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Sawon Pratiher

***Abstract*—Closed-loop specifications that constrain settling rate, damping ratio and natural frequency simultaneously define a semialgebraic, non-convex region in the space of characteristic polynomial coefficients, which is what makes fixed-structure synthesis against such specifications hard. We show that in *modal pair coordinates*, obtained from the canonical real factorization of the characteristic polynomial into irreducible sections, the same region becomes a product of polyhedra. Each complex section contributes a pair (A_i, ω_i), the real part of the pole pair and its modulus, and each real root contributes one coordinate. Hurwitz stability, a spectral abscissa bound, a damping cone and a bandwidth bound then become *linear* inequalities, the exact identity $|s| = \omega_i$ being what linearizes cone and disk simultaneously. All non-convexity is thereby isolated in the Vieta map returning modal coordinates to coefficients. We pose fixed-structure synthesis as a projection-residual least-squares problem whose residual vanishes exactly on the specification set and whose inner projection is solved in closed form by active-set enumeration. Four supporting analyses are given: a closed-form modal sensitivity, verified against finite differences to a median relative error of 1.3×10^{-7}, showing the modal map to be well conditioned with condition numbers that *decrease* with degree; a complexity account showing that a residual evaluation is projection-dominated rather than root-finding-dominated up to degree about 32; a convergence study establishing that the observed order is linear, empirical exponent 1.06, because the projection active set changes along the trajectory; and an exact single-vertex robustness test for box uncertainty in the modal parameters, the natural uncertainty model for flexible structures, decided in $O(m)$ arithmetic rather than over 2^{2m} vertices. On a lightly damped fourth-order plant the method meets a specification that spectral abscissa minimization misses by a wide margin in damping, and over 120 randomly generated lightly damped plants it succeeds on 99% within twelve restarts at a median of 17 residual evaluations. A specification that is infeasible at the controller order considered is reported rather than removed.**

***Index Terms*—Fixed-structure control, pole placement, damping ratio, D-stability, characteristic polynomial, non-convex optimization, root clustering, robust stability.**

I. Introduction

MANY practical closed-loop specifications are stated directly in terms of pole location: a settling rate through a bound on the spectral abscissa, an overshoot limit through a bound on the damping ratio, and an actuator bandwidth limit through a bound on the natural frequency. When a full-order controller and a state-space description are available, such specifications are handled convexly by the LMI region machinery of Chilali and Gahinet [1], extended in [2]. When the controller is constrained to fixed low order or fixed structure such as PID, that machinery no longer applies directly and the problem becomes non-convex and in general NP-hard [3], [4]. Practical approaches include nonsmooth optimization [5], [6], structured search [7], and coefficient-space methods based on Routh-Hurwitz or on parametric robustness results such as Kharitonov's theorem [8], [9].

The difficulty in coefficient space is geometric. The set of Hurwitz polynomials of a given degree is not convex, and once damping and bandwidth constraints are added the admissible set becomes semialgebraic of high degree with no useful convex description. Our starting observation is that this awkwardness is an artifact of the chosen coordinates.

A. Contribution

Let P be real monic of degree n with canonical factorization over $\mathbb{R}$

$$P(s) = \prod_{i=1}^{m} (s^2 - 2A_i s + \omega_i^2) \prod_{k=1}^{q} (s - x_k), \tag{1}$$

$2m + q = n$, each quadratic irreducible, $\omega_i > 0$. We call $(A_i, \omega_i)_{i \le m}$ with $(x_k)_{k \le q}$ the *modal pair coordinates*. Our contributions are:

1) Hurwitz stability is the linear condition $A_i < 0$, $x_k < 0$ (Theorem 1); the sign of the *pair spread* $\Lambda_i = A_i^2 - \omega_i^2$ classifies the damping regime (Proposition 1).
2) The specification set is a *polyhedron* in modal coordinates, a Cartesian product across sections (Theorem 2), whereas the corresponding coefficient-space set is not convex (Remark 1).
3) A closed-form modal sensitivity (Proposition 3) with an empirical conditioning study (Section III).
4) A projection-residual synthesis formulation (Section IV) with exact inner projection (Lemma 1), a complexity account (Proposition 4) and a convergence analysis explaining the observed linear rate (Section IV-C).
5) An exact single-vertex robustness test for modal box uncertainty (Theorem 3).
6) A mechanism-resolved stability margin separating Hopf from real-root loss of stability (Proposition 2).

B. Relation to prior work and scope

The factorization (1) is elementary and the identification of (ζ, ω_n) with a second-order section is textbook. What is new is not the factorization but the observation that the

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standard D-region specifications become *jointly linear* in these coordinates, that the feasible set is a product polyhedron, and that this supports both a synthesis algorithm and an exact robustness test. Second-order sections are standard in digital filter realization [10] and modal analysis, but to our knowledge have not been used as the working coordinate system for D-region synthesis with fixed structure. The algebra of pair centres and spreads is developed in [12].

We state clearly what is not claimed. Synthesis does not become convex. The map from modal coordinates to coefficients is the Vieta map, polynomial and non-injective, and the achievable set of characteristic polynomials for a fixed structure is an affine subspace of coefficient space. Convexity holds in the lifted coordinates only. The claim is that this is a better place to put the difficulty, because the constraint set becomes trivial to project onto and the lift admits an exact robustness test.

II. Modal Pair Coordinates

A. Definitions and modal identification

Definition 1 (Modal pair coordinates)**.** For each irreducible quadratic $Q_i(s) = s^2 - 2A_i s + \omega_i^2$ in (1), call $A_i \in \mathbb{R}$ the *pair centre*, $\omega_i > 0$ the *pair modulus* and

$$\Lambda_i = A_i^2 - \omega_i^2 \tag{2}$$

the *pair spread*. The roots of Q_i are $A_i \pm \sqrt{\Lambda_i}$.

Real roots are kept as separate first-order sections rather than combined in pairs. This is essential rather than cosmetic: a product of two real roots of opposite sign gives a quadratic with negative constant term, for which ω_i is undefined, and a stability test inspecting only the centre would then accept an unstable polynomial.

Proposition 1 (Modal identification)**.** *Write $Q_i(s) = s^2 + 2\zeta_i\omega_{n,i}s + \omega_{n,i}^2$. Then $\omega_{n,i} = \omega_i$, $\zeta_i = -A_i/\omega_i$, and $\Lambda_i = \omega_i^2(\zeta_i^2 - 1)$. The sign of Λ_i classifies the damping regime: underdamped if $\Lambda_i < 0$, critically damped if $\Lambda_i = 0$, overdamped if $\Lambda_i > 0$.*

Proof: Matching coefficients gives $2\zeta_i\omega_{n,i} = -2A_i$ and $\omega_{n,i}^2 = \omega_i^2$; substituting into (2) gives $\Lambda_i = \omega_i^2(\zeta_i^2 - 1)$, whose sign is that of $\zeta_i^2 - 1$. ∎

B. Stability

Theorem 1 (Hurwitz stability)**.** *P is Hurwitz if and only if $A_i < 0$ for every $i \le m$ and $x_k < 0$ for every $k \le q$.*

Proof: A product is Hurwitz iff every factor is. For $s - x_k$ the condition is $x_k < 0$. For irreducible $s^2 - 2A_i s + \omega_i^2$ with $\omega_i > 0$, the degree-two Routh array gives $-2A_i > 0$ and $\omega_i^2 > 0$; the second holds by construction, so the condition is $A_i < 0$, consistent with the roots being $A_i \pm \mathrm{j}\sqrt{|\Lambda_i|}$. ∎

We verified this on 8000 random monic real polynomials of degrees 2 to 8 with coefficients from $\mathcal{N}(0, 4)$, comparing against direct computation of root real parts. There were no mismatches.

C. The specification set is polyhedral

Theorem 2 (Polyhedrality)**.** *Fix $\alpha \ge 0$, $\bar\zeta \in [0, 1)$, $\bar\omega > 0$ and let $\mathcal{D} = \{s \in \mathbb{C} : \mathrm{Re}\, s \le -\alpha,\ |\mathrm{Re}\, s| \ge \bar\zeta|s|,\ |s| \le \bar\omega\}$. All roots of P lie in $\mathcal{D}$ iff*

$$\begin{aligned} A_i \le -\alpha, \quad -A_i - \bar\zeta\,\omega_i \ge 0, \quad \omega_i \le \bar\omega, \quad i \le m, \\ x_k \le -\alpha, \quad |x_k| \le \bar\omega, \quad k \le q. \end{aligned} \tag{3}$$

The admissible set is a polyhedron, a Cartesian product of two-dimensional polyhedra with intervals for the real roots.

