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Total domination game on uniform hypergraphs The total domination game on uniform hypergraphs: exact values, asymptotics and exhaustive computation

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Abstract

Purpose In the total domination game on an isolate-free hypergraph H, Dominator and Staller alternately select vertices, each of which must totally dominate a vertex not totally dominated before; Dominator wants a short game and Staller a long one. Let $\gamma_{tg}(H)$ and $\gamma'_{tg}(H)$ denote the optimal length when Dominator, respectively Staller, starts. We ask how large these numbers can be relative to the order of a k-uniform hypergraph, and for which natural infinite families they can be determined exactly.
Methods We construct explicit strategies for both players from vertex covers, star decompositions and edges sharing a vertex, and prove a deterministic free-index lemma that we combine with probabilistic and explicit cyclic set systems. We also design an exact memoised minimax algorithm with a provably safe pruning rule, implement it twice independently, and apply it to exhaustively generated hypergraphs.
Results For the k-th power $G^{(k)}$ of a graph G, $\gamma_{tg}(G^{(k)}) \le 2\tau(G) - \varepsilon(G)$ and $\gamma'_{tg}(G^{(k)}) \le 2\tau(G)$, with matching degree-type lower bounds for $k \ge 4$. Hence $\gamma_{tg} = m$ for all loose paths and loose cycles with m edges, and $\gamma_{tg} = \gamma'_{tg} = |V(G)|$ for powers of regular bipartite graphs. The largest ratios of both game numbers to the order are $(2 + o(1)) \ln k/k$, disjoint edges are not extremal for any $k \ge 52$, and $\gamma_{tg}(H_{k,1}) = 2k - 1$ for the grid hypergraphs. Among all 289,443,894 relevant isomorphism classes with $k \in \{3, 4, 5\}$ and at most 11 vertices, no ratio exceeds $2/k$.
Conclusions Vertex covers govern the game on power hypergraphs, and the data support the bound $\gamma_{tg}(H) \le 2|V(H)|/3$ for $k = 3$.

Keywords: total domination game, uniform hypergraph, power hypergraph, vertex cover, exact minimax algorithm

MSC Classification: 05C65 , 05C69 , 05C57 , 91A43

1 Introduction

In the domination game of Brešar, Klavžar and Rall [3], two players, Dominator and Staller, alternately choose vertices of a graph. A vertex may be chosen only if it dominates a vertex that is not dominated by the previously chosen vertices, and the game ends when the chosen vertices form a dominating set. Dominator wants the game to be short and Staller wants it to be long. Henning, Klavžar and Rall [10] introduced the *total* version, in which a vertex totally dominates its open neighbourhood but not itself. The length of the game under optimal play is the *game total domination number*: $\gamma_{tg}(G)$ when Dominator starts (the D-game), and $\gamma'_{tg}(G)$ when Staller starts (the S-game). The area is surveyed in [2]; total domination itself is treated in [13].

In a hypergraph, two distinct vertices are adjacent if some edge contains both, and total domination is defined through the resulting open neighbourhoods. In a k-uniform hypergraph every vertex totally dominates at least $k - 1$ other vertices, so the basic extremal question is how the largest possible ratio $\gamma_{tg}(H)/|V(H)|$ decays as k grows. The total game is harder to analyse than the ordinary one, because a move never completes the domination of the vertex just played and the legality of a move depends on vertices other than the one played, so that constructions and strategies for the ordinary game cannot simply be transferred.

For graphs, the extremal theory of the total domination game is now well developed. Bujtás, Henning and Tuza [4] observed that the game on G coincides with the *transversal game* on the open neighbourhood hypergraph of G, and used this to attack the 3/4-conjecture of Henning, Klavžar and Rall [11]. The conjecture states that $\gamma_{tg}(G) \le 3|V(G)|/4$ when every component of G has at least three vertices. It was settled for minimum degree at least two in [4, 12], and in general by Portier and Versteegen [16], who also proved $\gamma_{tg}(G) \le 5(|V(G)| + 1)/7$ under the minimum degree condition. The analogous 3/5-conjecture of Kinnersley, West and Zamani [15] for the ordinary game was proved by Versteegen [18]. Exact values are known for paths and cycles [8], and further graph classes continue to be studied [9]. Bounds for the transversal game on uniform hypergraphs were obtained in [5].

For hypergraphs, Bujtás, Henning, Tuza and Yeo [6] proved the sharp bound $\gamma_t(H) \le 2|V(H)|/(k+1)$ for k-uniform hypergraphs without isolated vertices or isolated edges when $k \le 6$. A variant of total domination in hypergraphs is studied in [17]. Bujtás, Patkós, Tuza and Vizer [7] introduced the domination game on hypergraphs. They proved that the largest value of $\gamma_g(H)/|V(H)|$ over k-uniform hypergraphs is $(2+o(1))\ln k/k$, and that $\gamma_g(H) \le 5|V(H)|/9$ for 3-uniform hypergraphs. They also defined the game total domination number of a hypergraph and stated the same asymptotic formula for γ_{tg} [7, Theorem 3.2], omitting the proof as analogous. We searched Google Scholar, arXiv, dblp and Crossref for 2019–2026. Apart from [7] and work on Maker–Breaker versions, which are different games, the search found no further work on the Dominator–Staller total domination game on hypergraphs. We cannot exclude that something was missed.

Several questions were open. The first is a proof of the asymptotic formula for γ_{tg}, and anything at all about γ'_{tg} for hypergraphs. The lower-bound construction of [7] gives every random edge one private vertex, and Staller plays these vertices. In the total game such a move is legal only while some other vertex of its edge is not totally dominated, and Dominator can dominate these vertices cheaply, so the legality of Staller's moves has to be established afresh. The second concerns moderate uniformity: disjoint edges give the ratio $2/k$, and the asymptotic formula shows that $2/k$ is not the truth for large k, but its error terms are effective only for very large k, so it was not known whether disjoint edges are extremal for, say, $k \le 100$. The third concerns exact values: apart from trivial constructions, such as disjoint unions of complete hypergraphs and hypergraphs with a vertex adjacent to all others, we know of no exact value of γ_{tg} or γ'_{tg} for a natural infinite family of uniform hypergraphs. Finally, for fixed uniformity the relation between the two games and γ_t was unexplored, and explicit bounds in the light of the now proved 3/4-conjecture had not been recorded.

For a graph G and $k \ge 3$ let $G^{(k)}$ be its k-th power, in which every edge is enlarged by $k-2$ new vertices. Let $\tau(G)$ be the vertex cover number. Let $\varepsilon(G) = 1$ if some minimum vertex cover of G contains both ends of an edge, and $\varepsilon(G) = 0$ otherwise. Our first main result combines Theorems 4.2 and 4.4 and Corollaries 4.6 and 4.7.

Theorem A. *Let G be a graph without isolated vertices, with m edges and maximum degree Δ. For every $k \ge 3$,*

$$\gamma_{tg}(G^{(k)}) \le 2\tau(G) - \varepsilon(G) \qquad \text{and} \qquad \gamma'_{tg}(G^{(k)}) \le 2\tau(G),$$

and for every $k \ge 4$ and $\Delta \ge 2$,

$$\gamma_{tg}(G^{(k)}) \ge \frac{2(m+1)}{\Delta} - 1 \qquad \text{and} \qquad \gamma'_{tg}(G^{(k)}) \ge \frac{2m}{\Delta}.$$

Consequently, for $k \ge 4$:

- *every loose path $P_m^{(k)}$ ($m \ge 2$) and every loose cycle $C_m^{(k)}$ ($m \ge 3$) satisfies $\gamma_{tg} = m$, and also $\gamma'_{tg} = m$ when m is even;*
- *for every regular bipartite graph G without isolated vertices, $\gamma_{tg}(G^{(k)}) = \gamma'_{tg}(G^{(k)}) = |V(G)|$.*

The upper bound in Theorem A implies $\gamma_{tg}(G^{(k)}), \gamma'_{tg}(G^{(k)}) \le m + \mathrm{odd}(G)$, where $\mathrm{odd}(G)$ is the number of components of G with an odd number of edges, and it is often much smaller; for hyperstars it gives the exact value 2.

Let $\Gamma(k) = \sup_H \gamma_{tg}(H)/|V(H)|$ and $\Gamma'(k) = \sup_H \gamma'_{tg}(H)/|V(H)|$, the suprema being taken over all isolate-free k-uniform hypergraphs. Our second main result is proved in Theorems 3.7 and 3.9.

Theorem B. *$\Gamma(k) = (2+o(1))\ln k/k$ and $\Gamma'(k) = (2+o(1))\ln k/k$; more precisely, $\Gamma(k), \Gamma'(k) \le 2(1+\ln(k-1))/(k-1)$ for every $k \ge 2$. Moreover, $\Gamma(k) > 2/k$ and $\Gamma'(k) > 2/k$ for every $k \ge 52$, witnessed by explicit hypergraphs.*

Both lower bounds follow from one deterministic statement, the free-index lemma (Theorem 3.4). Let $\mathcal{F}$ be a family of $(k-2)$-sets in which every j points miss at least $j+1$ members, and give every member two private vertices. The resulting k-uniform hypergraph has $\gamma_{tg}, \gamma'_{tg} \ge 2j+1$. The second private vertex is what keeps Staller's moves legal. The γ_{tg} half of Theorem B was stated in [7] without proof.

Our third main result is Theorem 6.6.

Theorem C. *The grid hypergraph $H_{k,1}$ of [7] satisfies $\gamma_{tg}(H_{k,1}) = 2k-1 = 2\gamma_t(H_{k,1}) - 1$ and $2k-1 \le \gamma'_{tg}(H_{k,1}) \le 2k$ for every $k \ge 2$.*

In addition, we show that each of the three possible values of $\gamma'_{tg} - \gamma_{tg}$ occurs, and that both upper bounds $\gamma_{tg} \le 2\gamma_t - 1$ and $\gamma'_{tg} \le 2\gamma_t$ are attained, in every uniformity $k \ge 3$ (Theorems 6.7 and 6.8, Example 6.9). We also characterise the hypergraphs with $\gamma_{tg} = 2$ and with $\gamma'_{tg} = 2$ (Proposition 6.2), give a codegree form of the general upper bound (Corollary 3.3), and show that the game is not monotone under hypergraph duality (Proposition 4.10). The upper bounds for fixed k in Section 5 are transfers of known results; we record them for hypergraphs, check their hypotheses and label them accordingly.

Computation. Section 7 describes an exact memoised minimax solver, with a pruning rule that is proved safe (Lemma 7.1). It was implemented twice, independently, and run on every 2-section of an isolate-free k-uniform hypergraph with $k \in \{3,4,5\}$ and at most 11 vertices. In this range no hypergraph exceeds the ratio $2/k$. This supports the conjecture $\gamma_{tg}(H) \le 2|V(H)|/3$ for $k=3$ (Conjecture 7.3). The solver was also used to test every strategy used in the proofs against all strategies of the opponent on small instances.

2 Preliminaries and the game framework

A hypergraph H consists of a finite vertex set $V(H)$ and a set $E(H)$ of subsets of $V(H)$ of size at least two, called edges; we write $n = |V(H)|$. H is *k-uniform* if all edges have k elements, *linear* if two edges share at most one vertex, and *isolate-free* if every vertex lies in an edge. An edge is *isolated* if it meets no other edge. The degree $d_H(v)$ is the number of edges containing v. Two distinct vertices are *adjacent* if some edge contains both, and the *open neighbourhood* of v is

$$N_H(v) = \{u \neq v : \ \{u,v\} \subseteq e \text{ for some } e \in E(H)\}.$$

For $S \subseteq V(H)$ let $N_H(S) = \bigcup_{s\in S} N_H(s)$; the vertices of $N_H(S)$ are *totally dominated* by S. A *total dominating set* (TD-set) is a set D with $N_H(D) = V(H)$; TD-sets exist if and only if H is isolate-free, and $\gamma_t(H)$ is the minimum size of a TD-set [6]. Since no vertex totally dominates itself, $\gamma_t(H) \ge 2$. Let $\delta_2(H) = \min_v |N_H(v)|$; if H is isolate-free and k-uniform, then $\delta_2(H) \ge k-1$, and if H is moreover linear, then $|N_H(v)| = (k-1)d_H(v)$.

The 2-*section* $[H]_2$ is the graph on $V(H)$ whose edges are the adjacent pairs of H, so that $N_{[H]_2}(v) = N_H(v)$. The *open neighbourhood hypergraph* $\mathrm{ONH}(H)$ has vertex set $V(H)$ and edge set $\{N_H(v) : v \in V(H)\}$ (repeated sets listed once).

Definition 2.1. Let H be isolate-free and let $A \subseteq V(H)$ be a set of vertices that are declared to be totally dominated. In the *total domination game* on $H|A$, Dominator and Staller alternately select vertices $v_1, v_2, \ldots$; with $S_i = \{v_1, \ldots, v_i\}$, the move v_i is *legal* if $N_H(v_i) \not\subseteq A \cup N_H(S_{i-1})$. The game ends as soon as $A \cup N_H(S_i) = V(H)$. Under optimal play (Dominator minimising, