

Stability and Smoothness of Navier-Stokes Equations

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Research Article

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Posted Date: October 28th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11518852/v1>

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Additional Declarations: The authors declare no competing interests.

Stability and Smoothness of Navier-Stokes Equations *

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Abstract

By applying discrete mathematical analysis and Direct Numerical Simulations (DNS), the present study aims to determine quantitative criteria for stability and smoothness of Navier-Stokes equations (NS) as functions of scale, defined by the grid Reynolds numbers $Re \in [10^{-4}, 10^3]$. These criteria are examined in conjunction with possible development of infinite velocity singularities as recently proposed with a vortex model (OpenAI, 2026). The current study examines a physically similar case of singular jets with or without external power. The results indicate that NS are stable below the local critical Reynolds number $Re_c = 1.5$, which define fundamentally the transition from laminar-to-turbulent flow regimes. It is shown that smooth physical solutions to NS always exist at the limit $Re \rightarrow 0$ and include the general case of initial discontinuous fields of velocity and pressure, $(\mathbf{u}, P)$, and smooth external forces: $\|\mathbf{f}(\mathbf{r}, t)\| < \infty$. At the microscale $Re \ll Re_c$, any singularities or discontinuities are mathematically and physically incompatible with the NS structure and law of conservation of energy

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1 Introduction

Singularity concept is directly linked to the existence and smoothness of NS solutions and their physical nature. Any singularity in continuous flow constitutes an unphysical result which is contained in the phenomenon of finite time velocity blowup [1]. Infinite spatial energy density at singularity is also unphysical, albeit formally satisfying the overall energy constraints formulated in the description of existence and smoothness hypothesis [2]. This is a principal problem for the entire field of fluid mechanics and

*Modified and expanded version 2

computational fluid dynamics (CFD), which are both based on NS. Moreover, from the first principles, NS directly follow from the fundamental laws of classical physics. If an unphysical situation manifests itself through NS, this inevitably contradicts the Newtonian dynamics, under the general conditions for which these laws have been formulated. Any counterarguments that discrete molecular structure, thermodynamics or statistical energy considerations may prevent physical solutions for continuous fluids are missing the point: fundamentally, NS computations involve nothing but the second Newton's law with inclusion of the viscosity terms and conservation of mass balance [3]. Thus, a detailed examination of this claim is of high scientific methodological significance and very much warranted.

Investigation into the OpenAI claims does not necessary entail reproducing every and each computational step. A more efficient approach employed here is two-tiered: first, the physics of NS is analyzed assuming existence of singularities, drawing conclusions about compatibility of NS with such solutions; second, the diametrically opposite thesis (*i.e.*, the smoothness of NS solutions at the microscale $Re \ll 1$) is verified. In this way, the singularity problem is naturally included in the analysis of NS stability and smoothness at the decreasing scales. NS are converted into a discrete form, which is completely defined by Re . Then mathematical analysis applied to show that singularities are prohibited by the momentum and energy transfer limits imposed by viscous forces at $Re \rightarrow 0$. Furthermore, the principal features of OpenAI vortex model are analyzed for their physical constrains with forces and pressure, whereas a more realistic (but physically similar to the vortex) singularity of growing velocity jet is examined using DNS, confirming that such systems are stabilized at maximum velocity defined by Re and external power. Phenomena of energy dissipation, kinetic energy spatial distribution and velocity oscillations are briefly discussed. The next section presents a numerical analysis of NS leading to determination of the critical local Reynolds number, Re_c , which defines the transition between viscous regime at small scales, to developed turbulence at $Re \rightarrow \infty$. Finally, it is proved that decreasing Re is also responsible for the velocity and pressure profiles being inherently smooth at the scale limit $h \rightarrow 0$.

The current paper is a significantly modified and expanded version of the previous preprint [4]. The following publications will expand the present methodology into the key aspects of quantitative turbulent theory, based entirely on NS analysis, including: DNS methodology for $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$ applications; NS averaging; energy dissipation, conversion and transfer; spectral dependencies and Kolmogorov hypothesis; origin of oscillations and intermittency; impact of boundary conditions and initial pressure-velocity profiles; and, finally, propagation of errors with criteria for energy blowup and control.

2 Methods

The present technique is centered on conversion of NS into a discrete form suitable for analysis of discontinuous functions as described in the next section. The detailed DNS framework will be presented in later publications. Briefly, it utilizes a finite-difference method applied for a cubic grid with length unit h , serving as an ultimate measure

of scale. Following computational stability criteria developed, the dimensionless time increment, Δt , (also known as Courant - Friedrichs - Lewy number) was $\sim 10^{-5}$. The computational steps involved a modified central difference algorithm, with accuracy similar to the fourth-order Runge - Kutta (RK4), but significantly faster. The boundary conditions in the present study enforced fixed velocity. Singular jets were generated by velocity pulses perpendicular to the (x,y) plane at the central point of 40^3 grid. This was an intentional feature of the structural analysis designed for fast computations at widely different scales, in order to systematically isolate and investigate the local singularity formations. Because the onset of localized gradient blowups depends strictly on the micro-structural interactions across neighboring grid nodes, a massive computational domain is unnecessary for such structural verification.

The primary boundary configuration utilizes an enforced constant velocity profile (Dirichlet condition) to establish steady-state momentum injections. However, the modular nature of the underlying asymptotic mean value stencil allows seamless translation to alternative boundary configurations, including periodic boundary for isolated bulk flow analysis, and zero-gradient outflow to eliminate artificial boundary pressures and velocities. The system also allowed for a variety of force implementations, including velocity and pressure pulses, external bulk forces and/or specific time variations of boundary conditions. The discrete matrix interactions were fully formalized and verified using displayed matrix protocols prior to translation into high-performance algorithmic computations. Associated dynamic measurements comprised velocity/pressure profiles, kinetic and potential energies, rate of energy dissipation, momentum/energy transfer rates, spatial velocity profile roughness and fractal dimensions, spatial and temporal correlations, and accompanied by spectral characterization, spatial averaging and running error monitoring. Air flow under normal conditions was used as a model medium. The method was validated for its adherence to the laws of conservation of energy and momentum.

