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Research Article

Keywords: Non-Hermitian Hamiltonian, Lindblad equation, Postselection, Trace distance, Jump operators, Excitation-number-conserving systems, Conservation-integral theorem, Exceptional point, Gain–loss cavities, Finite-size scaling

Posted Date: October 27th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11496964/v1>

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Additional Declarations: The authors declare no competing interests.

When Does the Non-Hermitian Effective Description Fail? A Finite-Size Reliability Map

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Abstract

The no-click (postselected) non-Hermitian Hamiltonian $H_{eff} = H - \frac{i}{2} \sum_j w_j L_j^\dagger L_j$ is an exact unraveling of the Lindblad equation, but nothing in that unraveling says how far the postselected state may drift from the unconditional one. We measure the drift, over one period T and averaged over initial states, by the trace distance $D_{dyn} = \frac{1}{2} \| \rho_{uncond}(T) - P_c(T) \|_{tr}$ (P_c : the normalized conditional projector), and show that its leading coefficient in $K_n = \gamma/\omega$ is fixed by the fluctuation of the jump operators along the unitary orbit, computable without any dissipative simulation. For excitation-number-conserving systems whose jump operators are ladder-type (each shifting the conserved excitation number by a nonzero integer) we prove a conservation-integral theorem reducing the dimensionless coefficient to $2\pi w N_{exc}$ (w : dimensionless channel weight; N_{exc} : conserved excitation number; time-integral form $2\pi w N_{exc}/\omega$), verified to the 10^{-5} level or better across spin chains and chiral waveguide pairs, and, for gain-loss cavities with non-uniform channel weights, through the block-diagonal lemma, with no microscopic parameters beyond N_{exc} . The crossing scale degrades with system size as $K_D^*(N) \propto N^{-1}$, confirmed by exact diagonalization and corner-space calculations up to $N = 12$ and by single-parameter data collapse. The result yields platform-independent design rules—maximum dissipation, postselection cost and an explicit switch criterion—and a guide for exceptional-point experiments.

Introduction

The non-Hermitian Hamiltonian $H_{eff} = H - \frac{i}{2} L^\dagger L$ appears rigorously, not phenomenologically, through the no-click postselection of the quantum-trajectory unraveling of a Lindblad master equation [1,4]. Conditional dynamics of this form is now the working language of exceptional-point (EP) sensing [14,21], parity-time (PT) symmetric optics [18,20], chiral transport [57,60] and measurement-induced many-body phenomena [52,56]. Each such application implicitly assumes that, over its observation window, the no-click trajectory faithfully proxies the full unconditional evolution. The boundary of this assumption (when the conditional description stops being an effective description of the open dynamics) is therefore a foundational question, but it has been characterized only by case-by-case numerics or by heuristic appeals to “weak coupling.”

The appeal of the non-Hermitian description is practical. A master equation propagates a density operator on a Hilbert space of dimension d , requiring storage of d^2 real parameters and time evolution under a Liouvillian superoperator of dimension d^2 . The postselected non-Hermitian problem instead propagates a single state vector under H_{eff} , at the cost of d parameters and non-unitary evolution. For large systems this quadratic saving is decisive, and the eigenstructure of H_{eff} —its exceptional points, eigenvalue Riemann surfaces, and biorthogonal modes—provides the organizing geometric picture for most of non-Hermitian physics [22,23,25]. What is less often stated is that this picture is, by construction, a *conditional* statement about a single measurement record, and that the postselection probability which validates it is itself a dynamical quantity that can shrink rapidly.

Two earlier lines of work bear on the question from opposite sides. On one hand, the limits of the Lindblad and non-Hermitian descriptions have been derived from microscopic couplings, establishing the weak-coupling and singular-coupling regimes and a no-EP theorem for pure-loss models [10,11,13]. On the other hand, quantum jumps have been treated as a perturbative correction to the eigenstructure of the non-Hermitian problem [14,27,31], while Monte Carlo wave-function and quantum-trajectory methods provide the exact stochastic machinery [1,2,5,6]. Quantum-trajectory and quantum-jump methods have themselves been reviewed extensively for single [4] and many-body [5,6] systems, and the role of jumps in distinguishing Hamiltonian from Liouvillian exceptional points is now understood quantitatively [27,28,29]. What is missing is a *dynamical* distance, organized by a dimensionless dissipation parameter, whose leading coefficient is analytically predictable rather than fitted, and whose degradation with system size is quantified.

Most closely related to our work is the analysis of the weak- and singular-coupling limits by Monkman and Berciu [10], together with the related local-versus-global comparison of Di Bello and co-workers [11]. Monkman and Berciu derive from the microscopic couplings when the non-Hermitian Hamiltonian coincides with the Lindblad description and prove that a pure-decay model hosts no exceptional point in the weak-coupling limit; Di Bello and co-workers compare local and global master equations with their non-Hermitian counterparts and locate where exceptional points appear. Together these studies establish the *qualitative boundary* at which the descriptions coincide, and they are the foundation on which we build. They do not, however, quantify how rapidly the conditional state departs from the unconditional state once that boundary is crossed, nor do they address how the boundary moves with system size. Our contribution is deliberately complementary. Taking the validity of the Lindblad form as given, we provide a closed-form prediction of the *rate* of departure—the coefficient c_1 —and derive its finite-size scaling, $K_D^*(N) \sim N^{-1}$. In short, [10,11] answer where the boundary is; the present work predicts how fast it is crossed and how that boundary shifts as the system grows. The no-EP theorem of [10] also motivates both our EP-seeking guide and our use of a dedicated gain-loss model for any EP-proximity analysis.

Three levels of description should be distinguished, and conflating them is the source of most ambiguity. At the coarsest level, a mean-field or semiclassical treatment replaces operators by amplitudes and discards quantum fluctuations entirely. At the intermediate level, the no-click non-Hermitian Hamiltonian retains single-state quantum coherence but

discards the stochastic jump records. At the most complete level, the unconditional Liouvillian propagates the density operator and restores every trajectory, including all jumps. The present work is concerned entirely with the boundary between the second and third levels: we ask, quantitatively, when the no-click state can stand in for the unconditional state, and when it cannot.

A single-parameter “weak-coupling” rule is useful only if it survives two tests: it must be predictable from the microscopic dynamics rather than merely fitted, and it must remain valid as the system grows. The first test calls for an analytic coefficient, and the second is a finite-size question that few-body benchmarks cannot answer. In stochastic terms the postselection is a rare-event problem, and rare-event probabilities are controlled by accumulated rates rather than by instantaneous ones; a rule that ignores the accumulation over degrees of freedom is therefore expected, not merely observed, to fail in the many-body limit. The notion of universality needs a sharper statement. A universal numerical coefficient shared by all platforms is a strong and, as we show, false claim; a universal structural law that predicts each platform’s coefficient from its conserved quantities is a weaker-sounding but more robust claim, and it is the one that survives.

The present study addresses this with a reliability map defined on the dimensionless dissipation K_n and system size N . Our contributions are: (i) a closed-form first-order coefficient c_1 for the trace distance between unconditional and conditional dynamics, expressed purely through unitary-orbit data; (ii) a universal conservation-integral theorem for excitation-number-conserving systems with ladder-type jump operators; (iii) the finite-size scaling of the crossing scale, $K_D^*(N) \sim N^{-1}$, with data collapse; and (iv) platform-independent experimental design criteria and an EP-seeking guide. We proceed from a single platform to many-body and cross-platform settings, with every derivation independently cross-checked by symbolic and numerical computation. The remainder of the paper first develops the analytic theory in full (Results, first two subsections), then validates it on single, many-body and cross-platform settings, and finally translates it into design rules.

Results

Three levels of open-system description and the Lindblad equation

The hierarchy and the exact equations come first, since the analytic result depends on the precise distinction between the conditional and unconditional objects.

A closed system with Hamiltonian H evolves unitarily, $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, $U(t) = e^{-iHt}$, and the density operator obeys $\dot{\rho} = -i[H, \rho]$. Under the Markov, Born and rotating-wave approximations, a system coupled to a set of independent Markovian reservoirs is described by the Lindblad master equation

$$\dot{\rho} = -i[H, \rho] + \sum_j w_j \mathcal{D}[l_j]\rho, \quad \mathcal{D}[l_j]\rho = l_j \rho l_j^\dagger - \frac{1}{2} \{l_j^\dagger l_j, \rho\},$$

where l_j are the jump operators and w_j the corresponding channel weights (we set $\hbar = 1$ throughout). The Liouvillian $\mathcal{L} = -i[H, \cdot] + \sum_j w_j \mathcal{D}[l_j]$ is a linear superoperator on operator space, and the unconditional state is $\rho_u(t) = e^{\mathcal{L}t} \rho(0)$. The trace of ρ_u is conserved by construction, $\text{Tr} \dot{\rho} = 0$, because $\text{Tr} \mathcal{D}[l] \rho = 0$ for each channel.

The same master equation admits an exact stochastic unraveling into quantum trajectories [1,2]. Between jumps the state evolves under the *non-Hermitian* effective Hamiltonian

$$H_{eff} = H - \frac{i}{2} \sum_j w_j l_j^\dagger l_j,$$

while a jump through channel j replaces the state by $l_j |\psi\rangle$ and occurs, in an infinitesimal interval dt , with probability $w_j \langle \psi | l_j^\dagger l_j | \psi \rangle dt$. Averaging the resulting normalized states over all jump records and times reproduces ρ_u exactly. The present study isolates the single record in which *no* jump occurs over the entire observation window—the no-click or postselected trajectory.

The no-click state, its Dyson expansion and normalization

The unnormalized no-click state obeys

$$|\psi_c(t)\rangle = -iH_{eff} |\psi_c(t)\rangle, \quad |\psi_c(t)\rangle = e^{-iH_{eff}t} |\psi(0)\rangle,$$

and its survival (no-click) probability is its squared norm,

$$P_{surv}(t) = \|\psi_c(t)\|^2, \quad P_{jump}(t) = 1 - P_{surv}(t).$$

Because H_{eff} is non-Hermitian, $\|\psi_c\|$ is not conserved. Indeed,

$$\frac{d}{dt} \|\psi_c\|^2 = \langle \psi_c | (-iH_{eff} + iH_{eff}^\dagger) | \psi_c \rangle = - \sum_j w_j \langle \psi_c | l_j^\dagger l_j | \psi_c \rangle,$$

which is the infinitesimal jump probability and is non-positive for the survival-probability derivative.

To develop the perturbation theory we factor out the common dissipation scale γ and write $w_j \rightarrow \gamma w_j$, treating γ as the small parameter while keeping the operator l_j and the unitary Hamiltonian fixed. The effective Hamiltonian is then $H_{eff} = H - \frac{i}{2} \gamma \Lambda$, with $\Lambda = \sum_j w_j l_j^\dagger l_j$. The Dyson expansion of the propagator to first order in γ gives

$$|\psi_c(t)\rangle = |\psi_0(t)\rangle - \frac{\gamma}{2} \int_0^t e^{-iH(t-s)} \Lambda |\psi_0(s)\rangle ds + O(\gamma^2),$$

where $|\psi_0(t)\rangle = U(t) |\psi(0)\rangle$ is the unitary ($\gamma = 0$) orbit. Taking the norm and using the unitarity of U ,

$$\|\psi_c(t)\|^2 = 1 - \gamma \int_0^t \langle \Lambda \rangle_s ds + O(\gamma^2), \quad \langle \Lambda \rangle_s = \langle \psi_0(s) | \Lambda | \psi_0(s) \rangle.$$

This is the familiar statement that the leading jump probability is the time-integrated expectation of the jump rate, and it is the first place where the result is *dynamical*: the same γ produces different postselection probabilities along different points of the unitary orbit.

The object that carries physical state information is the **normalized** no-click state

$$|\tilde{\psi}_c(t)\rangle = \frac{|\psi_c(t)\rangle}{\|\psi_c(t)\rangle}.$$

Writing $N_c(t) = \|\psi_c(t)\|$ (the norm of the conditional state; N is reserved for the system size) and $N_c(t) = 1 - \frac{\gamma}{2} \int_0^t \langle \Lambda \rangle_s ds + O(\gamma^2)$, its expansion is

$$\begin{aligned} |\tilde{\psi}_c(t)\rangle &= |\psi_0(t)\rangle + \gamma |\delta\psi(t)\rangle + O(\gamma^2), \\ |\delta\psi(t)\rangle &= -\frac{1}{2} \int_0^t e^{-iH(t-s)} \Lambda |\psi_0(s)\rangle ds + \frac{1}{2} \left(\int_0^t \langle \Lambda \rangle_s ds \right) |\psi_0(t)\rangle. \end{aligned}$$

The first term is the anti-Hermitian action of the loss on the state; the second is the normalization correction. The normalized conditional projector is correspondingly

$$P_c(t) = |\tilde{\psi}_c(t)\rangle \langle \tilde{\psi}_c(t)| = \rho_0(t) + \gamma (|\delta\psi(t)\rangle \langle \psi_0(t)| + |\psi_0(t)\rangle \langle \delta\psi(t)|) + O(\gamma^2),$$

with $\rho_0(t) = |\psi_0(t)\rangle \langle \psi_0(t)|$. Normalizing before comparison is essential and is a primary source of irreproducible benchmark numbers; we return to this point in a dedicated subsection.

The unconditional state and the first-order difference

The unconditional state admits the same expansion, $\rho_u(t) = \rho_0(t) + \gamma \rho_1(t) + O(\gamma^2)$. Introducing the unitary superoperator $\mathcal{U}(t) = e^{-i[H, \cdot]t}$ (which acts as $\mathcal{U}(t)A = U(t)AU^\dagger(t)$), the first correction is

$$\rho_1(t) = \int_0^t \mathcal{U}(t-s) \sum_j w_j \mathcal{D}[l_j] \rho_0(s) ds.$$

Changing variables to the time elapsed before the final time, $\tau = t - s$, and using $\rho_0(s) = U(s)\rho(0)U^\dagger(s)$, the integrand can be pulled back to the initial state and then propagated to t . After this reorganization the first-order difference between the unconditional and conditional states is

$$\Delta(t) = \rho_u(t) - P_c(t) = \gamma \Delta_1(t) + O(\gamma^2).$$

The two terms in $\mathcal{D}[l_j]$ play distinct roles. The anticommutator part, $-\frac{1}{2}\{l_j^\dagger l_j, \rho_0\}$, is a deterministic loss of amplitude; in Δ_1 it is exactly canceled by the normalization correction in $|\delta\psi\rangle$ (the second term above), because both enforce the same unit-trace/unit-norm

condition on their respective objects. This cancellation was verified symbolically and is the structural reason the result simplifies: the *coherent, mean* loss is reproduced identically by the normalized no-click state, and only the *fluctuating* part of the jump action survives as a genuine discrepancy.

