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Claude Petit

`claude.petit@univ-rennes.fr`

Institut de Recherche en Informatique et Systèmes Aléatoires

Aline Roumy

Centre Inria de l'Université de Rennes

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Asymptotic Approximation of a Gaussian Integral

Claude Petit ¹ and Aline Roumy ²

¹Irisa, University of Rennes.

²Inria, Center at Rennes University, 263 avenue du général Leclerc, Rennes, 35700, France.

Contributing authors: claude.petit@univ-rennes.fr; aline.roumy@inria.fr;

Abstract

We obtain an asymptotic approximation, as $n \rightarrow +\infty$, for the deficit $1 - J_n$ of the Gaussian integral

$$J_n(\alpha, \gamma) = k \int_0^{+\infty} G(\alpha x)^{n-k} G(x)^{k-1} g(x) dx$$

where $k = \gamma n$, $0 < \gamma < 1$, $\alpha > \sqrt{2}$, G and g are the cumulative distribution function and the density function, respectively, of a standard half-Gaussian random variable:

$$1 - J_n(\alpha, \gamma) \stackrel{n \rightarrow +\infty}{\sim} \alpha(1 - \gamma) \pi^{(\alpha^2 - 1)/2} \frac{\Gamma(\alpha^2)}{\gamma^{\alpha^2}} \frac{(\ln n)^{(\alpha^2 - 1)/2}}{n^{\alpha^2 - 1}}$$

where Γ is the Euler's gamma function. The evaluation of this integral is motivated by approximating the probability of an event that compares the maxima of absolute values from two independent centered Gaussian vectors. This type of integral arises, for example, in the winner's problem or in sparse signal recovery algorithms in the context of high-dimensional compressed sensing, where Gaussian approximations help simplify the evaluation of the involved probabilities.

Keywords: Gaussian maxima, half-normal distribution, uniform asymptotic approximation, generalized Watson lemma.

MSC Classification: 60G70 , 41A60.

1 Introduction and motivation

Let $X = (X_1, \dots, X_k)$ and $Y = (Y_1, \dots, Y_{n-k})$ be the vectors of absolute values of independent centered Gaussians with variance σ_1^2 and σ_2^2 respectively, with

$$\sigma_1^2 = 1 + 1/\mu \text{ and } \sigma_2^2 = 1/\mu. \quad (1)$$

X and Y are independent. Let $\mu > 1$ and $\gamma \in (0, 1)$ be real parameters, and assume that $k = \gamma n$ is an integer (we first establish the asymptotic equivalence without imposing this integrality assumption). Let

$$\alpha = \alpha(\mu) = \sigma_1/\sigma_2 = \sqrt{1 + \mu} > \sqrt{2}. \quad (2)$$

Let $X_\bullet = (X_{(1)}, \dots, X_{(k)})$ and $Y_\bullet = (Y_{(1)}, \dots, Y_{(n-k)})$ denote the order statistics of the two vectors ($X_{(1)}$ is the minimum of the X_i and $X_{(k)}$ is the maximum) and let

$$S_n = [X_{(k)} > Y_{(n-k)}] = \left[\max_{i=1}^k X_i > \max_{j=1}^{n-k} Y_j \right]. \quad (3)$$

The event S_n occurs when the maximum of $X_1, \dots, X_k$ and $Y_1, \dots, Y_{n-k}$ occurs within the X_i . X_i and Y_j follow half-normal distributions of density $g(x/\sigma)/\sigma$ and cumulative distribution function $G(x/\sigma)$ with $\sigma = \sigma_1$ or $\sigma = \sigma_2$. With this notation,

$$\mathbb{P}(S_n) = \mathbb{P}[X_{(k)} > Y_{(n-k)}] = \int_0^{+\infty} \mathbb{P}[Y_{(n-k)} < x] \mathbb{P}_{X_{(k)}}(dx) \quad (4)$$

$$= k \int_0^{+\infty} G(\alpha x)^{n-k} G(x)^{k-1} g(x) dx = J_n(\alpha, \gamma). \quad (5)$$

This type of integral arises, for example, in sparse signal recovery greedy algorithms like Orthogonal Matching Pursuit, within the context of high-dimensional compressed sensing ([Foucart and Rauhut, 2013](#), pp.65-69).

In this article, we derive an asymptotic equivalent of this integral. In high-dimensional compressed sensing, n may represent the sample size, $\gamma = k/n$ the sparsity ratio, and $\mu = m/k$ the overmeasure rate. In this setting, n, m, k must be integers. We therefore examine under what conditions the asymptotic equivalent remains valid when these integrality constraints are imposed and the exact relation $k = \gamma n$ is relaxed to $k/n \sim \gamma$ and $m/k \sim \mu$, as $n \rightarrow \infty$.

The article is organized as follows: in the next section, we state and prove the main theorem. The third section establishes conditions under which the asymptotic equivalent remains valid when n, m , and k are integers, while the fourth section is devoted to the related work.

2 The asymptotic equivalent

To simplify the notation, we will often omit the dependence of the functions under consideration on the parameter of interest. By a slight abuse of notation, we may also write J_n or λ as functions of different variables in order to make their dependence on these variables explicit; for example $\lambda(\mu, \gamma) = \lambda(\alpha(\mu), \gamma)$ specifies the dependence on μ rather than on α . Let $v_n(\mu) = (\ln n)^{\mu/2} n^{-\mu}$. Our main result is the following theorem.