Proof: For a complex section the roots are $A_i \pm \mathrm{j}\sqrt{|\Lambda_i|}$, so $\mathrm{Re}\, s = A_i$ and $|s|^2 = A_i^2 + |\Lambda_i| = A_i^2 + \omega_i^2 - A_i^2 = \omega_i^2$, giving $|s| = \omega_i$ exactly. The half-plane condition is $A_i \le -\alpha$; the disk condition is $\omega_i \le \bar\omega$; the cone condition with $A_i < 0$ is $-A_i \ge \bar\zeta\omega_i$. All three are affine. For a real root $|s| = |x_k|$ and the damping ratio is unity, so the cone is vacuous. No constraint couples sections. ∎

The identity $|s| = \omega_i$, with no square root and no absolute value, is what linearizes disk and cone simultaneously. This is special to the modal parameterization: in terms of the two roots individually, or of the coefficients, both constraints are quadratic.

Remark 1 (Coefficient space is not convex). With $\alpha = 0.5$, $\bar\zeta = 0.6$, $\bar\omega = 6$, the monic quartics $a^{(1)} = (1, 6.256, 16.153, 20.228, 10.516)$ and $a^{(2)} = (1, 11.536, 68.783, 188.246, 219.455)$ both have all roots in $\mathcal{D}$, while $0.2\,a^{(1)} + 0.8\,a^{(2)} = (1, 10.480, 58.257, 154.642, 177.667)$ does not. The lifted region is convex by construction; we confirmed this on 2×10^4 random convex combinations of admissible lifted points with zero violations.

D. Mechanism-resolved stability margin

Proposition 2 (Margin decomposition)**.** *Let P be Hurwitz. Under continuous variation of the coefficients, stability is first lost either when some A_i reaches zero, a conjugate pair crossing the imaginary axis at $\pm\mathrm{j}\omega_i$, or when some x_k reaches zero, a real root crossing the origin. The quantity*

$$\delta(P) = \min\Big\{\min_{i\le m}|A_i|,\ \min_{k\le q}|x_k|\Big\} \tag{4}$$

is the spectral abscissa margin, and the minimizing index identifies both the critical mode and the mechanism.

Proof: By Theorem 1 the Hurwitz region in modal coordinates is $\{A_i < 0\} \cap \{x_k < 0\}$, with boundary the union of the hyperplanes $A_i = 0$ and $x_k = 0$. Roots depend continuously on coefficients, so a path leaving the region crosses one of these. On $A_i = 0$ the section is $s^2 + \omega_i^2$ with roots $\pm\mathrm{j}\omega_i$; on $x_k = 0$ a real root sits at the origin. Since $\mathrm{Re}\, s = A_i$ for a complex section, the spectral abscissa is $\max_i\{A_i, x_k\}$ and its distance to zero is (4). ∎

This is finer than a scalar abscissa: it names the critical mode and says whether the impending instability is oscillatory or monotone, indicating whether added damping or added integral authority is the remedy.

Table I: Median relative condition numbers of the modal coordinates, 250 random monic polynomials per degree.

n	median κ_A	median κ_ω
4	2.28	1.09
6	1.99	0.95
8	1.81	0.89
10	1.62	0.84
12	1.47	0.78

III. Modal Sensitivity and Conditioning

Because modal coordinates are obtained from coefficients by root-finding, their conditioning governs the reliability of the whole scheme.

Proposition 3 (Modal sensitivity)**.** *Let $P(s) = \sum_k a_k s^{n-k}$ and let s_i be a simple root of complex section i. Then*

$$\frac{\partial s_i}{\partial a_k} = -\frac{s_i^{n-k}}{P'(s_i)}, \qquad \frac{\partial A_i}{\partial a_k} = \operatorname{Re}\frac{\partial s_i}{\partial a_k}, \tag{5}$$

$$\frac{\partial \omega_i}{\partial a_k} = \frac{1}{\omega_i}\operatorname{Re}\left(\overline{s_i}\,\frac{\partial s_i}{\partial a_k}\right). \tag{6}$$

Proof: Differentiating $P(s_i;a) = 0$ implicitly gives $s_i^{n-k} + P'(s_i)\,\partial s_i/\partial a_k = 0$, which is (5); $A_i = \operatorname{Re} s_i$ gives the second identity. For the modulus, $\omega_i^2 = s_i\overline{s_i}$, so $2\omega_i\,\partial\omega_i/\partial a_k = 2\operatorname{Re}(\overline{s_i}\,\partial s_i/\partial a_k)$. ∎

We verified (5) and (6) against central finite differences on 400 random polynomials of degrees 4 to 8, giving 4754 scalar comparisons: median relative error 1.33×10^{-7}, 90th percentile 6.22×10^{-7}, consistent with the step size used.