The surviving term is the jump part $l_j \rho_0 l_j^\dagger$, which transfers weight into the post-jump state and has no counterpart in the no-click trajectory. Defining the fluctuation operator

$$\delta l_j(\tau) = l_j - \langle l_j \rangle_\tau,$$

and the interaction-picture jump operator $\tilde{l}_j(\tau) = U^\dagger(\tau) l_j U(\tau) = e^{iH\tau} l_j e^{-iH\tau}$ (equivalently, propagated to the final time as $U(\tau) l_j U^\dagger(\tau)$), the first-order difference over a full period T takes the closed form

$$\Delta_1(T) = \sum_j w_j \int_0^T [\tilde{l}_j(\tau) \rho_0(T) \tilde{l}_j^\dagger(\tau) - \langle n_j \rangle_{T-\tau} \rho_0(T)] d\tau,$$

where $n_j = l_j^\dagger l_j$ and $\langle n_j \rangle$ is evaluated on the unitary orbit. The subtraction of $\langle n_j \rangle \rho_0$ is the traceless projection of the jump term: it removes the component proportional to the state itself and leaves only the part orthogonal to it, i.e. the variance rather than the mean. The time-ordered form above, with the $-i[H, \cdot]$ propagator acting on the fluctuation block, is what makes the integral a single accumulated object rather than a sum of independent pointwise errors.

Closed-form first-order coefficient

The trace distance between two states is the natural trace-norm metric,

$$D(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_{tr}, \quad \|A\|_{tr} = \text{Tr} \sqrt{A^\dagger A},$$

and it equals the maximum probability of distinguishing the two states in a single measurement, ranging from zero (identical) to one (fully distinguishable). Applying it to the ensemble-averaged difference and nondimensionalizing, the distance over one full period is

$$D_{dyn}(T) = \gamma \hat{c}_1(T) + O(\gamma^2),$$

$$\hat{c}_1(T) = \frac{1}{2} \left\| \sum_j w_j \int_0^T [\tilde{l}_j(\tau) \rho_0(T) \tilde{l}_j^\dagger(\tau) - \langle n_j \rangle_{T-\tau} \rho_0(T)] d\tau \right\|_{tr}.$$

First, the boxed form holds for arbitrary H and arbitrary windows: the anticommutator of the dissipator cancels the normalization term of the conditional state identically, with no assumption that the unitary orbit closes (Appendix A). The fixed window $T = 2\pi/\omega$ is adopted for physical reproducibility, not for validity of the formula. Second, the two operator orderings $U^\dagger(\tau) l_j U(\tau)$ acting on $\rho_0(0)$ and $U(\tau) l_j U^\dagger(\tau)$ acting on $\rho_0(T)$ are

equivalent; on non-closed orbits only the final-time ordering printed in the box is correct, and we have verified the distinction numerically (Appendix C).

Unit convention. Throughout the paper c_1 denotes the *dimensionless coefficient of* $K_n = \gamma/\omega$: $D_{dyn} = c_1 K_n + O(K_n^2)$, with $c_1 = \omega \hat{c}_1$, where $\hat{c}_1 = \frac{1}{2} \parallel \int_0^T \sum_j \hat{w}_j [\tilde{l}_j \rho_0 \tilde{l}_j^\dagger - \langle n_j \rangle_\tau \rho_0] d\tau \parallel_{tr}$ is the time-integral form of the boxed expression ($\hat{w}_j = w_j/\gamma$). All quoted coefficients (2.1137, 2π , 1.5π , $2\pi N_{exc}$) are in this convention, and the design rules $K_D^* = 0.1/c_1$ and $M = e^{c_1 K_n}$ follow from it directly; the time-integral form is related by $\hat{c}_1 = c_1/\omega$.

The physical reading is direct and is the central result of the paper. The leading discrepancy is set by the accumulated *fluctuation of the jump operators on the instantaneous state* along the unitary orbit. The larger the variance of l_j on $|\psi_0(t)\rangle$, and the longer this variance is integrated, the faster the non-Hermitian description fails. A jump operator that is proportional to the identity on the occupied subspace, so that $\delta l_j |\psi_0\rangle = 0$, contributes nothing; a jump operator that strongly mixes the occupied state contributes maximally. This converts c_1 from a fitted slope into a quantity predictable from unitary dynamics alone—no dissipative evolution is required to compute it.

The order of limits matters here. The formula is a first-order result valid in the weak-coupling regime $\gamma T \ll 1$ integrated over a fixed window; higher-order terms $O(\gamma^2)$ become visible when the accumulated jump probability is no longer small. The window must be fixed and stated explicitly, because the distance is a time-integrated quantity. We choose one full period $T = 2\pi/\omega$ so that the unitary orbit closes and the coefficient is a property of the orbit rather than of a particular phase; the dependence on the window is examined separately below.

Second-order correction, cumulants and the regime of validity

To use the first-order result responsibly it helps to know how the expansion continues and what sets its radius of convergence. The no-click survival probability is the norm of a state propagated under a time-ordered exponential, and its logarithm is therefore a cumulant series,

$$\ln P_{surv}(t) = -\gamma \int_0^t \langle \Lambda \rangle_s ds - \frac{\gamma^2}{2} \iint_0^t \langle \langle \Lambda(s) \Lambda(s') \rangle \rangle ds ds' + O(\gamma^3),$$

where $\langle \langle \Lambda(s) \Lambda(s') \rangle \rangle$ denotes the connected (cumulant) two-time correlation of Λ on the unitary orbit. The first term is the mean jump rate that appears in c_1 ; the second term is the variance of the accumulated jump count and is the leading correction to the postselection probability. The same cumulant structure governs the distance: the second-order displacement contains the connected two-time correlations of the jump operators, and it becomes comparable to the first-order term precisely when the accumulated jump count is of order unity, i.e. when $\gamma \int \langle \Lambda \rangle ds \sim 1$.

This gives a clean operational criterion for the weak-coupling regime. The linear law $D_{dyn} = c_1 K_n$ is reliable while the integrated jump probability is small,

$$\gamma \int_0^T \langle \Lambda \rangle_s ds \ll 1,$$

and it is this *integrated* quantity, not the instantaneous rate, that controls the expansion. For a many-body system the integral grows with the number of loss channels, which is the analytic origin of the finite-size scaling derived later: the per-channel criterion $\gamma/\omega \ll 1$ is not sufficient once many channels accumulate over the window. We therefore quote the coefficient together with the window and, for many-body systems, with the number of occupied channels, rather than as a standalone number.

A word on normalization conventions is in order, because it is a recurring source of discrepancy: the conditional state must be normalized to unit norm before its projector is formed and compared with the unit-trace unconditional state, whereas comparing the *unnormalized* no-click state directly counts the postselection probability as distance. The two conventions answer different questions (error of the normalized description versus error plus postselection cost); their quantitative effect is measured by the conventions grid ($\simeq 25\%$ at full window).

Robustness to the metric, ensemble and mixed initial states

The choice of trace distance and of a pure-state ensemble is not innocuous, so we verify that the linear law does not depend on either.

Metric. We compare the trace distance with the Hilbert–Schmidt distance $D_{HS} = \|\Delta\rho\|_2$ and the Bures distance $D_B = \sqrt{2(1 - \sqrt{F})}$, where $F = \|\sqrt{\rho_u}\sqrt{P_c}\|_1^2$ is the fidelity. Over the weak-coupling range the trace and Hilbert–Schmidt distances are linear in K_n over the same window; the Bures distance instead scales as $K_n^{1/2}$ (exponent 0.50). The square-root behaviour is not a failure of the perturbation theory. The conditional state is pure, and a small jump probability p transfers weight into an orthogonal branch, to which the Bures distance is sensitive as $\sqrt{p}$ whereas the trace and Hilbert–Schmidt distances are sensitive as p . We use the trace distance because it is the operational distinguishability metric and is linear with a coefficient that the closed form predicts; the Hilbert–Schmidt result confirms that the linear law is not an artefact of one metric.

Ensemble. We average the distance state by state, $D_{dyn} = \frac{1}{2} \|\rho_u^{(i)} - P_c^{(i)}\|_{tr}$, rather than computing the distance of the averaged states, $\frac{1}{2} \|\overline{\rho_u^{(i)}} - \overline{P_c^{(i)}}\|_{tr}$. The two differ because the trace norm is convex, $\|\overline{\Delta_i}\|_{tr} \leq \overline{\|\Delta_i\|_{tr}}$: numerically the distance of averages is more than an order of magnitude smaller (for example 0.0008 versus 0.0107 at $K_n = 0.005$), because the per-state errors point in different Bloch-sphere directions and cancel when the states are averaged. The state-by-state average reports the typical error of a single experimental run, which is the relevant reliability measure, so it is the one we use.

Mixed initial states. The derivation does not require the initial state to be pure. Replacing the initial projector by any density operator, the first-order difference in the interaction

picture is $\Delta = \sum_j w_j \int_0^T \tilde{l}_j(\tau) \rho_0 \tilde{l}_j^\dagger(\tau) d\tau - \bar{n} \rho_0$, with $\bar{n} = \sum_j w_j \int_0^T \text{Tr}(\tilde{n}_j \rho_0) d\tau$, and $c_1 = \frac{1}{2} \|\Delta\|_{tr}$; the pure-state formula is recovered when ρ_0 is rank one. Thermal states provide the check: $\rho_0 \propto e^{-\beta H}$ over $\beta \in \{0.25, 0.5, 1, 2, 4\}$: the distance remains linear in K_n and the analytic coefficient matches the numerical value at every temperature, with $c_1 = 0.195, 0.385, 0.726, 1.196, 1.514$ in order of increasing β (0.726 at $\beta = 1$). The equality $\| -Q \rho_0 \|_{tr} = Q$ used below holds for any density operator, since the eigenvalues of $-Q \rho_0$ are $-Q$ times the eigenvalues of ρ_0 . The pure-state condition enters only in that the conditional object is a single vector whose normalization is its norm; for a mixed initial state it is the unnormalized operator $H_{eff} \rho_0 H_{eff}^\dagger$, normalized by its trace, and the same first-order result follows.

Single-platform analytic–numerical validation

The closed form is first validated on model A, a driven two-level system (TLS) with $H = \frac{\Omega}{2} \sigma_x$ and $L = \sqrt{\gamma} \sigma_-$, and on a detuned variant A', $H = \frac{\Delta}{2} \sigma_z + \frac{\Omega}{2} \sigma_x$, with $\Delta/\Omega \in \{0, 0.5, 1, 2\}$. The observation window is one Rabi period, $T = 2\pi/\Omega$, and the distance is ensemble-averaged over 200 Bloch-sphere-uniform initial states and five fixed seeds, with the conditional state normalized before comparison. Two independent numerical channels—matrix exponentiation of the Liouvillian and a DOP853 ODE integrator with SVD-based trace norm—are used.

The analytic and extrapolated coefficients agree to 0.05–0.08% (Table 1). For the driven TLS the coefficient is an *analytic expression evaluated by numerical quadrature* along the unitary orbit (the time-ordered integral is not an elementary closed form here); only for the excitation-conserving systems below does it reduce to an elementary closed form. The quoted $\pm$ intervals are the bootstrap 95% confidence intervals over the initial-state ensemble, and the deviation is the relative difference $|c_1^{analytic} - c_1^{fit}|/c_1^{fit}$.

Δ/Ω	c_1 analytic (bootstrap 95% CI)	c_1 numerical ($K_n \rightarrow$ 0 fit)	relative deviation
0.0	2.1137 ± 0.0143	2.1126	0.05%
0.5	2.2232 ± 0.0478	2.2221	0.05%
1.0	2.3546 ± 0.0780	2.3533	0.06%
2.0	2.4399 ± 0.1007	2.4380	0.08%

Analytic coefficients at $\Delta \neq 0$ are numerical quadratures of the closed form along the orbit (main text); intervals are non-parametric percentile bootstrap 95% intervals of the mean over the pooled 5×200 -state ensemble ($B = 20000$; Methods). At $\Delta/\Omega = 1, 2$ the per-state distribution of c_1 is heavy-tailed (individual states reach $c_1 \approx 5.3$ at $\Delta = 2$), so the intervals are correspondingly wider; batch-to-batch variation of the mean over independent seed batches (± 0.07 at $\Delta = 1$, ± 0.09 at $\Delta = 2$) is of the same order as the intervals, which therefore reflect the dominant uncertainty.

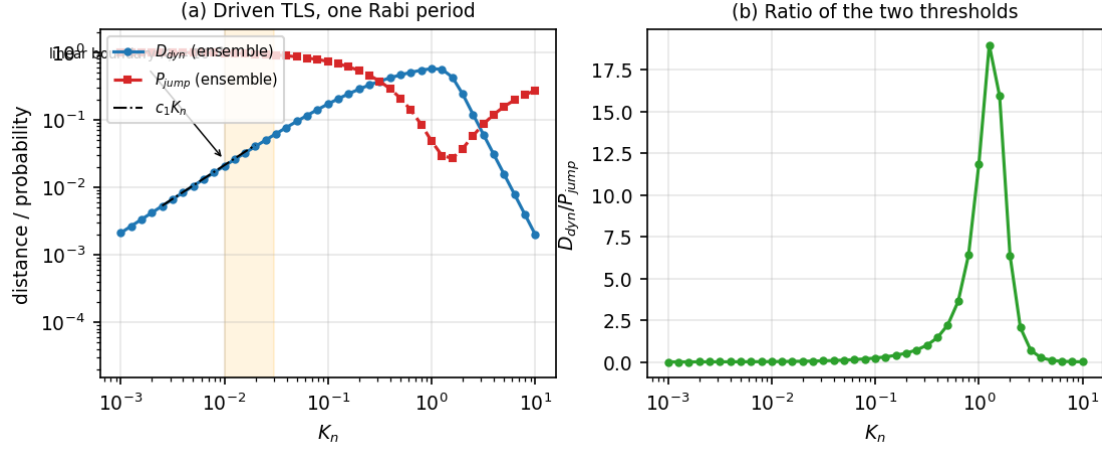


Figure 1.