Theorem 1 (A uniform asymptotic expansion)

$$1 - J_n \stackrel{n \rightarrow +\infty}{\sim} \lambda \frac{(\ln n)^{(\alpha^2-1)/2}}{n^{\alpha^2-1}} \quad (6)$$

where

$$\lambda = \lambda(\alpha, \gamma) = \alpha(1-\gamma)\pi^{(\alpha^2-1)/2} \frac{\Gamma(\alpha^2)}{\gamma^{\alpha^2}} \quad (7)$$

and Γ is the Euler gamma function. $k = \gamma n$ and n are not assumed to be integers. Moreover, for every fixed $\gamma \in (0, 1)$, the asymptotic equivalent for $1 - J_n$ holds uniformly for $\mu = \alpha^2 - 1 \in K$ compact in $(1, +\infty)$ and

$$\sup_{\mu \in K} \left| \frac{1 - J_n(\mu, \gamma)}{\lambda(\mu, \gamma) (\ln n)^{\mu/2} / n^\mu} - 1 \right| = \mathcal{O}\left(\frac{\ln \ln n}{\ln n}\right). \quad (8)$$

Proof The proof is divided into four steps. The first step examines the integrand and several auxiliary functions. The second step introduces a change of variables that reduces the asymptotic analysis to a neighborhood of 0^+ in a uniform manner, with the aim of applying the Laplace–Watson method to derive an asymptotic equivalent. The third step revisits Wong and Wyman’s generalization of the Laplace–Watson lemma to functions with singularities at 0^+ , establishing that each step of their argument holds uniformly in μ . Eventually, the fourth step establishes the desired uniform asymptotic expansion.

- First step: properties of the underlying functions. We write J_n in a more convenient form:

$$J_n = k \int_0^{+\infty} e^{nL(x)} \frac{g(x)}{G(x)} dx \quad (9)$$

with

$$L(x) = (1-\gamma) \ln G(\alpha x) + \gamma \ln G(x). \quad (10)$$

L is a negative, increasing and concave function on $(0, +\infty)$ (cf. Fig. 1). g , G and L are C^∞ functions on $(0, +\infty)$. L has a singularity in 0^+ . We will need the functions h_1 and h_2 defined by

$$h_1(x) = \frac{g(x)}{G(x)L'(x)} \text{ and } h_2(x) = \frac{h_1'(x)}{L'(x)}. \quad (11)$$

A direct expansion shows that h_1 is a C^∞ function on $(0, +\infty)$, not defined at 0, but such that :

$$h_1(x) = 1 + \frac{1}{3}(\alpha^2 - 1)(1-\gamma)x^2 + o(x^2), \quad (12)$$

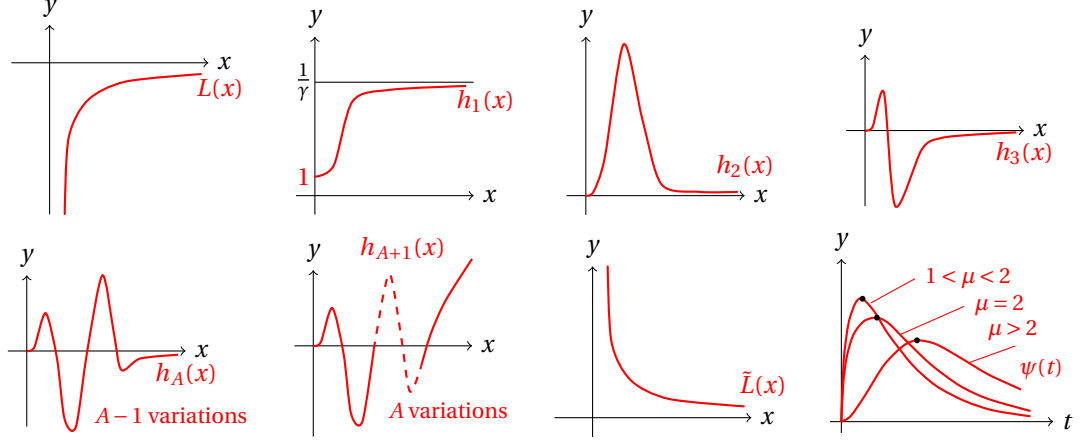


Fig. 1 Sketch (not to scale) of the representative curves of functions L , h_i , $\tilde{L}$, ψ

$$h_1(x) = \frac{1}{\gamma} - \frac{\alpha(1-\gamma)}{\gamma^2} e^{-(\alpha^2-1)x^2/2} + \frac{1}{\gamma^2} \sqrt{\frac{2}{\pi}} \alpha(1-\gamma) \frac{e^{-\alpha^2 x^2/2}}{x} \left[1 - \frac{1}{x^2} + o\left(\frac{1}{x^2}\right) \right]. \quad (13)$$

Therefore, h_1 can be extended in the neighborhood of 0, into a C^1 function by letting $h_1(0) = 1$ and $h_1'(0) = 0$. For the sake of simplicity, we will still call h_1 this new function. It is defined on $[0, +\infty)$ and is bounded above and below (cf. Fig. 1). Likewise, since $L'(x)$ is C^∞ on $(0, +\infty)$ and is always nonzero, $h_2(x)$ is C^∞ on $(0, +\infty)$ and

$$h_1'(x) \stackrel{+\infty}{\sim} -\lambda_1(\alpha^2 - 1)x \exp\left(-(\alpha^2 - 1)x^2/2\right), \text{ with } \lambda_1 = -\alpha \frac{1-\gamma}{\gamma^2}, \quad (14)$$

$$h_2(x) \stackrel{+\infty}{\sim} \lambda_2 x \exp\left(-(\alpha^2 - 2)x^2/2\right), \text{ with } \lambda_2 = \alpha(\alpha^2 - 1) \frac{1-\gamma}{\gamma^3} \sqrt{\frac{\pi}{2}}. \quad (15)$$

