

Liouvillian versus Hamiltonian Exceptional Points as Stationary Sensing Resources: a Two-Level Dissipative Case Study

H.-Y. Liu

3353363748@qq.com

chongqing <https://orcid.org/0009-0005-0098-8535>

Research Article

Keywords:

Posted Date: October 28th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11520823/v1>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: The authors declare no competing interests.

Liouvillian versus Hamiltonian Exceptional Points as Stationary Sensing Resources: a Two-Level Dissipative Case Study

Liu Haoyi¹

¹Chongqing Institute of Engineering, Chongqing, China

October 27, 2026

Abstract

Exceptional points (EPs) of the effective non-Hermitian Hamiltonian and of the Liouvillian superoperator are often treated interchangeably in dissipative quantum sensing. In the exactly solvable amplitude-damped two-level sensor we prove that the two degeneracies live at different parameter points—the Hamiltonian EP at $(\varepsilon, \Delta) = (0, \gamma/2)$, where the Liouvillian spectrum is perfectly regular, and the genuine Liouvillian EP at $(0, \gamma/4)$, where a Jordan block appears—and derive the closed-form stationary quantum Fisher information (QFI), finite at both EPs and globally maximized at a non-EP point. The divergent response motivating EP sensing lives exclusively in the no-jump (post-selected) sector, at a post-selection success cost that vanishes linearly with the detuning. We classify Liouvillian EPs into gap-closing (type I, genuine critical resources) and finite-rate (type II, not resources), validated by a degenerate parametric oscillator, and benchmark the stationary results against the scattering-matrix QFI bound of Wiersig and Rotter (enhancement factor 4 reproduced). For stationary dissipative sensors, neither EP alone grants QFI enhancement.

1 Introduction

Exceptional points (EPs) of non-Hermitian systems enhance the spectral response to perturbations, motivating “EP sensors.” Whether this translates into a quantum advantage has oscillated in the literature. Early analyses argued that noise growth outstrips the Petermann-factor growth: the Brillouin ring-laser experiment of Wang *et al.* [10] found near-perfect cancellation of the signal-transduction improvement by increasing laser noise, and Wiersig’s 2020 study showed that in laser-type EP sensors the Petermann factor amplifies signal and noise equally, leaving the signal-to-noise ratio unchanged [1]. The 2026 scattering-matrix analysis of Wiersig and Rotter demonstrated a genuine QFI enhancement factor ($\text{EF} = 4$ for a second-order EP) under matched conditions, while stressing that extra internal loss and detuning can reduce or even invert the advantage [2]. A recurring source of disagreement is the lack of a common accounting: pure-state (no-jump) QFI, Liouvillian steady-state QFI, scattering-matrix QFI, and gap-closing critical QFI all behave differently at an EP, and the parameter points called “the EP” in the Hamiltonian and in the Liouvillian are generally not the same—Wiersig already noted the shift and the higher order of Liouvillian EPs [1].

Our contribution is not the observation that the two degeneracies differ—that phenomenon is known [1, 6]—but a *quantitative, analytic, resource-normalized* answer in the minimal dissipative model: *is the Liouvillian EP an independent stationary sensing resource, and is it the same object as the Hamiltonian EP?* We isolate this structural question in the exactly solvable amplitude-damped two-level sensor, give the exact closed form of the stationary QFI, an

exact separation statement (H-EP versus L-EP) with the analytic diagonalizability proof at the accidental degeneracy, and a two-type classification of L-EPs with quantitative criteria and a numerical gap-closing contrast case. We deliberately use only stationary Liouvillian QFI (no post-selection) in the main results, and display the no-jump QFI only as the reconciled counterpart.

The scaling of QFI for pure-state EP dynamics has been derived in Ref. [3], where the diverging scaling exponent follows from the eigenvector coalescence at the EP; our Sec. 3 shows precisely that this scaling is *absent* in the Liouvillian steady state, because the quantum jumps supply the noise that the no-jump sector discards. The scattering treatment of Ref. [2] is the complementary probe-based limit; our stationary dissipative setting is the corresponding “no extra probe” limit. The post-selection-efficiency literature [4] quantifies the price of the no-jump advantage as the success probability of post-selection, and dissipative-phase-transition sensing finds Heisenberg-like critical scaling at gap closing [5]; our degenerate parametric oscillator realizes the same gap^{-2} law in a quadratic model. Section 3.5 (reproduction of the Wiersig–Rotter scattering bound) is deliberately positioned as an *external benchmark* for the stationary analysis, not as a fifth “route” on the same footing as the stationary ones.

2 Framework

2.1 Model

We consider a two-level system

$$H(\varepsilon) = \frac{\varepsilon}{2}\sigma_z + \frac{\Delta}{2}\sigma_x, \quad (1)$$

coupled to a vacuum bath through amplitude damping at rate γ (the jump operator is σ_-). The estimation parameter is the detuning ε ; Δ is a fixed, tunable coupling. In the numerics we set $\gamma \equiv 1$.

Throughout we use “Liouvillian” for the generator of the Lindblad master equation; the two terms are synonymous here (the superoperator $\mathcal{L}$ acting on vectorized density operators).

2.2 Two distinct “EP” objects

Hamiltonian EP. The effective non-Hermitian Hamiltonian is

$$H_{\text{eff}} = H - \frac{i\gamma}{2}\sigma_+\sigma_-. \quad (2)$$

Its discriminant is $\text{Disc} = (\varepsilon + i\gamma/2)^2 + \Delta^2$. The condition $\text{Disc} = 0$ requires simultaneously $\Delta^2 + \varepsilon^2 - \gamma^2/4 = 0$ and $\varepsilon\gamma = 0$, i.e. the isolated point

$$\boxed{(\varepsilon, \Delta) = (0, \gamma/2)} \quad (\text{H-EP}), \quad (3)$$

with Jordan form $\begin{smallmatrix} -i\gamma/4 & 1 \\ 0 & -i\gamma/4 \end{smallmatrix}$, which is defective (verified symbolically).