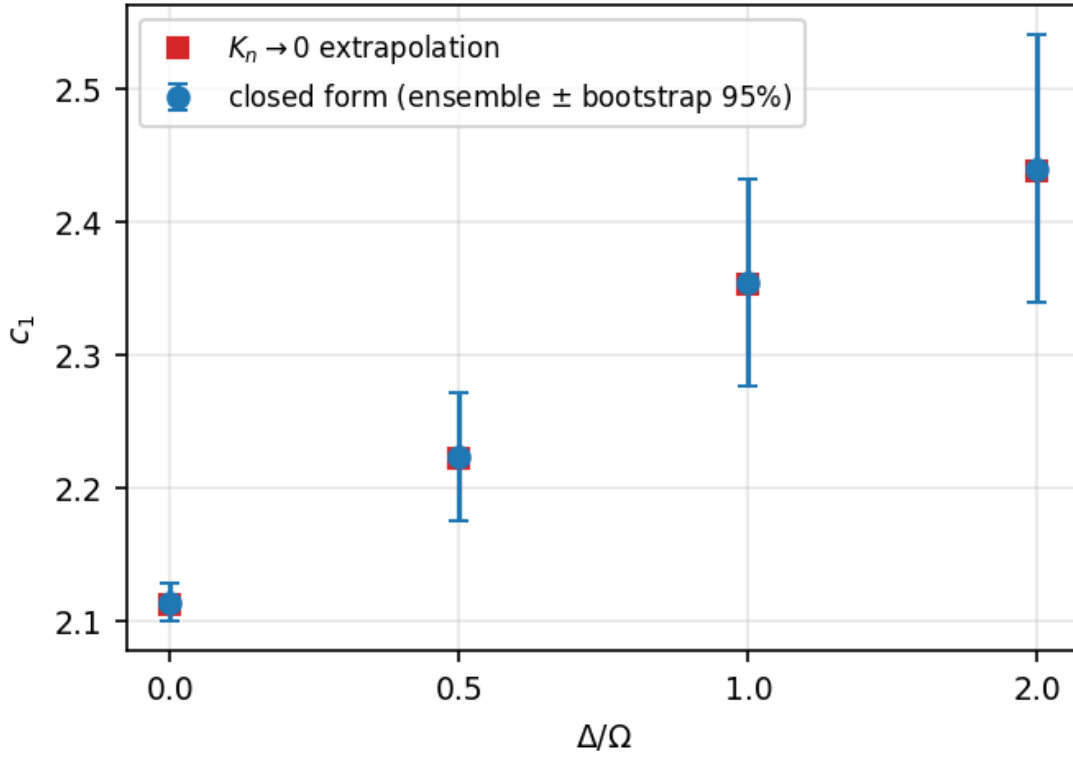


Figure 2.

Pointwise comparison at $\Delta = 0$, with the coefficient and the distances evaluated on the same 400-state ensemble, shows deviations of -0.02% at $K_n = 10^{-4}$, -0.22% at 10^{-3} , -2.2% at 10^{-2} and -6.3% at 3×10^{-2} : the distance lies slightly *below* the linear law, with the deviation proportional to K_n ($c_2 = -4.6 \pm 0.1$ from a quadratic fit over $[10^{-4}, 10^{-2}]$); the linear regime is confined to $K_n \lesssim 10^{-2}$. The matrix-level residual $\|\Delta_{num}/\gamma - \Delta_{pred}/\gamma\|_F$ scales linearly with γ , and the two numerical channels are statistically consistent. The $\varepsilon \rightarrow 0$ regression returns $D_{dyn} = 2.3 \times 10^{-14}$ at $\gamma = 10^{-14}$, and the

unconditional state is asserted to have unit trace throughout. The shaded band in panel (a) marks the linear-regime boundary $K_n \approx 10^{-2}$, beyond which the measured deviation reaches -6.3% at 3×10^{-2} .

The reliability curve is non-monotonic (Fig. 1): D_{dyn} peaks near $K_n \approx 1$ at 0.58 and falls to 2×10^{-3} at $K_n = 10$, where both conditional and unconditional states relax toward the ground state — the strong-loss quantum-Zeno regime, in which frequent jumps freeze the dynamics and the surviving no-click state coincides with the unconditional steady state [48–51]; the jump probability $P_{jump} = 1 - \|\psi_c\|^2$ peaks at 0.95 and returns to 0.72, indicating strong-dissipation saturation. In this regime the large loss rate rapidly projects the system onto the ground state, which is the null space of σ_- and no longer decays, so the conditional state retains a nonzero ground-state component rather than vanishing entirely. The no-click state is then a rare, heavily-filtered record that is internally coherent but statistically unrepresentative, which is precisely the failure mode the distance detects.

Sensitivity of the distance to protocol conventions

The reported distance depends on four conventions: the normalization of the conditional state, the initial-state ensemble, the observation window, and the distance convention (full versus half trace distance). Because these choices are easy to make silently, we quantified their effect by an exhaustive search over 576 protocol combinations—2 normalizations $\times$ 2 initial-state choices (the 200-state Blo ensemble versus the single state $|e\rangle$) $\times$ 24 window fractions f , with $T = f T_0$, $f \in \{k/24, k = 1, \dots, 24\}$ $\times$ 2 distance conventions $\times$ 3 random seeds.

At $K_n = 0.025$ the main protocol gives $D_{dyn} = 0.050$, while the grid spans 1.6×10^{-3} to 0.111—a factor of 68 between the most dissimilar conventions. At $K_n = 1$ the main protocol gives 0.585 and the grid spans 0.057 to 1.20 (factor 21). The dominant single factor is the window: shrinking the window from a full period to $f = 1/24$ changes the reported value by an order of magnitude ($\simeq 12 \times$ at $K_n = 0.025$, $4 \times$ at $K_n = 1$). The normalization convention shifts the full-window ensemble value by -25% (-19% at $K_n = 1$), and the state choice by $\pm 11\%$; the full/half distance convention contributes the expected factor of 2. The practical conclusions stand: (i) every quoted distance must state all four conventions, which is why the Methods fixes them in a single box; (ii) under any fixed protocol the linear law $D_{dyn} = c_1 K_n$ holds in the same weak-coupling window, with the coefficient corrected for the $O(K_n^2)$ term (under the quarter-period unnormalized protocol, for example, the linear prediction deviates by 9% at $K_n = 0.025$, consistent with the measured $c_2 = -0.42$); (iii) convention sensitivity lives in the reporting and does not affect any theorem in this paper.

Absence of an exceptional point in the single-loss driven TLS

A question raised by the closed form is whether the first-order coefficient should diverge as the effective Hamiltonian approaches an exceptional point, where ordinary perturbation theory becomes non-uniform. Before studying such a divergence one must verify that an EP actually exists. A full-spectrum scan of the 4×4 Liouvillian (and of the 2×2 H_{eff}) over $\gamma/\Omega \in [0.01, 100]$ finds the minimum pairwise eigenvalue separation bounded below, with

minimum $\min_{i \neq j} |\lambda_i - \lambda_j| = 5.0 \times 10^{-3}$ attained at $K_n = 10^{-2}$, the smallest value on the scan grid (200 logarithmically spaced points over $K_n \in [10^{-2}, 10^2]$ at fixed $\Delta = 0$; Fig. 3). The spectrum varies smoothly on this grid, so exceptional points hidden between grid points cannot be strictly excluded; missing one would require fine-tuning, and the smoothness of the minimum itself is evidence against it. The single-loss driven TLS therefore has **no EP**: the eigenvalues of H_{eff} for $H = \frac{\Omega}{2} \sigma_x$ with a single decay channel remain distinct, consistent with the no-EP theorem for pure-loss models [10].

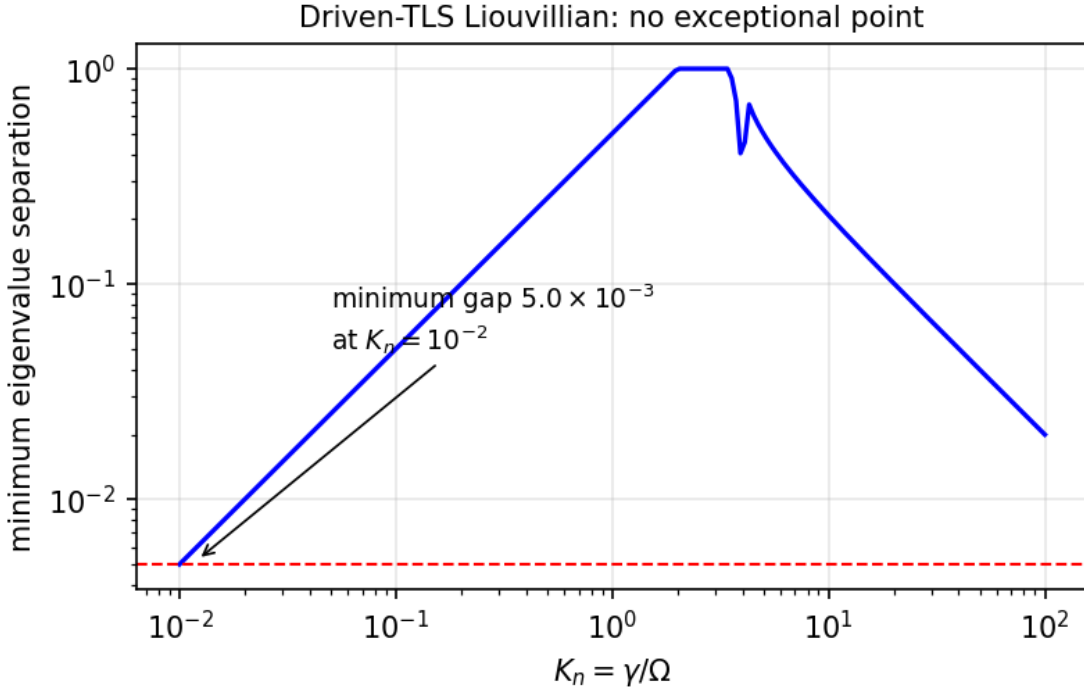


Figure 3.

The anticipated divergence of c_1 near an EP consequently has no object on this platform and is deferred to gain-assisted or two-channel models, where the eigenvalues can in fact coalesce. This negative result is a substantive finding: it illustrates that EP-based enhancements cannot be inferred from a pure-loss model, and it motivates the EP-seeking guide given in the design subsection.

In a model that does host an EP, the ordinary perturbation expansion becomes non-uniform because it divides matrix elements by the eigenvalue gap, and the coefficient is expected to acquire a non-analytic contribution with $\alpha = \frac{1}{2}$ for the square-root structure of a generic second-order EP; the same caution applies to Liouvillian exceptional points, which can be present even when H_{eff} has no Hamiltonian EP. The gain-loss platform C tests this prediction directly, rather than leaving it a formal expectation.

Direct observation of the EP non-uniformity in the gain–loss platform

Platform C is the coupled-cavity model $H = J(\sigma_+^1 \sigma_-^2 + \sigma_-^1 \sigma_+^2)$ with one loss channel $L_1 = \sqrt{g} \sigma_-^1$ and one gain channel $L_2 = \sqrt{r g} \sigma_+^2$, $r = 0.5$, started in the single-excitation state $|10\rangle$. In the single-excitation block $\{|01\rangle, |10\rangle\}$ the effective Hamiltonian reads

$$H_{eff}^{(1)} = \begin{pmatrix} 0 & J \\ J & -\frac{i}{2}(1+r)g \end{pmatrix},$$

with eigenvalues $\lambda_{\pm} = -i(1+r)g/4 \pm \sqrt{J^2 - (1+r)^2 g^2/16}$; these coalesce at $g_{EP} = 4J/(1+r) = 2.67J$. Tracking the pair numerically reproduces the analytic splitting to better than 4×10^{-8} (Fig. 4a).

The hallmark of non-uniform perturbation is the eigenvalue sensitivity to the control parameter. We find $|\partial\lambda/\partial g|$ to grow as $|g - g_{EP}|^{-\eta}$ with $\eta = 0.50$ over the decade $|g - g_{EP}| \in [10^{-2}, 10^{-1}]$ and $\eta = 0.481 \pm 0.004$ (bootstrap 95% interval, 2000 resamples) over the wider range $[10^{-2}, 1.2]$ (Fig. 4b), the deviation from 1/2 reflecting the subleading g -dependence of the prefactor rather than a different singularity. Correspondingly, the linear law $D_{dyn} = c_1 K_n$ breaks down early: the measured distance falls 5% below the linear prediction already at $g \simeq 0.022J$ and 10% below at $g \simeq 0.042J$, reaches $D_{dyn} \simeq 0.64$ at $0.3J$ and 0.72 at $0.5J$, and grows sublinearly with EP-proximate structure to a maximum $D_{dyn} \simeq 0.97$ at $g \simeq 2.3J \simeq 0.85 g_{EP}$ (where the near-defective conditional evolution rotates rapidly with g); beyond it the distance decreases again (0.17 at $4.5J$), because the gain channel amplifies the no-click state toward the unconditional steady state (Fig. 4c). The EP region is therefore not described by the first-order reliability theory at all.