3 Discrete NS form

The greatest advantage of utilizing discrete form of NS is that the system of non-linear partial differential equations is modified into a less complex set of second order polynomials, which can be solved using algebraic analysis. It also enables discontinuous functions to be analyzed. This transformation starts with the continuity equation for compressible fluids:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) \quad (3.1)$$

In the case of isotropic Newtonian fluids and adiabatic process, if the density fluctuations are relatively small, pressure change is defined by its first derivative:

$$\Delta P = K_s \Delta \rho; \quad K_s = \left(\frac{\partial P}{\partial \rho} \right)_s \quad (3.2)$$

For incompressible fluids, the modulus $K_s \rightarrow \infty$, and in the case of ideal gases, $K_s = \gamma P / \rho$, but in the present case, this coefficient can simply be considered a large constant. To computationally track the evolution of pressure fields without solving a globally

coupled Poisson equation at each computational step, the "nearly incompressible fluid" regime was implemented ¹. This borderline mode is defined by the scale separations:

$$|\mathbf{u} \cdot \nabla \rho| \ll |\rho \nabla \cdot \mathbf{u}|; \quad \left| \frac{\Delta \rho}{\rho} \right| \ll 1; \quad (\nabla \cdot \mathbf{u}) \rightarrow 0; \quad (\nabla \cdot \mathbf{u}) \neq 0 \quad (3.3)$$

so that under these constraints, the mass conservation law can be rewritten via the adiabatic bulk modulus, K_s , making pressure and velocity fluctuations directly proportional to each other:

$$\frac{\partial P}{\partial t} = -K_s \rho (\nabla \cdot \mathbf{u}) \quad (3.4)$$

Conditions above correspond to a borderline between more general NS for compressible fluids [3], and incompressible NS [1-3]. In the present work, incompressible fluids are treated as a limit case. The reason for using this approach is that, whilst fluid density can be considered practically constant, the continuity mass-balance equation still contains a general pressure-velocity relationship required for computations. This also means no loss of generality, because all principal conclusions related to the definition of scale, Re and energy functions can also be extended to compressible NS. As it will be discussed later, conditions (3.3) satisfy most practical applications such as air or liquid flows at normal pressure and temperature. Applying the vector form of incompressible NS [3]:

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f} \quad (3.5)$$

Energy form of this equation can be produced by scalar multiplication of all parts with $\mathbf{u}$:

$$\frac{\partial E}{\partial t} = -(\mathbf{u} \cdot \nabla) (E + U) + \nu \nabla^2 E + \varepsilon + w \quad (3.6)$$

where $E = (\mathbf{u}^2)/2$; $U = P/\rho$; $\varepsilon = -\nu (\nabla \mathbf{u})^2$ and $w = (\mathbf{u} \cdot \mathbf{f})$.

Equation (3.6) contains some hidden parameters. Firstly, heat exchange is not considered, because the motion is adiabatic, and the effect of dissipation on local entropy is also neglected, *i.e.* fluid is considered isentropic. This approximation is not critical because it practically makes coefficient K_s a weak function of time, but the mechanism described by equations (3.4-3.6) stands. Secondly and more importantly, there is a periodic conversion of kinetic energy, E , into elastic potential energy of fluid element deformation. This conversion may seem to be unimportant for incompressible fluids, but neglecting it would be a mistake. Whatever the magnitude of K_s in equation (3.2), the kinetic energy must convert into potential energy at least partially, affecting its behavior. The present DNS indicate that kinetic-to-potential energy conversions play an important role in the total energy balance, intermittency and propagation of sonic waves in turbulent fluids. Thus, one should consider underlying potential energy

¹This method was independently developed for this work, however during manuscript preparation priority references [4,5] were found, which consider a similar concept of artificial compressibility and pseudo-compressibility methods

terms in equation (3.6). This can be achieved by splitting pressure into two parts:

$$P = \bar{P} + \tilde{P} \quad (3.7)$$

where the fluctuating part, $\tilde{P}$, describes relative deviation of pressure from its base value, $\bar{P}$, caused by the velocity fluctuations in equation (3.4). Calculating the work of deformation, $d\tilde{U} = \tilde{P}dV$, for a fluid element with specific volume, $V = 1/\rho$, integration yields:

$$\tilde{U} = -\frac{1}{2K_s} \left(\frac{\tilde{P}}{\rho} \right)^2 \quad (3.8)$$

where $\tilde{U}$ is the potential energy of fluid deformation, similar to the energy of elastic strain of solids. By applying equation (3.4), the rate of elastic energy change is:

$$\frac{\partial \tilde{U}}{\partial t} = \frac{\tilde{P}}{\rho} (\nabla \cdot \mathbf{u}) \quad (3.9)$$

The next step is to convert equations (3.4-3.6) into discrete and dimensionless form suitable for computations and further analysis. Because NS equations represent the universal conservation laws of momentum (3.5) and energy (3.6), they are directly valid for any discrete values of these parameters, provided that non-linear advection terms $(\mathbf{u} \cdot \Delta) \mathbf{u}$ and $(\mathbf{u} \cdot \Delta) E$ are treated as momentum and energy transfer rates from one spatial point to another, and that spatial averages are taken for all variables, which can also be discontinuous functions. This theorem can be proven in the general case of finite variables, but in the present work this is unnecessary because limit $\Delta x_i \rightarrow 0$ implies differentials. The index form of converted equations is:

$$\frac{\Delta_t u'_i}{\Delta t'} = -u'_j \Delta_j u'_i - \Delta_i P' + \frac{1}{Re} \Delta_j^2 u'_i + f'_i \quad (3.10)$$

$$\frac{\Delta_t E'_i}{\Delta t'} = -u'_j \Delta_j (E'_i + U' \delta_{ij}) + \frac{1}{Re} (\Delta_j^2 E'_i + \varepsilon'_i) + w'_i \quad (3.11)$$

$$\frac{\Delta_t P'}{\Delta t'} = -K'_s \Delta_i u'_i \quad (3.12)$$

Where the following applies:

$Re = u_0 h / \nu$; $K'_s = K_s / u_0^2$; $\Delta t' = u_0 \Delta t / h$; $\Delta x'_i = \Delta x_i / h$; $u'_i = u_i / u_0$; $P' = P / \rho u_0^2$; $f'_i = f_i h / u_0^2$; $E'_i = u_i^2 / 2 = E_i / u_0^2$; $U' = U / u_0^2$; $\varepsilon'_i = -(\Delta_i u'_i)^2 = \varepsilon_i h / u_0^3$; $w'_i = w_i h / u_0^3$; $\Delta x_i = h$.

The last assignment leads to $\Delta x'_i = 1$ which is then omitted from equations (3.10-3.12) altogether. Parameter h now simultaneously represents the smallest scale, differential step and primary unit of computational grid. By normalizing the spatial differential step directly to unity, the most important length scale h is absorbed into the coordinate mapping. u_0 is the absolute value of characteristic reference velocity, which is typically taken as the steady, non-fluctuating part of velocity field. Symbols Δ_t and Δ_j denote discrete changes with variations of time, $\Delta t'$, or coordinate, $\Delta x'_i$, whereas quantities E'_i , U' , ε'_i and w'_i correspond to the dimensionless kinetic energy,

potential energy, energy dissipation rate and external power. The prime symbol (') is dropped from the following text.

Let us consider development of a singularity, $u \rightarrow \infty$, when $h \rightarrow 0$ and, correspondingly, $Re \rightarrow 0$. The last two conditions can be associated with viscous regime below the Kolmogorov microscale, $Re < 1$ [3]. The reference velocity, u_0 , must be the same for each coordinate and at each scale, otherwise the normalization of equations (3.10-3.12) is not longer applicable. As a consequence, u_0 cannot be the velocity at singularity, as suggested in work [1]. Futility of constructing blowup cases can already be seen from the form of equations (3.10-3.12): with $Re \rightarrow 0$, any singularity leads to infinite derivatives of velocity, meaning infinite energy dissipation rate and infinite pressure gradients, unachievable in real physical systems or, in fact, during a stable simulation. In particular, any momentum transfer towards developing singularity would require unbounded pressure gradients and/or external forces and still negated by infinitely increasing friction at $Re \rightarrow 0$, as exhibited by equation (3.10). The physical mechanism of this phenomenon is contained within NS: because for all Newtonian fluids, the molecular transfer of momentum and heat generation both depend on the magnitude of derivatives, $\partial u_i / \partial x_k$ or, in the discrete case, of differentials, $\Delta_j u_i$, such molecular transfer rate per unit mass becomes infinite near singularity, preventing any singularity to occur. This property also contradicts the major assumption in work [1] where $Re \rightarrow \infty$ at singularity (see p.4).

4 A case against singularity

In the singularity model proposed [1], a vortex rotates with continuously decreasing radius and length, contained by centripetal pressure and external forces. By retaining its angular momentum per unit mass, it is projected by the authors (at least in theory if not in practice) to reach infinite angular and axial velocities at finite time, suggesting a physical breakup of NS at the axial singularity point [1]. Here, the motion of inner core can be approximated by a fluid cylinder rotating around its central axis passing through singularity and with radius, r , decreasing to zero with time. The momentum per unit mass, $L = \frac{1}{2}u_\theta r$, is constant, and therefore $u_\theta \sim r^{-1} \rightarrow \infty$. Kinetic energy of rotation per unit mass $E = u_\theta^2/4 \sim r^{-2} \rightarrow \infty$. In order to balance this rotation speed, the centripetal pressure gradient can be calculated as: $grad P = \rho u_\theta^2 r^{-1} \sim r^{-3}$, directed from the axis radially and fast approaching infinity. Correspondingly, pressure must also rise to infinity near singularity as $P \sim r^{-2}$, acting against the incoming radial flux of momentum and energy. The same effect applies to the ideal fluid of Euler equation, which can be obtained from equation (3.10) at $Re \rightarrow \infty$.

Considering this highly undesirable pressure spike, one may suggest replacing vortex with a singular jet of increasing velocity but decreasing scale. Fig.1 shows velocity profiles of such jets originated with a constant initial velocity and no external energy supplied. They vary dramatically as the scale and Re decrease. This evolution starts with a sharp and intense peak at higher $Re > Re_c$ (Fig.1a), passing through a transitional scale where this peak becomes smaller and wider (Fig.1b) and ending up with very shallow morphologies (Figs.1c and d) which, at certain time points, are

