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Abstract

A planar piecewise-linear electronic neuron with six switching regions produces large spike-like excursions without an imposed reset but has no internal memory variable. We add one exponentially filtered voltage state w (time constant τ , feedback gain g) and leave the switching terms unchanged. Starting from rest, the current required to reach the voltage threshold has a closed form, $I_{\text{first}} = v_{\text{th}}/s_{\text{max}}$, where s_{max} is the peak of the linear step response. It lies well below the current $I_{\text{eq}} = (A + g)v_{\text{th}}$ at which the resting equilibrium becomes inadmissible, and the first-spike latency approaches a finite limit with a square-root cusp. The onset of persistent spiking is a saddle-node bifurcation of limit cycles. On the upward section $v = v_{\text{th}}$ the dynamics reduce to a one-dimensional return map $w_{n+1} = \Phi(w_n)$ whose stable and unstable fixed points merge at $I_{\text{fold}} = 0.0348814 \text{ V/ms}$ for $g = 0.05 \text{ ms}^{-1}$ and $\tau = 50 \text{ ms}$. Rest and spiking coexist for $I_{\text{fold}} < I < I_{\text{eq}}$, which spans 71% to 100% of I_{eq} ; the rate at onset is finite (about 12.6 Hz) and the cycle branch is hysteretic. Below the fold the circuit emits a finite burst of $N \simeq \pi/\sqrt{|a|b(I_{\text{fold}} - I)}$ spikes, with a and b measured from the return map; this agrees with simulation from $N = 2$ to $N = 837$. Over a 5×5 grid in (g, τ) the ratio $I_{\text{fold}}/I_{\text{eq}}$ lies between 0.595 and 0.957. Results beyond the first-switch analysis are numerical and were checked with an independent matrix-exponential solver and a matched instantaneous-feedback control. The saddle-node scaling itself is generic; the contribution is its quantitative realization in this discontinuous circuit.

Keywords: piecewise-smooth dynamics; electronic neuron; saddle-node of cycles; Poincaré return map; first-spike latency; hysteresis.

1 Introduction

A spike-shaped voltage waveform does not by itself identify the vector-field geometry that produces it. Sonnino *et al.* compared a three-variable electronic neuron, the “neurosimulator” [2], with a planar piecewise-linear reformulation. They found that the first-spike latency is logarithmic in both models although the onset mechanisms differ: a saddle-node limit point in the three-dimensional parent, and a stable node reaching a switching boundary in the planar model [1]. They also note that the reduction removes a memory degree of freedom, and they leave delayed or kernel-based feedback for future work [1].

We study the simplest such extension, a single exponentially filtered voltage w that feeds back negatively onto $\dot{v}$. Three questions guide the analysis. (Q1) How does the filter change the current needed to reach the first switching surface, and the latency of that first crossing? (Q2) Can the full hybrid system hold a stable rest state and a repetitive spiking orbit at the same current, and what ends the spiking branch as the current is lowered? (Q3) Does that termination follow a quantitative law?

Rest-spike bistability, resonance and saddle-node bifurcations of cycles are well-known phenomena. Rest-spike coexistence has been analyzed in slow-fast conductance-based models [5] and

in piecewise-linear neuron models [3, 6]; nonsmooth methods for spiking models are reviewed in [4], and for piecewise-smooth systems in general in [12]. A saddle-node bifurcation of limit cycles has been proved to exist in planar continuous piecewise-linear systems with three zones, with a piecewise-linear FitzHugh–Nagumo neuron as one application [7], and inverse-square-root laws for burst duration near saddle-node-type bifurcations of periodic orbits have been reported for a conductance-based neuron model [8]. Subthreshold resonance and the resonate-and-fire neuron [9, 10] account for overshoot of a linear step response, and adaptation variables of the same linear form as w appear in the adaptive exponential integrate-and-fire model [11]. We therefore do not claim that bistability, overshoot or a saddle-node scaling law is new. What this paper provides is an exact first-switch threshold and latency law for this model, the identification of the spiking onset as a fold of cycles in the full six-region discontinuous system with a filtered feedback state, a measured normal form that gives the burst length in terms of return-map iterations, and the dependence of the fold on (g, τ) .

2 Model

With voltage-like variable v and recovery variable k , the planar model of [1] is

$$\dot{v} = I - Av + PH(v - v_{\text{th}}) - Gk - E(k - v_{\text{th}})H(k - v_{\text{th}}), \quad (1)$$

$$\dot{k} = Fv_s H(v - v_{\text{ref}}) - Fk + F(v_{\text{th}} - k)H(k - v_{\text{th}}), \quad (2)$$

with $P = F(v_s - V_0)$ and parameters $A = 0.02$, $E = 20$, $F = 0.4$, $G = 0.2 \text{ ms}^{-1}$, $v_{\text{th}} = 0.7$, $v_{\text{ref}} = 3.3$, $v_s = 7.5$, $V_0 = 1 \text{ V}$. The drive I is in V/ms. The surfaces $v = v_{\text{th}}$, $v = v_{\text{ref}}$, $k = v_{\text{th}}$ divide the plane into six regions with affine dynamics and a firing cycle runs $R_1 \rightarrow \dots \rightarrow R_6 \rightarrow R_1$. There is no reset. We add