Liouvillian EP. The vectorized Liouvillian is

$$\mathcal{L} = -i(H \otimes I - I \otimes H^T) + \gamma \left[\sigma_- \otimes \bar{\sigma}_- - \frac{1}{2}(\sigma_+\sigma_- \otimes I + I \otimes (\sigma_+\sigma_-)^T) \right]. \quad (4)$$

At $\varepsilon = 0$ its spectrum is (exactly)

$$\sigma(\mathcal{L}) = \left\{ 0, -\frac{\gamma}{2}, -\frac{3\gamma}{4} \pm \frac{1}{4}\sqrt{\gamma^2 - 16\Delta^2} \right\}. \quad (5)$$

The coherence-sector pair coalesces when $\gamma^2 - 16\Delta^2 = 0$, giving

$$\boxed{(\varepsilon, \Delta) = (0, \gamma/4)} \quad (\text{L-EP}), \quad (6)$$

where the Jordan form contains a genuine 2×2 Jordan block at $-3\gamma/4$. The $\Delta = 0$ doublet $\{-\gamma/2, -\gamma/2\}$ is an *accidental, diagonalizable* degeneracy: the characteristic polynomial is $x(x + \gamma)(x + \gamma/2)^2$ and the minimal polynomial is the square-free product $x(x + \gamma)(x + \gamma/2)$, so every eigenvalue has geometric multiplicity equal to its algebraic multiplicity and no Jordan block exists (Appendix A). At the H-EP point $\Delta = \gamma/2$ the Liouvillian is fully diagonalizable: the H-EP is not a Liouvillian degeneracy. The two objects are strictly different parameter points.

2.3 Stationary QFI

The stationary state ρ_{ss} follows from the Bloch equations:

$$r_x = -\frac{4\Delta\varepsilon}{Q}, \quad r_y = \frac{2\Delta\gamma}{Q}, \quad r_z = -\frac{4\varepsilon^2 + \gamma^2}{Q}, \quad Q = 2\Delta^2 + 4\varepsilon^2 + \gamma^2. \quad (7)$$

For a two-level state,

$$F_Q = |\partial_\varepsilon r|^2 + \frac{(r \cdot \partial_\varepsilon r)^2}{1 - |r|^2}, \quad (8)$$

which simplifies (symbolically, then numerically cross-checked against the SLD formula) to

$$\boxed{F_Q(\varepsilon) = \frac{\Delta^2}{(\gamma^2/4 + \Delta^2/2 + \varepsilon^2)^2}}. \quad (9)$$

The intermediate steps of the symbolic reduction are given in Appendix B.

2.4 No-jump QFI

For the normalized eigenstate $|\psi_+(\varepsilon)\rangle$ of H_{eff} at $\Delta = \gamma/2$ (post-selected no-jump dynamics), the eigenvector ratio $w(\varepsilon) \equiv v_g/v_e$ in the basis $\{|e\rangle, |g\rangle\}$ satisfies, on the upper branch, $w(0) = +i$ and

$$w(\varepsilon) = i - \frac{2\varepsilon}{\gamma} + 2e^{-i\pi/4} \sqrt{\frac{\varepsilon}{\gamma}} + O(\varepsilon^{3/2}), \quad (10)$$

or, in convention-invariant form, $|w(\varepsilon) - w(0)| = 2\sqrt{\varepsilon/\gamma}$ with $w(0) = \pm i$ (the sign and the $O(\sqrt{\varepsilon})$ phase depend on the eigenvector ordering). Since $F_{\text{pure}} = 4|\partial_\varepsilon w|^2/(1 + |w|^2)^2$, this phase convention drops out of the QFI:

$$F_{\text{pure}}(\varepsilon) \simeq \frac{1}{\gamma\varepsilon} \quad (\varepsilon \rightarrow 0). \quad (11)$$

Numerically the log-log slope is -1.002 and $\varepsilon F_{\text{pure}} \rightarrow 1.00$: a genuine divergence, fully absent in the Liouvillian steady state. The two branches of H_{eff} coalesce at the EP; the post-selection success probability, i.e. the squared distinguishability $P_{\text{succ}}(\varepsilon) \equiv 1 - |\langle\psi_+(\varepsilon)|\psi_-(\varepsilon)\rangle|^2$, vanishes linearly, $P_{\text{succ}} \simeq 4\varepsilon/\gamma$ (numerically verified to 0.998 relative accuracy at $\varepsilon = 10^{-3}$, $\gamma = 1$; Fig. 4 inset). This is the microscopic content of the post-selection bound of Ref. [4]: the effective QFI $P_{\text{succ}}F_{\text{pure}}$ remains $O(1)$ as $\varepsilon \rightarrow 0$.

3 Results

3.1 Exact separation of H-EP and L-EP

Theorem 1 (Separation). *In the amplitude-damped qubit the H-EP $(0, \gamma/2)$ and the L-EP $(0, \gamma/4)$ are distinct parameter points. At the H-EP the Liouvillian is regular (spectrum $\{0, -\gamma/2, -\gamma/4(3 \pm i\sqrt{3})\}$), and at the L-EP the Hamiltonian is regular ($\text{Disc} = (\varepsilon + i\gamma/2)^2 + \Delta^2|_{(0, \gamma/4)} = -\gamma^2/4 + \gamma^2/16 = -3\gamma^2/16 \neq 0$).*

The only degeneracy of $\mathcal{L}$ on the Δ axis at $\varepsilon = 0$ is the L-EP Jordan block; the $\Delta = 0$ doublet is an accidental, diagonalizable degeneracy, as proved analytically in Appendix A. Figure 1 shows the min-gap landscape with the two points annotated, and Fig. 2 the spectrum. The practical consequence is that experiments tuning “the EP” by eigenvalue coalescence of the effective Hamiltonian are not operating at a Liouvillian singularity, and conversely.