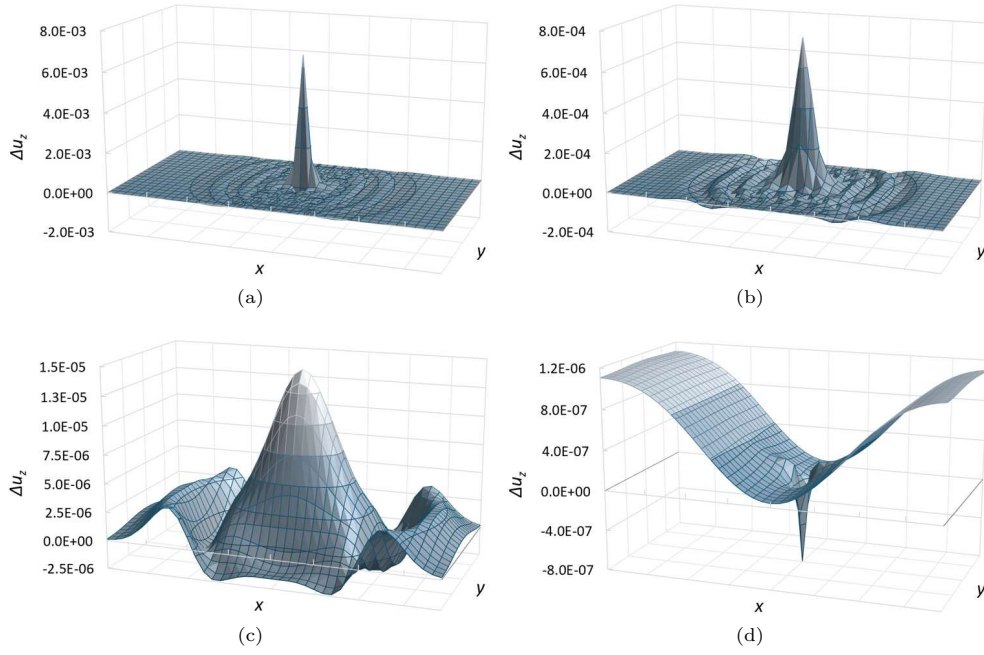


Fig. 1: Jet velocity profiles $\Delta u_z(x, y)$ originated with the same value and propagated at different Reynolds numbers: (a) $Re = 2$, (b) $Re = 0.1$, (c) $Re = 0.01$, and (d) $Re = 0.001$. These profiles were captured at the end of time interval in Fig.2

reversed. The transformations observed illustrate three principal phenomena occurring with introduction of any velocity gradients: increasing dissipation of energy as scale decreases, spread of kinetic energy in fluids starting with sonic waves visible in Figs.1a and b, and periodic conversion between kinetic and elastic energies responsible for the peak fluctuations and reversals at lower Re . The energy diagram of this process is shown in Fig.2. The energy dissipation and periodic energy conversions are clearly seen as functions of Re . At high Re however, another phenomenon emerges: a gradual increase in total energy due to computational errors associated with the finite values of dimensionless time increment, Δt . The later effect, although not physical, manifests itself as an additional energy source inherent to any computation. Consequently, it has to be addressed in order to prevent a real (for computer simulations) blowup of velocity and energy. The supplemental recordings (S1-S4) illustrate jet dynamics at different Re .

When energy per unit mass is supplied at a constant rate, one can write the kinetic energy balance from equation (3.11), neglecting the conservative energy transition terms:

$$\frac{\partial E}{\partial t} = \frac{\varepsilon}{Re} + w \quad (4.1)$$

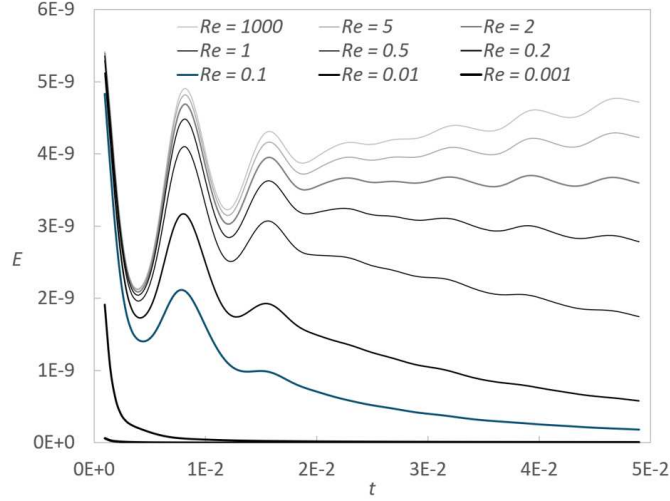


Fig. 2: Time-dependence of jet kinetic energy, $E(t)$, as a function of Reynolds number

In the present case of singular jet:

$$\varepsilon \approx -2E \quad (4.2)$$

Integration of equations (4.1) and (4.2) yields:

$$E(t) = (E_o - \frac{w Re}{2})e^{-\frac{2t}{Re}} + \frac{w Re}{2} \quad (4.3)$$

This function plateaus at $E_{t \rightarrow \infty} = w Re/2$ and requires an infinite power to sustain at $Re \rightarrow 0$. The evolution of such jet at a constant external power, w , but decreasing scale is presented in supplemental video recording S5.

5 NS stability

Within the Eulerian framework of NS, the local and convective accelerations are separate terms. As a consequence, in any simulation a fluid system needs time to recover its balance after a disturbance occurred. This recovery proceeds as follows: if a small (theoretically infinitesimal) perturbation is made at any spatial point, the reactions of pressure and friction forces are always negative (*i.e.* resisting change), whereas the reaction of inertia (first term on the right in equation (3.10)) is positive, formally supporting the instability. This classical reaction mechanism is exploited below.

The elemental unit for computation here is characterized by the von Neumann neighborhood of six cells surrounding the central cell [7,8] as shown in Fig.3. Let us consider a case whereby small perturbations of velocity, $\partial_{ij}u_i$, may appear randomly in any cell of this neighborhood. The index i is assigned to a specific velocity component

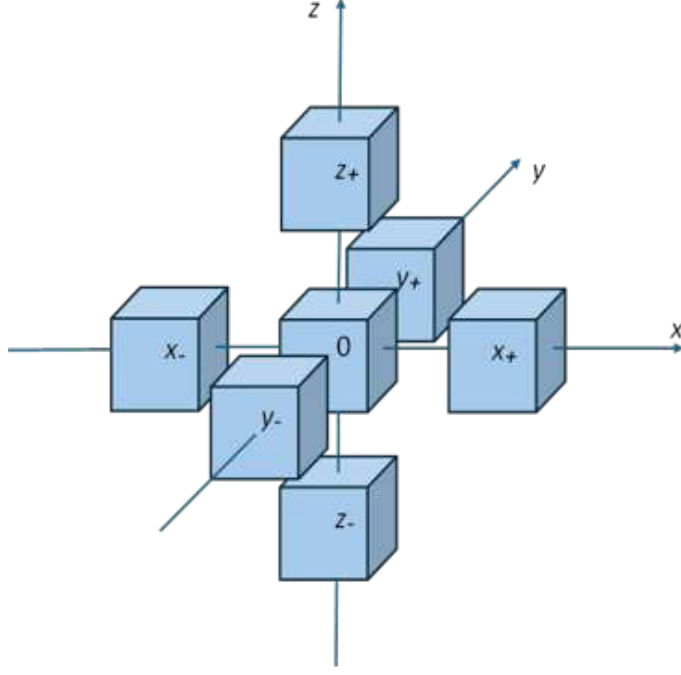


Fig. 3: The elemental unit of computation for discrete NS consisting of octahedral von Neumann neighborhood where the indexes (-) and (+) denote cell positions in respect to the central cell (0)

