2(u\gamma_2)+\beta_2\right)-(1-\beta_1)\mathrm{th}(u\gamma_1)(1-\beta_2)\mathrm{th}(u\gamma_2)\right] \\
c_{12}^1(u) &= r(u)\left[-\left(-\mathrm{th}^2(u\gamma_1)+\beta_1\right)(1-\beta_2)\mathrm{th}(u\gamma_2)-(1-\beta_1)\mathrm{th}(u\gamma_1)\left(1-\beta_2\,\mathrm{th}^2(u\gamma_2)\right)\right] \\
c_{21}^1(u) &= r(u)\left[(1-\beta_1)\mathrm{th}(u\gamma_1)\left(-\mathrm{th}^2(u\gamma_2)+\beta_2\right)+\left(1-\beta_1\,\mathrm{th}^2(u\gamma_1)\right)(1-\beta_2)\mathrm{th}(u\gamma_2)\right] \\
c_{22}^1(u) &= r(u)\left[-(1-\beta_1)\mathrm{th}(u\gamma_1)(1-\beta_2)\mathrm{th}(u\gamma_2)+\left(1-\beta_1\,\mathrm{th}^2(u\gamma_1)\right)\left(1-\beta_2\,\mathrm{th}^2(u\gamma_2)\right)\right]
\end{aligned} \tag{8.6}$$

$r(u)=\mathrm{ch}^2(u\gamma_1)\mathrm{ch}^2(u\gamma_2)$

The integral equation of the contact problem of shear of a three-layered coating, therefore, has the form (5.9), in which $N=3$

$$\int_{-1}^{1}\varphi(\xi)k_{3-1}\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi\mu_1\frac{\delta}{a}\ \text{for}\ -1\le x\le 1,\ \lambda=\frac{h}{a} \tag{8.7}$$

$$k_{3-1}(t)=\int_0^{\infty}K_{3-1}(u)\cos(ut)du \tag{8.8}$$

$$K_{3-1}(u)=\frac{c_{11}^1(u)\mathrm{sh}(u)+c_{21}^1(u)\mathrm{ch}(u)}{u\left(c_{11}^1(u)\mathrm{ch}(u)+c_{21}^1(u)\mathrm{sh}(u)\right)} \tag{8.9}$$

where $c_{ij}^1(u)\quad i,j=1,2$ are matrix elements $C_1^{3-1}(u)$, defined by formulas (8.6).

To obtain the displacement formulas $w_k(x,y_k)$ for $k=1,2$ from (8.1) at the first ($k=1$) and second ($k=2$) interfaces, we calculate the functions $W_k(u,\gamma_k)$ from (8.2) for $k=1,2$. The elements $c_{ij}^1(u)\quad i,j=1,2$ are defined in (8.6), and the elements $c_{ij}^2(u)\quad i,j=1,2$ of the matrix $C_2^{3-1}(u)$ from (4.27) take the form

$$C_2^{3-1}(u)=\begin{pmatrix}c_{11}^2(u) & c_{12}^2(u)\\ c_{21}^2(u) & c_{22}^2(u)\end{pmatrix} \tag{8.10}$$

$$\begin{aligned}
&c_{11}^2(u)=\rho(u)\left[-\mathrm{th}^2(u\gamma_2)+\beta_2\right]r(u)\\
&c_{12}^2(u)=\rho(u)\left[-(1-\beta_2)\mathrm{th}(u\gamma_2)\right]\\
&c_{21}^2(u)=\rho(u)\left[(1-\beta_2)\mathrm{th}(u\gamma_2)\right]=-c_{12}^2(u)\\
&c_{22}^1(u)=\rho(u)\left[1-\beta_2\,\mathrm{th}^2(u\gamma_2)\right]\\
&\rho(u)=\mathrm{ch}^2(u\gamma_2)
\end{aligned} \tag{8.11}$$

Studies have shown that $W_k(u,\gamma_k)\quad k=1,2$ exhibit exponential decay as $u\to\infty$ and have no singular points on $u\in[0,\infty)$, and the following estimates hold true

$$W_k(u,\gamma_k)=O\left(\frac{e^{-u(1-\gamma_k)}}{u}\right)\ \text{for}\ u\to\infty,\ k=1,2 \tag{8.12}$$