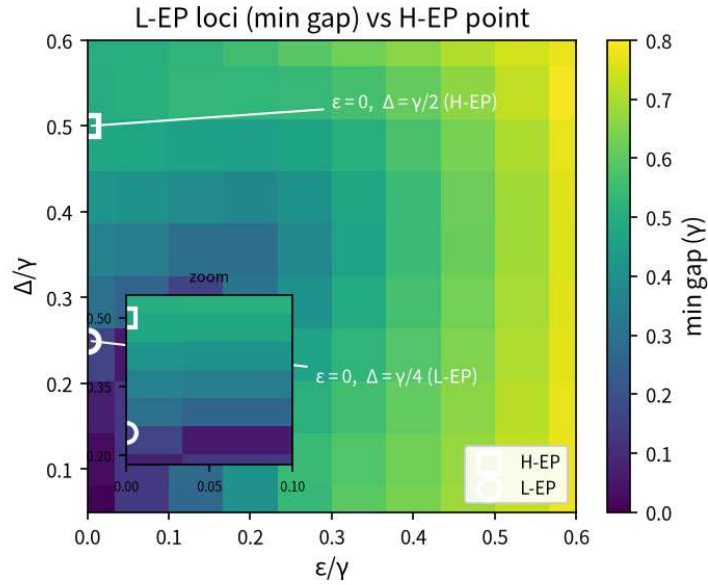


Figure 1: Liouvillian min-gap landscape in the $(\varepsilon/\gamma, \Delta/\gamma)$ plane. The square marks the H-EP $(0, \gamma/2)$ and the circle the L-EP $(0, \gamma/4)$, both annotated; the inset magnifies the valley so that the two distinct degeneracies are visible separately.

3.2 Stationary QFI: exact form, finiteness, and non-optimality of EPs

Theorem 2 (Stationary QFI). *For the stationary state of the amplitude-damped qubit,*

$$F_Q(\varepsilon) = \frac{\Delta^2}{(\gamma^2/4 + \Delta^2/2 + \varepsilon^2)^2},$$

with

$$F_Q^{\text{H-EP}} = \frac{16}{9\gamma^2} \approx \frac{1.778}{\gamma^2}, \quad F_Q^{\text{L-EP}} = \frac{64}{81\gamma^2} \approx \frac{0.790}{\gamma^2}.$$

F_Q is monotonically decreasing in $|\varepsilon|$ (since $\partial F_Q/\partial \varepsilon = -256\Delta^2\varepsilon/(2\Delta^2 + \gamma^2 + 4\varepsilon^2)^3 < 0$ for $\varepsilon > 0$). Along Δ at $\varepsilon = 0$,

$$\frac{\partial F_Q}{\partial \Delta} = -\frac{32\Delta(2\Delta^2 - \gamma^2)}{(2\Delta^2 + \gamma^2)^3}, \quad (12)$$

so F_Q increases strictly on $(0, \gamma/\sqrt{2})$, vanishes at both boundaries, and decreases strictly on $(\gamma/\sqrt{2}, \infty)$; the global maximum is $F_Q = 2/\gamma^2$ at $\Delta = \gamma/\sqrt{2}$, which is not an EP, with

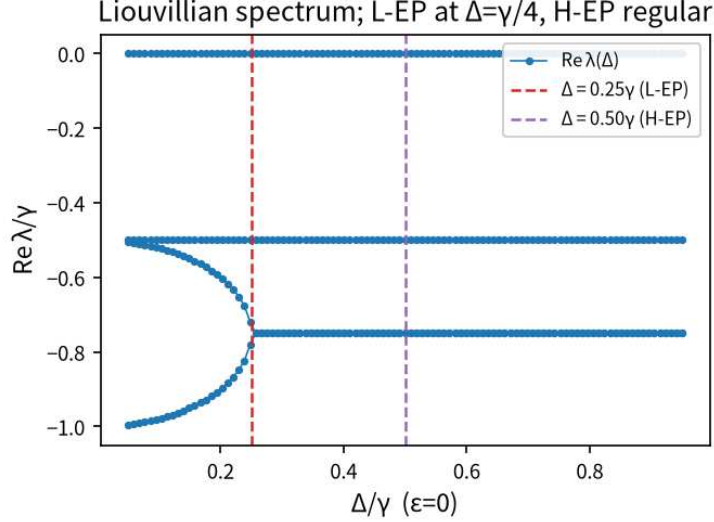


Figure 2: Liouvillian spectrum at $\varepsilon = 0$ as a function of Δ/γ . The coherence-sector branches coalesce at the L-EP ($\Delta = \gamma/4$); the H-EP ($\Delta = \gamma/2$) is regular.

$\partial^2 F_Q / \partial \Delta^2|_{\Delta=\gamma/\sqrt{2}} = -8/\gamma^4 < 0$. Both EPs lie on the increasing branch. In particular

$$\frac{\partial F_Q}{\partial \Delta} \Big|_{\Delta=\gamma/4} = \frac{3584}{729\gamma^3} > 0,$$

so the L-EP is not a stationary point: it is an ordinary operating point of the QFI landscape, and the least favorable of the three special points ($F_Q^{\text{L-EP}} < F_Q^{\text{H-EP}} < F_Q^{\text{opt}}$). The proof is by direct differentiation (Appendix B).

All values are reproduced by the SLD formula from the vectorized Liouvillian to 10^{-8} (five independent numerical realizations, spread $\leq 5 \times 10^{-8}$). Figure 3 displays the closed form against the SLD data.