Define relative condition numbers $\kappa_A = \|\nabla_a A_i\|\,\|a\|/|A_i|$ and $\kappa_\omega = \|\nabla_a \omega_i\|\,\|a\|/\omega_i$. Table I reports medians over 250 random polynomials per degree.

Two features deserve comment. The condition numbers are of order unity, so for generic polynomials the modal coordinates are not an ill-conditioned function of the coefficients. They also *decrease* with degree over the range tested, in contrast with the growth of root condition numbers familiar from Wilkinson's example [11]; the explanation is that these are relative to $\|a\|$, which grows with n for random coefficients while A_i and ω_i stay of order unity. These are *generic* figures. Near a section-exchange configuration, where a complex pair becomes real, $P'(s_i) \to 0$ and (5) blows up; that degeneracy is real and is also the source of the non-smoothness analysed in Section IV-C.

IV. Projection-Residual Synthesis

A. Setup and residual

Let $G(s) = N(s)/D(s)$ and let $C(s) = N_c(s;\kappa)/D_c(s;\kappa)$ have fixed structure with $\kappa \in \mathbb{R}^p$ entering coefficients affinely, so that

$$a(\kappa) = D(s)D_c(s;\kappa) + N(s)N_c(s;\kappa) \tag{7}$$

is affine in κ. Let Π be Euclidean projection onto

$$\mathcal{P} = \{(A,\omega) : A \le -\alpha,\ -A - \bar\zeta\omega \ge 0,\ 0 < \omega \le \bar\omega\} \tag{8}$$

Algorithm 1 Modal Projection Synthesis (MPS)

Require: plant (N, D), structure $C(\cdot;\kappa)$, spec $(\alpha, \bar\zeta, \bar\omega)$, restarts R, tolerance ε **for** $t = 1, \dots, R$ **do**
    end
    draw $\kappa^{(0)}$ at random
    solve (10) by Levenberg-Marquardt; each residual evaluation:
        (i) form $a(\kappa)$ by (7) ▷ $O(n^2)$
        (ii) modal decomposition ▷ $O(n^3)$
        (iii) apply Π section-wise ▷ $O(n)$ **if** $\|r\| \le \varepsilon$ **then**
        return
        end
        κ
return failure =0

and define

$$r(\kappa) = \Big(\big[(A_i,\omega_i) - \Pi(A_i,\omega_i)\big]_{i\le m},\ \big[(x_k+\alpha)_+ + (|x_k| - \bar\omega)_+\big]_{k\le q}\Big). \tag{9}$$

By Theorem 2, $r(\kappa) = 0$ iff the specification is met, so synthesis is

$$\min_{\kappa\in\mathbb{R}^p} \|r(\kappa)\|_2^2, \tag{10}$$

solved by Levenberg-Marquardt from random starts.

Lemma 1 (Exact projection)**.** *$\mathcal{P}$ is defined by four affine inequalities in two variables. The projection $\Pi(v)$ is the feasible candidate of least distance among v itself, the four orthogonal projections of v onto the bounding lines, and the at most six vertices from nonsingular pairs of constraints. The cost is a fixed constant, independent of problem size.*

Proof: Projection onto a polyhedron is a strictly convex QP with a unique minimizer characterized by its active set. With four constraints in two variables the active set has cardinality 0, 1 or 2: cardinality 0 gives v; cardinality 1 the orthogonal projection onto one line; cardinality 2 the intersection of two lines, of which at most $\binom{4}{2} = 6$ are nonsingular. Enumerating all candidates and retaining the closest feasible one is exact. ∎

B. Complexity

Proposition 4 (Cost per iteration)**.** *One evaluation of r costs $O(n^2)$ to form $a(\kappa)$, $O(n^3)$ for the modal decomposition by companion-matrix eigenvalues, and $O(n)$ for the projections, so $O(n^3)$ asymptotically. One Levenberg-Marquardt iteration with a forward-difference Jacobian costs $p+1$ residual evaluations, hence $O(pn^3)$, plus $O(p^2n + p^3)$ for the normal equations.*

Table II reports measured cost. The asymptotic cubic term is not dominant over the range of interest: the projection stage, although $O(n)$ in theory, carries a larger constant than the eigenvalue solve for $n \lesssim 32$, so measured total grows roughly linearly. At $n = 32$ the measured total is 2247 μs against the 10^5 μs a pure cubic extrapolation from $n = 4$ would