To determine $W_3(u,\gamma_3)$, we use formulas (8.1) and (8.2), and for $k=3$, we obtain

$$w_3(x,y_3)=-\frac{a}{\pi\mu_1}\int_{-1}^{1}\varphi(\xi)d\xi\int_0^{\infty}W_3(u,\gamma_3)\cos\left(u\frac{\xi-x}{\lambda}\right)du\ \text{for}\ -\infty<x<\infty \tag{8.13}$$

where

$$W_3(u,\gamma_3)=\frac{c_{11}^3(u)\mathrm{sh}(u\gamma_3)+c_{21}^3(u)\mathrm{ch}(u\gamma_3)}{u\left(c_{11}^1(u)\mathrm{ch}(u)+c_{21}^1(u)\mathrm{sh}(u)\right)} \tag{8.14}$$

in which $y_3=\gamma_3=0$, and elements $c_{11}^3(u)=1$, $c_{21}^3(u)=0$ according to (4.27) are elements of the identity matrix E. Substituting the parameters into (8.14) we obtain that

$$W_3(u,\gamma_3)=0$$

and, consequently,

$$w_3(x,y_3)=0 \text{ for } -\infty<x<\infty,\ y_3=0 \tag{8.15}$$

Formula (8.15) coincides with the boundary condition (2.4) of the problem under consideration, which implies that the coating is bonded to the non-deformable base at the surface of the bottom (third) layer.

The shear stresses in the coating layers are determined by formula (3.12) in terms of displacements or by formulas (4.46)-(4.48) in terms of contact stresses $\varphi(x)$. In formula (4.46) for $\sigma_y^k(x,y)$, we switch to dimensionless variables by making the substitutions $\xi=a\xi'$, $d\xi=ad\xi'$, $x=ax'$, $\alpha=h^{-1}u$, $d\alpha=h^{-1}du$, and obtain the resulting formula after substitutions $y=y_k$ $k=1-3$

$$\sigma_y^k(x,y_k)=-\frac{1}{\pi}\frac{\mu_k}{\mu_1}\frac{a}{h}\int_{-1}^{1}\varphi(\xi)d\xi\int_{-\infty}^{\infty}V_k(u,\gamma_k)\cos\left(u\frac{\xi-x}{\lambda}\right)du \text{ for } -\infty<x<\infty,\ k=1-3 \tag{8.16}$$

$$V_k(u,\gamma_k)=\frac{c_{11}^k(u)\mathrm{ch}(u\gamma_k)+c_{21}^k(u)\mathrm{sh}(u\gamma_k)}{c_{11}^1(u)\mathrm{ch}(u)+c_{21}^1(u)\mathrm{sh}(u)} \text{ for } k=1-3, \tag{8.17}$$

$$y_k=h-\sum_{i=1}^{k}\Delta h_i=h\gamma_k,\ \gamma_k=1-\sum_{i=1}^{k}\frac{\Delta h_i}{h},\ \gamma_0=1 \tag{8.18}$$

For $y=y_0=h$, the formula $V_1(u,\gamma_0)$ differs from (8.17) and, taking into account $\gamma_0=1$, assumes the form

$$V_1(u,\gamma_0)=\frac{c_{11}^1(u)\mathrm{ch}(u\gamma_0)+c_{21}^1(u)\mathrm{sh}(u\gamma_0)}{c_{11}^1(u)\mathrm{ch}(u)+c_{21}^1(u)\mathrm{sh}(u)}=1 \tag{8.19}$$

Substituting (8.19) into (8.16), we obtain, in dimensionless form

$$\sigma_y^1(x,y_0)=-\frac{a}{h}\int_{-1}^{1}\varphi(\xi)\delta(\xi-x)d\xi=-\frac{a}{h}\begin{cases}\varphi(x) & \text{at } -1\le x\le 1\\ 0 & \text{at } -\infty<x<-1,\ 1<x<\infty\end{cases} \tag{8.20}$$

which is equivalent to the boundary condition (2.5) of the problem in dimensional form, and $\delta(\xi)$ is the Dirac delta function.

When deriving formulas for the stresses $\sigma_y^k(x,y_k)$ for $k=1-3$ at the first ($k=1$) and second ($k=2$) interfaces, as well as at the base of the coating ($k=3$) bonded to a rigid foundation, it should be noted that in all these cases, the function $V_k(u,\gamma_k)$ for $k=1-3$ from (8.17) decays exponentially as $u\to\infty$, has no singular points on $[0,\infty)$, and the following estimates hold true

$$V_1(u,\gamma_1)=O\left(e^{-u(1-\gamma_1)}\right),\ V_2(u,\gamma_2)=O\left(e^{-u(1-\gamma_2-\gamma_1)}\right),\ V_3(u,\gamma_3)=O\left(e^{-u(1-\gamma_2-\gamma_1)}\right) \text{ for } u\to\infty \tag{8.21}$$

Numerical integration schemes are used to evaluate the improper integrals in (8.1) and (8.16).