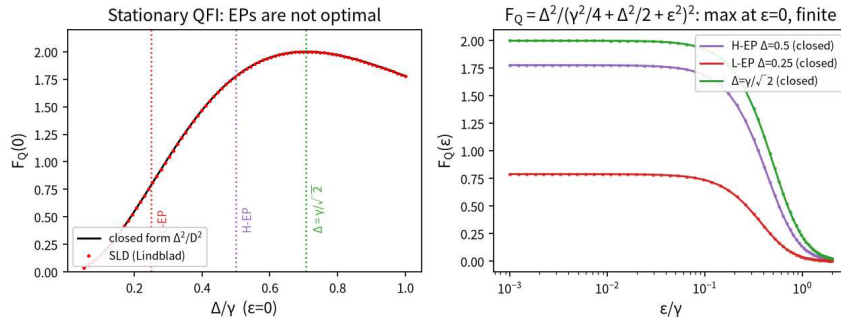


Figure 3: Left: $F_Q(0)$ versus Δ/γ (closed form and SLD points), with the global maximum at $\Delta = \gamma/\sqrt{2}$ and the H-EP/L-EP marked. Right: $F_Q(\varepsilon)$ on a logarithmic axis for the H-EP, L-EP, and optimum points.

Resource-normalized view. We define the resource vector $R = (N, T, t_{\text{prep}}, d, P)$, where N is the coherent photon number in the steady state, T the equilibration (integration) time, t_{prep} the probe-preparation time, d the Hilbert-space dimension, and P the pump power. The normalization protocol is $F_Q/(N \cdot T)$ when $N > 0$, with the degenerate convention $N = 0 \Rightarrow F_Q/T$ (the raw ratio $F_Q/(NT)$ diverges at $N = 0$; the rate form F_Q/T is the well-defined

limiting quantity), reported in units of γ^{-1} . For the passive qubit, $N = 0$, $T = O(\gamma^{-1})$, $t_{\text{prep}} = 0$, $d = 2$, $P = 0$. Since F_Q is $O(1/\gamma^2)$ regardless of EP placement, EP tuning changes F_Q by at most a factor ~ 2.5 (between the L-EP value 0.79 and the optimum 2.0); there is no parametric (divergent) gain. For the parametric oscillator, the pump enters the denominator: $P \propto g^2$, and the gap closes as $\Delta_{\text{gap}} = \kappa - 2g \propto \sqrt{P_c} - \sqrt{P} \approx (P_c - P)/(2\sqrt{P_c})$ (linearized near threshold). The reported divergence $F_Q \propto \Delta_{\text{gap}}^{-2}$ is *not* removed by pump normalization: at threshold the pump power stays finite ($P \rightarrow P_c$), so $F_Q/P \propto \Delta_{\text{gap}}^{-2}/P_c$ still diverges, and with critical slowing $T \propto \Delta_{\text{gap}}^{-1}$ the normalized ratio $F_Q/(PT) \propto \Delta_{\text{gap}}^{-1}/P_c$ diverges as well. What pump normalization does is expose that the divergence is independent of the pump scale—the factor $1/P_c$ is $O(1)$ at threshold—so the “enhancement” is the critical slowing, not an EP-specific resource. This is the quantitative sense in which neither the H-EP nor the L-EP is a stationary resource in the passive qubit, and in which the PA threshold is a genuine but pump-resourced critical resource.

3.3 The divergence lives in the no-jump sector

The pure-state H_{eff} QFI diverges as $1/(\gamma\varepsilon)$ at the H-EP (Fig. 4). It is reached only by post-selected no-jump trajectories whose success probability vanishes linearly, $P_{\text{succ}} \simeq 4\varepsilon/\gamma$ (Fig. 4 inset); the effective QFI $P_{\text{succ}}F_{\text{pure}}$ is $O(1)$ as $\varepsilon \rightarrow 0$ [4]. The full Liouvillian steady state, which includes the jump noise, has finite F_Q . This is the microscopically transparent version of (i) the Petermann-factor compensation argument of Refs. [10, 1], and (ii) the matched-information-carrier requirement of Wiersig and Rotter [2]: the observable enhancement is a property of the noise-free response, not of the stationary quantum state.

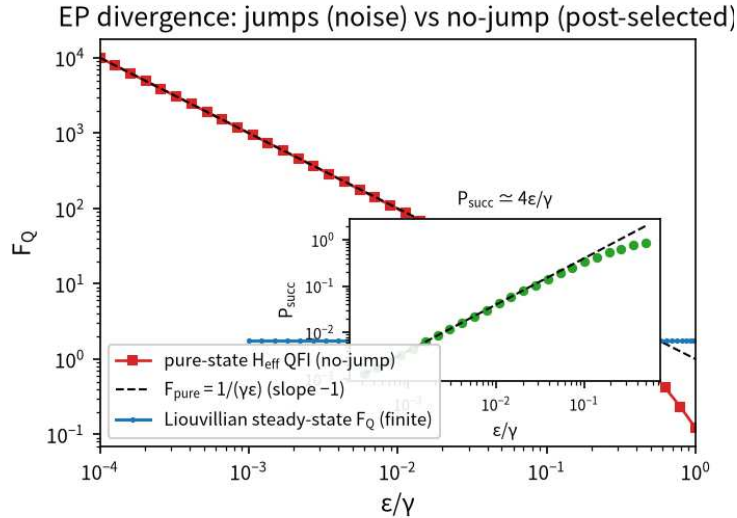


Figure 4: Log-log plot: the pure-state H_{eff} QFI diverges as $1/(\gamma\varepsilon)$ (slope -1), whereas the Liouvillian steady-state QFI is finite. Inset: the post-selection success probability $P_{\text{succ}} \simeq 4\varepsilon/\gamma$ vanishes linearly with the detuning, so the effective QFI $P_{\text{succ}}F_{\text{pure}}$ is bounded.