$$\dot{v} \rightarrow \dot{v} - gw, \quad \tau \dot{w} = v - w, \quad g \geq 0, \tau > 0, \quad (3)$$

so $w(t) = w_0 e^{-t/\tau} + \tau^{-1} \int_0^t e^{-(t-s)/\tau} v(s) ds$ is an exponentially weighted voltage history. The system is three-dimensional, $x = (v, k, w)$, and is *not* the three-dimensional parent model of [1]. For $g = 0$ it reduces to (1)–(2). The variables are circuit quantities, not ionic conductances. Main example: $g = 0.05 \text{ ms}^{-1}$, $\tau = 50 \text{ ms}$, which satisfies $\tau = 1/A$.

3 Rest state and first switch

3.1 Resting equilibrium

For $v < v_{\text{th}}$, $k < v_{\text{th}}$ the switching terms vanish. The equilibrium is $(v^*, k^*, w^*) = (\frac{I}{A+g}, 0, \frac{I}{A+g})$, admissible for $I < I_{\text{eq}} = (A + g)v_{\text{th}}$. The linearization has eigenvalue $-F$ and the 2×2 block with characteristic polynomial $\lambda^2 + (A + \tau^{-1})\lambda + (A + g)/\tau$; trace < 0 and determinant > 0 give asymptotic stability wherever the point is admissible. The block has complex eigenvalues $-\alpha \pm i\omega$ if and only if $(A - \tau^{-1})^2 < 4g/\tau$, with $\alpha = (A + \tau^{-1})/2$, $\omega = \sqrt{(A + g)/\tau - \alpha^2}$. At $\tau = 1/A$ this holds for every $g > 0$ ($\omega = \sqrt{Ag}$). At the main example, $I_{\text{eq}} = 0.049 \text{ V/ms}$ and the eigenvalues are -0.400 and $-0.020 \pm 0.031623 i \text{ ms}^{-1}$.

3.2 Exact threshold for the first switch

Start from $x(0) = 0$ with a current step I . While $v < v_{\text{th}}$ we have $k \equiv 0$ and (v, w) obeys a linear system. Write $\rho = \alpha^2 + \omega^2 = (A + g)/\tau$ and $c = (\alpha - A)/\omega$.

Proposition 1. *Let $(A - \tau^{-1})^2 < 4g/\tau$. From the zero state, $\dot{v}(t) = I e^{-\alpha t} [\cos \omega t + c \sin \omega t]$ and $v(t) = I s(t)$ with*

$$s(t) = \frac{1}{\rho} \left[\alpha + e^{-\alpha t} (\omega \sin \omega t - \alpha \cos \omega t) + c (\omega - e^{-\alpha t} (\alpha \sin \omega t + \omega \cos \omega t)) \right]. \quad (4)$$

The first maximum is at $t_p = \omega^{-1} \operatorname{atan}2(\omega, A - \alpha) \in (0, \pi/\omega)$ and is the global maximum, $s_{\max} = s(t_p)$. A strictly transverse upward crossing of $v = v_{\text{th}}$ occurs if and only if

$$I > I_{\text{first}} := v_{\text{th}}/s_{\max}. \quad (5)$$

Proof (Appendix A). For $\tau = 1/A$: $\alpha = A$, $c = 0$, $t_p = \pi/(2\sqrt{Ag})$ and

$$I_{\text{first}} = \frac{(A + g)v_{\text{th}}}{1 + \sqrt{g/A} \exp[-\frac{\pi}{2}\sqrt{A/g}]}. \quad (6)$$

At the main example $t_p = 49.67294$ ms, $s_{\max}(A + g) = 1.58548573$, $I_{\text{first}} = 0.03090536$ V/ms, i.e. 36.9% below I_{eq} . Table 1 compares (5) with a numerical bisection (exact-branch solver) for six (g, τ) pairs, including $\tau \neq 1/A$. For $I_{\text{first}} < I < I_{\text{eq}}$ the rest state is stable and the zero state nevertheless reaches the switching surface. Thus I_{first} is an activation threshold that depends on the initial state, not a stability boundary, and it does not by itself imply repetitive firing.

g (ms ⁻¹)	τ (ms)	I_{first} from Eq. (5) (V/ms)	numerical bisection (V/ms)
0.050	50	0.030905355	0.030905356
0.050	35	0.033908465	0.033908465
0.035	65	0.025863902	0.025863902
0.080	20	0.048241422	0.048241422
0.020	80	0.021385556	0.021385556
0.065	50	0.033917076	0.033917077

Table 1: First-switch threshold: closed form versus numerical bisection (tolerance $\sim 10^{-9}$).