Returning to J_n , integration by parts is justified because this generalized integral is well defined and the boundary term of the expression has a finite limit. Moreover, $\exp(nL(x))$ is bounded between 0 and 1, $h_1'(x)$ is C^∞ on $(0, +\infty)$, tends to 0 in 0 and in $+\infty$ and is integrable thanks to (14). So,

$$J_n = \frac{k}{n} \left[e^{nL(x)} \times \frac{g(x)}{G(x)L'(x)} \right]_0^{+\infty} - \frac{k}{n} \int_0^{+\infty} e^{nL(x)} \times h_1'(x) dx \quad (16)$$

$$= \frac{k}{n} \left[e^{nL(x)} \times h_1(x) \right]_0^{+\infty} - \frac{k}{n} \int_0^{+\infty} e^{nL(x)} \times h_1'(x) dx. \quad (17)$$

We have

$$\lim_{x \rightarrow 0} e^{nL(x)} \times h_1(x) = 0 \text{ and } \lim_{x \rightarrow +\infty} e^{nL(x)} \times h_1(x) = \frac{1}{\gamma} \quad (18)$$

so that

$$J_n = \frac{k}{n\gamma} - \frac{k}{n} \int_0^{+\infty} e^{nL(x)} \times h_1'(x) dx = \frac{k}{n\gamma} - \frac{k}{n} I_n. \quad (19)$$

By the Lebesgue dominated convergence theorem:

$$\lim_{n \rightarrow +\infty} I_n = \lim_{n \rightarrow +\infty} \int_0^{+\infty} e^{nL(x)} h_1'(x) dx = \int_0^{+\infty} \lim_{n \rightarrow +\infty} e^{nL(x)} h_1'(x) dx = 0. \quad (20)$$

So far, we have proven that for all $\mu > 1$ and $\gamma \in (0, 1)$, $\lim_{n \rightarrow +\infty} J_n = 1$.

Let us now consider $L(x) = L_\mu(x)$ for $\mu \in K$, where K is a compact subset of $(1, +\infty)$. Since $\alpha^2 = 1 + \mu$, α also ranges over a compact interval, which we denote by $[\alpha_1, \alpha_2]$, with $\sqrt{2} < \alpha_1 < \alpha_2$. Let $\bar{G}(x) = 1 - G(x)$. Using Mills' ratio (Wong, 1989, p.42), we obtain:

$$\bar{G}(x) = \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{x} (1 + \mathcal{O}(\frac{1}{x^2})) \text{ and } \bar{G}(\alpha x) = \sqrt{\frac{2}{\pi}} \frac{e^{-\alpha^2 x^2/2}}{\alpha x} (1 + \mathcal{O}(\frac{1}{x^2})). \quad (21)$$

Since $\alpha > \alpha_1 > 1$, the $\mathcal{O}$ -term is uniform with respect to μ (the coefficients of the powers of x appearing in the remainder of the expansion are bounded by a constant depending only on α_1 and α_2). Similarly,

$$\sup_{\mu \in K} \left| \frac{\bar{G}(\alpha x)}{\bar{G}(x)} \right| = \sup_{\mu \in K} \left| \frac{1}{\alpha} e^{-(\alpha^2-1)x^2/2} (1 + \mathcal{O}(\frac{1}{x^2})) \right| \leq C_K e^{-(\alpha_1^2-1)x^2/2} \rightarrow 0 \quad (22)$$

as $x \rightarrow +\infty$. Here, C_K denotes a generic uniform constant, which may vary from one formula to another, but depends only on K , or equivalently on α_1 and α_2 . Since $\bar{G}$ does not depend on α ,

$$\ln G(x) = \ln(1 - \bar{G}(x)) = -\bar{G}(x) + \mathcal{O}(\bar{G}(x)^2) \quad (23)$$

implies

$$-L(x) = (1 - \gamma) \bar{G}(\alpha x) + \gamma \bar{G}(x) + \mathcal{O}(\bar{G}(x)^2), \quad (24)$$

$$-\frac{L(x)}{\gamma \bar{G}(x)} = 1 + \frac{1 - \gamma}{\gamma} \frac{\bar{G}(\alpha x)}{\bar{G}(x)} + \mathcal{O}(\bar{G}(x)), \quad (25)$$

uniformly with respect to $\mu \in K$. Hence,

$$\lim_{x \rightarrow +\infty} \sup_{\mu \in K} \left| \frac{-L(x)}{\gamma \bar{G}(x)} - 1 \right| = 0. \quad (26)$$

- Second step : change of variables. Let us consider in I_n the change of variables given by

$$t = -L(x). \quad (27)$$

It is valid because L is one-to-one on $(0, +\infty)$ and differentiable. Since $x = L^{-1}(-t)$ and $dt = -L'(x)dx$,

$$I_n = \int_0^{+\infty} e^{-nt} h_2(L^{-1}(-t)) dt = \int_0^{+\infty} h_2 \circ \tilde{L}(t) \times e^{-nt} dt \quad (28)$$

with $\tilde{L}$ function defined by $\tilde{L}(t) = L^{-1}(-t)$. This is exactly the Laplace transform of $h_2 \circ \tilde{L}$. Denote this function by ψ . $\tilde{L}$ is C^∞ , decreasing from $(0, +\infty)$ to $(0, +\infty)$. It tends to $+\infty$ as $t \rightarrow 0^+$ and toward 0 in $+\infty$. h_2 is also C^∞ , tends to zero as $x \rightarrow 0$ and as $x \rightarrow +\infty$. $h_2 \circ \tilde{L}$ is then a C^∞ function on $(0, +\infty)$ and tends to 0 in 0^+ . ψ is not defined at 0, but the existence of a finite limit at that point allows us to extend ψ into a continuous function. Using again Mills' ratio, we can then give an asymptotic expansion and show that the change of variable from x to t preserves uniformity with respect to μ :

$$t = -L(x) = \gamma \sqrt{\frac{2}{\pi}} \frac{e^{-x(t)^2/2}}{x(t)} (1 + \mathcal{O}(1/x^2)) \quad (29)$$

uniformly with respect to μ . It follows immediately that

$$-2 \ln t = x(t)^2 + 2 \ln x(t) - 2 \ln \left(\gamma \sqrt{\frac{2}{\pi}} \right) + \mathcal{O}(1/x^2). \quad (30)$$