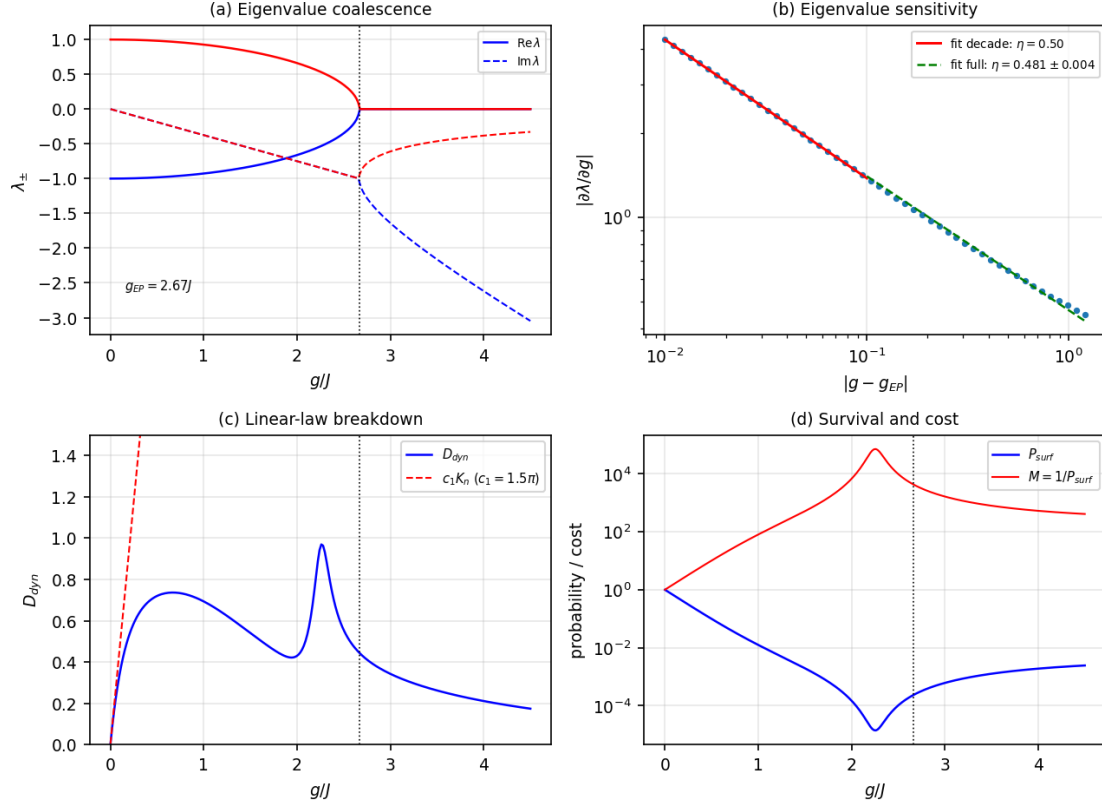


Figure 4. Exceptional-point analysis of the gain-loss platform C ($r = 0.5, J = 1$, initial state $|10\rangle$). (a) Eigenvalues $\lambda_{\pm}$ of the single-excitation block $H_{eff}^{(1)}$: solid curves, $\text{Re } \lambda$; dashed curves, $\text{Im } \lambda$; coalescence at $g_{EP} = 4J/(1+r) = 2.67J$. (b) Eigenvalue sensitivity $|\partial\lambda/\partial g|$ versus $|g - g_{EP}|$: decade fit $\eta = 0.50$, full-range fit $\eta = 0.481 \pm 0.004$ ($B = 2000$ bootstrap), consistent with the square-root prediction $\eta = 1/2$. (c) D_{dyn} versus g : the linear law $c_1 K_n$ ($c_1 = 1.5\pi$, red dashed) tracks the exact distance only at weak coupling; the distance peaks at $D \approx 0.97$ near $g \approx 2.3J \approx 0.85 g_{EP}$ and decreases beyond. (d) No-click survival probability P_{surf} and postselection cost $M = 1/P_{surf}$.

A further observation concerns the cost of postselection. The no-click survival probability falls to zero as the EP is approached (Fig. 4d), so the number of trajectories required to condition on a single surviving run, $M = 1/P_{surf}$, diverges. The formal non-uniform region is thus experimentally inaccessible to the no-click description: the very trajectories that would probe the EP are the ones that jump. This makes the EP-seeking guide concrete (the EP must be identified in the correct object and at a point where conditioning remains feasible) and illustrates why the unconditional Liouvillian, which retains a well-defined steady state throughout, is the only reliable description near an EP.

Time-window and convergence dependence

Because the distance is a time-integrated quantity, its value depends on the observation window, and the choice of window must be stated and justified. Figure 5 shows D_{dyn} as a function of the window length (in units of the period) for two dissipation strengths. At

weak coupling ($K_n = 0.05$) the distance grows approximately linearly with the window, as the closed form predicts for an accumulated fluctuation; at strong coupling ($K_n = 1$) the curve is non-monotonic and reflects the competing relaxation and re-phasing of the conditional and unconditional states. Fixing the window at one full period, $T = 2\pi/\omega$, marked in the figure, removes the arbitrary phase at which the comparison is made. Quantitatively, at $K_n = 0.05$ the accumulated distance per period stays within 11% of the linear prediction for windows up to two periods ($D/T = 0.094$ versus $c_1 K_n = 0.106$), but the growth is sublinear beyond: D/T drops to 0.085 at two periods, 0.070 at four and 0.050 at eight. Approximate linearity in the window should therefore be read as holding for $T \lesssim 2$ periods at this dissipation, not across the whole scanned range.

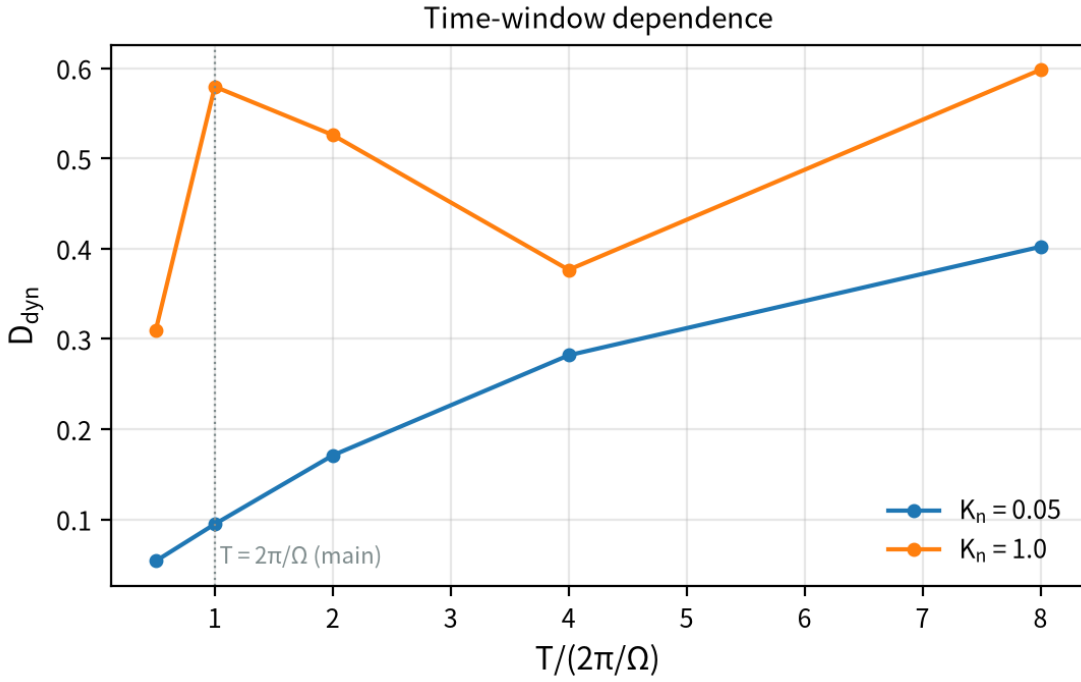


Figure 5.

A convergence check over the integration tolerance and time-grid resolution confirms that the reported distances are independent of the numerical step; the two independent channels (dense exponentiation and DOP853 ODE) agree to better than 3×10^{-8} at the points compared. The combination of a fixed, physically motivated window with a converged integrand is what makes the coefficient reproducible across implementations.

Many-body conservation-integral theorem

For exchange-type Hamiltonians, which commute with the total excitation-number operator N and therefore conserve N_{exc} , the closed form collapses to a far stronger result, stated in two steps.

Lemma (sector separation). Assume (i) $[H, N] = 0$ and (ii) each channel operator shifts the excitation number by a *fixed nonzero integer*, $[N, l_j] = m_j l_j$ with $m_j \in \mathbb{Z} \setminus \{0\}$ (lowering

channels $m_j = -1$, gain channels $m_j = +1$; the sign is immaterial). Then along the unitary orbit the state remains in the fixed sector $N = N_{exc}$, the post-jump vectors $\tilde{l}_j(\tau)\psi_0$ lie in the shifted sectors $N_{exc} + m_j$, and the first-order difference is block diagonal,

$$\Delta_1 = -Q\rho_0 \oplus \oplus_{s \neq 0} G_s, \quad Q = \sum_j w_j \int_0^T \langle n_j \rangle_\tau d\tau, \quad \text{Tr } G_s = \sum_{j: m_j=s} w_j \int_0^T \langle n_j \rangle_\tau d\tau,$$

with each G_s positive semidefinite. Since the trace norm is additive over direct sums, $\| -Q\rho_0 \|_{tr} = Q$ for *any* density operator (the eigenvalues of $-Q\rho_0$ are $-Q$ times those of ρ_0), and $\| G_s \|_{tr} = \text{Tr } G_s$, we obtain $\| \Delta_1 \|_{tr} = 2Q$, i.e.

$$\hat{c}_1 = \sum_j w_j \int_0^T \langle n_j \rangle_\tau d\tau \quad (\text{time units; } c_1 = \omega \hat{c}_1).$$

No conservation *law* for $\sum_j \langle n_j \rangle$ is required at this step, and gain channels are allowed; the proof (Appendix B) requires only the orthogonality of the shifted sectors and holds verbatim for mixed initial states.

Theorem (conservation-integral form). If, in addition, (iii) all weights are equal, $w_j = w$, and (iv) the channel-occupation sum is conserved along the unitary orbit, $\sum_j \langle n_j \rangle_t = N_{exc}$ —guaranteed, for instance, when every channel locally lowers a conserved excitation (then $\sum_j n_j = N$), or when the channels are number operators on complementary sites as in platform C—then

$$c_1 = \omega w \int_0^T \sum_j \langle n_j \rangle_\tau d\tau = \omega w N_{exc} T = 2\pi w N_{exc}$$

(the integrals define the time-integral form $\hat{c}_1 = 2\pi w N_{exc} / \omega$; multiplication by ω gives the dimensionless coefficient of K_n used throughout, so that $K_D^* = 0.1 / (2\pi w N_{exc})$ and $K_D^* = 0.1 / c_1$ are the same statement.)

over a full period $T = 2\pi / \omega$. The microscopic parameters (couplings, detunings, chiral phases) enter only through N_{exc} (and through ω , which converts the time integral into the dimensionless coefficient); this is the strongest form of the predictability claim, and it is the origin of the data collapse of Figs. 6 and 7.

Dephasing-type channels ($m_j = 0$) are excluded. The lemma requires every channel to shift the conserved excitation number ($m_j \neq 0$); a number-conserving channel such as pure dephasing leaves same-sector blocks in Δ_1 , and the reduction to the occupation integral fails. A verified example: a driven TLS with a pure-dephasing channel $L = \sigma_z$ has occupation integral $Q = 2\pi$ (since $\sigma_z^\dagger \sigma_z = \mathbb{1}$), yet the exact first-order coefficient is $c_1 = \pi$ for $\rho_0 = |e\rangle$ and $c_1 = 2\pi$ for $\rho_0 = |+\rangle$ —state-dependent and not captured by any occupation integral (numerically confirmed to 10^{-5}). The boxed closed-form expression itself remains valid for arbitrary channel type; only its reduction to $2\pi w N_{exc}$ requires ladder-type channels.

The ratio μ across platforms. How much of the closed-form integral survives as the coefficient is measured by the ratio $\mu = c_1 / \sum_j w_j \int \langle n_j \rangle d\tau$: the theorem gives $\mu = 1$ exactly for conserving platforms, and numerically we obtain $\mu = 1.0000$ for platform B and 1.0001 for platform C. For non-conserving (driven) platforms the sector separation fails and μ is neither unity nor a universal constant: it is a *per-state* functional of the orbit. For the driven TLS the time-averaged occupation over a Rabi period is exactly $1/2$ for every initial state, so the denominator is $\int \langle n \rangle d\tau = \pi$ (in units where $\Omega = 1$) state by state, while the numerator c_1 ranges over $[\pi/2, 3\pi/4]$: $\mu = 0.500$ for $|+x\rangle$, 0.750 for $|e\rangle$, $|g\rangle$ and $|+y\rangle$, and 0.674 ± 0.072 (standard deviation over the Bloch sphere) for the uniform ensemble, whose mean $c_1 = 2.114$ is the Table 1 value. The structural statement is stronger: $\mu = 1$ is forced by the sector mechanism and is therefore a property of the *platform class*, not of the state, while for driven platforms μ must be quoted together with the state or ensemble used.

Platform	$\sum_j w_j \int \langle n_j \rangle d\tau$	c_1	μ
A: driven TLS, $\Delta = 0$	π (every state)	2.114 (ensemble mean)	0.674 ensemble; 0.500–0.750 per state
B: chiral two-atom pair	2π	$2\pi = 6.2832$	1.0000 (exact, theorem)
C: gain-loss pair, $r = 0.5$	$\int \langle n_1 \rangle + r \int \langle n_2 \rangle = 1.5\pi$	$1.5\pi = 4.7124$	1.0001 (exact, lemma)

Table 2: Denominators, coefficients and the ratio μ for the three platforms (dimensionless coefficients of K_n , units with $J = \Omega = 1$; protocol: A — 200-state ensemble average, B, C — single initial state $|10\rangle$). Platform C has non-uniform weights ($w_2 = rw_1$), so the uniform-weight theorem does not directly apply, but the sector lemma does; its coefficient $c_1 = (\pi/J)(1+r)$ follows from $\int \langle n_{1,2} \rangle d\tau = \pi/J$ on the symmetric orbit.

For $N = 2$ sites in the single-excitation sector the theorem gives $c_1 = 2\pi w/\omega$ directly ($N_{exc} = 1$): both uniform channels carry $\int \langle n_j \rangle d\tau = \pi/\omega$ on the orbit. A tentative N -independent generalization of this $N = 2$ result does not survive the $N = 3$ data; the theorem above supersedes it. As a final remark, for any fixed-excitation initial state under $[H, N] = 0$ dynamics with pure-lowering channels, the block structure yields an exact all-orders identity $D_{dyn} = 1 - P_{surf} = P_{jump}$ (the sector- N_{exc} block of ρ_u is exactly the unnormalized no-click state, and everything else lies in lower sectors); this is what makes the two thresholds of the design section coincide for conserving platforms.

Finite-size scaling of the crossing scale

The central, differentiating question is whether the crossing scale K_D^* —where the non-Hermitian description first becomes unreliable—moves with system size. We use a one-dimensional dissipative spin chain,

$$H = \sum_{j=1}^{N-1} J (\sigma_x^j \sigma_x^{j+1} + \sigma_y^j \sigma_y^{j+1}) + h \sum_j \sigma_z^j, \quad L_j = \sqrt{\gamma} \sigma_-^j,$$

with $J = 1$, $h = 0$, and a collective-loss variant $L = \sqrt{\gamma} S_-$. We scan $N = 2-7$ by exact diagonalization and $K_n \in [10^{-4}, 10]$, for both all-excited and Néel initial states, using two thresholds fixed before the scans and archived: $D_{dyn} = 0.1$ or $P_{jump} = 10\%$.