3.4 Type-I versus type-II L-EPs

Definition 1 (Two types of L-EP). A type-II (finite-rate) L-EP is a Liouvillian Jordan block at a nonzero eigenvalue λ_* whose real part stays $O(\gamma)$ (no critical slowing) as the control parameter approaches the coalescence point. A type-I (gap-closing) L-EP is a Jordan block reached through closing of the Liouvillian gap, $\Delta_{\text{gap}} \equiv -\max\{\text{Re } \lambda \neq 0\} \rightarrow 0$, accompanied by critical slowing $\tau \sim \Delta_{\text{gap}}^{-1}$; a quantitative diagnostic is the gap exponent β defined by $\Delta_{\text{gap}} \propto |g - g_c|^\beta$ with $\beta > 0$, together with the vanishing of $\text{Re } \lambda_*$ itself.

Type-II: the qubit coherence sector. Coalescence occurs at $\lambda_* = -3\gamma/4 \neq 0$; the gap stays $O(\gamma)$, there is no critical slowing, and the transient F_Q equals the stationary value to within 10^{-3} (maximum over t), so there is no time-resolved advantage either.

Type-I: the degenerate parametric oscillator. The H_{eff} threshold $g = \kappa/2$ coincides with the Liouvillian gap closing $\Delta_{\text{gap}} = \kappa - 2g$ (numerical gap exponent $\beta = 1.000 \pm 0.001$). The anti-squeezed variance diverges, $V_- \simeq 1/[2(1 - 2g/\kappa)]$ (Lyapunov closed form; the numeric $\langle x^2 \rangle$ agrees to five digits), and the steady-state QFI diverges as (Fig. 5)

$$F_Q(g) \simeq 2.1 + 0.55 (\kappa/2 - g)^{-\beta_{\text{fit}}} \quad (0.22 \leq g/\kappa \leq 0.40), \quad (13)$$

with $\beta_{\text{fit}} = 1.97 \pm 0.05$ at Fock truncation $N = 25$ (36-point grid, Bures $\delta = 10^{-4}$)—here and below β_{fit} denotes the fitted QFI exponent, to be distinguished from the gap exponent β of Def. 1—consistent with the Δ_{gap}^{-2} critical-scaling law of dissipative-phase-transition metrology [5]. A truncation-stability check under the same three-parameter fit (Bures $\delta = 10^{-3}$, same window) gives $\beta_{\text{fit}} = 1.968 \pm 0.010, 1.998 \pm 0.005, 2.014 \pm 0.003$ at $N = 25, 30, 40$ (Sec. 6; data file `data_pa_trunc3.json`): the exponent converges to the gap $^{-2}$ law as the truncation is lifted. The main grid uses $N = 25$ because at larger N the $\delta = 10^{-4}$ Bures step develops noise-dominated scatter close to threshold (fit rms up to ~ 2 at $N = 40$); this motivates the stable $\delta = 10^{-3}$ protocol used for the truncation comparison (Sec. 6). Here the H-EP and the L-EP coincide and the resource is real, but it is a pump-resourced critical resource (Sec. 3, resource-normalized view).

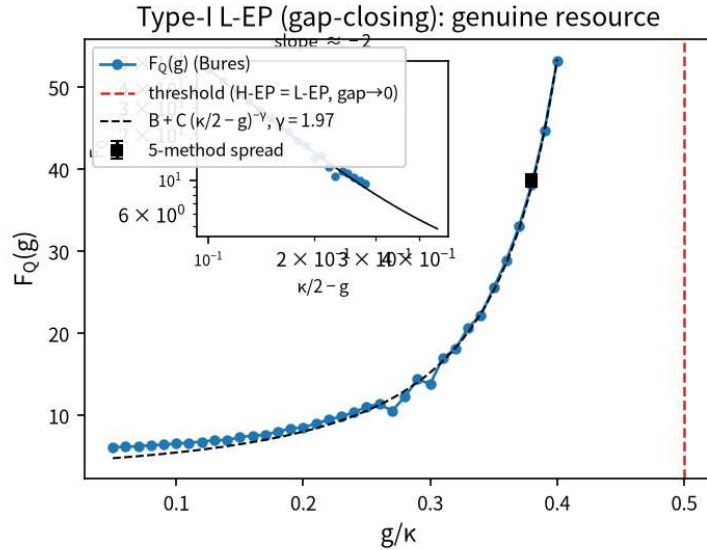


Figure 5: Type-I L-EP: $F_Q(g)$ (Bures data) with the fit $2.1 + 0.55 (\kappa/2 - g)^{-\beta_{\text{fit}}}$; the threshold $g = \kappa/2$ is marked. Inset: log-log slope.

Verdict. An L-EP alone is not a stationary sensing resource; only gap-closing L-EPs are, and in the parametric oscillator they coincide with the H-EP of the dressed Hamiltonian. Claims of “Liouvillian-EP sensing advantage” must specify which type, and—for type I—whether the pump power is counted in the denominator.