The numerical implementation of the relative displacements $\delta^{-1}w_1(x,y_0)$ of the contact surface and the relative displacements $\delta^{-1}w_k(x,y_k)$ $k=1,2$ on the interlayer surfaces was carried out using the obtained formulas (8.1)-(8.14) for a three-layer coating, in which $y_0=h$, $y_1=\frac{2}{3}h$, $y_2=\frac{1}{3}h$, $y_3=0$. At the same time, two coating options were considered:

- homogeneous when $\mu_1=\mu_2=\mu_3$,
- inhomogeneous when $\mu_1=\mu_3$, $\mu_2=0.5\mu_1$.

Figure 2 shows a plot of the projection of the boundary of the region of relative displacements $\delta^{-1}w_1(x,y_0)$ of the upper surface of the coating, resulting from its displacement by a strip indenter over a segment $-1\le x\le 1$ by a

distance δ, onto the plane xOz. This figure also presents, in graphical form, the projections of the boundaries of the regions of relative displacements $\delta^{-1}w_k(x, y_k)$, $k = 1,2$ at the interfaces, resulting from shearing of the top surface by a strip indenter, onto the plane xOz.

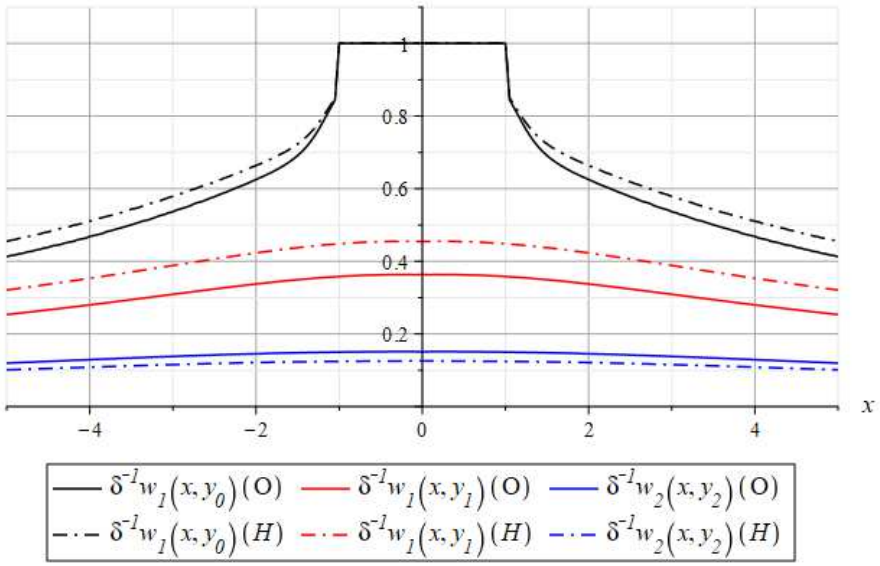


Fig. 2. Plots of the projections of the boundaries of regions of relative displacements onto a plane xOz, for the upper surface of the coating $\delta^{-1}w_1(x, y_0)$ and the interface surfaces $\delta^{-1}w_k(x, y_k)$ $k = 1,2$

The displacements $\delta^{-1}w_1(x, y_0)$ of the shear zone boundary at the upper surface of the coating and the displacements $\delta^{-1}w_k(x, y_k)$ $k = 1,2$ of the shear zone boundaries at the interface surfaces are calculated using formulas (8.1)-(8.6), based on the dimensionless contact stresses $a\delta^{-1}\mu_1^{-1}\varphi(x)$ obtained by solving the integral equation (8.7)-(8.11). The plots shown as solid lines correspond to the case of a homogeneous three-layer coating ($\mu_1 = \mu_2 = \mu_3$), whereas the plots shown as dash-dotted lines represent the inhomogeneous case, where the stiffness of the middle ($k = 2$) layer $\mu_2 = 0.5\mu_1$ when $\mu_1 = \mu_3$. Calculations were performed for $\lambda = \frac{h}{a} = 10$. A 50% reduction in the stiffness of the middle layer (compared to a homogeneous coating) resulted in a 25% change in the interface between the first and second layers, and a 17% change in the interface between the second and third layers. Calculations show that the relative shear force is 0.9698 in the homogeneous case and 0.8456 in the inhomogeneous case — that is, it decreases by 13%.

Figure 3 shows the projection of the boundary of the region of relative contact stresses $a\delta^{-1}\mu_1^{-1}\varphi(x)$, arising on the upper surface of the coating in contact with the base of a strip indenter over the segment $-1 \le x \le 1$ upon its displacement by the distance δ, onto the plane xOz. The figure also shows the boundaries of the regions of relative stresses $a\delta^{-1}\mu_1^{-1}\sigma_y^k(x, y_k)$, $k = 1-3$, arising at the interfaces and projected onto the plane xOz.

