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A circle and a confocal ellipse have at most ten distance-stationary points

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Research Article

Keywords: Keplerian distance, MOID, critical points, trigonometric polynomials, computer-assisted proof, interval arithmetic

Posted Date: November 9th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11496437/v1

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Additional Declarations: No competing interests reported.

Abstract

Let a circle and an ellipse in $\mathbb{R}^3$ have a common focus at the centre of the circle. We prove that the squared distance between a point of the circle and a point of the ellipse has at most ten stationary points, and that ten stationary points do occur. This answers the second question of Problem 8 in the list of Albouy, Cabral and Santos (2012). The question, posed by G. F. Gronchi, asks whether the bound 12 for two confocal ellipses can be lowered to 10 when one of them is a circle. Counting the stationary points amounts to counting the real zeros of a trigonometric polynomial of degree six whose coefficients depend on four parameters, and our proof of the bound is computer-assisted. We subdivide the compactified parameter space into boxes and certify each box in one of two ways: by certified signs of derivatives on arcs of the circle, which bound the number of real zeros directly, or by a winding-number argument that counts the zeros of an associated polynomial inside a contour. The degenerate limits, among them the parabolic limit in which the polynomial acquires twelve real zeros, are treated with exact weighted blow-ups. All floating-point errors are bounded explicitly, and every certificate is recorded, replayed and checked independently.

Mathematics Subject Classification (2020): 70F15 · 26C10 · 42A05 · 65G30

1 Introduction

The squared distance $d^2(u, v) = |X(u) - C(v)|^2$ between points of two Keplerian orbits is the function behind the minimum orbit intersection distance (MOID), the standard first filter for the impact hazard of asteroids and comets. Algorithms for the MOID compute *all* stationary points of d^2 on the torus (Gronchi 2002, 2005; Baluev and Mikryukov 2019), so it matters to know how many there can be. Kholshevnikov and Vassiliev (1999) reduced the question to the real zeros of a trigonometric polynomial. Gronchi (2002) obtained the bound 16 for two ellipses, with examples reaching 12; in the circle–ellipse case he obtained 12, with examples reaching 10. Gronchi, Baù and Grassi (2023) proved sharper bounds in the planar case and conjectured 10 for two coplanar ellipses.

In their list of problems on the *n*-body problem, Albouy, Cabral and Santos (2012) included as Problem 8 a question of Gronchi on the stationary points of d^2 for two distinct confocal ellipses in $\mathbb{R}^3$. It has two parts. If both eccentricities are positive, is 12 the maximum number? And if one of the eccentricities is zero, “can this upper bound be reduced to 10?” The configurations with infinitely many stationary points, namely coinciding conics and concentric coplanar circles, are excluded. In this paper we answer the second question.

Theorem 1

Let a circle of radius $R > 0$ and an ellipse with eccentricity $e \in (0, 1)$ lie in $\mathbb{R}^3$ with a common focus at the centre of the circle. Then d^2 has at most ten stationary points. The bound is attained, for instance by $e = 0.673$, $\sin I = 0.992$, $\omega = 0.338$, $R/a = 0.7772$.

Our proof is computer-assisted, and we have tried to make it easy to check. Section 2 reduces the count to the real zeros of a trigonometric polynomial, and Section 3 describes the two kinds of certificate that bound this number on a box of parameters. Section 4 describes the atlas of charts that covers the parameter space, Section 5 the treatment of the degenerate corners, and Section 6 the control of rounding errors; Section 7 reports on the computation and its verification. The complete technical documentation, that is, the reduction, the full atlas, the degenerate corners, the floating-point analysis and the results for each run, is given in the Supplementary Information (Online Resource 1). The code, the recorded certificates and the verification outputs are public; see the Data availability statement.

2 Reduction to a trigonometric polynomial

Coordinates. We scale the ellipse to semi-major axis $a = 1$ and put the circle in the plane $z = 0$, so that $C(v) = s(\cos v, \sin v, 0)$ with $s = R/a$. The ellipse is $X(u) = (\cos u - e)P + \beta \sin u\, Q$, with $\beta = \sqrt{(1 - e^2)}$ and

$$P = (\cos \omega, \sin \omega \cos I, \sin \omega \sin I), \qquad Q = (-\sin \omega, \cos \omega \cos I, \cos \omega \sin I).$$

The symmetries $\omega \mapsto -\omega$, $\omega \mapsto \omega + \pi$ and $I \mapsto \pi - I$ reduce the parameters to $e \in (0, 1)$, $I = \sin I \in [0, 1]$, $\omega \in [0, \pi/2]$ and $s \in (0, \infty)$. We put $p = I \sin \omega$ and $q = I \cos \omega$; these are the components P_z and Q_z of P and Q along the axis of the circle.

Branch functions. For fixed u, $\partial_v d^2 = 0$ exactly at the nearest and at the farthest point of the circle, provided the horizontal distance $\rho(u)$ of $X(u)$ from the axis is positive. The stationary points are therefore the critical points of $g_{\mp}(u) = r^2 + s^2 \mp 2s\rho$, where $r = 1 - e \cos u$. Squaring the conditions, we obtain

$$\mathrm{Pol}(u) = A^2\rho^2 - s^2((\rho^2)')^2 = h_- h_+, \qquad h_{\mp} = \rho\, g_{\mp}'(u), \qquad A = (r^2)',$$

which is a real trigonometric polynomial of degree at most six. Its coefficients are polynomials in (e, β, p, q, s^2).

Lemma 2.1

(counting). $\mathrm{Pol} \not\equiv 0$ *for* $e > 0$. *The number of stationary points of* d^2 *is at most the number of real zeros of* Pol *on the circle, counted with multiplicity. This holds also at points where* $A = (\rho^2)' = 0$ *or* $\rho = 0$: *each such point carries at most two stationary points and a zero of multiplicity at least two.*

Charts. Every chart used below certifies a function $c \cdot (\mathrm{Pol} \circ \varphi)$, where $c > 0$ is smooth and φ is a diffeomorphism of the circle or of a sub-arc, so that the real zeros and their multiplicities are those of Pol. Besides the eccentric anomaly we use the compactification $t = s^2/(1 + s^2)$, with $f_t = (1 - t)\Phi - t\Psi$, $\Phi = A^2\rho^2$, $\Psi = ((\rho^2)')^2$. We also use the true anomaly, with pericentre distance 1; in this chart $P_f = F_- F_+$, where $F_{\mp} = \mu k \sin f\, \sigma \mp DB$ is a positive multiple of $g_{\mp}'(f)$ and $\mu = q/R$. Finally we use the tangent-half-angle form

$Q(\tau) = P_f\,(1+\tau^2)^6$, with $\tau = \tan((f-\pi)/2)$, and division by the positive normalisations of the blow-up charts.

3 Certificates

Lemma 3.1

(Rolle runs). *If $F^{(j)}$ has no zero on a closed arc, then F has at most j zeros there, counted with multiplicity.*

For a parameter box we cover the circle by overlapping arcs. On each arc the program certifies a strict sign of $F^{(j)}$ for some $j \le 6$, uniformly over the box. A run of consecutive arcs on which $F^{(j)}$ has the same certified sign contains at most j zeros, and a dynamic program minimises the total over all partitions into runs.

Lemma 3.2

(parity). *The number of real zeros of a real trigonometric polynomial $F \not\equiv 0$, counted with multiplicity, is even.*

Proof

The zeros of odd multiplicity are exactly the sign changes of F, and a continuous periodic function changes sign an even number of times. □

We therefore accept a box when the certified total over the whole circle is at most 11: the true count is even, hence at most 10. A certified Rolle run also shows that $F^{(j)} \not\equiv 0$ on its arc, so that $F \not\equiv 0$ and Lemma 3.2 applies.

Lemma 3.3

(winding). *Write $F(u) = \sum_{|k|\le 6} c_k e^{iku}$ with $c_{-k} = \bar{c}_k$, and $\tilde{Q}(z) = \sum_k c_k z^{k+6}$. Then $\tilde{Q}$ is self-inversive, that is, $z^{12}\,\overline{\tilde{Q}(1/\bar{z})} = \tilde{Q}(z)$. If $\tilde{Q}$ has no zero on $|z| = \rho < 1$ and has $W \ge 1$ zeros in $|z| < \rho$, then it has W zeros in $|z| > 1/\rho$ (counting zeros at ∞ for a degree drop), and hence F has at most $12 - 2W \le 10$ real zeros.*

On $|z| = e^L$ we have $\tilde{Q} = e^{6L} e^{6i\theta} G(\theta)$, where

$G(\theta) = \sum_k \cosh(kL)\,(a_k \cos k\theta + b_k \sin k\theta) + i \sum_k \sinh(kL)\,(a_k \sin k\theta - b_k \cos k\theta)$.

The winding certificate is applied in three steps. First, on every arc of an overlapping cover, the program certifies a half-plane label for G, uniformly over the box, namely one of $\operatorname{Re} G > 0$, $\operatorname{Re} G < 0$, $\operatorname{Im} G > 0$, $\operatorname{Im} G < 0$. Second, it evaluates the winding number at the box centre, measuring both endpoint arguments of each arc in the half-plane branch of that arc. Finally, $W = 6 + \operatorname{wind}(G)$.

The two certificates complement each other. Rolle runs need a box small enough to resolve every real zero, which is expensive where real zeros cluster: at pitchfork bifurcations on the symmetric faces $\omega \in \{0, \pi/2\}$, and at six-fold points where the circle becomes large. The winding certificate never looks at the real zeros and is not affected by these clusters.

Lemma 3.4

(tau-winding). *Let* $Q(\tau) = P_f(1 + \tau^2)^6$ *with* $\tau = \tan((f - \pi)/2)$, *a real polynomial of degree at most* 12. *The real zeros of* P_f, *counted with multiplicity, are the real zeros of* Q *together with a zero of order* $12 - \deg Q$ *at* $f = 0$. *So the number of real zeros of* P_f *is exactly* 12 *minus the number of non-real zeros of* Q. *Suppose that, for every parameter of a box,* Q *has no zero on a circle contained in the open upper half-plane, and that* Q *has a zero inside this circle at the box centre. Then* P_f *has at most* 10 *real zeros on the whole box.*

Proof

The number of zeros inside the circle is constant on the connected box, since a zero can only leave by crossing the circle and a degree drop only moves zeros to infinity. So Q has a non-real zero, its conjugate is another one, and P_f has at most $12 - 2 = 10$ real zeros. □

The same holds with Q replaced by its reverse $\tau_p^{12} Q(-1/\tau_p)$, $\tau_p = \tan(f/2)$, or by the chart polynomials $Q(vy)/v^{10}$ (the C3 chart in which Q_z dominates) and $Q(sy)/s^6$ (the face charts), since $v, s > 0$ at every physical point: a non-real zero of any of these gives a non-real zero of Q.

Lemma 3.5

(disc-winding outer count). *With* Q *as in Lemma 3.4, suppose that* Q *has no zero on* $|\tau| = T$ *for any parameter of a box, and that* Q *has* W *zeros in* $|\tau| < T$ *at the box centre. Then at every point of the box,* P_f *has at most* $12 - W$ *real zeros, counted with multiplicity, with* $|\tau| \geq T$ *or* $f = 0$.

Proof

As in Lemma 3.4, Q has W zeros in the disc at every point of the box. The real zeros in question are the real zeros of Q with $|\tau| > T$, together with the zero of order $12 - \deg Q$ at $f = 0$. At most $\deg Q - W$ zeros of Q lie outside the closed disc. □

Both contour certificates use the same enclosure. On each arc of the circle, the value of the polynomial at the box centre and at the centre of the arc exceeds a certified bound on its variation along the arc and over the box. The winding number is then the sum of the principal argument steps at the box centre.

4 The atlas

The parameter space is covered by the following charts, whose boundaries overlap. The eccentric-anomaly chart, with $\eta = \arcsin e$, covers $e \in [0.199, 0.8362]$ and all l, ω, t. Three charts cover $e \le 0.2$: C_1, polar in (e, l^2); C_1', with $t = e^2\tau/(1-\tau)$, for $l \ge 0.4$; and C_1'', with $e = \varepsilon'' l^2$ and $t = \varepsilon''^2\tau/(1-\tau)$. Inside them lie the blow-up cones of the corners D1 and D2 (Section 5).

For $e \ge 0.835$ we work in the true-anomaly variables, with $t = \kappa^2/(1+\kappa^2)$, $\kappa = R/q$ and $\mu = q/R = \sqrt{((1-t)/t)}$. For $t \le 0.5$ the true-anomaly chart covers the whole circle. For $t \ge 0.5$ we use the engine KQ of Section 5, which counts the zeros on the aphelion arc $|f - \pi| \le 0.9$ through $Q(\tau)$, and the rest with the true-anomaly chart. Two cells need more care. In the cell $e \ge 0.99$, $\mu \le 0.0505$, $l \ge 0.95$, $\omega \ge 9\pi/20$ we use the C3 core charts near aphelion, a sign certificate in an intermediate range, and the true-anomaly chart away from aphelion. In the face neighbourhood $1 - e \le 4 \cdot 10^{-4}$, $\mu \le 4 \cdot 10^{-4}$, outside that cell, we use the face charts of Section 5 near aphelion, a sign certificate in a gap range, and the engine KQ restricted to $|\tau| \ge 0.06$. In the corner $K_{l\omega} = \{l \ge 0.88, \omega \ge 1.29\}$ of the face neighbourhood this construction is used only for blow-up radius $s \le 0.010005$, with the engine KQ restricted to $|\tau| \ge 0.030015$; there the shell $0.010005 < s \le 0.02$ is covered by whole-circle KQ counts.

Every whole-circle count is certified to be at most 11, hence at most 10 by Lemma 3.2. In the C3 cell the near, intermediate and away counts are at most 9, 0 and 2, and on the face neighbourhood they are at most 5, 0 and 6; in both cases the total is at most 11.

The coverage of each seam is elementary. For example, for $e \le 0.2$ and $l < 0.4$: either $l^2 \le 2e$, and then $\sqrt{(e^2 + l^4)} < 0.26$, which is the C_1 range; or $l^2 > 2e$, so $\varepsilon'' < 0.5$, which is the C_1'' range. The complete table of seams is in Section S2 of Online Resource 1.

5 Degenerate corners

We summarise here the treatment of the degenerate corners; the derivations are given in full in Sections S3 and S4 of Online Resource 1.

D1: nearly coincident orbits ($e, l \to 0, s \to 1$). In the coordinates $(e, \theta = l^2/e, \tau = t - \tfrac{1}{2})$,

$$f/e^2 = 2 \sin u \,[\theta \sin 2(u+\omega) - 4\tau \sin u - e \sin 2u] + O(2).$$

This is homogeneous of degree one, and three cone charts resolve it, with exact divisibility. The leading term vanishes identically only in the direction $l = e$, $\omega = 0$, $s = 1$. A second homogeneous blow-up there has leading term $2 \sin u \,[2a \sin 2u + 2p \cos 2u - 4\tau \sin u + e\, r_1(u)]$, where r_1 contains $\sin 3u$; it never vanishes identically.

D2: a nearly circular orbit perpendicular to a tiny circle ($e \to 0, s \sim e, l \to 1, \omega \to 0$). In the chart C_1' the function vanishes identically at the limit point. Its linear part is

$$-8\tau' s^2c^2 + 2\kappa(2s^4 + 4s^2c^2) - 4p\, sc^3 - 4e\, s^2c^3, \qquad s = \sin u,\ c = \cos u,\ \kappa = 1 - q.$$

Its four functions are independent, and one homogeneous blow-up resolves the corner. In the cone where τ' dominates, the leading term has a four-fold zero at $u = \pi$ along $p = 0$, $2\kappa + e = 2\tau'$, so that cone chart cannot certify boxes there. The boxes it leaves open are covered in one of two ways: either their eccentricity is larger than that of every leaf the corner charts must cover, or they lie in the cone where e dominates, extended to $\tau'/e \le 1.28$, which certifies them with the winding certificate.

C3: the limit e → 1 with the aphelion direction on the axis of the circle. In the tangent-half-angle form we substitute $\tau = vy$, $Q_z = va$, $1 - P_z = v^2K$, $1 - e = v^2E$ and $\mu = v^3m$. Then

$$Q = v^{10}[P_0(2y) + O(v^2)], \qquad P_0(x) = 16m^2x^2(\lambda + (x - a)^2) - (E + x^2/2)^2B(x)^2,$$

with $B(x) = 2E(x - a) - 4Kx + 3ax^2 - x^3$ and $\lambda = 2K - a^2$. The roots near aphelion are counted in the four parameter cones, a sign certificate covers the intermediate range, and an arc count handles the region away from aphelion. At $E = m = 0$, P_0 has a six-fold zero at $y = 0$; a second blow-up of (y, E, m) with weights (1, 2, 2) shows that at most four roots lie near $y = 0$.

The parabolic face ($e = 1$, $\mu = 0$, general direction). Here $P = -D^2B^2$, and for l above a threshold $l^*(\omega)$, which increases from 0.707 at $\omega = 0$ to 0.9985 at $\omega = 1.41$, it has twelve real zeros counted with multiplicity: a six-fold zero at aphelion and three double zeros of B^2. So no box touching the face can be certified with total at most 11 in the original variables, and we blow up. With weights (1, 2, 2) for (τ, ε, μ), $\varepsilon = 1 - e$, and radius s, the function $G = Q(sy)/s^6$ is a polynomial, with leading part

$$G_0 = 16c^2y^2[4m^2 - c^2(E + 2y^2)^2], \qquad c^2 = 1 - P_z^2, \quad E = \varepsilon/s^2, \quad m = \mu/s^2.$$

For $c > 0$ its real zeros are $y = 0$ (double) and at most one pair $y^2 = (2m/c - E)/2$, so at most four zeros lie near aphelion, against six from the rest of the circle. Two cone charts count the zeros on $|y| \le 3$ (at most 5); two charts certify that Q has no zero for $3s \le |\tau| \le 0.06$, outside the corner K described below; and the engine KQ counts $|\tau| \ge 0.06$ (at most 6), by Rolle runs or by Lemma 3.5 with $W = 6$, since the six zeros near aphelion stay inside $|\tau| < 0.06$. Outside the C3 cell, $c > 0.156$.

The corner K needs separate care. Near $l = 1$ with ω just below $9\pi/20$, c is close to its minimum 0.156 outside the C3 cell. There the next-order terms push the outer member of the near pair beyond $|y| = 3$, so the gap charts have genuine zeros, from $\sigma = |\tau|/3 \approx 0.015$ on. Let $K = \{P_z \ge 0.875, 0.125 \le Q_z \le 0.25\}$; it contains every face-chart box with a failed or open leaf, and its image lies in $K_{l\omega} = \{l \ge 0.88, \omega \ge 1.29\}$. Inside $K_{l\omega}$ we use the construction above only for $s \le 0.010005$, with the outer count taken from $|\tau| \ge 0.030015$; for these s the gap is empty, the near zeros stay within $|y| \le 2.79$, and the outer count is 6. The shell $0.010005 < s \le 0.02$ is covered by whole-circle KQ counts (at most 11).

The near-parabolic region (engine KQ). As $e \to 1$ the factor $D = 1 + e\cos f$ becomes small near aphelion. In the Fourier basis the true-anomaly polynomial then takes values of order ε^4 there, while its coefficients are of order one, and no enclosure of moderate width resolves its zeros near aphelion. The engine KQ

therefore writes $Q(\tau)$, for $|\tau| \le \tan(0.45)$, as a polynomial in τ in the monomial basis centred at aphelion; its coefficients are polynomials in $(\varepsilon, P_z, Q_z, \mu)$. In this form the pair of zeros near aphelion, at $\tau = O(\varepsilon^2)$, is an ordinary, well-conditioned cluster. The rest of the circle is counted in the true-anomaly chart, and the two ranges overlap. Since Q is polynomial in the parameters, the points where the orbit crosses the axis of the circle ($\sigma = 0$) are not singular, and the same engine also covers the band where the orbital plane nearly contains the axis. On each box KQ first tries the winding certificate of the true-anomaly polynomial, and counts as above only when that fails. The winding certificate does not resolve the real zeros, so it is effective near $e \approx 0.95$, where up to ten real zeros cluster while the remaining complex pair stays off the unit circle.

6 Floating-point rigour

The certificates run in IEEE binary64 arithmetic, and every source of error is bounded explicitly. The coefficient polynomials have at most 265 terms each, of degree at most 40, and are evaluated with error at most $\gamma_{305} P_{\mathrm{abs}} < 10^{-13} P_{\mathrm{abs}}$. The variation over a parameter box is bounded by the mean-value form, using absolute-value polynomials for the variation of the gradients, and the radii are padded for the floating-point centre. In the near-parabolic engine and in the hardest C3 chart the program also bounds the variation by the exact gradient and Hessian at the box centre plus a third-order Taylor remainder, and uses the smaller of the two bounds. Near $e \approx 0.95$ the absolute-value term of the mean-value form exceeds the true second-order term by factors of 25 to 500, so this saves several levels of subdivision.

The trigonometric arguments are smaller than 63 in modulus. Their rounding, and the error of the library sine and cosine, which we assume to be within two ulps, are covered by an explicit padding of $2 \cdot 10^{-14} k^j$ per term. Arcs overlap by 10^{-12}, and the final sums carry a relative padding of 10^{-15}. In the engine KQ the box is converted to $(\varepsilon, P_z, Q_z, \mu)$, with every bound rounded outward.

For the contour certificates, every point of an arc of a contour $|z - c| = r$ lies within $D_j = r h_j (1 + 10^{-12}) + 10^{-12}(|c| + r)$ of the floating-point centre of the arc. The deviation along the arc is bounded by the exact Taylor expansion at the centre of the arc, and the parameter variation by the second-order bound evaluated on that disc. An arc is accepted only if the value at its centre exceeds the resulting radius, which carries a relative slack of 10^{-12}. Each argument step between consecutive arcs is then below $\pi - 2 \cdot 10^{-6}$ in modulus, far beyond the rounding of the quotient and of the argument function, so the computed winding number is exact. The full analysis is in Section S5 of Online Resource 1.

7 The computation and its verification

Recording. Every run records, for each initial box, the complete split tree, with one character per box: a split, a leaf with its bound and method, a pruned box (unphysical, or covered by another chart), or a failed leaf. Winding leaves also store their test radius and tau-winding leaves their contour; the disc of a disc-winding leaf is fixed by its run. A replay tool re-derives every leaf exactly. Failed leaves and open boxes

are covered by continuation runs or by other charts (the D2 corner charts, and in the corner K the runs of Section 5), or they lie outside the range that a face chart needs. A continuation run may split first the coordinate that matters most (the size of the circle in the large-circle runs, one ratio in a C3 chart, the blow-up scale near the parabolic face); the split rule affects only efficiency.

Size. The final certificate consists of 51 run families with 16,048,175 certified leaves and 818,373 pruned leaves. Every family replays exactly, and every open or failed box of every family is covered. The statistics for each family are in Section S6 of Online Resource 1.

Independent checks. We checked the certificate in three further ways. First, sampled leaves were compared with the original polynomial Pol at 60 and 100 digits, at the centre and at random points of each sampled box, independently of every generated chart polynomial and of the certificate engines; no violation was found, and the largest count observed is 10. Second, samples of the Rolle-run and winding leaves of every run file, every tau-winding leaf (175,932 leaves) and every disc-winding leaf (4,451 leaves) were re-certified with the final code. Third, every derivation, generated formula, engine, coverage claim and run went through fifteen rounds of independent auditing. Among other things, the audits found a coverage seam, an unrecorded-leaf problem, rounding gaps in the trigonometric arguments, a branch error in the winding evaluation, and the twelve-root obstruction at the parabolic face, which led to the face charts; all of these were repaired. The final pass replays every family independently, re-certifies every tau- and disc-winding leaf and samples of all other leaves, and finds no violation in 100-digit spot checks.

Remark

The certificate engines are verified by audit and by independent re-checking, not formally. A formalisation in a proof assistant would have to re-check all 16 million leaves with a verified checker.

Declarations

Competing interests.

The authors declare no competing interests.

Ethics approval and consent.

Not applicable.

Use of artificial intelligence.

AI tools were used for development and validation.

Funding.

No funding was received for this work.

Author Contribution

A.M. contributed to the Mathematics of the paper, along with testing and data gathering. A.S. wrote the main manuscript text. All authors reviewed both the mathematics and the manuscript.

Data Availability

All code (checked for reproducability), is publicly availible at: https://github.com/anshM123/Albouy-Cabral-Santos-list-Problem-8We have used AI to make the repository more human-readable, along with making the code easier to understand. We have also spent a substantial amount of time checking each line of code, comment, etc. generated.

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