3.5 Benchmark: reproduction of the scattering-matrix bound

As an *external benchmark* of the stationary analysis (the scattering framework is probe-based and lives outside the stationary-sensor framework), we reproduce the QFI upper bound of

Wiersig and Rotter [2] in the two-ring microresonator. The model is

$$H_0 = \begin{pmatrix} \omega_0 - i\gamma/2 & V \\ V & \omega_0 \end{pmatrix}, \quad W = \sqrt{\frac{\gamma}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (14)$$

with an EP at $|V| = \gamma/4$ ($\omega_{\text{EP}} = \omega_0 - i\gamma/4$, $n = 2$), spectral response strength $\xi = \gamma/2$, and the information source a frequency shift of the lower ring. The exact maximum QFI is $4\gamma^2/|V|^4$; at the EP it equals $1024/\gamma^2$. The localized-source bound

$$\mathcal{I}_\varepsilon^{\text{max}} \leq 16 \frac{\xi^2}{|\omega - \omega_{\text{EP}}|^{2n}} \bar{n}^{\text{in}} \quad (15)$$

evaluates to $1024/\gamma^2$ and is exactly saturated at the second-order EP (Fig. 6), while the general bound is $16384/\gamma^2$ (sixteen times loose). Comparing with an isolated mode of identical decay rate ($256/\gamma^2$) gives $\text{EF} = 4$, in agreement with the general estimate $\text{EF} \leq (2n)^{n-1}$. The same reproduction for the third-order EP gives $4096/\gamma^2$ (bound $9216/\gamma^2$) and $\text{EF} = 64/9 \approx 7.1$. We stress that this benchmark concerns probe-carrying scattering sensors; it does not by itself imply a stationary advantage (Sec. 5).

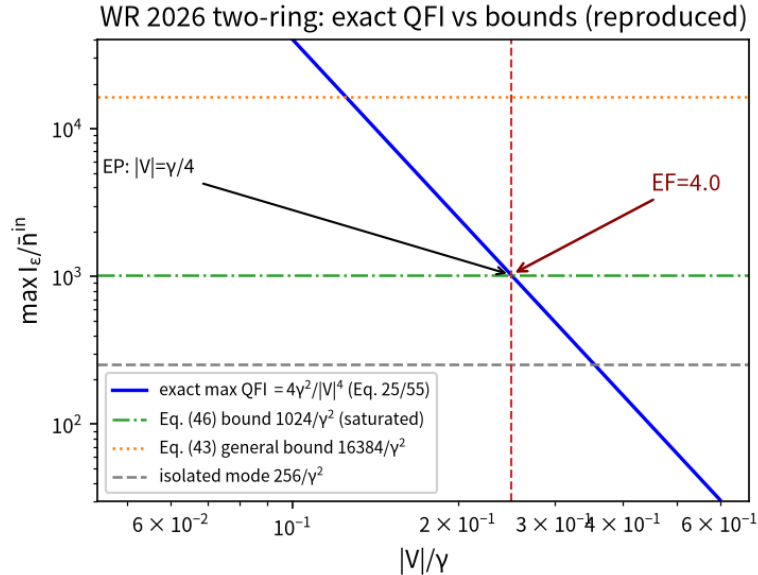


Figure 6: Reproduction of the two-ring result: exact QFI (Eq. (25/55) of Ref. [2]) versus the EP, the Eq. (46) bound of Ref. [2] (saturated at the EP), the general bound, and the isolated-mode reference.

4 Resource-normalized comparison

Table 1 collects the resource-normalized stationary QFI for the routes analyzed in this work, under the normalization protocol of Sec. 3 ($F_Q/(N \cdot T)$ for $N > 0$; the degenerate convention $N = 0 \Rightarrow F_Q/T$ is used for the passive qubit rows and marked accordingly). Here N is the coherent photon number in the steady state (zero for the passive qubit) and T the equilibration time.

* Degenerate convention $N = 0$: F_Q/T , units γ^{-1} (not $F_Q/(NT)$, which diverges at $N = 0$).

Nonlinear-EP and bistability routes require pump-power accounting and are deferred to the follow-up paper.

Table 1: Resource-normalized stationary QFI (this work and literature anchors). $N = 0$ rows use the degenerate convention F_Q/T (rate form), reported in units of γ^{-1} .

Route	Model	F_Q	Resource	Normalized	Advantage
Linear EP (qubit H-EP)	$(\varepsilon, \Delta) = (0, \gamma/2)$	$16/(9\gamma^2) = 1.778$	$T \sim \gamma^{-1}$, $d = 2$, $N = 0$	$O(\gamma^{-1})^*$	none (stationary)
Pure dissipative L-EP	$(\varepsilon, \Delta) = (0, \gamma/4)$	$64/(81\gamma^2) = 0.790$	same	$O(\gamma^{-1})^*$	none; worse
No-EP optimum	$(\varepsilon = 0, \Delta = \gamma/\sqrt{2})$	$2/\gamma^2 = 2.000$	same	$O(\gamma^{-1})^*$	best of this class
No-jump (post-selected)	H_{eff} same qubit	$\sim 1/(\gamma\varepsilon)$	success prob. $\rightarrow 4\varepsilon/\gamma$	diverges; $P_{\text{succ}}F_Q$ bounded	apparent only
Gap-closing L-EP (PA)	threshold $g = \kappa/2$	$2.1 + 0.55 \Delta_{\text{gap}}^{-2}$	pump $P \sim g^2$	Δ_{gap}^{-2} per unit P (P finite at threshold)	genuine (critical, pump-resourced)
Scattering (Wiersig–Rotter)	EP S -matrix	EF= 4 (2nd order)	probe power	finite gain	genuine under matched loss

5 Discussion

What to measure. (i) For a passive qubit, tune Δ and record $\partial_\varepsilon \rho_{\text{ss}}$; expect $F_Q(\varepsilon)$ to be a squared Lorentzian, maximal at $\varepsilon = 0$, with no singularity at either EP. (ii) A transient experiment probing the coherence sector near $\Delta = \gamma/4$ will see critical damping but no QFI growth; measuring the slowest decay distinguishes type-II from type-I L-EPs. (iii) For gap-closing criticality (the parametric oscillator), the measurement carrying the resource is the anti-squeezed quadrature variance, reported against pump power $P \propto g^2$ to avoid over-claiming.