When x tends to infinity, the dominant term of the right-hand side is $x(t)^2$. The first-order asymptotic relation $x(t)^2 \sim -2 \ln t$ is uniform with respect to μ . Combining this relation with the previous identity yields

$$x(t)^2 = -2 \ln t + \mathcal{O}(\ln(-2 \ln t)), \quad (31)$$

$$\ln(x(t)^2) = \ln(-2 \ln t) + \mathcal{O}\left(\frac{\ln(-2 \ln t)}{-2 \ln t}\right), \quad (32)$$

then, substituting this expression back into the previous identity, we obtain the following expansion, uniformly with respect to μ :

$$x(t)^2 = -2 \ln t - \ln(-2 \ln t) + \ln \frac{2\gamma^2}{\pi} + \mathcal{O}(\eta(t)) \text{ with } \eta(t) = \frac{\ln \ln(1/t)}{\ln(1/t)}. \quad (33)$$

We now show that h_2 admits a uniform asymptotic expansion as $x \rightarrow \infty$. To this end, let

$$\phi(x) = g(x)/G(x) \text{ and } \Delta(x) = \alpha \frac{1-\gamma}{\gamma} \phi(\alpha x)/\phi(x). \quad (34)$$

We then obtain successively:

$$h_1(x) = \frac{\phi(x)}{L'(x)} = \frac{1}{\gamma} [1 + \Delta(x)]^{-1} \text{ and } h_1'(x) = \frac{1}{\gamma} \Delta(x) A(x) [1 + \Delta(x)]^{-2}, \quad (35)$$

denoting by $A(x) = (\alpha^2 - 1)x - \phi(x) + \alpha \phi(\alpha x)$, using the relation $\phi'(x) = -x\phi(x) - \phi^2(x)$ and noting that $\Delta'(x) = -\Delta(x)A(x)$. We then obtain

$$h_2(x) = \frac{h_1'(x)}{L'(x)} = \frac{\Delta(x)A(x)}{\gamma^2 \phi(x) [1 + \Delta(x)]^3}. \quad (36)$$

$\lim_{x \rightarrow +\infty} G(x) = 1$ and ϕ is decreasing, so $0 < \Delta(x) \leq \alpha_2 \phi(\alpha_1 x)/\phi(x)(1 - \gamma)/\gamma$. The expansion

$$\Delta(x) = \alpha \frac{1-\gamma}{\gamma} e^{-(\alpha^2-1)x^2/2} \left(1 + \mathcal{O}(1/x^2)\right) \quad (37)$$

is therefore uniform with respect to μ , and $\Delta(x) \rightarrow 0$ as $x \rightarrow +\infty$. Similarly,

$$[1 + \Delta(x)]^{-3} = 1 + \mathcal{O}(\sup_{\mu \in K} \Delta(x)) \quad (38)$$

uniformly with respect to $\mu \in K$. Finally, $A(x) = (\alpha^2 - 1)x(1 + \mathcal{O}(\phi(x)/x))$ uniformly, since $|\phi(x) + \alpha \phi(\alpha x)| \leq (1 + \alpha_2)\phi(x)$ and $(\alpha^2 - 1)x \geq (\alpha_1^2 - 1)x$. Therefore,

$$h_2(x) = \lambda_2 x e^{-(\alpha^2-2)x^2/2} \left(1 + \mathcal{O}(1/x^2)\right) \quad (39)$$

is uniform in $\mu \in K$.

We finally show that the composition $\psi = h_2 \circ \tilde{L}$ yields a uniform approximation as $t \rightarrow 0^+$. We have two uniform approximations, $h_2(x) = H(x)(1 + R_2(x))$ and $x(t) = X(t)(1 + r(t))$ with

$$\lim_{x \rightarrow +\infty} \sup_{\mu \in K} |R_2(x)| = 0 \text{ and } \lim_{t \rightarrow 0^+} \sup_{\mu \in K} |r(t)| = 0. \quad (40)$$

$\psi(t) = H(x(t))(1 + R_2(x(t)))$ and since $x(t) \rightarrow +\infty$ is uniform in μ ,

$$\lim_{t \rightarrow 0^+} \sup_{\mu \in K} |R_2(x(t))| = 0. \quad (41)$$

it remains to consider $H(x(t))$:

$$x(t) = \sqrt{-2 \ln t} [1 + \mathcal{O}(\eta(t))], \quad (42)$$

$$e^{-(\alpha^2-2)x(t)^2/2} = t^{\alpha^2-2} (-2 \ln t)^{(\alpha^2-2)/2} \left(\frac{2\gamma^2}{\pi} \right)^{-(\alpha^2-2)/2} [1 + \mathcal{O}(\eta(t))], \quad (43)$$

and since $\eta(t) \rightarrow 0$ uniformly with respect to $\mu \in K$, $e^{\mathcal{O}(\eta(t))} = 1 + \mathcal{O}(\eta(t))$. Therefore, the asymptotic expansion of ψ is uniform as $t \rightarrow 0^+$ and

$$\psi(t) = \lambda'(\alpha, \gamma) (-\ln t)^{(\alpha^2-1)/2} t^{\alpha^2-2} (1 + \mathcal{O}(\eta(t))), \quad (44)$$

with

$$\lambda'(\alpha, \gamma) = \alpha(\alpha^2 - 1)(1 - \gamma)\pi^{(\alpha^2-1)/2} \gamma^{\alpha^2+1}. \quad (45)$$