and index j to the direction of this component change. Let differential for any function, $\varphi(u_i)$, be:

$$\partial_{ij}\varphi(u_i) = \varphi(u_i + \partial_{ij}u_i) - \varphi(u_i) \quad (5.1)$$

and calculate the following terms:

$$\partial_{ij}(\Delta_j u_i) = \partial_{ij}u_{i+} - \partial_{ij}u_{i0} \quad (5.2)$$

$$\partial_{ij}(u_j \Delta_j u_i) = u_{j0}(\partial_{ij}u_{i+} - \partial_{ij}u_{i0}) \quad (5.3)$$

$$\partial_{ij}(\Delta_j^2 u_i) = \partial_{ij}u_{i+} + \partial_{ij}u_{i-} - 2\partial_{ij}u_{i0} \quad (5.4)$$

Equation (5.3) ignores differentials of the higher order. For the pressure term:

$$\partial_{ij}(\Delta_i P \delta_{ij}) = \partial_{ij}(\Delta_t P \delta_{ij}) = -K_s(\partial_{ij}u_{i0} - \partial_{ij}u_{i-})\Delta t \quad (5.5)$$

and, therefore, this term is of the second order in respect to time. Moreover, $\delta_{ij}f_i = 0$, because the external force is considered independent of velocity fluctuations. The stability criterion is:

$$\partial_{ij}(\Delta_t u_i) = 0 \quad (5.6)$$

leading to the following relationship:

$$\chi_2 = Re_c \chi_1 \quad (5.7)$$

where $\chi_1 = u_{i0} (\partial_{ij} u_{i+} - \partial_{ij} u_{i0})$; $\chi_2 = (\partial_{ij} u_{i+} + \partial_{ij} u_{i-} - 2\partial_{ij} u_{i0})$.

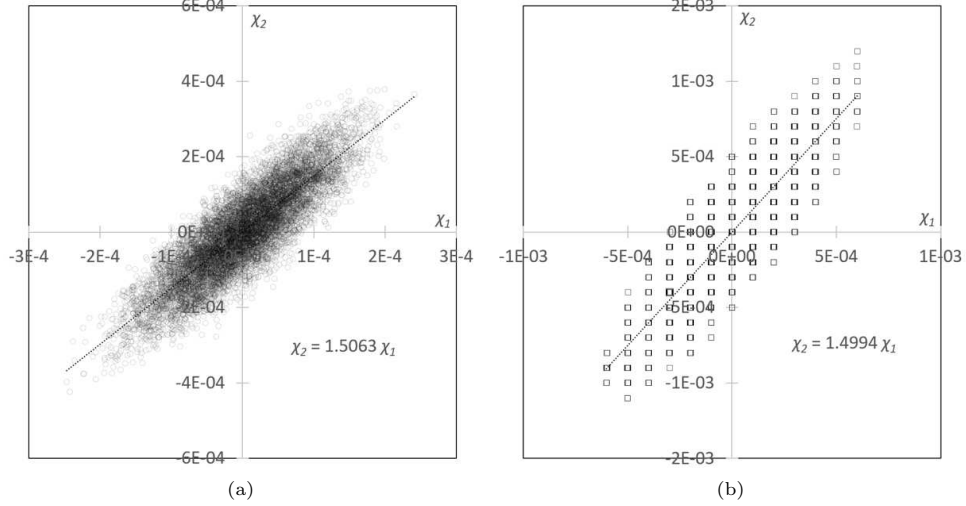


Fig. 4: Diagrams of NS stability response according to equation (5.7) with (a) random continuous and (b) random discrete perturbations of velocity in $\mathbb{R}^3$ space, generated with 10^4 data points

Functions χ_1 and χ_2 correlate with each other defining the critical Reynolds number, Re_c , below which the friction term, χ_2 , dominates the inertial (advective) term, χ_1 . Clearly, there is a fundamental analytical relationship between these functions defined by the distribution density of points (χ_2/χ_1). Equation (5.7) is insensitive to the base velocity profile or amplitude/frequency of fluctuations, $\partial_{ij} u_i$, because these parameters affect both parts of this equation equally. It can also be noted that pressure fluctuations transform into velocity fluctuations according to equation (3.10), resulting in the same effect but with some time delay. At $Re < Re_c$, NS are stable to small, random perturbations of velocity. The magnitude of Re_c is easiest to obtain by numerical analysis with a sufficient data set. Fig.4 exhibits distributions χ_2 (χ_1) obtained with $\partial_{ij} u_i$ pulses randomly generated from continuous interval $[-\frac{a}{2}, \frac{a}{2}]$ (Fig.4a) or from discrete number set $(-a, 0, a)$ (Fig.4b), where the amplitude $a = 10^{-4}$. In all cases, data points are distributed around trend lines with inclination $Re_c = 1.5$. This distribution graphically resembles elongated hexagon or oblong ellipse, or, in case of lower densities, a family of straight lines. The density pattern is defined by the frequency of generating similar numbers, but the inclination remains practically constant and robust, confirming that NS are statistically stable below Re_c .