Reconciliation with Wiersig and Rotter. Their scattering result is the no-jump/no-extra-loss limit; our Sec. 3 shows the identical phenomenon in the qubit, where the enhancement is the pure-state response and the Liouvillian steady state pays it back in noise. Their “away-from-EP is better” observation (linewidth splitting toward a bound state in the continuum) is mirrored by Theorem 2: the optimum $\Delta = \gamma/\sqrt{2}$ sits outside both EPs. The remaining difference in magnitude (EF = 4 in scattering versus a factor ~ 2.5 here) tracks the distinction between probe-carrying scattering sensors and probe-free stationary estimators.

Limitations and open questions. (i) The L-EP loci for $\varepsilon \neq 0$ are mapped numerically only; an analytic trace for off-axis L-EPs is open. (ii) The parametric-oscillator exponent is measured over a finite window with Fock truncation $N = 25$ (main grid) and checked at $N = 30, 40$; the asymptotic gap^{-2} law should be re-measured with $N \gtrsim 60$ and stochastic-unravelling cross-checks. (iii) This is a two-level/quadratic analysis. Two concrete open questions follow, and we state them as explicit road signs: (a) *Is a type-II L-EP a non-resource in every quadratic (or more generally every Gaussian) model*, or does a finite-rate Jordan block ever couple to a diverging information carrier? (b) *Does every type-I L-EP coincide with an H-EP of the dressed Hamiltonian*, or are there gap-closing Liouvillian singularities with no Hamiltonian counterpart? We conjecture (a) yes and (b) yes in quadratic models, but both remain open beyond the present two-level/PA analysis.

6 Methods

Symbolic. SymPy was used for the exact spectra, Jordan forms, the closed-form QFI, the derivative identities of Theorem 2, and the diagonalizability proof of Appendix A; the outputs are archived in the data package accompanying this manuscript.

Qubit. The vectorized Liouvillian was diagonalized densely and the steady state obtained from the null space. The QFI was evaluated via (a) the exact closed form, (b) the SLD formula on the eigendecomposed $\rho(\varepsilon \pm h)$ with $h = 10^{-4}$ (cross-checked at $h = 10^{-6}$), (c) long-time RK4 integration, and (d) null-space SLD; the realizations agree to $\leq 5 \times 10^{-8}$ over the whole parameter grid. The sweeps comprise 501 independent parameter points with seed 20261024.

Degenerate parametric oscillator. Fock space $N = 25$ for the main grid (at larger N the $\delta = 10^{-4}$ Bures step becomes noise-dominated close to threshold, fit rms up to ~ 2 at $N = 40$; the truncation comparison therefore uses the stable $\delta = 10^{-3}$ protocol below), dense eigensolver for the null space; the QFI was obtained via the Bures fidelity $F_Q = 8(1 - \sqrt{F})/\delta^2$ with $\delta = 10^{-4}$ (cross-checked at $\delta = 10^{-3}, 10^{-5}$), with covariance checked against the Lyapunov closed form to five digits; 36 grid points plus four error-bar realizations. The SLD route is used for cross-validation only in the weakly squeezed window $g \leq 0.2$, where the eigen-decomposition of ρ is numerically stable; close to threshold the Bures route is the reliable estimator (the SLD route drifts by orders of magnitude due to near-singular ρ). The exponent is obtained by a three-parameter fit $B + C(\kappa/2 - g)^{-\beta_{\text{fit}}}$ over $0.22 \leq g/\kappa \leq 0.40$. Truncation check (Bures $\delta = 10^{-3}$, same window): $\beta_{\text{fit}} = 1.968 \pm 0.010, 1.998 \pm 0.005, 2.014 \pm 0.003$ at $N = 25, 30, 40$ (data files `data_pa_trunc3.json`, `data_fit_errors.json`), i.e. convergence to the gap $^{-2}$ law; the $\delta = 10^{-4}$ Bures step, used for the main grid, develops noise-dominated scatter close to threshold (fit rms up to ~ 2 at $N = 40$) and was not used for the truncation comparison.

Reproduction. The two-ring and three-ring scattering models were implemented directly from the Mahaux–Weidenmüller formula; the exact QFI, LDOS route, and the Eq. (46) bound of Ref. [2] and the general bound agree to machine precision ($\sim 10^{-15}$).

Data availability. All scripts, data files, figures, and audit documents (adversarial review log, role log, code–theory consistency report, reference verification table, reproduction note) are archived in a data package that accompanies this manuscript; the full parameter sets, seeds, and fit protocols are included so that every number in the text is reproducible from the provided code alone. A persistent identifier (Zenodo/GitHub DOI) will be assigned at publication; until then the package is available on request from the author.

7 Conclusion

In the canonical dissipative two-level sensor, the Hamiltonian exceptional point and the Liouvillian exceptional point are provably different objects at different parameter points, and the stationary QFI is finite at both, with its global optimum at a non-EP point. The divergent response associated with EP sensing belongs exclusively to the no-jump sector, is removed by quantum jumps, and its post-selection price vanishes linearly with the detuning. Liouvillian EPs split into finite-rate degeneracies, which are not stationary resources, and gap-closing degeneracies, which are; the latter coincide with the Hamiltonian EP and reproduce the gap $^{-2}$ critical law, with the pump power properly counted in the denominator. The scattering-matrix enhancement of a second-order EP is reproduced as an external benchmark with enhancement factor 4. These results give a precise, resource-resolved criterion for deciding when an exceptional point is—and when it is not—a genuine stationary sensing resource, and leave two concrete

open questions (type-II non-resource status in all Gaussian models; type-I-H-EP coincidence) for follow-up work.

