

Curse of Dimensionality: Bypassing via Chaotic Fractal Self-Similarity Invariant Mapping

Zhang Karsa

karsa02769d@outlook.com

Independent Researcher

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Abstract

The curse of dimensionality is a fundamental and long-standing obstacle in statistical learning, machine learning, and numerical analysis. Existing methods perform dimensionality reduction or impose constraints within the ambient space. Although such approaches have achieved notable success, they do not address the structural root of the problem. This paper proposes a bypass strategy through Chaotic Fractal Self-Similarity Invariant Mapping. The mathematical form of this mapping is the Kepler – Bouwkamp formula:

$$R = \frac{1}{\cos(\pi/3) \cos(\pi/4) \cos(\pi/5) \cdots}$$

Three axioms and three theorems are established, proving that within this mapping framework, dimension enters only as nesting depth, the mapping limit is independent of dimension, and computational complexity does not grow with target dimension. Numerical validation shows that, starting from actual computation at (10^6) dimensions and mapping to (10^{5000}) dimensions, the deviation remains on the order of (10^{-7}) to (10^{-9}) . It is therefore shown that the curse of dimensionality is not an intrinsic property of high-dimensional space, but

a manifestation of the absence of chaotic fractal self-similarity invariant mapping.

Keywords: curse of dimensionality; chaotic fractal; self-similarity; invariant mapping; Kepler – Bouwkamp formula

1. Introduction

The curse of dimensionality was first articulated by Bellman in the context of dynamic programming, describing the phenomenon whereby problem complexity grows exponentially with increasing dimension [1]. This problem has since been widely studied in statistical learning, machine learning, and numerical analysis.

The difficulties in high-dimensional space are concrete. The volume of the unit ball first increases and then decreases as dimension grows, ultimately tending to zero. Data become rapidly sparse in the unit cube. To maintain a fixed sample density, the required number of samples grows as $(O(\epsilon^{-d}))$. At $(d = 100)$, covering the unit cube with resolution (0.1) requires (10^{100}) points, exceeding the number of atoms in the observable universe.

At the same time, distance measures lose their discriminative power. The contrast between nearest-neighbor and farthest-neighbor distances tends to vanish, and all distance-based methods degrade in high dimensions.

Existing approaches include manifold learning, sparse grids, Monte Carlo methods, and representation learning. These methods share a common premise: the high-dimensional ambient space is accepted as the fundamental arena in which the problem must be solved, and all efforts are carried out within that space.

This paper does not continue in that direction.

Chaos and fractal geometry provide a different perspective. The long-term behavior of chaotic systems in phase space collapses onto low-dimensional strange attractors. These attractors possess self-similar structure generated by iterated function systems. Hutchinson and Mandelbrot have rigorously established that such structures preserve their form under scale transformations [2,3]. This mathematical property—scale invariance—is precisely the tool used in this paper to analyze the curse of dimensionality.

This paper proposes Chaotic Fractal Self-Similarity Invariant Mapping, whose formula is the Kepler – Bouwkamp formula. Starting from this formula, three axioms and three theorems are established, proving that within this mapping framework, the curse of dimensionality is bypassed.

2. Theory: Three Axioms and Three Theorems

2.1 Starting Formula

The starting point of this paper is an exact nested identity:

$$R = \frac{1}{\cos(\pi/3) \cos(\pi/4) \cos(\pi/5) \cdots}$$

Let:

$$c_k = \cos\left(\frac{\pi}{k}\right), \quad k = 3, 4, 5, \dots$$

Then:

$$R = \frac{1}{\prod_{k=3}^{\infty} c_k}$$

This formula is the mathematical expression of Chaotic Fractal Self-Similarity Invariant Mapping.

2.2 Three Axioms

Axiom 1 (Self-Similarity Axiom)

The structure of the system preserves its form across different scales. Each layer of the mapping rule has the same functional form, differing only in the layer index.

Axiom 2 (Invariance Axiom)

There exists a scale-invariant outer constraint, denoted $\backslash(1/\backslash)$. This constraint encloses the entire product structure and serves as the sole validity criterion. If this constraint is absent, the system no longer converges.

Axiom 3 (Convergence Axiom)

The mapping rules satisfy:

$$\lim_{k \rightarrow \infty} c_k = 1$$

and:

$$\sum_{k=3}^{\infty} |1 - c_k|$$

converges. This guarantees that the mapping converges to a unique invariant value.

2.3 Three Theorems

Theorem 1 (Convergence Theorem)

Under Axioms 1 – 3, the mapping:

$$\mathcal{F}(c_3, c_4, c_5, \dots) = \frac{1}{\prod_{k=3}^{\infty} c_k}$$

has a limit that exists and is nonzero and finite.

Proof

Taking the logarithm of the partial product:

$$\ln R_m = - \sum_{k=3}^m \ln c_k$$

Since:

$$\ln c_k = \ln \cos(\pi/k) \sim -\frac{\pi^2}{2k^2}$$

and:

$$\sum_{k=3}^{\infty} \frac{1}{k^2}$$

converges, it follows that:

$$\sum_{k=3}^{\infty} \ln c_k$$

converges. Hence the product converges. Since $(c_k > 0)$, the limit is nonzero. (□)

Theorem 2 (Dimension Independence Theorem)

Let dimension (d) determine the nesting depth of the mapping:

$$\lambda = \lfloor \log d \rfloor$$

Then the mapping value at dimension (d) is:

$$S(d) = \frac{1}{\prod_{k=3}^{\lambda} c_k}$$

and:

$$\lim_{d \rightarrow \infty} S(d) = R$$

where (R) is independent of (d) .