• Third step: a uniform version of the generalized Laplace-Watson lemma. The hypotheses of the Laplace-Watson lemma are not satisfied (Dieudonné, 1968, p.121-125) because the local expansion of ψ has a logarithmic singularity in 0^+ , but there exists a generalization of this lemma proved by Wong and Wyman (Wong and Wyman, 1972), (Wong, 1989, p.69-70). Generalizing the identity

$$\int_0^{\infty e^{i\theta}} t^{s-1} e^{-zt} dt = \Gamma(s) z^{-s} \quad (46)$$

with $\operatorname{Re}(s) > 0$, $\operatorname{Re}(\sigma) > -1$ and $|\arg(ze^{i\theta})| < \pi/2$, Wong proves that, as $|z| \rightarrow +\infty$,

$$\int_0^{\infty e^{i\theta}} t^{s-1} (-\ln t)^\sigma e^{-zt} dt = (\ln z)^\sigma \Gamma(s) z^{-s} (1 + \mathcal{O}\left(\frac{1}{\ln z}\right)). \quad (47)$$

For the present real-valued application, fix $0 < \delta < 1$. The generalized Watson lemma is applied only to the local integral over $0 < t < \delta$, where $-\ln t > 0$. The contribution of $t \geq \delta$ is exponentially small as $n \rightarrow \infty$, hence no choice of a complex branch is required.

Wong's lemma in (Wong, 1989) provides an asymptotic approximation for functions of the form $t^{s-1}(-\ln t)^\sigma$, whereas (Wong and Wyman, 1972) considers the more general case in which $\psi(t)$ admits an expansion of the above form, in which case the remainder term must also be taken into account. This is precisely the situation encountered here, since

$$\psi(t) = \lambda'(\alpha, \gamma) (-\ln t)^{(\alpha^2-1)/2} t^{\alpha^2-2} (1 + \mathcal{O}(\eta(t))). \quad (48)$$

The following lines revisit the proof of Wong and Wyman's theorem, establishing that each step is uniform with respect to $\mu \in K$. We first note that

$$\int_0^\infty \psi(t) dt = \int_0^{+\infty} h'_1(x) dx = \frac{1}{\gamma} - 1. \quad (49)$$

This integral does not depend on μ and is thus uniform with respect to μ . Let $0 < \delta < e^{-1}$ be sufficiently small so that the uniform local expansion above holds for every $t \in (0, \delta)$. $-\ln t > 0$ and

$$\left| \int_\delta^\infty \psi(t) e^{-nt} dt \right| \leq e^{-n\delta} \int_\delta^\infty |\psi(t)| dt = \mathcal{O}(e^{-n\delta}). \quad (50)$$

Set $\sigma = (\alpha^2 - 1)/2$ and $s = \alpha^2 - 1$. Since $\mu \in K$, $\alpha \in [\alpha_1, \alpha_2]$, $\sigma \in [\sigma_1, \sigma_2]$ and $s \in [s_1, s_2]$ are bounded; any function depending only on these variables can therefore be absorbed into a constant C_K . For $0 < u < n\delta < n$, we perform the change of variable $u = nt$:

$$\int_0^\delta \psi(t) e^{-nt} dt = \lambda' \frac{(\ln n)^\sigma}{n^s} \int_0^{n\delta} u^{s-1} \left(1 - \frac{\ln u}{\ln n} \right)^\sigma \left(1 + \mathcal{O}\left(\eta\left(\frac{u}{n}\right)\right) \right) e^{-u} du. \quad (51)$$

The idea is to show that, after integration, the approximation has a remainder that is uniform in σ and s , of order $\mathcal{O}(\ln \ln n / \ln n)$. To bound the integral of the leading term, we split the integration interval $(0, n\delta]$ into the three subintervals $(0, n^{-1/2})$, $[n^{-1/2}, n^{1/2}]$, and $(n^{1/2}, n\delta]$. Let

$$g_n(u) = g_n(u, \sigma) = \left(1 - \frac{\ln u}{\ln n} \right)^\sigma. \quad (52)$$

- If $n^{-1/2} \leq u \leq n^{1/2}$, then $|\ln u| / \ln n \leq 1/2$, and the mean value theorem applied to $g_n(u)$ yields $|g_n(u) - 1| \leq C_K |\ln u| / \ln n \leq C_K/2$. Then,

$$\left| \int_{1/\sqrt{n}}^{\sqrt{n}} u^{s-1} (g_n(u) - 1) e^{-u} du \right| \leq \frac{C_K}{\ln n} \int_0^\infty u^{s-1} |\ln u| e^{-u} du = \frac{C'_K}{\ln n}, \quad (53)$$

because the latter integral is convergent and uniformly bounded with respect to s .