6 The existence and smoothness of NS solutions

The results of previous sections establish that NS solutions have to be continuous functions without singularities at the scale limit $\Delta_i x = h \rightarrow 0$. The same methodology can be applied to examine existence and smoothness of these functions. According to equation (3.10), there are 24 boundary conditions of velocity and pressure in the six neighborhood cells (Fig.3). Three quadratic equations are to be solved in respect to the three velocity components and pressure in the central cell, with given external forces, and in conjunction with the linear continuity equation (3.12). The boundary conditions are restricted and have to be physically compatible with the internal parameters of the central cell. Let us consider when a real solution of these equations can exist and how it presents itself at the limit $h \rightarrow 0$. For simplicity, the cell coordinate system is selected whereby the axis x is aligned with the velocity vector, $\mathbf{u}$, in the central cell. This means that axis x is tangential to the velocity streamline. This implies no loss of generality because such coordinates can always be assigned to a specific velocity on the streamline. Consequently, equations (3.10-3.12) convert to:

$$u_{10}^2 - (u_{1+} + \frac{6}{Re})u_{10} + \frac{\Sigma_1}{Re} + \psi_1 = 0 \quad (6.1)$$

$$u_{2+}u_{10} - \frac{\Sigma_2}{Re} + \psi_2 = 0 \quad (6.2)$$

$$u_{3+}u_{10} - \frac{\Sigma_3}{Re} + \psi_3 = 0 \quad (6.3)$$

$$u_{1-} + u_{2-} + u_{3-} - u_{10} - \xi = 0 \quad (6.4)$$

here, $\Delta_i P = (P_{i+} - P_{i0})$; $\Sigma_i = \sum_{j=1}^3 (u_{ij+} + u_{ij-})$; $\psi_i = (f_i - \Delta_i P - a_i)$; $a_i = \Delta_t u_i / \Delta t$; $\xi = -\Delta_t P / (K_s \Delta t)$.

Vector $\mathbf{a}$ designates local accelerations, whereas scalar ξ corresponds to the rate of pressure change according to equation (3.12). These four quantities vanish in the case of steady solutions but, in general, they are additional variables. Vector $\boldsymbol{\psi}$ can be called "force function" because it represents the balance between external forces, pressure and local acceleration (being considered as a pseudo-force). Linear equations (6.2-6.4) always have real solutions in respect to u_{10} and P , given compatible boundary conditions. However, for a real solution to exist in the quadratic equation (6.1), its discriminant must be non-negative:

$$D = (u_{1+} + \frac{6}{Re})^2 - 4(\frac{\Sigma_1}{Re} - \Delta_1 P + f_1 - a_1) \geq 0 \quad (6.5)$$

Looking for the stable physical root (-), this relationship is always valid at $Re \rightarrow 0$ ($h \rightarrow 0$) where it is dominated by the positive quantity $(\frac{6}{Re})^2$, while equation (6.1) becomes linear in the close proximity to that point.

$$u_{10} = \frac{\Sigma_1}{6} + \mathcal{O}_1(Re) \quad (6.6)$$

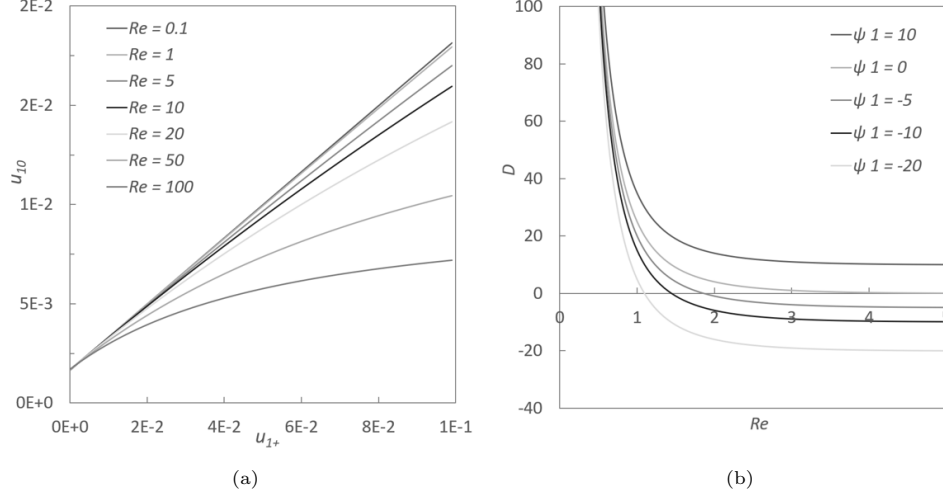


Fig. 5: (a) Local solution of equation (6.1) in coordinates $u_{10}(u_{1+})$ and for different Reynolds numbers, Re ; (b) Discriminant of equation (6.5) as function of Re , and the force parameter, ψ_1

Now, it is clear that above arguments can be easily extended to any coordinate system in which all of the velocity components, $u_i \neq 0$. In this case, with $Re \rightarrow 0$, only the linear friction terms survive, resulting in the solutions:

$$u_{i0} = \frac{\Sigma_i}{6} + \mathcal{O}_i(Re) \quad (6.7)$$

The dimensionless velocity field $\mathbf{u}(\mathbf{r}, t)$ on the symmetric cubic grid satisfies the Asymptotic Mean Value Property (AMVP) scaled exclusively by Re :

$$\left(\frac{1}{6} \sum_{y \in \mathcal{N}_1(\mathbf{r})} \mathbf{u}(y, t) \right) - \mathbf{u}(\mathbf{r}, t) = \frac{Re}{6} \left[(\mathbf{u}(\mathbf{r}, t) \cdot \nabla) \mathbf{u}(\mathbf{r}, t) \right] + \mathcal{O}(Re) \quad (6.8)$$

where $\mathcal{N}_1(\mathbf{r})$ represents the six-neighborhood of the central fluid cell. Accordingly, each velocity component at any grid node is the average of corresponding velocity components in the boundary nodes, plus the vanishing advection term and vanishing quantity $\mathcal{O}(Re)$ (scaled as $\sim h$) which contains the force function. With $Re \rightarrow 0$, the viscous dissipation term acts as a smoothing operator that dampens any fluctuations, guaranteeing that the velocity gradients remain bounded and finite across the entire grid. At higher Reynolds numbers, the nonlinear advective term is responsible for hyperbolic momentum (and energy) transfer, which naturally allows sharp fronts or directional shocks to persist. When $Re \rightarrow 0$, the governing continuous equation becomes purely elliptic ($\Delta \mathbf{u} = 0$). Elliptic systems enforce the strong maximum principle, meaning the velocity field cannot have localized spikes or numerical instabilities inside the grid domains; it must vary smoothly between boundaries [9,10].