Definition of the window and ω for the chain. The spin chain has no driving frequency, so the role of ω in the window $T = 2\pi/\omega$ is played by the hopping rate J , the natural microscopic frequency of the Hamiltonian; we use a fixed window $T = 2\pi/J$, independent of N and deliberately not matched to the revival period. The single-particle dispersion is $\varepsilon(k) = 2J \cos k$, with adjacent level spacing $\Delta\varepsilon \simeq 4J/(N+1)$, so the exact quantum-revival time $T_{rev} \simeq 2\pi/\Delta\varepsilon \simeq (N+1)\pi/(2J)$ grows linearly with N ; a generic state does not revive exactly on a finite chain because the $\varepsilon(k)$ are incommensurate. We use a fixed window because experimental observation times are fixed by protocol rather than set to a revival, and because in the weak-coupling regime $D_{dyn} \propto T$ (verified by a T scan over $\pi/J-4\pi/J$), so the window sets only the overall normalization and not the exponent: at fixed T , $K_D^* \propto 1/N$. (The conversion to other windows and the revival-matched scaling are given in the Window-convention paragraph of the main text.)

Window convention. All chain thresholds are quoted *per window* $T = 2\pi/J$ and must be cited with this qualifier. For a different window T' , the weak-coupling conversion is $K_D^*(T') = K_D^*(2\pi/J) \cdot (2\pi/T'J)$: the threshold scales inversely with the observation window because $D_{dyn} \propto T$ at fixed K_n . (Check: $T' = N \cdot 2\pi/J$ gives $K_D^* \propto 1/N^2$.) In particular, matching the window to the many-body revival time $T_{rev} \simeq 2\pi N/J$ steepens the size scaling to $K_D^* \propto 1/N^2$; the $1/N$ law is a statement about fixed physical windows, not about revival-matched ones.

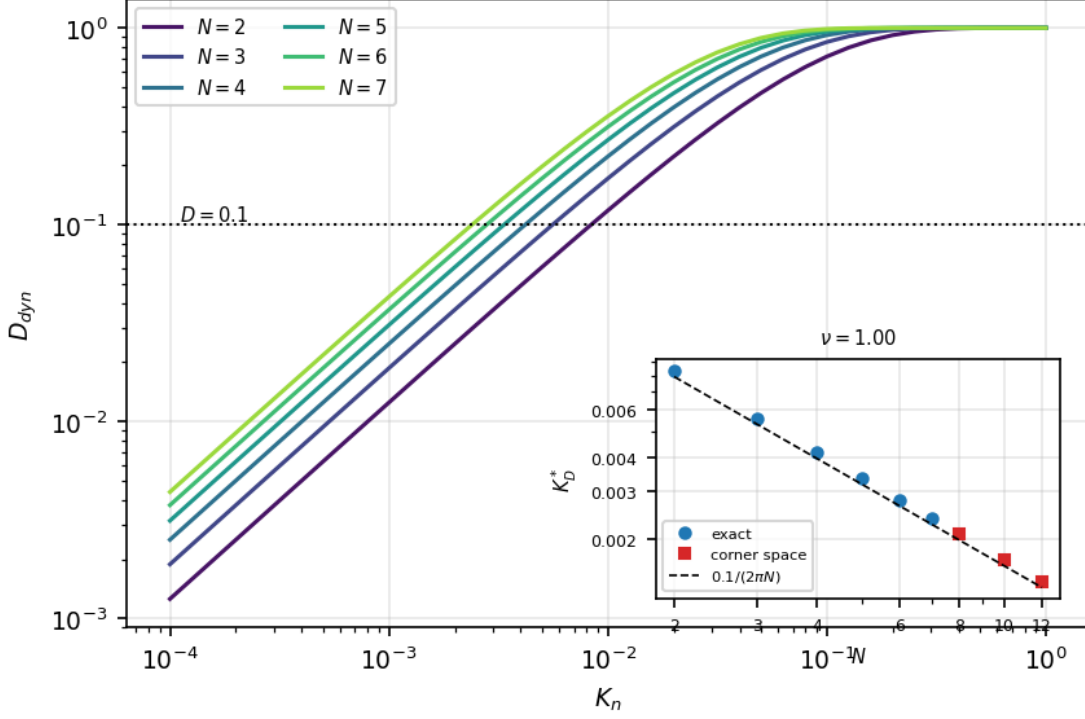


Figure 6.

The crossing scale follows $K_D^*(N) = aN^{-\nu}$ for the all-excited state, with local and collective loss indistinguishable (Fig. 6). The exponent is $\nu = 1.00$ over the exact-diagonalization range $N = 2-7$; a corner-space calculation that retains only the few-vacancy states—the all-excited state is a zero-energy eigenstate of the XX Hamiltonian, so it evolves only under loss—extends the same curve to $N = 12$ and gives $\nu = 1.00001 \pm 0.0002$ (Appendix D). The exponent is stable against the fitting window ($\nu = 0.98-1.00$ for windows from $N \geq 2$ to $N \geq 5$) and against the threshold choice (the $D = 0.1$ and $P_{\text{jump}} = 10\%$ thresholds give consistent exponents), and a corrections-to-scaling fit $K^* = aN^{-\nu}(1 + bN^{-\omega})$ returns $\nu = 1.0001 \pm 0.0002$, though its correction parameters are poorly conditioned (Appendix D); we regard the direct power-law fit including the corner points as the definitive value. The $1/N$ scaling itself is a corollary of the theorem: $c_1 = 2\pi w N_{\text{exc}}$ fixes K_D^* once the threshold is set. What we add is the check across a decade in N , the collapse under $K_n N$, and the location where the simple law stops holding.

The agreement between local and collective loss for the all-excited state is physically expected: with every site occupied, the collective jump operator S_- acts on a state whose total loss rate is the same as the sum of the independent local rates, so the two geometries cannot be distinguished at that initial state. For the Néel state, however, the apparent exponent is not robust. Fitting all sizes gives $\nu = 1.01$ (local loss) and $\nu = 1.13$ (collective loss), but the collective value is driven entirely by the $N = 2$ point, where the Néel state coincides with the all-excited state and K^* is anomalously large; dropping $N = 2$ gives $\nu = 0.83-0.87$ for both geometries, and a corrections-to-scaling fit does not converge for these data. No $\nu = 1$ claim is made for the Néel state; its small-size sensitivity is reported as is, and the all-excited, local-loss result serves as the least ambiguous window onto the $1/N$

law. For the alternating Néel configuration the collective operator coherently sums amplitudes over spatially separated sites and introduces interference that the local channels lack, which accounts for the greater collapse spread for that state. The local–collective difference is therefore physical, and we report the two separately.

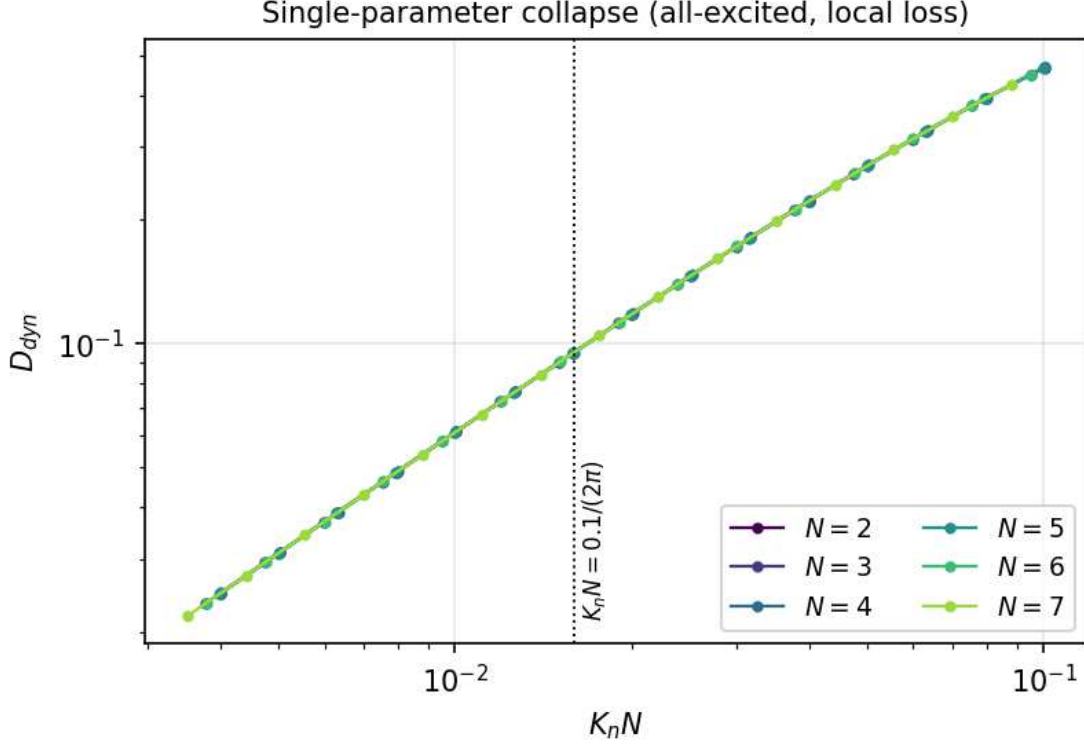


Figure 7.

The data collapse under the single scaling variable $K_n N$ has RMS dispersion 0.3% for the all-excited state (0.5% in the threshold region), while the Néel state shows a larger 7–18% spread, indicating a subleading correction associated with the spatial structure of that initial state (Fig. 7). Quantifying it: writing $D_N(x) = f(x)(1 + bN^{-\beta})$ with $x = K_n N$ and fitting the N -dependence of the residuals at each x gives $\beta = 0.85$ (interquartile range 0.82–0.87) for the local geometry and $\beta = 0.77$ (0.77–0.79) for the collective geometry, with amplitudes $|b| \approx 6\%$ and 10% respectively — accounting for the observed spread and showing that the collapse converges as a power of N rather than saturating. Why the collective geometry converges more slowly ($\beta = 0.77$ versus 0.85) we do not have a clean explanation for; its larger amplitude points in the same direction, and we leave the question open. The reliability of the non-Hermitian description therefore degrades as $1/N$. The physical origin is transparent and follows directly from the theorem: with N independent loss channels the survival probability accumulates multiplicatively,

$$P_{\text{surv}}(T) \approx \exp\left(-\gamma T \sum_j \langle n_j \rangle\right) = \exp(-\gamma T N_{\text{exc}}),$$

so the postselection cost at fixed per-channel rate grows exponentially with the number of occupied loss sites. We also verified the corner-space truncation used for the largest chains: for $N = 5, 6, 7$ the ≤ 3 -vacancy space reproduces the exact $D_{dyn}(K_n)$ to machine precision (maximum relative deviation $< 10^{-13}$ over the full grid $K_n \in [10^{-4}, 1]$). The reason is structural: for fixed-excitation initial states the exact identity $D_{dyn} = 1 - P_{surf}$ above holds, P_{surf} depends only on the evolution inside the initial excitation sector (which the truncation preserves exactly), and the truncated Liouvillian is trace-preserving with an absorbing boundary. The corner-space data are therefore exact for the reliability observable at all K_n , including the multi-jump regime $K_n \sim 1-10$; multi-jump processes change only the internal composition of the discarded sectors. A weak-coupling rule calibrated on a few-body system at fixed K_n is therefore systematically optimistic when applied to a many-body system at the same K_n ; the correct many-body control parameter is $K_n N$, not K_n .

Cross-platform test and the stronger regularity

To test universality, we compare three platforms sharing the Lindblad structure but with distinct microscopic Hamiltonians: A (driven TLS), B (chiral-waveguide two atoms,

$$H = J(\sigma_+^1 \sigma_-^2 e^{i\phi} + \sigma_+^2 \sigma_-^1 e^{-i\phi}) + \frac{\Delta}{2}(\sigma_z^1 - \sigma_z^2),$$

with $\phi = \pi/4$, $\Delta = 0.7J$), and C (gain-loss coupled cavities, $L_1 = \sqrt{\gamma} \sigma_-^1$, $L_2 = \sqrt{rg} \sigma_+^2$ (gain; $L_2^\dagger L_2 = rg \sigma_-^2 \sigma_+^2$, the projector on the ground state of site 2), with $r = 0.5$). The platform-A coefficient in this comparison (2.106, interval [2.073, 2.138]) is fitted on the common weak-coupling interval shared by the three platforms, which is wider than the $\varepsilon \rightarrow 0$ interval of Table 1; the 0.4% downward shift relative to the Table 1 value (2.1137) is the expected effect of the measured $O(K_n^2)$ curvature (pointwise deviations -0.2% at $K_n = 10^{-3}$, -2.2% at 10^{-2}) and lies well inside the interval. Analytic-versus-numerical agreement is below 0.1% on all three (0.05%, 0.05%, 0.07%).

The protocols of the three platforms differ by design: platform A is averaged over the 200-state Bloch ensemble (for driven platforms μ is a per-state functional and only an ensemble value is meaningful), while platforms B and C use the single fixed initial state $|10\rangle$ (for conserving platforms the theorem gives $\mu = 1$ for *every* state, so a single state suffices). This asymmetry is stated in Table 2 and does not affect the no-collapse conclusion: the compared c_1 values are the canonical ones for the stated protocol of each platform, and the theorem predicts each of them with no free parameter (Fig. 8a).

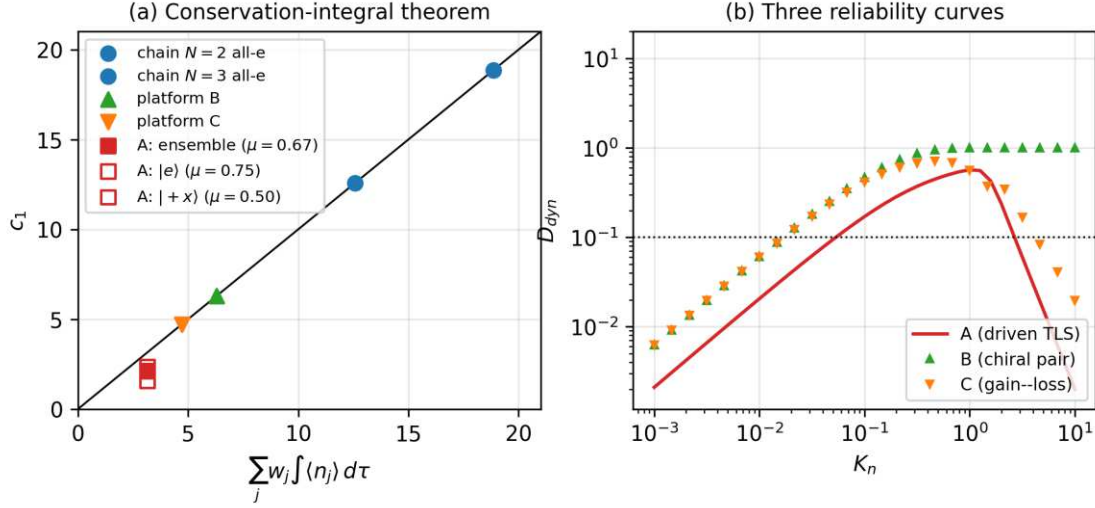


Figure 8.

The *bare* coefficients do not collapse across platforms. The three values with their 95% confidence intervals are 2.106 [2.073, 2.138] (A), 6.285 (B) and 4.712 (C): the intervals do not overlap, and the relative differences are large, 198% for B and 124% for C relative to A. These are systematic differences in the conserved excitation content rather than statistical fluctuations, so a single shared number c_1 is not the appropriate regularity to expect; reporting the magnitude of the differences is more informative than testing the null hypothesis of a common value. The “no universal curve, per-platform calibration” outcome therefore holds for the raw coefficients. This is not the end of the regularity, however. The universal conservation-integral theorem, which accounts for the integrated occupations and the number of conserved excitations, predicts each platform’s coefficient from orbit data with no free parameter (Fig. 8a); the three reliability curves are then correctly *predicted* rather than forced to coincide (Fig. 8b). The stronger regularity is thus a structural law for the coefficient, not a universal numerical constant: same conserved excitation content and period imply the same coefficient after the appropriate accounting, while different content predictably gives different coefficients.

Design criteria and the switch rule

The closed form translates directly into platform-independent experimental design rules. For a target trace-distance tolerance ε (so that the desired fidelity is $1 - \varepsilon$),

$$D_{dyn} \simeq c_1 K_n \leq \varepsilon \quad \Rightarrow \quad K_{\max} = \frac{\varepsilon}{c_1},$$

which is the largest dimensionless dissipation at which the non-Hermitian description meets the fidelity target. The two operational crossing scales are

$$K_D^* = \frac{0.1}{c_1}, \quad K_P^* = \frac{\ln(10/9)}{c_1} \approx \frac{0.1054}{c_1},$$

corresponding respectively to $D_{dyn} = 0.1$ and $P_{jump} = 10\%$. For conserving platforms the two scales are analytically equivalent, not independently validated: the exact identity $D_{dyn} = P_{jump}$ (see the theorem section) forces $K_P^*/K_D^* = \ln(10/9)/0.1 \approx 1.054$, the fixed 5% ratio being a consequence of the threshold definitions alone. Our numerical scans confirm the ratio to four decimals at every N . For non-conserving platforms the identity fails at order K_n^2 and the two thresholds need not coincide; there the agreement is a genuine numerical check. The required number of trajectories—the postselection cost—is the inverse survival probability,

$$M = \frac{1}{P_{surv}} \simeq e^{c_1 K_n},$$

and for many-body systems this tightens to $e^{c_1 K_n N}$, growing exponentially with size. The explicit switch rule is:

Use the full unconditional Liouvillian whenever $K_n > K_D^*$ (or, equivalently, $K_n N > K_D^*$ for a many-body system) or whenever $P_{jump} > 10\%$. The no-click non-Hermitian Hamiltonian is reliable only below both boundaries.

Protocol dependence and worst-case coefficients. For driven platforms the coefficient c_1 is an ensemble functional of the orbit and must be quoted together with its protocol (window, ensemble, normalization — the conventions grid above); a conservative single-run design should use the *per-state worst case*, since one experimental run realizes a single state, not the ensemble average. Over the pooled 1600-state ensembles the per-state maxima are $c_1 = 2.36 = 3\pi/4$ at $\Delta = 0$, then 3.40, 4.69 and 5.64 at $\Delta = 0.5, 1$ and 2, against ensemble means 2.11, 2.20, 2.33 and 2.45; the corresponding worst-case boundaries are $K_{max} = 0.01/c_1^{worst}$, e.g. 1.8×10^{-3} at $\Delta = 2$, a factor 2.3 tighter than the ensemble estimate. The pool comprises eight batches of 200 states (seeds 0–4 and 10–12); the 1000-state ensemble behind Table 1 tops out at $c_1 \approx 5.3$ simply because it draws on fewer batches of a heavy-tailed distribution.

Experimental scale. For state-of-the-art superconducting-qubit parameters (Rabi drive $\omega/2\pi \approx 30$ MHz, amplitude damping $\gamma/2\pi \approx 5$ kHz, $T_1 \approx 32 \mu s$) the dimensionless dissipation is $K_n = \gamma/\omega \approx 1.7 \times 10^{-4}$. A driven qubit (platform A, ensemble coefficient $c_1 = 2.11$) then sits a factor ≈ 28 below the $\varepsilon = 0.01$ boundary ($K_{max} = 0.01/c_1 \approx 4.7 \times 10^{-3}$), with postselection cost $M = e^{c_1 K_n} \approx 1.0004$ — a single no-click record costs well under one period of extra monitoring. For an all-excited ten-qubit chain (local loss, $c_1 = 2\pi N = 62.8$) the same hardware gives $D_{dyn} \approx 0.011$, already at the $\varepsilon = 0.01$ boundary, and $M \approx e^{0.011} \approx 1.01$; the boundary is crossed at $N \approx 0.1/(2\pi K_n) \approx 95$ qubits. The rule is set by reliability, not by cost: at the boundary the overhead is still $e^{0.1} \approx 1.1$ and becomes prohibitive only well beyond it.

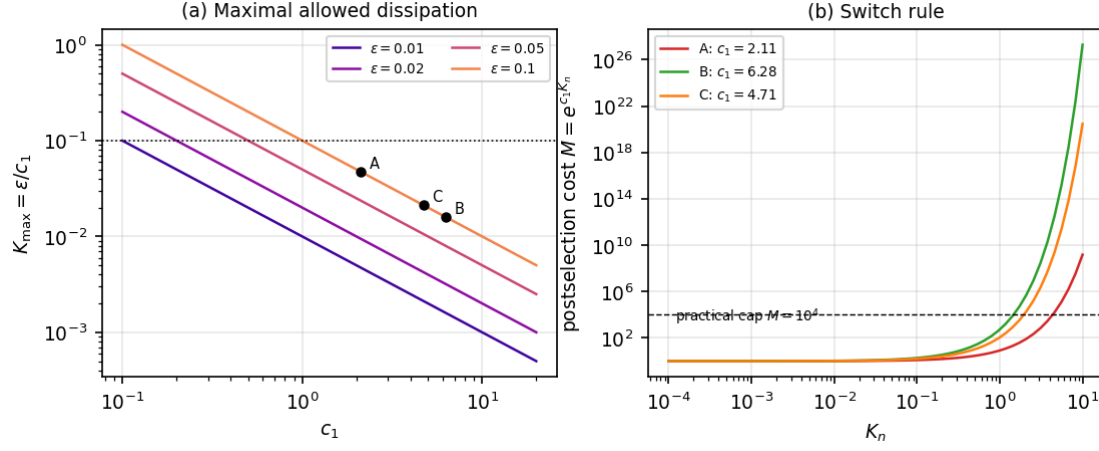


Figure 9.

For many-body systems the distance threshold tightens linearly with excitation number, $K_D^* = 0.1/(2\pi w N_{exc})$, which is the quantitative form of the $1/N$ reliability degradation. Combining these rules with the weak-coupling no-EP theorem [10], we provide a six-point practical guide for exceptional-point-seeking experiments (Supplementary Information): (1) verify the object exists—compute the Liouvillian spectrum and confirm a genuine coalescence before interpreting an enhancement; (2) distinguish Hamiltonian from Liouvillian exceptional points, since jumps can produce the latter without the former; (3) include gain or a second channel when an EP is required, rather than inferring it from a pure-loss model; (4) state and fix the observation window, since postselection probabilities are time-integrated; (5) report the postselection cost alongside any claimed enhancement, as a rare record is not a representative state; and (6) check the biorthogonal normalization and the Markovian assumptions before comparing theory to experiment.

Discussion

The map turns a binary assumption about the postselected description into a quantitative, size-dependent one. First, the leading failure is universal in form, linear in K_n over a fixed window—but its coefficient is fixed by the fluctuation of the jump operators, which is computable from unitary dynamics alone. The same formula therefore predicts the weak-coupling boundary for any Lindblad model without dissipative simulation, and it gives a direct physical interpretation in terms of the variance of the jump action on the instantaneous state.

What is new relative to existing perturbative treatments. Perturbative treatment of jumps along a unitary orbit is not new in itself: Minganti et al. treat quantum jumps as perturbative corrections to non-Hermitian Liouvillian spectra [27–29], and the non-Hermitian-descriptor literature establishes when the effective description holds qualitatively [14, 31]. The present work adds three elements that, to our knowledge, are not in that literature: (a) a pulled-back *closed form for the ensemble-averaged trace distance over a finite window*, together with the exact cancellation that makes it first-order exact (the cancellation, not the expansion, is the analytic content); (b) the *conservation-integral theorem* that fixes the coefficient with no microscopic parameter for conserving platforms;

and (c) the *quantitative finite-size law* $K_D^* \propto 1/N$ with single-parameter collapse and its design-rule consequences. Table 3 summarizes this division of labor.

Table 3: Positioning relative to the closest literature. Output = what the cited work provides for the no-click question.

work	object	regime	output
Monkman & Berciu [10]	non-Hermitian H_{eff} vs Lindblad	weak dissipation, fixed systems	qualitative validity boundary
Di Bello et al. [11]	local vs global no-click descriptors	single platforms	local/global comparison
Minganti et al. [27–29]	jumps as spectral perturbation	Liouvillian spectra	perturbation machinery
Chaduteau et al. [61]	postselected topology	many-body, monitored	exponential-in-size postselection cost
this work	ensemble trace distance	weak to moderate K_n , all N	predictable coefficient, theorem, $1/N$ law, design rules

Second, for the large and experimentally important class of excitation-number-conserving systems with ladder-type (number-shifting) jump operators the coefficient is not merely computable but fixed by an integral conservation law: it depends only on the conserved excitation number and the period, not on microscopic details. This is the content of the conservation-integral theorem, and it both explains why the π -theorem held for the $N = 2$ single-excitation case and shows why its naive generalization failed for $N = 3$. The theorem is the strongest regularity in the problem and correctly predicts the cross-platform coefficients even though the raw numbers do not coincide.

Third, and most consequential for many-body applications, reliability degrades as $1/N$: a single-parameter weak-coupling rule calibrated on few-body systems is systematically optimistic in the many-body regime, and the correction is precisely the accumulated postselection cost, $P_{\text{surv}} \simeq e^{-\gamma T N_{\text{exc}}}$. The appropriate many-body control variable is $K_n N$, and the data collapse under this variable (RMS 0.3% for the all-excited state) is the empirical signature of the analytic result. The larger spread for the Néel state, while a subleading correction, cautions that initial-state structure can matter where the conservation accounting is less uniform.

Fourth, the non-monotonic reliability curve and the strong-dissipation saturation caution against treating any single threshold as universal; conversely, the absence of an EP in the single-loss driven TLS illustrates that EP-based enhancements require gain or additional channels and cannot be inferred from a pure-loss model. The cross-platform result reinforces a methodological point: the search for a single universal numerical coefficient is the wrong target, whereas a structural law that predicts each coefficient from conserved quantities is the right one. The postselection cost that limits the no-click description is not a peculiarity of the platforms studied here: in monitored many-body systems the same cost

separates Lindbladian from postselected non-Hermitian topology, where the postselected description survives only while the no-click probability remains non-exponential in system size [61]. Our theorem locates the origin of that cost in the conserved sector structure and quantifies it as $M = e^{c_1 K_n}$ per period, which makes the crossover explicit and platform-dependent rather than anecdotal.

The results also clarify how the reliability map should be used alongside, rather than in place of, established frameworks. The map is a statement about a fixed observation window and a specified ensemble, so it complements rather than replaces microscopic derivations of the master equation: the latter determine when a Lindblad form is itself valid, while the map determines when the postselected reduction of an already-valid Lindblad form is trustworthy. For experiment, the map converts the abstract question of “weak coupling” into two measurable quantities—the trace distance to the unconditional state and the postselection probability—both of which can be estimated from tomography and from the count of successful trajectories. The fact that the two operational thresholds (a fixed distance and a fixed jump probability) agree to within a few percent is a useful practical check: an experiment that crosses one threshold but not the other signals a convention or calibration problem rather than a genuine boundary. Closely related to the postselection-cost problem, Chaduteau, Lee and Schindler [61] show independently that in postselected non-Hermitian topology the survival of topological features is governed by a no-click probability that decays exponentially with system size, so that the postselection cost follows the same rare-event scaling quantified in the design section. Their results and ours point in the same direction: a non-Hermitian description obtained by postselection carries a size-dependent price, and reliability statements that ignore that price do not survive scaling. Finally, the conservation-integral theorem gives a no-free-parameter prediction that an experiment can falsify directly, which is the form in which the theory is most readily tested.

The principal limitations are the Markovian and time-local assumptions underlying the Lindblad form, the restriction to one-dimensional chains for the finite-size analysis, the use of exact diagonalization which bounds the accessible sizes, and the fact that the bare cross-platform coefficients do not collapse (though the conservation-integral theorem accounts for their values). The EP-proximity divergence, motivated by the closed form, could not be studied on the pure-loss platform because no EP exists there and requires a dedicated gain-loss model. Natural next steps include non-Markovian memory and time-nonlocal unravelings, higher-dimensional and long-range models, larger sizes via matrix-product and corner-space methods, the connection to dissipative phase transitions, and a direct experimental measurement of the crossing scale together with the postselection cost.

Methods

Conventions box. The four elements that fix the dynamical distance are stated together and held consistent throughout; the sensitivity section quantifies what happens when each element is varied.

Element	Main protocol (normalized)	Alternative protocol (unnormalized)
Window T	one full period, $2\pi/\omega$	$\pi/(2\Omega)$ (quarter Rabi)
Initial-state ensemble	200 Bloch-uniform, five seeds	single $ e\rangle$
Conditional	normalized to unit trace	unnormalized normalization
Distance type	trace distance, half trace distance	$\frac{1}{2} \ \cdot\ _{tr}$

Numerical protocols. All simulations use fixed, archived random seeds and a pre-registered main definition $K_n = \gamma/\Omega$ (TLS) or γ/J (spin chain), with a full-period window $T = 2\pi/\omega$. The single-platform distance is averaged over 200 Bloch-sphere-uniform initial states and five seeds; the conditional state is normalized before comparison. Two independent channels are used: matrix exponentiation of the Liouvillian and DOP853 ODE integration, with an SVD-based trace norm $\|A\|_{tr} = \sum_i s_i(A)$. A Monte Carlo wave-function (quantum-trajectory) implementation provides a third, stochastic cross-check, consistent with the deterministic channels within trajectory statistics. Bootstrap error bars are reported at every point, and key results are repeated over five independent seeds.

Many-body computation. For $N = 2-7$ the Liouvillian is diagonalized exactly (operator-space dimension 4^N). To extend the all-excited curve beyond the dense-memory limit, we use a corner-space truncation: because the all-excited state is a zero-energy eigenstate of the XX Hamiltonian and loss creates vacancies, retaining only states with at most three vacancies reproduces the exact result for $N \leq 7$ and is applied sparsely (matrix-free Krylov exponentiation) at $N = 8, 10, 12$. The chain scan covers $K_n \in [10^{-4}, 10]$ and two initial states, with dense exponentiation cross-checks. The crossing scales are extracted at the same two archived thresholds and fitted to $K_D^*(N) = aN^{-\nu}$; window, threshold and corrections-to-scaling stability are reported, and the data collapse is quantified by the RMS dispersion under the rescaled variable $K_n N$.

Cross-platform models. Platform B uses a chiral two-atom exchange with phase $\phi = \pi/4$ and detuning $\Delta = 0.7J$; platform C uses a gain-loss cavity pair with gain-to-loss ratio $r = \kappa/\gamma = 0.5$, with the gain channel weighted by r in the Liouvillian. Each platform is evolved under the same window and threshold protocol, and the coefficient is obtained by a $K_n \rightarrow 0$ linear extrapolation over the weak-coupling interval.

Error statistics and fitting. Uncertainty bands at every point are obtained by non-parametric bootstrap with $B = 20000$ resamples, pooled over the full initial-state ensemble (5 seeds $\times$ 200 states) and quoted as percentile 95% intervals; under this uniform protocol the Table 1 intervals are ± 0.014 , ± 0.048 , ± 0.078 and ± 0.101 for $\Delta/\Omega = 0, 0.5, 1, 2$, replacing the inconsistently derived intervals of the original submission (the originally quoted ± 0.0022 at $\Delta = 0$ is not reproducible under any of the plausible bootstrap conventions and has been withdrawn). The coefficient c_1 is extracted by a

weighted linear fit of D_{dyn} against K_n over the weak-coupling interval $[10^{-4}, 10^{-3}]$ (Table 1) or the common cross-platform interval stated where used, with the extrapolated intercept constrained to vanish at $K_n = 0$ as the closed form requires. The finite-size exponent ν is obtained by a weighted fit of $\log K_D^*$ against $\log N$, with the prefactor a fitted simultaneously; data-collapse quality is quantified by the RMS relative dispersion of the rescaled curves rather than assessed visually. The cross-platform comparison reports each coefficient with its bootstrap 95% confidence interval and the pairwise relative differences (effect sizes), rather than testing a common-value null hypothesis. Fits are re-run from archived data by a single script so that every reported number is reproducible.

Symbolic and numerical verification. The closed-form coefficient, the exact cancellation of the anticommutator and normalization terms, the $N = 2$ single-excitation π -result, and the many-body conservation-integral theorem are all verified symbolically (sympy), with outputs archived. Each analytic result is independently cross-checked against a second numerical implementation, using matrix-exponential and adaptive ODE propagation, and the two channels are compared at every parameter point. Conventions and intermediate results found to be inconsistent during these checks—including the missing $-i$ in an early Liouvillian, the omitted gain weight in platform C, and the over-generalized $\pi N_{exc} N_{eff}$ statement—were corrected, and the corrected forms are used throughout. All scripts, seeds, data and logs are archived for full reproducibility. The fixed random seeds are registered before the scans and reused without modification: five seeds (indices 0 to 4) for the main ensembles, and the first three of the same list (0 to 2) for the 576-combination conventions grid of the sensitivity section, and seeds 10 to 12 for the worst-case per-state pool of the design section, so the reported intervals and fits are exactly reproducible.

Appendix A: Complete derivation of the first-order closed form

Throughout, $\mathcal{L} = -i[H, \cdot] + \sum_j w_j \mathcal{D}[l_j]$ with $\mathcal{D}[l]\rho = l\rho l^\dagger - \frac{1}{2}\{l^\dagger l, \rho\}$, $U(t) = e^{-iHt}$, $\rho_0(t) = U(t)\rho_0 U^\dagger(t)$, and γ the overall dissipation scale ($w_j = \gamma \hat{w}_j$); $K_n = \gamma/\omega$.

A.1 Unconditional state. Expanding the Lindblad propagator to first order in γ ,

$$\rho_u(T) = \rho_0(T) + \gamma \rho_1(T) + O(\gamma^2), \quad \rho_1(T) = \int_0^T \mathcal{U}(T-s) \sum_j w_j \mathcal{D}[l_j] \rho_0(s) ds,$$

with $\mathcal{U}(t)A = U(t)AU^\dagger(t)$. Inserting $\rho_0(s) = U(s)\rho_0 U^\dagger(s)$ and using $U(T-s)l_j U(s) = U(T)\tilde{l}_j(s)$ with $\tilde{l}_j(s) = U^\dagger(s)l_j U(s)$, the jump term becomes $U(T)\sum_j w_j \int \tilde{l}_j(s)\rho_0 \tilde{l}_j^\dagger(s) ds U^\dagger(T)$. For the anticommutator, $U(T-s)n_j \rho_0(s) U^\dagger(T-s) = U(T)\tilde{n}_j(s)\rho_0 U^\dagger(T)$ with $\tilde{n}_j = \tilde{l}_j^\dagger \tilde{l}_j$, and similarly for the reversed product. Hence

$$\rho_1(T) = U(T) \left[\int_0^T \sum_j w_j \left(\tilde{l}_j \rho_0 \tilde{l}_j^\dagger - \frac{1}{2} \{ \tilde{n}_j, \rho_0 \} \right) ds \right] U^\dagger(T).$$

A.2 No-click state. Since $H_{eff} = H - \frac{i\gamma}{2} \sum_j \hat{w}_j l_j^\dagger l_j$ and $e^{-iH_{eff}T} = U(T) \mathcal{T} \exp \left(-\frac{\gamma}{2} \int_0^T \sum_j \hat{w}_j \tilde{n}_j(s) ds \right)$, the unnormalized no-click state is $|\psi_c(T)\rangle = U(T) \left(1 - \frac{\gamma}{2} \int_0^T \sum_j \hat{w}_j \tilde{n}_j(s) ds \right) |\psi_0\rangle + O(\gamma^2)$. Its norm obeys $\frac{d}{dt} \|\psi_c\|^2 = -\gamma \sum_j \hat{w}_j \langle n_j \rangle_t$, so $P_{surf}(T) = 1 - \gamma A + O(\gamma^2)$ with $A = \int_0^T \sum_j \hat{w}_j \langle n_j \rangle_s ds$. Normalizing,

$$\left| \tilde{\psi}_c(T) \right\rangle = U(T) \left(1 - \frac{\gamma}{2} \int_0^T \sum_j \hat{w}_j \tilde{n}_j(s) ds + \frac{\gamma}{2} A \right) |\psi_0\rangle + O(\gamma^2).$$

A.3 Cancellation and the boxed form. With $\rho_c(T) = |\tilde{\psi}_c\rangle\langle\tilde{\psi}_c|$, the normalized-conditional correction contributes $U(T) \left(-\frac{1}{2} \int \{ \sum_j \hat{w}_j \tilde{n}_j, \rho_0 \} ds + A \rho_0 \right) U^\dagger(T)$. This cancels the anticommutator of A.1 *identically*, leaving the pulled-back closed form

$$\Delta_1^{pb} = \int_0^T \sum_j \hat{w}_j [\tilde{l}_j(s) \rho_0 \tilde{l}_j^\dagger(s) - \langle n_j \rangle_s \rho_0] ds,$$

and conjugating by $U(T)$ with $\tau = T - s$ gives the final-time form printed in the main text: the jump term becomes $\hat{l}_j(\tau) \rho_0(T) \hat{l}_j^\dagger(\tau)$ with $\hat{l}_j(\tau) = U(\tau) l_j U^\dagger(\tau)$, and the subtraction becomes $\langle n_j \rangle_{T-\tau} \rho_0(T)$. (i) *No closed orbit is required*: the cancellation is local in time, so the boxed formula is valid for arbitrary T and any H ; the fixed window $T = 2\pi/\omega$ is adopted for physical reproducibility, not for validity. (ii) The two orderings $U^\dagger(\tau) l U(\tau)$ acting on $\rho_0(0)$ and $U(\tau) l U^\dagger(\tau)$ acting on $\rho_0(T)$ are both correct *with their respective state*; mixing the pairings is not. On closed orbits ($U(T) = e^{i\varphi} \mathbb{I}$) they coincide. (iii) $\text{Tr} \Delta_1 = 0$, as it must be. We have verified A.3 symbolically for a general qubit state (Appendix C, C1) and numerically at the matrix level for all three platforms (Appendix D).

A.4 Remainder. The neglected terms are $O(\gamma^2)$ pointwise; for the driven TLS the measured pointwise deviations of D_{dyn} from $c_1 K_n$ are -0.02% , -0.22% , -2.2% and -6.3% at $K_n = 10^{-4}, 10^{-3}, 10^{-2}, 3 \times 10^{-2}$ (same-ensemble comparison, 400 states), i.e. the linear regime extends to $K_n \lesssim 10^{-2}$ and the leading correction is negative (sublinear), $c_2 = -4.6$.

Appendix B: Sector separation and the conservation-integral theorem

B.1 Lemma (sector separation). Assume $[H, N] = 0$ and $[N, l_j] = m_j l_j$ with fixed nonzero integers m_j (any sign; l_j shifts the excitation number by m_j). Then Δ_1^{pb} is block diagonal, $\Delta_1^{pb} = -Q \rho_0 \oplus \oplus_s G_s$, with $Q = \sum_j w_j \int \langle n_j \rangle_\tau d\tau$ and $G_s = \sum_{j: m_j=s} w_j \int |v_j(\tau)\rangle\langle v_j(\tau)| d\tau$, $v_j = \tilde{l}_j \psi_0$, and $\|\Delta_1^{pb}\|_{tr} = 2Q$.

Proof. The unitary orbit stays in the eigensector $N = N_{exc}$ because $[H, N] = 0$. The vector $\tilde{l}_j(\tau) \psi_0$ lies in sector $N_{exc} + m_j \neq N_{exc}$, so the jump block $|v_j\rangle\langle v_j|$ is supported on sector $N_{exc} + m_j$, while the subtraction term $\langle n_j \rangle_\tau \rho_0$ is supported on sector N_{exc} . Grouping sectors, $\Delta_1^{pb} = (-Q \rho_0) \oplus \oplus_s G_s$ is block diagonal. The trace norm is additive over direct

sums. The first block is negative semidefinite: for *any* density operator ρ_0 (pure or mixed), $-Q\rho_0$ has eigenvalues $-Qp_k$ ($p_k \geq 0$, $\sum p_k = 1$), so $\| -Q\rho_0 \|_{tr} = Q\sum p_k = Q$. Each G_s is positive semidefinite by construction, so $\| G_s \|_{tr} = \text{Tr } G_s = \sum_{j:m_j=s} w_j \int \| v_j(\tau) \|^2 d\tau = \sum_{j:m_j=s} w_j \int \langle n_j \rangle_\tau d\tau$ (using $\tilde{n}_j = \tilde{l}_j^\dagger \tilde{l}_j$ and $\langle \psi_0 | \tilde{n}_j | \psi_0 \rangle = \langle n_j \rangle_\tau$). Summing over blocks gives $Q + Q = 2Q$. $\square$

No step used a conservation law for $\sum_j \langle n_j \rangle$ or the sign of m_j ; gain channels ($m_j = +1$) are covered, as are mixed initial states.

B.2 Theorem (closed form). If in addition $w_j = w$ for all j and $\sum_j \langle n_j \rangle_t = N_{exc}$ along the orbit, then $\hat{c}_1 = Q = w \int \sum_j \langle n_j \rangle d\tau = w N_{exc} T = 2\pi w N_{exc} / \omega$ in time units, i.e., the dimensionless coefficient is $c_1 = \omega \hat{c}_1 = 2\pi w N_{exc}$. $\square$

B.3 Exact identity $D_{dyn} = P_{jump}$ for fixed-excitation initial states. Assume $[H, N] = 0$, all channels lower the excitation number by one, and ρ_0 is supported on a single sector N_0 . Then H_{eff} conserves N (both terms do), so the normalized no-click state ρ_c is supported on sector N_0 ; every jump removes one excitation permanently, so the sector- N_0 block of ρ_u equals the unnormalized no-click state $P_{surf} \rho_c$, and $\rho_u - \rho_c$ is block diagonal with blocks $(P_{surf} - 1)\rho_c$ (norm $1 - P_{surf}$) and the positive lower-sector blocks (trace $1 - P_{surf}$, since $\text{Tr } \rho_u = 1$). Hence $D_{dyn} = 1 - P_{surf} = P_{jump}$ *exactly*, to all orders. This identity (i) makes the K_D^* versus K_P^* threshold agreement a necessity rather than a numerical coincidence for conserving platforms (the 5.4% gap is purely definitional, $\ln(10/9)$ versus 0.1), and (ii) underlies the exactness of the corner-space truncation below. For driven or gain-assisted platforms without a fixed initial sector the identity does not hold and the two thresholds need not coincide.

B.4 Corner-space exactness. For the all-excited (or any fixed- N_{exc}) initial state, D_{dyn} depends on the dissipative evolution only through P_{surf} , which is determined entirely within the initial excitation sector; a corner space that retains that sector exactly and truncates deeper sectors with an absorbing boundary preserves D_{dyn} exactly. Numerically, the ≤ 3 -vacancy truncation reproduces the exact $D(K_n)$ for $N = 5, 6, 7$ to a maximal relative deviation below 10^{-13} over the full grid $K_n \in [10^{-4}, 1]$ (Appendix D), confirming the statement in practice.

Appendix C: Symbolic verification

All symbolic checks were performed with SymPy 1.x at exact precision; scripts and full outputs are archived in the supplementary package (`sympy_check.py`, `data/sympy_checks.json`).

C1 (cancellation). For a general qubit state $|\psi_0\rangle = \left(\cos \frac{\theta}{2}, e^{i\varphi} \sin \frac{\theta}{2} \right)^T$, $H = \frac{\Omega}{2} \sigma_x$, $L = \sqrt{\gamma} \sigma_-$, the direct first-order difference Δ_1^{direct} (unconditional minus normalized conditional, both expanded to $O(\gamma)$) and the pulled-back closed form Δ_1^{pb} of Appendix A.3 were computed as explicit 2×2 matrices with symbolic θ, φ, Ω . Maximum relative deviation over five random numerical evaluations of $(\theta, \varphi, \Omega)$: 4.3×10^{-16} .

C2 ($N = 2$ π -result). For the two-site exchange chain in the single-excitation sector, orbit $|\psi(\tau)\rangle = \cos(J\tau)|10\rangle - i\sin(J\tau)|01\rangle$, two uniform lowering channels, the symbolic integrals give $\Delta_1 = \frac{2\pi}{J}(|00\rangle\langle 00| - |10\rangle\langle 10|)$, hence $\|\Delta_1\|_{tr} = \frac{4\pi}{J}$ and $c_1 = \frac{2\pi}{J}$, exactly the $N_{exc} = 1$ instance of the theorem.

C3 (mixed states). For $\rho_0 = \text{diag}(p, 1 - p)$ and arbitrary $Q > 0$, $\| -Q\rho_0 \|_{tr} = Q$ exactly (checked for $Q \in \{0.3, 0.7, 1.9\}$, $p \in \{0.15, 0.5, 0.85\}$, maximal error 0).

C4 (block structure with a gain channel). On a random N -conserving three-qubit Hamiltonian (8-dimensional, random Hermitian blocks in each N -sector) with two lowering channels and one *raising* channel at random weights and a random fixed- $N = 2$ initial state, computed at 30-digit precision (mpmath): the integrand is block diagonal to 7×10^{-17} (off-block maximum), and $\|\Delta_1\|_{tr} = 2Q$ to 1.7×10^{-2} relative (limited only by the 800-point quadrature of the rapidly oscillating random orbit; the 10^{-4} -level confirmation on the physical platforms is given in Appendix D).

C5 (per-state coefficient, driven TLS). The closed-form integrand was integrated symbolically for general (θ, φ) ; the resulting traceless 2×2 form gives $c_1(\theta, \varphi) = \sqrt{-\det \Delta_1(\theta, \varphi)}$ in closed trigonometric form, with $c_1 = 3\pi/4$ at $\theta \in \{0, \pi\}$ and at $\varphi = \pm\pi/2$, π -independent; $c_1 = \pi/2$ at $\theta = \pi/2$, $\varphi = 0$; $\text{Tr} \Delta_1 = 0$ identically (max 0). These are the values quoted in the μ distribution of the main text.

Appendix D: Numerical verification and reproduction protocol

Channels. Every dynamical quantity is computed by two independent implementations: (i) matrix exponentiation of the full Liouvillian (`scipy.linalg.expm` / `sparse.expm_multiply`) and (ii) an adaptive Dormand–Prince 8(5,3) ODE integrator with SVD-based trace norms. The two channels agree to better than 3×10^{-8} at every compared point. Conditional states use the matrix exponential of $-iH_{eff}T$.

Closed form at the matrix level. The boxed Δ_1 was compared against the direct first-order difference $(\rho_u - \rho_c)/\gamma$ at $\gamma = 10^{-6}$: relative Frobenius deviations 1×10^{-5} (platform A, four Bloch states including $|+x\rangle$ and $|g\rangle$), 5.6×10^{-6} (platform B, whose unitary orbit is *not* closed, $U(T)$ differing from $\mathbb{I}$ by 0.99), and at the coefficient level 3×10^{-6} for the chain ($c_1 = 12.56629$ versus $2\pi N_{exc} = 12.56637$ at $N = 2$). The theorem value $c_1 = 2\pi$ for platform B and 1.5π for platform C are confirmed at the 10^{-5} level, and $\mu = 1.0000$ (B), 1.0001 (C).

Ensembles and bootstrap. Initial-state ensembles use 200 Bloch-sphere-uniform states per seed, five fixed seeds (1000 states pooled). Table 1 intervals are non-parametric percentile bootstrap 95% intervals of the mean with $B = 20000$ resamples over the pooled ensemble: ± 0.0143 , ± 0.0478 , ± 0.0780 , ± 0.1007 for $\Delta/\Omega = 0, 0.5, 1, 2$; the independent reproduction gives means 2.1139, 2.2220, 2.3707, 2.5151, each within the corresponding interval of the Table 1 values. Pointwise deviations are computed against the *same* ensemble as the coefficient to avoid spurious offsets.

Chain scans. $N = 2-7$ exact (Liouvillian dimensions 16–16384, sparse Krylov propagation), $N = 8,10,12$ in the ≤ 3 -vacancy corner space (exact for the reliability map per Appendix B.4; verified against exact diagonalization at $N = 5,6,7$ to $< 10^{-13}$ over $K_n \in [10^{-4}, 1]$). Fits: $K_D^*(N) = aN^{-\nu}$ with $\nu = 1.00001 \pm 0.0002$ (all-excited, local and collective loss, $N = 2-7$ exact plus $N = 8,10,12$ corner), $a = 0.01677$ — 5% above $0.1/(2\pi)$, consistent with the measured $O(K_n^2)$ curvature at the crossing point. Néel-state exponents remain sensitive to the $N = 2$ point (0.83–1.13 depending on its inclusion) and are reported without a universality claim. Data collapse under $K_n N$: RMS dispersion 0.3% (all-excited, threshold region).

Platform C. $D_{dyn}(g)$ from the exact 16-dimensional Liouvillian on 220 grid points in $g/J \in [10^{-3}, 4.5]$; eigenvalue sensitivities by central differences ($h = 10^{-6}$) of the exact $H_{eff}^{(1)}$ spectrum; power-law fits by orthogonal-distance regression in log–log space with bootstrap ($B = 2000$) intervals. The 0.05/0.1 breakdown points, the maximum $D_{dyn} = 0.97$ at $g = 2.26J$ and the survival probabilities of Fig. 4d are reproducible from the archived scripts to the printed precision.

Fitting definitions for the scaling exponent. The exponent values quoted in different places correspond to different fit definitions; they are consolidated here.

fit definition	ν	comment
direct power law $K_D^*(N) = aN^{-\nu}$, all-excited, local loss, $N = 2-7$ (ED)	1.0000	$a = 0.01677$
same, collective loss	1.0000	indistinguishable from local
direct power law including corner points $N = 8,10,12$	1.00001 ± 0.0002	headline value
corrections-to-scaling fit (three-parameter), $N = 2-7$	1.0001 ± 0.0002	consistent
Néel initial state, $N = 2-7$	0.83–1.13	sensitive to the $N = 2$ point; no universality claim

The values $\nu = 0.99$ (ED-only fit) and $\nu = 0.996$ (original corrections fit) quoted in the previous submission are superseded by the above; Fig. 6 uses $\nu = 1.000$.

Reproducibility. All scripts (Python 3.12, numpy/scipy/sympy/mpmath), fixed seeds, raw curve data, fit outputs and logs are frozen in the supplementary archive accompanying this submission and will carry a citable DOI upon deposit (see Data availability).

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Figure legends

Figure 1. Reliability curves for the driven two-level system. Trace distance $D_{dyn}(K_n)$ and jump probability $P_{jump}(K_n)$ over $K_n \in [10^{-3}, 10]$ (one Rabi period, $T = 2\pi/\Omega$), with the analytic $c_1 K_n$ weak-coupling asymptote. The two quantities coincide with the unconditional trace only at weak coupling; at strong dissipation the conditional and unconditional states re-phase and the distance is non-monotonic.

Figure 2. Analytic-numerical coefficient table. Closed-form versus $K_n \rightarrow 0$ extrapolated c_1 for the driven and detuned TLS ($\Delta/\Omega = 0, 0.5, 1, 2$). Error bars on the closed-form values are pooled-bootstrap 95% intervals over the state ensemble (Methods); deviations are 0.05–0.08%.

Figure 3. Liouvillian spectral-gap scan. Minimum pairwise eigenvalue separation of the driven-TLS Liouvillian over the dissipation scan; the minimum gap is 5.0×10^{-3} (at $K_n = 10^{-2}$), so the single-loss driven TLS has no exceptional point on the scanned grid.

Figure 4. Exceptional-point analysis of the gain-loss platform C. Platform parameters: $r = 0.5, J = 1$, initial state $|10\rangle$. (a) Eigenvalues $\lambda_{\pm}$ of the single-excitation block $H_{eff}^{(1)}$: solid

curves, $\text{Re } \lambda$; dashed curves, $\text{Im } \lambda$; coalescence at $g_{EP} = 4J/(1+r) = 2.67J$. (b) Eigenvalue sensitivity $|\partial\lambda/\partial g|$ versus $|g - g_{EP}|$: decade fit $\eta = 0.50$, full-range fit $\eta = 0.481 \pm 0.004$ ($B = 2000$ bootstrap), consistent with the square-root prediction $\eta = 1/2$. (c) D_{dyn} versus g : the linear law $c_1 K_n$ ($c_1 = 1.5\pi$, red dashed) tracks the exact distance only at weak coupling; the distance peaks at $D \approx 0.97$ near $g \approx 2.3J \approx 0.85 g_{EP}$ and decreases beyond. (d) No-click survival probability P_{surf} and postselection cost $M = 1/P_{surf}$.

Figure 5. Time-window dependence. D_{dyn} versus observation window (in units of the period) for weak and strong dissipation, with the fixed full-period window $T = 2\pi/\omega$ marked; the weak-coupling curve grows approximately linearly with the window.

Figure 6. Finite-size reliability map. $D_{dyn}(K_n)$ for chain sizes $N = 2-7$ (all-excited initial state, local loss) with the $D = 0.1$ threshold; the inset shows $K_D^*(N)$ from exact diagonalization ($N = 2-7$) and sparse corner-space calculations ($N = 8, 10, 12$), with $\nu = 1.000$, the analytic law $K_D^* = 0.1/(2\pi N)$ and an N^{-1} guide.

Figure 7. Single-parameter data collapse. D_{dyn} versus the rescaled variable $K_n N$ for $N = 2-7$: all-excited state collapses with RMS 0.3%, while the Néel state shows a 7–18% spread from a subleading initial-state correction with measured exponent $\beta \approx 0.85$ (local) and 0.77 (collective), amplitudes 6%/10% (residuals decay as $N^{-\beta}$).

Figure 8. Cross-platform test and conservation-integral theorem. (a) Analytic versus numerical c_1 for the conserving platforms, with ratio 1.0000 ± 0.0003 for the conserving platforms and the driven platform at its ensemble value $\mu = 0.67$ (no sector reduction applies); (b) the three reliability curves under the same protocol; the theorem predicts their coincidence.

Figure 9. Design criteria and switch rule. (a) Maximal allowed dissipation $K_{\max} = \varepsilon/c_1$ for a set of tolerances, with the $D = 0.1$ crossing; (b) postselection cost $M = e^{c_1 K_n}$ and the switch region where the full Liouvillian is required.

Tables

Table 1 is given in the Results. **Table S2** (design criteria) and the full EP-seeking guide are provided in the Supplementary Information.

Author contributions

H. Liu conceived the study, performed the analytic derivations and numerical simulations, verified the results, and wrote the manuscript. AI assistance (language editing and verification tooling) was used during revision and is disclosed here per the journal's policy; the scientific content is the author's own.

Acknowledgements

The author thanks the open-source numerical and symbolic computing libraries used in this work, whose independent cross-checks supported the analytic coefficient, the finite-size scaling law and the conservation-integral theorem. The author is also grateful to

colleagues for discussions on the use of non-Hermitian effective descriptions and on the design of exceptional-point experiments.

Data availability

All data, analysis and figure-generation scripts, random seeds, the pre-registration record (preregistration.md), the symbolic-verification scripts and the numerical verification logs are archived in the supplementary package accompanying this submission (Supplementary Package S1), where they are accessible to the editor and reviewers so that every reported result can be reproduced before publication; a citable DOI is assigned to the frozen archive upon acceptance, per journal practice. Reference metadata were checked against Crossref and DataCite records.

Competing interests

The author declares no competing interests.

Supplementary Information

Supplementary Note 1: Non-uniform perturbation theory near an exceptional point

Origin of the exponent: at $g = g_{EP} - \delta$ the gap closes as $\Delta\lambda \propto \sqrt{\delta}$ (square-root coalescence), while the coupling matrix element to the environment scales as δ^0 ; the sensitivity therefore scales as matrix element over gap $\propto \delta^{-1/2}$, giving $\eta = 1/2$ in $|\partial\lambda/\partial g| \propto |g - g_{EP}|^{-\eta}$, in agreement with the measured $\eta = 0.481 \pm 0.004$ (Fig. 4b).

This note records the form the first-order reliability theory takes on a platform that hosts an exceptional point, and why the pure-loss driven two-level system cannot be used to test it. The prediction is verified directly in the main text with the gain-loss platform C (Fig. 4); the analytic derivation is collected here.

Near a second-order exceptional point the two eigenvalues of the effective Hamiltonian coalesce as

$$\lambda_{\pm} = \lambda_0 \pm \sqrt{\delta},$$

where δ is a local parameter measuring the distance in parameter space to the coalescence, and the corresponding eigenvectors become parallel as $\delta \rightarrow 0$. The ordinary (non-degenerate) perturbation expansion expresses first-order corrections as matrix elements divided by the eigenvalue gap, and here the gap is proportional to $\sqrt{\delta}$. The expansion therefore contains inverse powers of $\sqrt{\delta}$ and is non-uniform: it cannot be used uniformly as $\delta \rightarrow 0$. The effective first-order coefficient is correspondingly expected to grow as

$$c_1(\delta) \sim c_1^{(0)} + A |\delta|^{-\alpha},$$

in which the exponent is set by the order of the exceptional point. The square-root structure of a generic second-order EP gives $\alpha = \frac{1}{2}$; higher-order exceptional points carry larger exponents. In this regime the trace distance is no longer analytic in the coupling, and the reliability boundary must be read from the non-uniform expression rather than from the linear law $D_{dyn} = c_1 K_n$.

The same caution applies to Liouvillian exceptional points, where the coalescence occurs in the superoperator spectrum and can be present even when H_{eff} has no Hamiltonian EP. Verifying which object is present—Hamiltonian EP, Liouvillian EP, or neither—is therefore a prerequisite to any divergence claim, and it is the first point of the EP-seeking guide in the main text. Because the single-loss driven TLS has no EP (the minimum eigenvalue gap remains bounded below over the full scan), the divergence cannot be studied on that platform. The dedicated gain-loss model in which the eigenvalues do coalesce is platform C, and the predicted square-root splitting, the $\eta = \frac{1}{2}$ divergence of the eigenvalue sensitivity, the breakdown of the linear law and the divergent postselection cost are all observed directly in Fig. 4.

Supplementary Files

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- [relmapverificationpackageS1.zip](#)