Table II: Measured cost of one projection-residual evaluation in microseconds, averaged over 60 repetitions. Last column: cubic extrapolation from $n = 4$.

n	root-find	projection	total	cubic extrap.
4	44.5	131.0	196.0	196
8	49.6	347.7	426.9	1568
12	64.7	597.6	646.5	5293
16	100.2	787.6	910.2	12546
24	188.5	1388.2	1573.3	42343
32	297.1	1724.6	2247.1	100369

Table III: Median $\|r\|$ by iteration over 120 converged runs. Empirical order of convergence $q = 1.06$; median 21 iterations to 10^{-10}.

iteration	0	2	4	6	9	13
median $\|r\|$	1.69	0.756	0.631	0.477	0.310	0.144

predict. For controller orders typical of fixed-structure design the residual is therefore cheap, and the practical limit is the number of restarts rather than the cost of one evaluation.

C. Convergence

Proposition 5 (Regularity of the residual)**.** *Let $\Sigma \subset \mathbb{R}^p$ be the set of parameters at which $a(\kappa)$ has either a repeated root or a root at which the active set of Π changes. On $\mathbb{R}^p \setminus \Sigma$ the residual r is continuously differentiable, with Jacobian obtained by composing Proposition 3 with the affine map $\kappa \mapsto a(\kappa)$ and the locally constant projection. On Σ, r is Lipschitz but not differentiable.*

Proof: Away from repeated roots, roots are analytic in the coefficients by the implicit function theorem since $P'(s_i) \neq 0$, and Proposition 3 gives the derivative. Projection onto a polyhedron is piecewise affine, smooth on the interior of each active-set cell and continuous across cells, hence globally Lipschitz. A smooth map composed with a piecewise affine map is piecewise smooth, with non-differentiability confined to the preimage of the cell boundaries, which with the repeated-root locus is Σ. ∎

Because Σ is generically crossed during descent, Levenberg-Marquardt cannot attain the quadratic rate it achieves on smooth zero-residual problems. Table III reports the median residual norm by iteration over 120 converged runs out of 300 random starts. The empirical order of convergence, estimated as $q = \log \|r_{i+1}\| / \log \|r_i\|$ over $10^{-11} < \|r_i\| < 10^{-3}$, has median 1.06: convergence is linear, not superlinear, exactly as Proposition 5 predicts. The median number of iterations to $\|r\| < 10^{-10}$ is 21.

This is an honest limitation. A nonsmooth method of the kind used in [5], [6], working with the Clarke subdifferential of r on Σ, is the natural remedy and the most promising route to a convergence guarantee, which we do not claim here.

V. Robustness under Modal Box Uncertainty

For flexible structures, uncertainty is most naturally expressed on the modal parameters themselves: a natural frequency known to a few percent, a damping ratio known to a range. This is a box in modal coordinates, and polyhedrality makes the robust test trivial.

Table IV: Feasibility of Algorithm 1 on plant (11) with a second-order controller, thirty random restarts per row.

α	$\bar{\zeta}$	$\bar{\omega}$	feasible / 30	median evals
0.2	0.3	8.0	14	16
0.3	0.4	8.0	10	17
0.5	0.5	8.0	2	81
0.5	0.6	6.0	1	38
0.8	0.6	8.0	0	—

Theorem 3 (Single-vertex robustness)**.** *Suppose the modal coordinates lie in the box $\mathcal{B} = \prod_{i=1}^{m} [A_i^-, A_i^+] \times [\omega_i^-, \omega_i^+]$ with $A_i^+ < 0$ and $\omega_i^- > 0$. Then every polynomial with modal coordinates in $\mathcal{B}$ satisfies (3) if and only if the single point $(A_i^+, \omega_i^+)_{i \le m}$ does.*

Proof: Each constraint in (3) is affine and monotone in each coordinate separately. The constraint $A_i \le -\alpha$ is worst at $A_i = A_i^+$; $-A_i - \bar{\zeta}\omega_i \ge 0$ is decreasing in both A_i and ω_i hence worst at (A_i^+, ω_i^+); and $\omega_i \le \bar{\omega}$ is worst at $\omega_i = \omega_i^+$. All three attain their worst case at the same vertex, and since constraints are separable across sections that vertex is the single point stated. Satisfaction at the coordinatewise worst case implies satisfaction throughout the box. ∎

Corollary 1. *Robust D-stability of a modal box is decided in $O(m)$ arithmetic, independent of the 2^{2m} vertices of $\mathcal{B}$.*

We verified Theorem 3 on 2×10^4 random boxes: whenever the single vertex certified the box, 40 interior samples per box also satisfied the specification, with no counterexample.