Thus, the baseline solutions $u_{i0} = \Sigma_i/6$ conform to the discrete harmonic mean value property [9,10] which is the definitive proof that discrete harmonic functions on a grid converge uniquely and smoothly to continuous harmonic functions at $h \rightarrow 0$. In the forced system represented by vector equation (6.8), the structural smoothness of the continuous velocity field $\mathbf{u}(\mathbf{r}, t)$ is determined asymptotically by Re . As $Re \rightarrow 0$, the non-linear advective transport term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ becomes subdominant to the stabilizing diffusion/viscosity term on the left, and the asymptotic quantity $\mathcal{O}(Re)$ retains a well-defined weak structure. Following the generalized framework of viscosity solutions [11,12], the presence of such a smooth forcing function maintains bounded solution gradients, whilst preventing localized numerical singularity blowups and, therefore retaining smoothness C^∞ of the underlying continuous function, provided that the boundary conditions are reasonably well-behaved. It should be emphasized that equations (6.7-6.8) hold for any finite values of velocity, pressure, external forces, and accelerations, because there is always a scale below which all these terms are negligible compared to $1/Re$.

The spatial smoothness of the resulting pressure field follows directly from the conditions (3.3-3.4). Taking the spatial gradient (∇) of the pressure equation (3.4) yields $\frac{\partial}{\partial t}(\nabla P) = -K_s \rho \nabla(\nabla \cdot \mathbf{u})$. Because the time-evolution of the pressure gradient ∇P depends linearly on the secondary spatial derivatives of the velocity field ($\nabla(\nabla \cdot \mathbf{u})$), the pressure function is smooth, provided the discrete velocity field variations remain bounded (the term $\nabla(\nabla \cdot \mathbf{u})$ remains finite). This guarantees that the spatial pressure gradients ∇P cannot instantaneously spike to infinity. This specific formulation mathematically acts as an acoustic wave equation. Localized pressure spikes do not instantly form singular discontinuities: they are naturally radiated away to the boundaries as acoustic waves at the speed of sound $c = \sqrt{K_s}$, while smoothing out localized numerical artifacts [4,5]. This effect was observable through the DNS in Fig. 1.

The smoothness problem can also be considered from a different perspective. The local solution at $Re \rightarrow 0$ depends strongest on the neighborhood values of velocity, u_{ij} , (see equations (6.1-6.4)) and also analytically related to the scale through Re , and to the force function, ψ . All these functions, by their structure, are smooth when $Re \rightarrow 0$ and equation (6.5) is fulfilled. For example, Fig.5a shows dependence $u_{10}(u_{1+})$ at different Re and fixed boundary condition Σ_1 and force ψ_1 . For small velocity fluctuations, $u_{1+} \approx 1$; $\Sigma_1 \approx 6$, the discriminant in equation (6.5) converts into: $D = (1 - 6/Re)^2 + \psi_1$. If $\psi_1 \geq 0$, a solution to equations (6.1-6.4) does exist in general. However, at the higher Re (*i.e.* larger scales, finite variables), this solution is not guaranteed to be smooth or even continuous, representing turbulence on a larger scale. Obviously, solutions always exist with undefined acceleration $\mathbf{a}$ (*i.e.* unsteady solutions) because function ψ can always be made positive in such cases. When $\psi_1 < 0$, there is still a maximum value of Re below which a solution can still be found as illustrated in Fig.5b. Again, not finding a steady solution with specific boundary conditions means that solution is not steady, *e.g.* observed when external forces are time-dependent or when velocity or pressure undergo fluctuations at the boundary surface. Consequently, the following flow regimes can be distinguished at different scales and conditions:

- $Re \rightarrow 0$; $h \rightarrow 0$: On the microscale, smooth, global solutions always exist. These are generally time-dependent but convert into steady solutions when the force function, ψ , is stationary, the boundary conditions are fixed and physically consistent, and both parameters $\mathbf{a} \rightarrow 0$ and $\xi \rightarrow 0$. The steady solution also implies uniqueness, because at $Re \rightarrow 0$, NS are transformed into a system of linear equations which, under the fixed physical boundary conditions should have only one solution. There is no place for singularities.

- $D \geq 0$, $Re \approx Re_c$: On this transitional scale, flow becomes unstable, but smooth and steady solutions may still endure under some boundary conditions and with uniform external forces.

- $D < 0$, $Re \gg Re_c$: On the larger scales, the flow is turbulent, meaning that only non-stationary, discontinuous solutions exist for the discrete NS (equations (3.10-3.12)).

It may seem paradoxical that, at the scale limit $h \rightarrow 0$, there is always a smooth solution to NS, which, dependently on the boundary conditions and forces applied, may become steady with time, whereas at a sufficiently large scale, fluid is unsteady and turbulent. However, as it will be presented in following publications, this fact represents the nature of turbulence which is defined by a superposition of small-scale motions over large domains. This superposition is always smooth under high magnification, although generally fast changing. At the large scales however, it can be viewed as highly irregular, with discontinuities and sharp boundaries.