Non-oscillatory block. For $(A - \tau^{-1})^2 > 4g/\tau$ the roots are distinct and real, $\lambda_{1,2} = -\alpha \pm \delta$ with $\delta = \sqrt{\alpha^2 - \rho}$, and $\dot{v}/I = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$ with $a_1 = (-A - \lambda_2)/(\lambda_1 - \lambda_2)$, $a_2 = 1 - a_1$. The derivative $\dot{v}$ has a positive zero if and only if $A > 1/\tau$, because $(A - \alpha)^2 - \delta^2 = g/\tau > 0$. In that case the zero is at $t_p = \ln(-a_2/a_1)/(2\delta)$, the maximum of $s(t) = a_1(e^{\lambda_1 t} - 1)/\lambda_1 + a_2(e^{\lambda_2 t} - 1)/\lambda_2$ is $s_{\max} = s(t_p)$, and Eq. (5) holds unchanged. If $A \leq 1/\tau$, s increases monotonically to $1/(A + g)$ and the first crossing occurs if and only if $I > I_{\text{eq}}$. The repeated-root boundary $(A - \tau^{-1})^2 = 4g/\tau$ requires the corresponding continuous limit of the formulas above; the expressions with $1/\delta$ must not be evaluated directly at $\delta = 0$. Numerical check against the exact flow: for $(g, \tau) = (0.0005, 80)$ and $(0.0005, 100)$ the formula gives $s_{\max} = 48.81673$ and 48.86655 , against 48.81673 and 48.86655 from the flow and $1/(A + g) = 48.7805$; for $(0.001, 30)$ and $(0.0003, 25)$ the response is monotone and $\sup s = 1/(A + g)$ to all printed digits.

Corollary 1 (latency cusp). *As $I \downarrow I_{\text{first}}$ the first-crossing time satisfies*

$$t_{\text{first}} = t_p - \sqrt{\frac{2s_{\max}(I - I_{\text{first}})}{I|u'_0(t_p)|}} + o(\sqrt{I - I_{\text{first}}}), \quad |u'_0(t_p)| = \omega(1 + c^2)e^{-\alpha t_p} \sin \omega t_p. \quad (7)$$

Equation (7) follows from a quadratic expansion of v about its maximum (Appendix A). The latency is bounded by t_p and does not diverge at threshold, in contrast to the planar model, where the first-crossing time $A^{-1} \ln[I/(I - Av_{\text{th}})]$ diverges logarithmically at $I = Av_{\text{th}}$ [1]. Figure 1 shows agreement with simulation: at a relative overdrive of 10^{-6} the numerical latency is 49.61077 ms and Eq. (7) gives 49.61074 ms.

3.3 Matched instantaneous-feedback control

Replacing $-gw$ by $-gv$ gives a two-variable system with the same equilibrium voltage $I/(A + g)$. Before any switch $k = 0$ and $v_{\text{inst}}(t) = \frac{I}{A+g}(1 - e^{-(A+g)t}) < \frac{I}{A+g}$. For $I = 0.040$ V/ms this bound is 0.5714 V, below v_{th} , so no switch occurs from the zero state, whereas the filtered system crosses

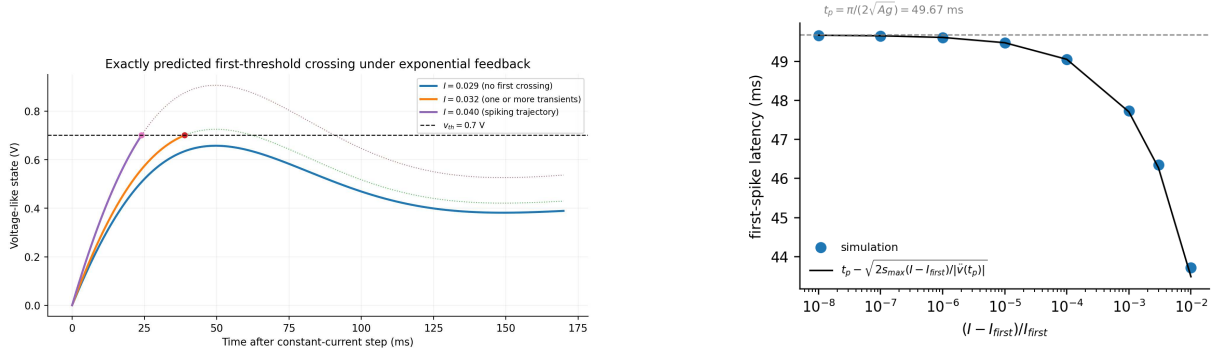


Figure 1: Left: linear step response up to the first crossing (solid); dotted continuations are not solutions of the switched system. Curves below I_{first} never reach v_{th} . Right: first-spike latency near I_{first} (main example) against the cusp law (7); the limit is $t_p = 49.67$ ms.

at 24.0218 ms. The lag is therefore essential: a first-order voltage equation cannot overshoot its equilibrium, while the filter introduces complex eigenvalues that give $s_{max} > 1/(A + g)$, the familiar resonance mechanism [9, 10]. The comparison is specific to this initial state and does not exclude cycles of the memoryless system from other initial conditions.