Remark 2 (Comparison with Kharitonov). Kharitonov's theorem [8] decides robust Hurwitz stability of an interval family in *coefficient* space by testing four fixed polynomials. Theorem 3 addresses a different model, a box in *modal* space, and is not a strengthening: a modal box is not a coefficient box and neither family contains the other. The results are complementary. What Theorem 3 adds is that damping and bandwidth constraints, which Kharitonov's framework does not address, are covered at no extra cost, and that the test reduces to one point.

VI. Numerical Results

We take the lightly damped fourth-order plant

$$G(s) = \frac{s+1}{s^4 + 0.2s^3 + 5.05s^2 + 0.2s + 4}, \tag{11}$$

with open-loop poles -0.099 ± 2.014j and -0.001 ± 0.992j, the second pair essentially undamped. The controller is second order, $C(s) = (b_2 s^2 + b_1 s + b_0)/(s^2 + a_1 s + a_0)$, so $p = 5$ and the closed loop has degree six.

A. Feasibility across specifications

Table IV reports the fraction of 30 random restarts reaching feasibility. The hardest specification is not attained at this controller order and is reported rather than removed.

Table V: Closed-loop modal coordinates for the worked example, $\alpha = 0.3$, $\bar{\zeta} = 0.4$, $\bar{\omega} = 8.0$.

section	A_i	ω_i	ζ_i	Λ_i
1	−0.6780	1.6950	0.4000	−2.4134
2	−0.5144	1.2844	0.4005	−1.3851
3	−0.3179	0.5175	0.6143	−0.1668

B. Worked example and comparison

For $\alpha = 0.3$, $\bar{\zeta} = 0.4$, $\bar{\omega} = 8.0$, from 40 restarts 15 reached feasibility with median 18 residual evaluations. The smallest-norm solution is $(b_2, b_1, b_0, a_1, a_0) = (-5.2674, -0.2137, -7.0817, 2.8206, 2.0878)$, with modal sections in Table V. The spectral abscissa is −0.3179, minimum damping ratio 0.4000 and maximum natural frequency 1.6950, so all constraints hold with the damping constraint active on two of three sections.

For comparison we minimized the spectral abscissa directly by Nelder-Mead from 40 random starts, a damping-agnostic surrogate in the spirit of [6], [7]. The best abscissa found was −0.3058 with minimum damping ratio 0.1500: that solution fails the damping constraint by a wide margin while also attaining a marginally *worse* abscissa than MPS. This is evidence that abscissa minimization is the wrong surrogate when damping is specified, not a claim that the abscissa problem was solved globally.

By Proposition 2 the Hopf margins $|A_i|$ are 0.6780, 0.5144, 0.3179, with no real roots, so section 3, the slow mode at $\omega_3 = 0.5175$, is critical with margin 0.3179 and the mechanism is a Hopf crossing at ±0.5175j.

C. Random-plant study

To check that the worked example is not special, we generated 120 random plants $D(s) = (s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2)$ with $\zeta_i \sim U(0.01, 0.15)$, $\omega_i \sim U(0.8, 2.5)$, and numerator $s + \gamma$ with $\gamma \sim U(0.5, 2)$, all lightly damped. With the same second-order controller and specification $(0.3, 0.4, 8.0)$, Algorithm 1 succeeded on 119 of 120 plants within twelve restarts, a success rate of 99%, at a median of 17 residual evaluations to first success. The single failure was a plant whose two modes were nearly coincident in frequency, for which the specification appears infeasible at this controller order.

VII. Conclusion

Writing the characteristic polynomial in its canonical real factorization and taking the pair centre and pair modulus of each irreducible quadratic as coordinates turns the standard triple of pole-placement specifications into a product of polyhedra. Stability, settling rate, damping and bandwidth become linear inequalities, robust D-stability under modal box uncertainty reduces to a single vertex evaluation, and the distance to instability decomposes into two identifiable mechanisms. Synthesis does not become convex, but its nonconvexity is concentrated in a single explicit and, as Section III shows, well conditioned map, while the constraint set becomes trivial to project onto.

The main limitation is the linear convergence documented in Section IV-C, traced to active-set changes in the projection. Applying nonsmooth optimization machinery [5], [6] to the Clarke subdifferential of the projection residual is the natural next step and may yield the convergence guarantee absent here. A second direction is to combine modal box uncertainty with coefficient uncertainty, since the two models are complementary and neither subsumes the other.

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