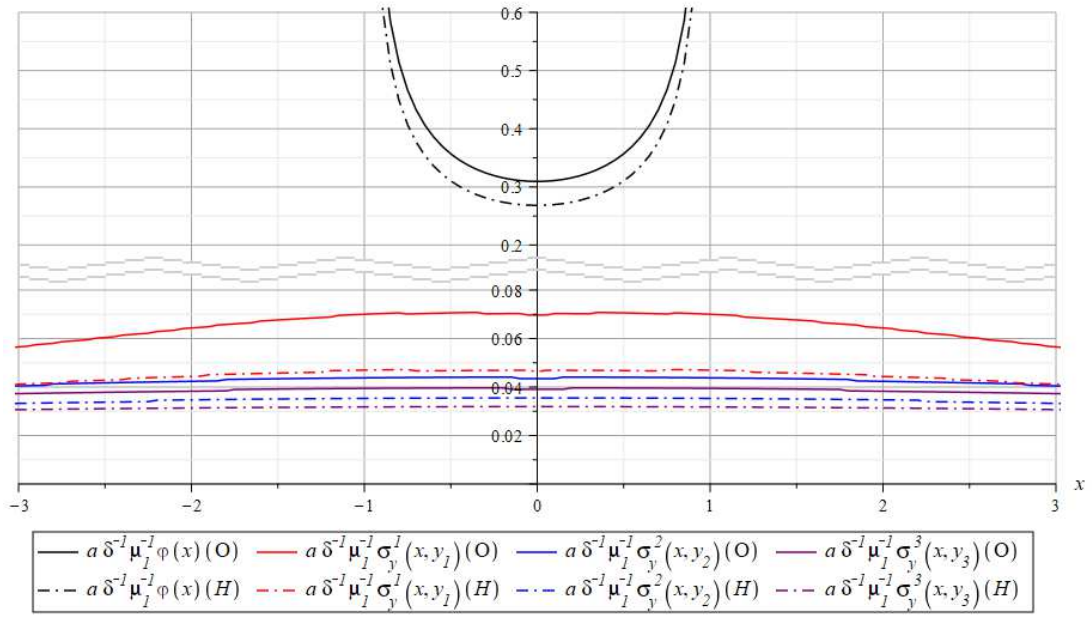


Fig. 3. Plots of the projections of the boundaries of regions of relative contact stresses $a\delta^{-1}\mu_1^{-1}\varphi(x)$ on the upper surface of the coating and relative stresses $a\delta^{-1}\mu_1^{-1}\sigma_y^k(x,y_k)$ $k=1-3$ on the interfaces onto a plane xOz

Contact stresses are calculated using formulas (7.1)-(7.3), obtained by solving the integral equation (5.9)-(5.12) for $N=3$. The shear stresses $a\delta^{-1}\mu_1^{-1}\sigma_y^k(x,y_k)$ $k=1-3$, arising at the interfaces, are calculated using formulas (8.16)-(8.18). As in the previous analysis of displacements, the plots shown as solid lines correspond to the case of a homogeneous three-layered coating at $\mu_1=\mu_2=\mu_3$, whereas the plots shown as dash-dotted lines correspond to the inhomogeneous case where $\mu_2=0.5\mu_1$, $\mu_1=\mu_3$. A 50% reduction in the stiffness of the middle layer (compared to a homogeneous coating) resulted in a 13% decrease in contact stresses, a 34% decrease in stresses at the first interface, and an 18% decrease at both the second and third interfaces.

The presented calculations demonstrate the capabilities of selecting parameters for multilayered coatings to achieve an optimal configuration in terms of both mechanical and geometric characteristics.

Conclusion

1 The use of matrix functions to describe interface connections makes it possible to derive integral equations for contact problems involving stratified materials and coatings in a compact analytical form, wherein the transforms of the kernels are expressed as a ratio of matrix products.

2 The integral equations governing contact problems for multilayered coatings are Fredholm equations of the first kind of the Fourier convolution type with a difference kernel, the transform of its kernel incorporates all the mechanical and geometric parameters of the coating layers, represented within the elements of the matrix functions defining the interface boundary conditions.

3 The asymptotic properties of the kernel transform for the integral equation governing anti-plane shear contact problems involving multilayered coatings coincide (up to constants) with the corresponding properties of the kernel transforms for the integral equations describing pure shear of an elastic homogeneous layer by a rigid indenter within the framework of elasticity theory.

4 The analytical solutions derived for contact problems involving the shearing of multilayered coatings enable effective monitoring of the stress-strain state at interfaces, as well as the solution of parametric optimization problems related to the selection of both the geometric and mechanical properties of the layers and the characteristics of the interfaces.

5 The proposed method for simulating multilayered coatings using interface coupling matrix functions makes it possible to analyze stratified structures with a large number of thin layers, characterizing the corresponding biological or medical tissues (perceived at the cellular level) as substantially stratified, continuously inhomogeneous media.

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