4 Numerical methods

Event-resolved integration. The hybrid system is integrated region by region. Two implementations are used and agree: (a) adaptive DOP853 (rtol 5×10^{-10} , atol 3×10^{-12}) with event functions $v - v_{th}$, $v - v_{ref}$, $k - v_{th}$; (b) an independent solver that propagates the exact matrix-exponential flow of each affine region and locates crossings by bracketing and Brent refinement. Both use the same equations, so their agreement guards against coding errors but does not validate the model itself.

Return map. Let $\Sigma = \{v = v_{th}\}$ with $\dot{v} > 0$ on the approach side. A trajectory leaving Σ at (k, w) spends ~ 50 – 80 ms in the sequence of regions before returning; since k relaxes at rate $F = 0.4$ ms $^{-1}$, the value at return is $k \leq 1.2 \times 10^{-8}$ V on the stable cycles just above the fold at all 25 grid points of Section 5.4. The map $(k, w) \mapsto (k', w')$ therefore contracts strongly along k (second multiplier $\approx 6.8 \times 10^{-8}$ at the main orbit) and is effectively one-dimensional, $w_{n+1} = \Phi(w_n; I)$ with $k = 0$. A crossing is transverse only if $g w < I - A v_{th} - G k$; Φ is defined where the trajectory returns to Σ transversally and has a finite domain $[0, w_{edge})$. Set $D(w; I) = \Phi(w; I) - w$. A cycle is a zero of D ; a saddle-node of cycles is $D = 0$ with $\partial_w D = 0$.

Fold detection. For given (g, τ, I) let $m(I) = \min_w D(w; I)$ over the domain, computed by a 40-point scan followed by bounded refinement. Since $D(0; I) > 0$, a cycle exists whenever $m < 0$ and none exists when $m > 0$. For $m < 0$ at least one fixed point lies inside the domain; just above the fold there are exactly two, while for larger currents the second root may lie outside the domain (Section 5.2). I_{fold} is the root of m , found by bisection on $[0.2I_{eq}, 1.05I_{eq}]$ under the assumption of a single sign change, and the minimizer is checked to be interior to the domain. At the main example this gives $I_{fold} = 0.0348814193$ V/ms, which differs from an independent DOP853-based computation by 10^{-10} V/ms.

5 Results

5.1 Coexistence of rest and spiking

At $I = 0.040$ V/ms ($g = 0.05$, $\tau = 50$) the zero state first crosses v_{th} at 24.0218 ms (exact formula and solver agree to 3×10^{-14} ms) and then approaches a periodic orbit of period 52.87476 ms

(18.9126 Hz) with late-period s.d. 4×10^{-8} ms, while an initial state at or near the equilibrium returns to rest. Perturbing v by +0.01, +0.05, +0.10 V from rest leaves the trajectory at rest, whereas +0.20 V (across v_{th}) enters the cycle (14 spikes in 800 ms); the basin of rest was probed only along this direction. The orbit crosses Σ at $(k, w) \approx (1.3 \times 10^{-9}, 0.26943)$ V with return-map multipliers 0.39375 and 6.8×10^{-8} , stable under finite-difference steps 5×10^{-6} – 8×10^{-5} V.

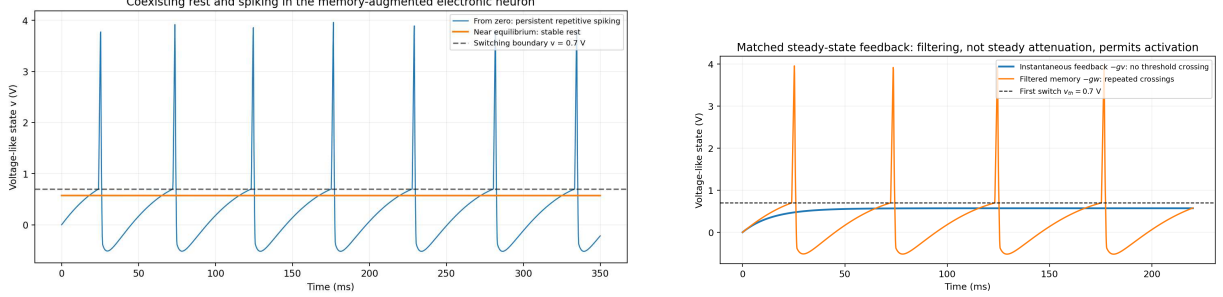


Figure 2: Left: coexisting rest and spiking at $I = 0.040$ V/ms. Right: matched instantaneous-feedback control stays below v_{th} ; the filtered system fires.