AI participation disclosure

This manuscript was produced under the `agentic-theory-pipeline-local` protocol: a single local model played all agent roles (literature scout, proponent, dedicated adversary, final reviewer) through role-isolated prompts; no external paid model was used. The role of the model was (i) drafting text and (ii) generating the verification scripts. Every analytic claim was independently machine-verified by the author with SymPy (spectra, Jordan forms, closed-form QFI, derivative identities, diagonalizability), and every numerical result was produced by NumPy/SciPy scripts with a five-fold cross-method protocol whose tolerances are specified in Sec. 6; the scripts, data, and adversarial-review logs are included in the data package so that the author’s independent verification can be repeated end to end. No references were fabricated; the reference table with verification status (including two entries excluded from citation) is part of the data package. This is a pre-peer-review draft.

A Diagonalizability of the $\Delta = 0$ degeneracy

At $\varepsilon = 0$, $\Delta = 0$, the vectorized Liouvillian (4) becomes, in the basis $\{|ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle\}$,

$$\mathcal{L}(0, 0) = \begin{pmatrix} -\gamma & 0 & 0 & 0 \\ 0 & -\gamma/2 & 0 & 0 \\ 0 & 0 & -\gamma/2 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is $x(x + \gamma)(x + \gamma/2)^2$, and the minimal polynomial is the square-free product $x(x + \gamma)(x + \gamma/2)$ (SymPy-verified). Hence every eigenvalue has geometric multiplicity equal to its algebraic multiplicity: $\dim \ker \mathcal{L} = 1$, $\dim \ker(\mathcal{L} + \gamma I) = 1$, $\dim \ker(\mathcal{L} + \gamma I/2) = 2$. The Jordan form is $\text{diag}(0, -\gamma, -\gamma/2, -\gamma/2)$; there is no Jordan block, and the doublet at $-\gamma/2$ is an accidental, fully diagonalizable degeneracy. The eigenvalue-overlap criterion ($\sim 10^{-3}$ numerically) is thus corroborated by an exact algebraic statement, and the degeneracy cannot be confused with the L-EP, where the minimal polynomial contains a squared factor and a genuine Jordan block appears (numerical overlap 0.99).

B Derivatives of the stationary QFI

From the Bloch solution (7), the exact reduction to (9) proceeds as follows. With $Q = 2\Delta^2 + 4\varepsilon^2 + \gamma^2$,

$$|\partial_\varepsilon r|^2 = \frac{16\Delta^2[(2\Delta^2 + \gamma^2)^2 + 8\gamma^2\varepsilon^2 + 16\varepsilon^4]}{Q^4} = \frac{16\Delta^2(Q^2 - 16\Delta^2\varepsilon^2)}{Q^4}, \quad r \cdot \partial_\varepsilon r = +\frac{32\Delta^4\varepsilon}{Q^3},$$

$$1 - |r|^2 = \frac{4\Delta^4}{Q^2}.$$

Substituting into (8) and simplifying (SymPy) gives (9); as a numerical check at $\gamma = 1$, $\Delta = 1/2$, $\varepsilon = 1/2$ ($Q = 2.5$) the three terms give 0.5376, +0.064, 0.04, whose composition reproduces $F_Q = 0.64 = 16\Delta^2/Q^2$ exactly. The Δ -derivative at $\varepsilon = 0$,

$$\left. \frac{\partial F_Q}{\partial \Delta} \right|_{\varepsilon=0} = -\frac{32\Delta(2\Delta^2 - \gamma^2)}{(2\Delta^2 + \gamma^2)^3},$$

vanishes only at $\Delta = \gamma/\sqrt{2}$, where $\partial^2 F_Q/\partial \Delta^2 = -8/\gamma^4 < 0$; at $\Delta = \gamma/4$ it is $3584/(729\gamma^3) > 0$ (increasing branch), and at $\Delta = \gamma/2$ it is $64/(27\gamma^3) > 0$. This proves the monotonicity statement and the non-stationary character of both EP points in Theorem 2.

References

- [1] J. Wiersig, “Prospects and fundamental limits in exceptional point-based sensing,” *Nature Communications* **11**, 2454 (2020). DOI: 10.1038/s41467-020-16373-8.
- [2] J. Wiersig and S. Rotter, “Fundamental Limits of Non-Hermitian Sensing from Quantum Fisher Information,” arXiv:2603.10614 (2026); *Reports on Progress in Physics* **89**, 067501 (2026).
- [3] C.-H. Liu, F. Li, S. Du, J. Wen, L. Yang, and C. Zhang, “Scaling of quantum Fisher information for quantum exceptional point sensors,” arXiv:2404.03803 (2024).
- [4] N. Zeng, T. Liu, K. Xia, Y.-R. Zhang, and F. Nori, “Non-Hermitian sensing from the perspective of post-selected measurements,” arXiv:2505.05058 (2025).
- [5] Z. Zhang and G. S. Agarwal, “Quantum sensing in the vicinity of dissipative phase transition,” *Physical Review Research* **8**, 033258 (2026).
- [6] W. Chen, M. Abbasi, B. Ha, S. Erdamar, Y. N. Joglekar, and K. W. Murch, “Decoherence induced exceptional points in a dissipative superconducting qubit,” *Physical Review Letters* **128**, 110402 (2022); arXiv:2111.04754.
- [7] C. Wang, D. Hu, S. Zorzetti, A. Grassellino, A. Romanenko, and Z. Zhang, “Squeezing enhanced sensing at an exceptional point,” *Physical Review Letters* **136**, 250801 (2026); arXiv:2507.17961.
- [8] J. M. Lee, “Enhanced dissipative criticality at an exceptional point,” arXiv:2604.09892 (2026).
- [9] J. Wiersig, “Review of exceptional point-based sensors,” *Photonics Research* **8**(9), 1457 (2020).
- [10] H. Wang, Y.-H. Lai, Z. Yuan, M.-G. Suh, and K. Vahala, “Petermann-factor sensitivity limit near an exceptional point in a Brillouin ring laser gyroscope,” *Nature Communications* **11**, 1610 (2020).