- If $0 < u \leq n^{-1/2}$, then, on the one hand, $g_n(u) \geq (3/2)^\sigma$, while, on the other hand, $g_n(u) \leq (1 + |\ln u|)^{\sigma_2}$. $u^\varepsilon |\ln u|^{\sigma_2}$ tends to 0 as $u \rightarrow 0^+$ for every $\varepsilon > 0$. We may therefore choose ε such that $0 < \varepsilon < s_1$ and $u^{s-1} |\ln u|^\sigma \leq C_K u^{s_1-1-\varepsilon}$, so that

$$\left| \int_0^{1/\sqrt{n}} u^{s-1} (g_n(u) - 1) e^{-u} du \right| \leq C_K \int_0^{1/\sqrt{n}} u^{s_1-1-\varepsilon} du = \mathcal{O}(n^{-(s_1-\varepsilon)/2}). \quad (54)$$

- If $n^{1/2} \leq u \leq n\delta < n$, $0 < 1 - \ln u / \ln n < 1/2$, thus $|g_n(u) - 1| \leq 1$ and the integral is $\mathcal{O}(n^{(s_2-1)/2} e^{-\sqrt{n}})$, uniformly with respect to σ and s . If $u > n\delta$,

$$\sup_{s_1 \leq s \leq s_2} \int_{n\delta}^{+\infty} u^{s-1} e^{-u} du = \mathcal{O}\left(n^{s_2-1} e^{-n\delta}\right), \quad (55)$$

which is exponentially negligible.

The integral of the leading term of ψ is therefore $\Gamma(s) + \mathcal{O}(1/\ln n)$.

To bound the remainder term, we split the integration domain into only two subintervals:

- If $0 < u \leq \sqrt{n}$, then $\ln(n/u) \geq \ln n/2$, and the monotonicity of $\ln y/y$ implies that $|\eta(u/n)| \leq C \ln \ln n / \ln n$ uniformly in u and in $\mu \in K$.
- If $\sqrt{n} \leq u \leq n\delta$, then $u^{s-1}e^{-u} \leq u^{s_2-1}e^{-u}$. $\eta(u/n)$ is uniformly bounded on $0 < u \leq n\delta$, thus

$$\left| \int_{\sqrt{n}}^{n\delta} u^{s-1} g_n(u) \left(1 + \mathcal{O}\left(\eta\left(\frac{u}{n}\right)\right)\right) e^{-u} du \right| \leq C_K \int_{\sqrt{n}}^{\infty} u^{s_2-1} e^{-u} du. \quad (56)$$

which is $\mathcal{O}(n^{(s_2-1)/2} e^{-\sqrt{n}}) = o(1/\ln n)$.

Finally, combining the leading term and the remainder term yields

$$\int_0^{\infty} \psi(t) e^{-nt} dt = \lambda'(\alpha, \gamma) \frac{(\ln n)^\sigma}{n^s} \Gamma(s) \left(1 + \mathcal{O}\left(\frac{\ln \ln n}{\ln n}\right)\right), \quad (57)$$

with $s = \alpha^2 - 1$, $\sigma = (\alpha^2 - 1)/2$ and this expansion is uniform in $\mu \in K$.

• Last step: the uniform asymptotic expansion. Fix $\gamma \in (0, 1)$ and consider $v \in K$, where K is a compact neighborhood of μ in $(1, +\infty)$. By (57), substituting $s = \alpha^2 - 1$ and $\sigma = (\alpha^2 - 1)/2$ where $\alpha = \alpha(v) = \sqrt{1+v}$,

$$1 - J_n = 1 - J_n(\alpha(v), \gamma) = \gamma \int_0^{\infty} \psi(t) e^{-nt} dt = \lambda(\alpha(v), \gamma) v_n(v) \left(1 + \mathcal{O}\left(\frac{\ln \ln n}{\ln n}\right)\right), \quad (58)$$

with $v_n(v) = (\ln n)^{v/2} n^{-v}$, so that

$$\frac{1 - J_n}{\lambda v_n(v)} - 1 = \mathcal{O}\left(\frac{\ln \ln n}{\ln n}\right) \quad (59)$$

and $\lambda = \gamma \lambda' \Gamma(\alpha^2 - 1)$. This asymptotic expansion as $n \rightarrow +\infty$ is uniform with respect to v . In particular,

$$1 - J_n \sim \alpha(1 - \gamma) \pi^{(\alpha^2-1)/2} \frac{\Gamma(\alpha^2)}{\gamma^{\alpha^2}} \frac{(\ln n)^{(\alpha^2-1)/2}}{n^{\alpha^2-1}}. \quad (60)$$

This concludes the proof. $\square$

3 Equivalence with integrality constraints

In this section, we examine the conditions under which the asymptotic equivalent for $1 - J_n$ remains valid when n , $k = k_n$ and $m = m_n$ are integers and the equalities are relaxed to $k/n \sim \gamma$ and $m/k \sim \mu$ as $n \rightarrow \infty$.

In the remainder of this section, we write $\gamma_n = k_n/n$, $\mu_n = m_n/k_n$ and $\alpha_n = \sqrt{1 + \mu_n}$. We also let

$$L_n(x) = (1 - \gamma_n) \ln G(\alpha_n x) + \gamma_n \ln G(x), \quad (61)$$

$$\tilde{J}_n = \tilde{J}_n(\alpha_n, \gamma_n) = k_n \int_0^{+\infty} e^{nL_n(x)} \frac{g(x)}{G(x)} dx. \quad (62)$$

The theorem establishing the asymptotic equivalence relies on continuity properties of the functions involved, monotonicity of the integrand in J_n , and the uniform bound on the asymptotic approximation. We record the monotonicity property in the following lemma, whose proof is deferred to the appendix.