7 The final note: critique of singularity models

Considering high publicity surrounding the OpenAI publication [1], claiming a counterexample to the NS Existence and Smoothness hypothesis [2], one of the famous Millennial Prize Problems, it would be beneficial to list major arguments, known at this time, on the validity of such claim. It should be said that the currently dominating opinion seems to be that OpenAI has constructed a mathematically sound model, albeit unphysical, artificial and methodologically unhelpful to the related research communities of mathematicians, physicists and engineers. The main goal of the present work, however, is to demonstrate why any model proposing a NS blowup solution (singularity) must be, in general, fundamentally flawed.

a. Relationship between physics and mathematics of NS. Contrary to a popular belief, the NS Smoothness and Existence is not a purely mathematical problem. It is just as much a question of physics and computational science, constituting an important part of fluid mechanics, computational fluid dynamics (CFD), and the theory of turbulence. The latter was famously characterized by Richard Feynman as "the most important unsolved problem of classical physics" because of its notable lack of self-consistent, quantitative mathematical description, despite the fact that NS certainly provide the first principles. The physical origin of NS makes search for smooth solutions (or, alternatively, singularities) objectively a physical, non-abstract problem affecting many practical applications. The opposite is also true: as a consistent physical framework, NS must result in non-contradictory mathematical solutions. Therefore, it is impossible to separate here a pure mathematical proof from its physical meaning:

failure of one leads to the failure of the other. Consequently, any mathematical conclusions correctly derived from NS (including those produced by a valid computational technique) have to conform the physical reality and allow for potential experimental verification. This scientific methodology obviously breaks down in the case of Open AI solution.

b. Physical limitations of NS. Because a singularity implies infinitesimal scale, it has been often suggested that NS may fail at that extreme due to molecular structure of any fluid, local heat generation leading to evaporation, cavitation, *etc*, and generally restricted by the thermodynamics / statistical physics of the "real world fluids". Of course, there is no question about the reality of molecular / atomic structure of matter or role of thermodynamics, but all of these processes are completely irrelevant in the current context. NS is the product of classical Newtonian mechanics, and all other effects are simply not a part of its physics. Theoretically, the continuous fluid concept is valid for indiscernible distances and volumes imposed by the differential structure of NS. There is no mathematical limit here but zero. OpenAI did not use any additional principles (at least not to the author's best knowledge) but only the NS system. This fact means that Newtonian physics, a validated theory applied for centuries, suddenly failed in the OpenAI simulation.

c. The nature of external force. Another common critique is that OpenAI applied an external force in a highly artificial, engineered manner which cannot "naturally" occur. The analysis of such force is beyond the scope of current discussion, but most importantly it is also irrelevant, because no physical law strictly limit the origin of force, whether it is artificially created or widespread in nature, provided that it can be strictly defined, measurable and have a known physical source (even in the hypothetical scenario). Engineering must be considered here a part of nature, and therefore such an objection is illogical. However, any external force must be constrained and not infinite when approaching singularity, because that is, indeed, unnatural (except perhaps in a black hole) and excluded by the Millennium Problem description [2], specifying a smooth, divergence-free force field (see below on the root cause).

d. Code verification. Semantic discrepancies are noted between the OpenAI natural language and its accompanying autoformalization code, whereby the Lean's verification does not guarantee the mathematical correctness of the core text [13]. This is an important point because it opens a real possibility that the computational procedures were not implemented as described and/or somewhat different, perhaps mathematically flawed computations were not verified by Lean due to the "mistranslation gap". This however does not address the root cause why any singularity is fundamentally incompatible with NS.

e. The root cause. The mathematical structure and physical nature of NS both assure that any increase in momentum or kinetic energy is compensated by the friction forces generated within viscous fluids. This is, of course, a well-known fact fundamentally linked to the energy dissipation. The main goal of present work, however, was to find the exact quantitative criteria defining stability and smoothness of NS solutions as functions of scale. With $Re \rightarrow 0$, the friction terms scale as $\sim 1/Re$, always dominating external forces, pressure and momentum inertia, and thus resulting in suppression of any discontinuities (including singularities) on the microscale. Reiterating

this statement, one can say that no bounded external force (engineered or natural), pressure enforcement or "concentration of kinetic energy" can alter this fact. Unfortunately, the manuscript [1] does not provide sufficient data in terms of verifiable, scale- and time-dependent velocity-pressure profiles, and energy balance diagrams to pinpoint the exact location and reasons for such discrepancies. The issues may potentially appear in the wrong physical assumptions or conditions (*e.g.* incorrect definition of the Reynolds numbers given on p.4), implementing procedures with large computational errors, and/or running unverified, distorted (and potentially erroneous) protocols as recently analyzed in work [13].

8 Conclusion

Newtonian fluids are stable to random perturbations of velocity and pressure below the critical local Reynolds number $Re_c = 1.5$ which is directly derived from the behavior of NS. These equations always result in smooth global solutions $(\mathbf{u}, P)$ at the microscale $Re \rightarrow 0$, which automatically coincides with the NS differential limit, and also uniquely defined by their boundary conditions and force functions. This statement extends to a general case of initially discontinuous pressure-velocity fields and bounded external forces ($\|\mathbf{f}(\mathbf{r}, t)\| < \infty$), whereas the existence of singularities contradicts the NS mathematical structure and violates the law of conservation of energy. Consequently, singularities in Navier-Stokes equations are impossible.

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Funding

No funding was received for conducting this study.

Conflict of interest

The author declares no conflict of interest.

Data availability

All necessary data supporting the findings are included with this manuscript. Any additional data can be provided upon a reasonable request.

Supplementary Recordings

Supplementary materials and recordings are available online. The supplemental recordings associated with this manuscript are listed below:

- S1:** Development of singular jet at $Re = 1000$
- S2:** Development of singular jet at $Re = 2$
- S3:** Development of singular jet at $Re = 0.1$
- S4:** Development of singular jet at $Re = 0.01$
- S5:** Kinetics of jet propagation at constant power input and $Re \rightarrow 0$

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