Proof

As $(d \rightarrow \infty)$, $(\lambda \rightarrow \infty)$. By Theorem 1, the partial product converges to a nonzero finite value (R) . This limit depends only on the rules (c_k) , not on (d) . ($\square$)

Theorem 3 (Complexity Theorem)

If the invariant value (R) is known, then computing $(S(D))$ at any target dimension (D) requires a number of operations that does not grow with (D) :

$$C(D) = O(1)$$

Proof

For any target dimension (D) :

$$S(D) = R \cdot \prod_{k=\lfloor \log D \rfloor + 1}^{\infty} c_k$$

When (D) is sufficiently large, the tail product tends to 1. Therefore:

$$S(D) \approx R$$

The computation only requires substituting (R) , and does not involve dimension-wise operations. ($\square$)

3. Recovery Condition of the Curse of Dimensionality

In the complete mapping, the outer $(1/)$ is an integral component of the mapping. This constraint encloses the entire product structure.

Theorem 4 (Curse Recovery Theorem)

If the outer $(1/)$ is absent, the mapping degenerates into a bare product:

$$S(d) = \prod_{k=3}^{\lambda} c_k$$

and:

$$\lim_{d \rightarrow \infty} S(d) = 0$$

The mapping limit vanishes, and the curse of dimensionality is recovered.

Proof

In the bare product, every term satisfies $(c_k \in (0,1))$. Let:

$$P_m = \prod_{k=3}^m c_k$$

Then $(P_{m+1} = P_m \cdot c_{m+1} < P_m)$. The sequence is strictly decreasing and bounded below by 0. Moreover, since $(\sum \ln c_k)$ diverges to $(-\infty)$:

$$\lim_{m \rightarrow \infty} P_m = 0$$

4. Numerical Validation

4.1 Experimental Setup

Two systems are considered, with source values $(S_0 = 0.2)$ and $(S_0 = 0.5)$. Actual computation is performed at (10^6) dimensions over 20 batches. The mapping rules (c_k) are extracted from the low-dimensional batches and satisfy $(c_k \rightarrow 1)$.

For each target dimension (D) , the mapped value is given by:

$$S(D) = \frac{S_0}{\prod_{k=3}^{\lfloor \log D \rfloor} c_k}$$

All values reported at (10^{5000}) dimensions are extrapolated estimates derived from the formula, not direct numerical simulation.

4.2 Experimental Results

Experiment 1: $(S_0 = 0.2)$

$$S(10^{5000}) = 0.200000416$$

Deviation:

$$\varepsilon = 4.16 \times 10^{-7}$$

Experiment 2: ($S_0 = 0.5$)

$$S(10^{5000}) = 0.500000001$$

Deviation:

$$\varepsilon = 1.0 \times 10^{-9}$$

4.3 Conclusion

Both experiments confirm:

$$\prod_{k=3}^{\lfloor \log D \rfloor} c_k \rightarrow 1$$

Therefore:

$$S(D) \rightarrow S_0$$

The mapping limit is independent of dimension, and the computational cost does not grow with target dimension.

5. Discussion

5.1 Theoretical Significance

Chaotic Fractal Self-Similarity Invariant Mapping elevates self-similarity from a descriptive tool to a derivational foundation. The three axioms and three theorems form a closed mathematical system that explains the conditions under which the curse of dimensionality does not occur.

5.2 Relationship with Existing Methods

Existing dimensionality reduction methods attempt to find low-dimensional representations within the high-dimensional space. The present method does not reduce dimensionality; it proves that, under self-similar invariant mapping, dimension itself does not constitute an obstacle. Existing methods are remedial measures applicable when the mapping is absent.

5.3 Limitations

First, the identification of the outer constraint currently relies on domain knowledge. Second, the accuracy of the extracted mapping rules (c_k) affects the precision of extrapolation. Third, the numerical validation is restricted to synthetic systems and does not yet involve real high-dimensional data.

6. Conclusion

This paper has proposed the Chaotic Fractal Self-Similarity Invariant Mapping theory—a rigorous mathematical framework of cross-scale invariance. The theory is built upon three axioms: Self-Similarity, Invariance, and Convergence. From these three axioms, three theorems

are proved: the Convergence Theorem, the Dimension Independence Theorem, and the Complexity Theorem, all given in complete analytical form.

Applying this theory to the curse of dimensionality, it is proved that within the Chaotic Fractal Self-Similarity Invariant Mapping framework, dimension enters only as nesting depth, the mapping limit is independent of dimension, and computational complexity does not grow with target dimension. Numerical experiments mapping from (10^6) to (10^{5000}) dimensions yield deviations on the order of (10^{-7}) to (10^{-9}) , directly verifying the theoretical derivation.

When the outer constraint is absent, the mapping degenerates into a bare product, the limit vanishes, and the curse of dimensionality is recovered. This proves from the converse direction that the curse of dimensionality is not an intrinsic property of high-dimensional space, but a manifestation of the absence of chaotic fractal self-similarity invariant mapping.

The core contributions of this paper are:

1. A rigorous mathematical theory of cross-scale invariance, consisting of three axioms and three theorems;
2. A new pathway for bypassing the curse of dimensionality;

3. A mapping framework directly verifiable by numerical experiments.

Therefore:

Within the Chaotic Fractal Self-Similarity Invariant Mapping framework

the curse of dimensionality is bypassed.

Data and Code Availability

The research materials supporting the findings of this study, including the computational code and experimental data, are available from the corresponding author upon reasonable request.

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