5.2 Onset of persistent spiking is a fold of cycles

Figure 3a shows $D(w; I)$ near the transition. For $I = 0.0348$ V/ms, $D > 0$ on the whole domain (no cycle). For currents just above the fold ($I - I_{fold} \leq 10^{-5}$ V/ms, Table 2) D has two zeros: D crosses zero downwards at the stable fixed point (multiplier < 1) and upwards at the unstable one (multiplier > 1). For larger currents such as $I = 0.0350$ V/ms ($I - I_{fold} = 1.2 \times 10^{-4}$ V/ms) only the stable zero lies inside the domain of Φ (Fig. 3a). The unstable zero leaves the domain at $I - I_{fold} \approx 9.7 \times 10^{-5}$ V/ms ($w_u = 0.4172$ V), where the return trajectory becomes tangent to $v = v_{th}$; this is consistent with a grazing event of the unstable cycle, which we did not analyze further. Approaching the fold from above, the two zeros merge at

$$I_{fold} = 0.0348814193 \text{ V/ms} = 0.7119 I_{eq}, \quad w_{fold} = 0.39353 \text{ V}. \quad (8)$$

Table 2 lists the roots and multipliers just above I_{fold} . The multipliers approach 1 as $\lambda_{s,u} \simeq 1 \mp 2\sqrt{|a|b(I - I_{fold})}$ with $a = \partial_I \min_w D = -87.0$ ms and $b = \frac{1}{2}\partial_w^2 D = 16.18 \text{ V}^{-1}$ (measured by finite differences at the fold). For $I - I_{fold} = 10^{-7}$ V/ms the prediction 1 ∓ 0.0237 matches the measured 0.9770 and 1.0244.

$I - I_{fold}$ (V/ms)	w_s (V)	w_u (V)	λ_s	λ_u	$1 \mp 2\sqrt{ a b(I - I_{fold})}$
10^{-8}	0.39330	0.39376	0.99258	1.00756	0.9925, 1.0075
10^{-7}	0.39280	0.39426	0.97696	1.02442	0.9763, 1.0237
10^{-6}	0.39122	0.39586	0.93141	1.08260	0.9250, 1.0750
10^{-5}	0.38627	0.40095	0.81689	1.33446	0.7627, 1.2373

Table 2: Stable and unstable fixed points of Φ above the fold. The last column is the leading-order normal-form prediction.

The one-dimensional reduction was tested directly. At each fixed point the full 2×2 Jacobian of the return map in (k, w) was computed by central differences (steps 2×10^{-9} V in k and 2×10^{-6} V in w) and compared with the multiplier $1 + \partial_w D$ (Table 3). The dominant eigenvalue agrees with the one-dimensional multiplier to at least five digits for $I \leq 0.045$ V/ms and to 3×10^{-5} in relative terms at 0.052 V/ms, and the second eigenvalue never exceeds 2.3×10^{-5} in modulus, so the strongly contracting k direction does not affect the fold. Along the stable branch, from just above the fold to 0.052 V/ms, which is beyond I_{eq} , the dominant multiplier stays positive and

below one: it falls from 0.93 near the fold to a minimum of about 0.39 near 0.040 V/ms and rises to 0.47 at 0.052 V/ms. We therefore found no period doubling or other loss of stability of the cycle at the sampled currents; the branch was not sampled more densely.

I (V/ms)	$1 + \partial_w D$	dominant eigenvalue of 2×2 Jacobian	second eigenvalue
stable cycle			
0.0348824	0.931403	0.931404	2.6×10^{-12}
0.0349000	0.769577	0.769577	4.6×10^{-12}
0.0360000	0.419151	0.419151	4.4×10^{-10}
0.0400000	0.393748	0.393748	6.8×10^{-8}
0.0450000	0.424014	0.424013	1.6×10^{-6}
0.0490000	0.450452	0.450447	8.4×10^{-6}
0.0520000	0.469353	0.469341	2.3×10^{-5}
unstable cycle ($I = I_{\text{fold}} + \Delta$)			
$\Delta = 10^{-7}$	1.024419	1.02442	2.1×10^{-12}
$\Delta = 10^{-6}$	1.082596	1.0826	1.9×10^{-12}
$\Delta = 10^{-5}$	1.334456	1.33446	1.3×10^{-12}

Table 3: Multipliers from the one-dimensional map and eigenvalues of the full two-dimensional return map.

Sweeping the current down along the stable cycle (each step started from the previous orbit) the cycle persists to $I_{\text{fold}} + 10^{-7}$ V/ms, at an event rate of 12.59 Hz (period 79.4 ms), and is lost at $I_{\text{fold}} - 10^{-7}$ V/ms (Fig. 3b). The rest branch exists for $I < I_{\text{eq}} = 0.049$ V/ms and for $I > I_{\text{eq}}$ the equilibrium is inadmissible. Rest and spiking therefore coexist over $I_{\text{fold}} < I < I_{\text{eq}}$. The cycle disappears at finite amplitude and at a finite rate. A finite-window scan from the zero state shows recurrent firing from about 0.035 V/ms and very long bursts just below it. The fold accounts for both; the long bursts are the bottleneck of the map (Section 5.3).

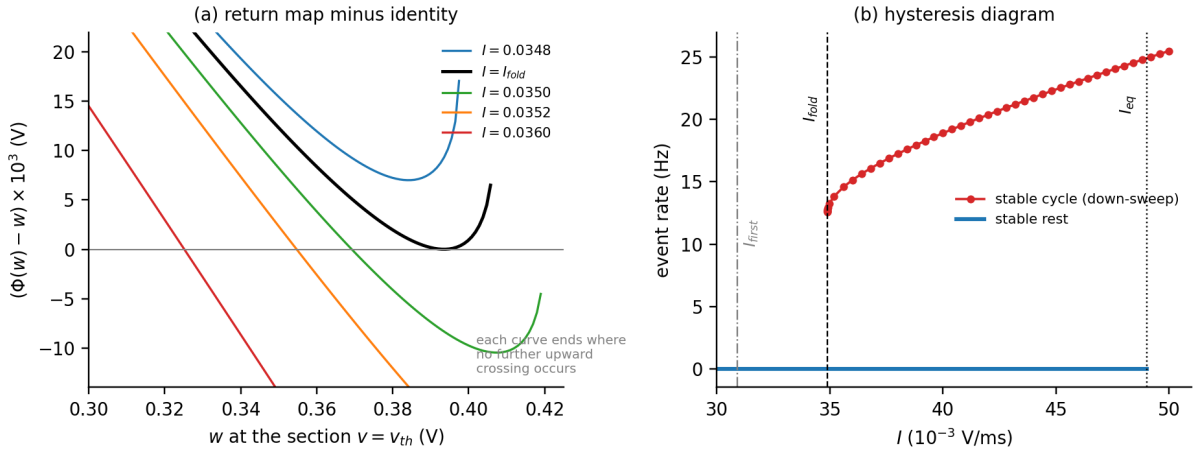


Figure 3: (a) $\Phi(w) - w$ on the section for several currents; each curve ends where no further upward crossing occurs. The bold curve touches zero at I_{fold} . (b) Hysteresis diagram: stable cycle branch (down-sweep) and rest branch.

5.3 Burst length below the fold

For $I < I_{\text{fold}}$ the map $w \mapsto w + D$ has no fixed point, and a trajectory started from rest-like states passes through the narrow channel at $w \approx w_{\text{fold}}$, where $D \simeq a(I - I_{\text{fold}}) + b(w - w_{\text{fold}})^2 = |a|\varepsilon + b x^2$

with $\varepsilon = I_{\text{fold}} - I$. The number of iterations to cross the channel is

$$N \simeq \frac{\pi}{\sqrt{|a|b\varepsilon}} + O(1), \quad (9)$$

the standard bottleneck (“ghost”) law for a saddle-node of a map (Appendix B). Table 4 compares (9) with iteration of the exact return map from $w = 0.18$ V (the first return from the zero state), using a, b from Section 5.2 without fitting. Agreement spans $N = 2$ to 837 and the fitted slope 0.08383 in $N = c_0 + c_1\varepsilon^{-1/2}$ equals the theoretical $\pi/\sqrt{|a|b} = 0.08374$. An alternative $N = c_0 + c_1 \ln(1/\varepsilon)$ fit has an r.m.s. error of 140 against 0.23 for the square root (fit range $\varepsilon \leq 3 \times 10^{-4}$). Full simulations from the zero state give 26 spikes at $\varepsilon = 10^{-5}$ (both solvers) and 83 at $\varepsilon = 10^{-6}$ (DOP853), against 25 and 83 returns after the first spike from the map, i.e. agreement within one spike; at $I_{\text{fold}} + 10^{-5}$ firing persists over the whole 9000 ms window (DOP853).

$\varepsilon = I_{\text{fold}} - I$ (V/ms)	N (return map)	$\pi/\sqrt{ a b\varepsilon}$
10^{-3}	2	2.6
10^{-4}	7	8.4
10^{-5}	25	26.5
10^{-6}	83	83.7
10^{-7}	264	264.8
10^{-8}	837	837.4

Table 4: Number of returns before the cycle is lost.

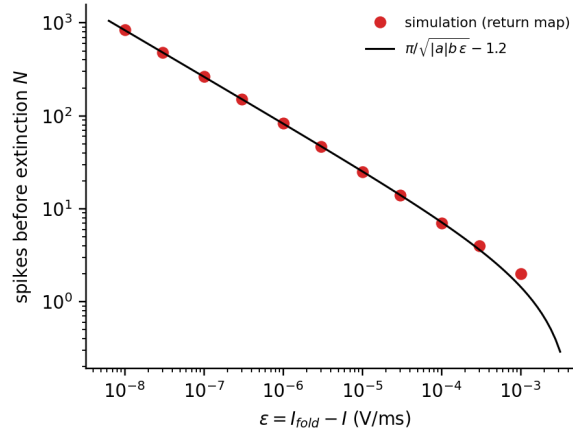


Figure 4: Burst length versus distance below the fold; the line is Eq. (9) with $c_0 = -1.2$ and measured a, b .

5.4 Dependence on (g, τ)

The fold current was computed on the grid $g \in \{0.020, 0.035, 0.050, 0.065, 0.080\} \text{ ms}^{-1}$, $\tau \in \{20, 35, 50, 65, 80\} \text{ ms}$ (Table 5, Fig. 5). In every cell the minimizer of D is interior to the domain, so the transition is a fold and not a grazing event. The ratio $I_{\text{fold}}/I_{\text{eq}}$ decreases monotonically with g and with τ over the grid, from 0.957 ($g = 0.020$, $\tau = 20$) to 0.595 ($g = 0.080$, $\tau = 80$): stronger and slower feedback widens the coexistence window, and the dependence on τ weakens beyond $\tau \approx 50$ ms. As an independent consistency check, direct simulations from the zero state at $I = 0.8 I_{\text{eq}}$ in each cell (data file `phase_map.csv`) show persistent firing in exactly the 17 cells with $I_{\text{fold}}/I_{\text{eq}} < 0.8$ and in none of the other 8 (25 of 25 cells agree). The limit $g \rightarrow 0$ lies outside the grid.

τ (ms) \ g (ms ⁻¹)	0.020	0.035	0.050	0.065	0.080
20	26.796 (0.957)	33.850 (0.879)	39.770 (0.812)	45.002 (0.756)	49.754 (0.711)
35	25.235 (0.901)	31.244 (0.812)	36.401 (0.743)	41.030 (0.690)	45.284 (0.647)
50	24.402 (0.871)	30.011 (0.780)	34.881 (0.712)	39.289 (0.660)	43.369 (0.620)
65	23.906 (0.854)	29.308 (0.761)	34.031 (0.695)	38.329 (0.644)	42.326 (0.605)
80	23.582 (0.842)	28.857 (0.750)	33.493 (0.684)	37.727 (0.634)	41.677 (0.595)

Table 5: Fold current $I_{\text{fold}} \times 10^3$ (V/ms) and, in parentheses, $I_{\text{fold}}/I_{\text{eq}}$ with $I_{\text{eq}} = (A + g)v_{\text{th}}$.

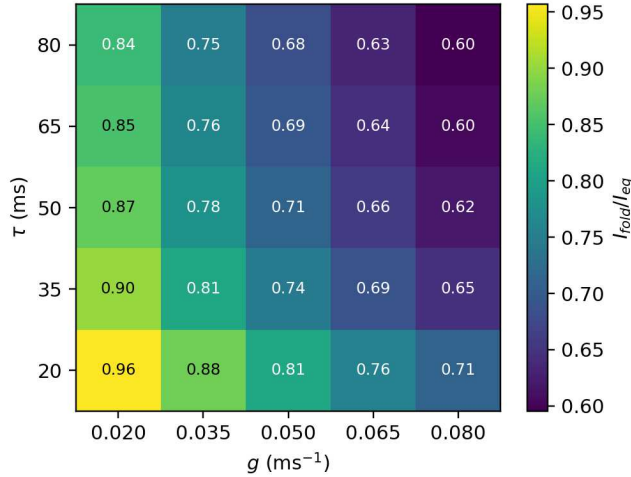


Figure 5: Ratio $I_{\text{fold}}/I_{\text{eq}}$ over the (g, τ) grid. Bistability exists for $I_{\text{fold}} < I < I_{\text{eq}}$ and is wider where the ratio is smaller.

5.5 Solver cross-checks

The independent matrix-exponential solver gives 1, 3, 10, 13, 16 upward crossings in 700 ms at $I = 0.032, 0.034, 0.035, 0.040, 0.045$ V/ms, with event times agreeing with DOP853 to 1.3×10^{-9} ms. In a 1000 ms scan it gives 14, 19, 23, 26 events at 0.035, 0.040, 0.045, 0.050 V/ms and late rates 13.35, 18.91, 22.42, 25.52 Hz. A finite-window classification of runs as recurrent or transient is therefore a consequence of the fold: $I = 0.034$ V/ms lies about 8.8×10^{-4} V/ms below I_{fold} and gives a burst of three spikes. Smoothing the Heaviside functions with sigmoids of width 0.005–0.010 V preserves the qualitative outcome over 350 ms but changes the system and is not used for any quantitative claim.

6 Discussion

What is established. For this model, Section 3 gives exact results and Sections 4–5 give numerically verified ones: the first-switch threshold and latency cusp, the identification of the spiking onset as a fold of cycles on a one-dimensional return map, its location, the hysteresis, the measured normal form and the burst-length law.

Relation to prior work. Overshoot of a linear step response and subthreshold resonance are classical [9, 10], and the bottleneck scaling near a saddle-node is textbook material [13, 14]. Saddle-node bifurcations of cycles in piecewise-linear systems have been proved for continuous planar systems with three zones [7], and inverse-square-root laws for burst duration were found for a smooth conductance-based model [8]; rest-spike bistability has been analyzed in smooth and piecewise-linear models [3, 5, 6]. Our w has the same linear form as the adaptation variable in adaptive exponential integrate-and-fire models [11]. What is specific here is the quantitative, region-resolved treatment of a particular discontinuous switching circuit with a filtered feedback

state, including the contrast with the planar model’s logarithmic latency, and the expression of the burst-length law through return-map iterations with coefficients measured from the map. We found no earlier analysis of this memory-extended model, but the literature search behind this statement was limited.

Limitations. (1) The fold is located numerically; no closed form for I_{fold} is derived (the return map needs transcendental crossing times in each region) and the normal-form coefficients are finite-difference estimates. (2) The reduction to one dimension relies on $k \lesssim 10^{-8}$ V at the section near the folds (k reaches 6×10^{-7} V at $I = 0.052$ V/ms). It was checked numerically, including by the full two-dimensional Jacobian at eight currents (Table 3), but not proved. (3) The fold current was found by bisection assuming a single sign change of m on the bracket; the (g, τ) grid has 25 points. (4) The unstable cycle disappears from the return map at $I - I_{\text{fold}} \approx 9.7 \times 10^{-5}$ V/ms through the boundary of its domain; this grazing-type termination was located but not analyzed, and the full basin boundary of rest is not computed. (5) Noise and parameter perturbations were not studied. (6) No hardware measurements are used, and the added state is a theoretical perturbation of the circuit, not an identified component. (7) Both solvers share the same equations and switching conventions.

Possible extensions. Continuation of the cycle branches with a bifurcation package that handles discontinuities, a closed-form return map from the six exact flows, kernels other than exponential, and comparison with the authors’ three-dimensional parent model once its equations and code are available.

7 Conclusions

Adding one filtered-voltage state to the planar switching neuron lowers the first-switch threshold to I_{first} , which has a closed form, and bounds the first-spike latency with a square-root cusp. It also turns the onset of persistent spiking into a saddle-node bifurcation of cycles at I_{fold} , with a finite onset rate, hysteresis over $I_{\text{fold}} < I < I_{\text{eq}}$, and a burst length $N \simeq \pi / \sqrt{|a|b(I_{\text{fold}} - I)}$ below the fold. For the main parameters $I_{\text{fold}} = 0.0348814$ V/ms, or $0.71 I_{\text{eq}}$. The results are specific to this model and, beyond the first-switch analysis, numerical.

Declarations

Data and code. All results are from deterministic simulations; no experimental data were used. The code, numerical outputs and figures are provided as ancillary files with this submission.

Source model. The planar model and its parameters are taken from [1], a preprint that has not been peer reviewed.

Ethics. The study involves no human participants, animals or biological specimens.

A Derivation of Proposition 1 and Eq. (7)

With $k = 0$, $\dot{v} = I - Av - gw$, $\dot{w} = (v - w)/\tau$, $v(0) = w(0) = 0$. Let $u = \dot{v}$. Differentiating, $\ddot{u} + (A + \tau^{-1})\dot{u} + \frac{A+g}{\tau}u = 0$ with $u(0) = I$ and $\dot{u}(0) = \ddot{v}(0) = I/\tau - (A + \tau^{-1})I = -AI$. The roots are $-\alpha \pm i\omega$, so $u = Ie^{-\alpha t}[\cos \omega t + c \sin \omega t]$ with $c = (\alpha - A)/\omega$. Integrating gives (4). $u = 0$ if and only if $\tan \omega t = \omega/(A - \alpha)$, which gives t_p . Since $u(0) > 0$ the first zero is a maximum of v . Writing $s(t) = \frac{1}{A+g} + e^{-\alpha t}R \cos(\omega t + \phi)$ the successive maxima differ by factors $e^{-2\pi\alpha/\omega} < 1$, so the first is global. The crossing occurs if and only if $I s_{\text{max}} > v_{\text{th}}$. At $\tau = 1/A$ this gives (6). For (7): $v(t) \approx v_p - \frac{1}{2}|\ddot{v}(t_p)|(t - t_p)^2$, $v_p - v_{\text{th}} = (I - I_{\text{first}})s_{\text{max}}$, $\ddot{v}(t_p) = Iu'_0(t_p)$, and $u'_0(t_p) = -\omega(1 + c^2)e^{-\alpha t_p} \sin \omega t_p$ using $\cos \omega t_p = -c \sin \omega t_p$.

B Normal form and burst-length law

Near the fold, $w_{n+1} = w_n + a(I - I_{\text{fold}}) + b(w_n - w_{\text{fold}})^2 + \dots$ with $a < 0 < b$. For $\varepsilon = I_{\text{fold}} - I > 0$ set $x = w - w_{\text{fold}}$, $\mu = |a|\varepsilon$: $x_{n+1} = x_n + \mu + bx_n^2$. In the continuum limit $\dot{x} = \mu + bx^2$, the passage from

$-L$ to $+L$ takes $\frac{1}{\sqrt{\mu b}} [\arctan(L\sqrt{b/\mu})] \cdot 2 \rightarrow \pi/\sqrt{\mu b}$ for $L \gg \sqrt{\mu/b}$, giving (9). For $\varepsilon < 0$ fixed points $x_{\pm} = \pm\sqrt{|a||\varepsilon|/b}$ have multipliers $1 + 2bx_{\pm} = 1 \pm 2\sqrt{|a|b|\varepsilon|}$.

C Reproducibility

`fastmap.py` (exact-branch return map), `foldfind.py` (fold detection), `sweep.py` ((g, τ) grid), `fold_analysis.py` (fixed points, burst law), `hysteresis.py`, `latency_law.py`, `direct_2d_check.py` and the earlier `full_switching.py`, `independent_exact_solver.py`. Python ≥ 3.10 with NumPy, SciPy, Matplotlib; no random numbers are used.

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