Lemma 1 (Monotonicity of $1 - J_n$) *Fix n . The function $(\alpha, k) \mapsto 1 - J_n(\alpha, k)$ is non-increasing in k on $(0, n)$ and decreasing in α on $(1, +\infty)$.*

Theorem 2 (Asymptotic equivalence for $k_n/n \sim \gamma$ and $m_n/k_n \sim \mu$) *Let $\gamma \in (0, 1)$, $\mu > 1$ and $\alpha^2 = 1 + \mu$. Assume that:*

$$\frac{k_n}{n} \overset{n \rightarrow +\infty}{\sim} \gamma \text{ and } \frac{m_n}{k_n} = \mu + o\left(\frac{1}{\ln n}\right).$$

Then

$$1 - \tilde{J}_n(\alpha_n, \gamma_n) \overset{n \rightarrow +\infty}{\sim} \lambda \frac{(\ln n)^{(\alpha^2-1)/2}}{n^{\alpha^2-1}}. \quad (63)$$

Proof Fix $0 < \varepsilon < \min(\gamma, 1 - \gamma)$ so that $(\gamma - \varepsilon)n \leq k_n \leq (\gamma + \varepsilon)n$ for n large. Lemma 1 in the second variable applied at α_n gives

$$1 - J_n(\alpha_n, (\gamma + \varepsilon)n) \leq 1 - \tilde{J}_n(\alpha_n, \gamma_n) \leq 1 - J_n(\alpha_n, (\gamma - \varepsilon)n). \quad (64)$$

Fix $c \in \{\gamma - \varepsilon, \gamma + \varepsilon\}$. Since $\mu_n \rightarrow \mu$, $\mu_n \in K$ for n large, the uniformity in (8) applies along the sequence $v = \mu_n$ and yields

$$1 - J_n(\alpha_n, cn) = \lambda(\mu_n, c) v_n(\mu_n) (1 + o(1)). \quad (65)$$

by continuity of λ in its second variable, $\lambda(\mu_n, c) \rightarrow \lambda(\mu, c)$ and since $(\mu_n - \mu) \ln n \rightarrow 0$ by assumption,

$$\lim_{n \rightarrow +\infty} \frac{v_n(\mu_n)}{v_n(\mu)} = \lim_{n \rightarrow +\infty} \exp\left(-(\mu_n - \mu) \ln n + \frac{1}{2}(\mu_n - \mu) \ln \ln n\right) = 1. \quad (66)$$

Therefore $1 - J_n(\alpha_n, cn) \sim \lambda(\mu, c) v_n(\mu)$ for each $c = \gamma \pm \varepsilon$. Dividing (64) by $v_n(\mu)$,

$$\lambda(\mu, \gamma + \varepsilon) \leq \liminf_n \frac{1 - \tilde{J}_n}{v_n(\mu)} \leq \limsup_n \frac{1 - \tilde{J}_n}{v_n(\mu)} \leq \lambda(\mu, \gamma - \varepsilon). \quad (67)$$

Eventually, letting $\varepsilon \rightarrow 0$ completes the proof. $\square$

Remark 1 $v_n(\mu_n)/v_n(\mu) \rightarrow 1$ if and only if $(\mu_n - \mu) \ln n \rightarrow 0$. The condition $\mu_n - \mu = o(1/\ln n)$ is therefore necessary as well as sufficient. Taking $\mu_n = \mu + c/\ln n$ with $c \neq 0$ gives $(1 - \tilde{J}_n)/(\lambda(\mu, \gamma) v_n(\mu)) \rightarrow e^{-c} \neq 1$.

Remark 2 The choice $k_n = \lfloor \gamma n \rfloor$ and $m_n = \lfloor \mu k_n \rfloor$ implies $\mu_n - \mu = \mathcal{O}(1/n) = o(1/\ln n)$, so that the required condition is automatically satisfied.

Remark 3 If $\mu_n = \mu$ is constant, no additional rate assumption on k_n/n is required beyond $k_n/n \rightarrow \gamma$.

Remark 4 Instead of performing the change of variables starting from h_2 , we could have continued the integrations by parts by defining $h_i(x) = h'_{i-1}(x)/L'(x)$, $2 \leq i \leq A$, where A is the integer part of α^2 if α^2 is not an integer, $A = \alpha^2 - 1$ otherwise. An asymptotic expansion of h_i at infinity then gives, by recurrence, for $2 \leq i \leq A$,

$$h_i(x) \stackrel{+\infty}{\sim} \lambda_i x^{i-1} \exp\left(-(\alpha^2 - i)x^2/2\right), \quad (68)$$

$$h'_i(x) \stackrel{+\infty}{\sim} -\lambda_i(\alpha^2 - i)x^i \exp\left(-(\alpha^2 - i)x^2/2\right) \quad (69)$$

with

$$\lambda_i = (-1)^i \left(\frac{\pi}{2}\right)^{\frac{i-1}{2}} \frac{\alpha(1-\gamma)}{\gamma^{i+1}} \prod_{j=1}^{i-1} (\alpha^2 - j). \quad (70)$$

For $2 \leq i \leq A$, $h_i(0) = 0$ and the asymptotic equivalents above show that h_i is integrable. Let $a = \alpha^2 - A \in (0, 1]$. Then $h'_A(x) \sim -\lambda_A a x^A e^{-ax^2/2}$ and since $L'(x) \sim \gamma\sqrt{2/\pi}e^{-x^2/2}$, $|h_{A+1}(x)| \rightarrow +\infty$. If $a < 1$ the divergence is exponential; if $a = 1$ ($\alpha^2 \in \mathbb{N}$), it is polynomial, hence, no further integration by parts can be performed. A is the last index for which h_i is integrable, so:

$$1 - J_n = \mathcal{O}\left(1/n^{A-1}\right). \quad (71)$$

Setting $\psi = h_A \circ \tilde{L}$, the change of variables $t = -L(x)$ can still be performed, and the application of the Wong-Wyman theorem yields exactly the same asymptotic equivalent as in the case of h_2 . It is noteworthy that the accuracy of the asymptotic approximation does not depend on the number of integrations by parts performed in the analysis of J_n . The same asymptotic equivalent is obtained whether it is derived from h_A or from h_2 , and the quantity A eventually disappears altogether.

Related Work

In (McClure and Wong, 1986), McClure and Wong derive an asymptotic approximation for a different integral involving the normal distribution. Related Laplace integrals with logarithmic factors and generalized Watson-type expansions are studied in (Wong, 1970; Wong and Wyman, 1972; Wong, 1989). Zhang, Miao and Mao (Zhang et al., 2026) study the same comparison of maxima by extreme-value methods (Resnick, 1987), within the framework of the winner problem (Davydov and Rotar, 2024, 2026; Engelke et al., 2015). For n_1 variables of unit variance against n_2 variables of variance α^2 , their Theorem 1 shows that the behaviour is governed by $R_n = n_1 (\log n_2)^{\mu/2} / n_2^{1+\mu}$, the probability that the first group holds the overall maximum converging to 1, to a non-degenerate limit $\mathcal{L}(C, \alpha)$, or to 0, according as $R_n \rightarrow +\infty$, $R_n \rightarrow C \in (0, +\infty)$, or $R_n \rightarrow 0$. The linear regime $k \sim \gamma n$ studied here is the third case. Our variables being absolute values, a group of N of them behaves like $2N$

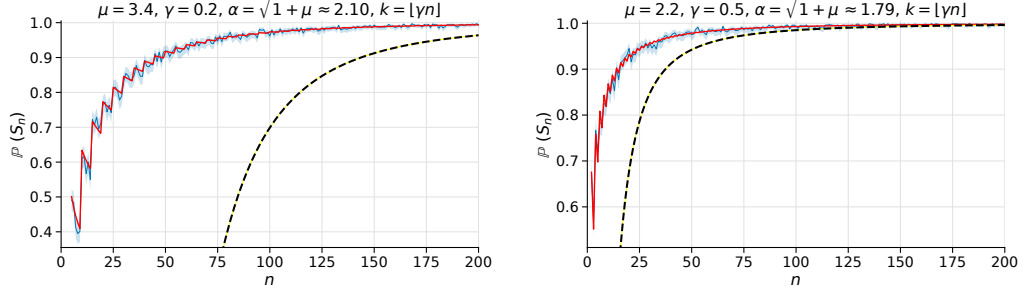


Fig. 2 Comparison of the probability of the event $S_n = [X_{(k)} > Y_{(n-k)}]$ by simulation of 1000 independent samples (blue line and 95% confidence band), numerical evaluation of the integral $J_n(\alpha, \gamma)$ (red line) and asymptotic equivalent (black dotted line). Left: case $\mu = 3.4$ and $\gamma = 0.2$. Right: case $\mu = 2.2$ and $\gamma = 0.5$

signed ones; the comparison is made at $n_1 = 2(1 - \gamma)n$ against $n_2 = 2\gamma n$, the probability in question is $1 - J_n$, and

$$R_n \stackrel{n \rightarrow +\infty}{\sim} 2^{-\mu} \frac{1-\gamma}{\gamma^{1+\mu}} v_n(\mu), \quad (72)$$

so that $R_n \rightarrow 0$. This case is the object of their Proposition 1, which gives the limit 0 without a rate. Expanding their limiting function at the origin gives $\mathcal{L}(C, \alpha) \sim \alpha \Gamma(\alpha^2) (4\pi)^{(\alpha^2-1)/2} C$, $C \rightarrow 0^+$. Combined with (72), this reproduces the multiplicative constant in (6). This agreement does not, however, allow Theorem 1 of (Zhang et al., 2026) to be applied directly with $C = R_n \rightarrow 0$, since that theorem is pointwise in C and does not provide the uniform control near $C = 0$ required for a substitution.

The two approaches are complementary: (Zhang et al., 2026) identify the critical scale, the three regimes it induces and the limit law in the non-degenerate case, whereas the present paper supplies the rate on the degenerate branch $R_n \rightarrow 0$, together with the uniformity in μ that integrality constraints require.

Conclusion

We have obtained an asymptotic equivalent, as n tends to infinity, for an integral involving powers of the cumulative distribution function of a standard half-Gaussian variable. This equivalent allows us to prove that, for two independent families of centered Gaussian variables of respective sizes k and $n - k$ and respective variances $\sigma_1^2 > 2\sigma_2^2$, the probability that the largest absolute value in the first family exceeds the largest in the second tends to 1 in the linear regime $k \sim \gamma n$. The rate is governed by the parameter $\mu = \sigma_1^2/\sigma_2^2 - 1 > 1$, and the probability differs from 1 by a quantity of order $(\ln n)^{\mu/2}/n^\mu$. The analytic asymptotic results are valid for both integer and real values of the parameters k , n , and m , whereas their probabilistic interpretation naturally requires integer sample sizes.

A Proof of Lemma 1

Proof Let $u = G(x)^k$. The inverse change of variable is $x = G^{-1}(u^{1/k})$, where u runs over $(0, 1)$ as x runs over $(0, +\infty)$, so that

$$1 - J_n(k, \alpha) = \int_0^1 \left[1 - \Lambda(u, \alpha, k)^{n-k} \right] du, \quad (73)$$

with

$$\Lambda(u, \alpha, k) = G(\alpha G^{-1}(u^{1/k})) \in (0, 1). \quad (74)$$

Fix $u \in (0, 1)$. Monotonicity in k : $k \mapsto u^{1/k}$ is increasing with k and G^{-1} is increasing, so Λ is increasing in k . As k increases, $n - k > 0$ decreases and Λ^{n-k} increases, so the integrand in (73) therefore decreases. Monotonicity in α : $G^{-1}(u^{1/k}) > 0$ is fixed and the exponent $n - k > 0$ is fixed, while Λ increases with α . Thus, Λ^{n-k} increases and the integrand decreases. $\square$

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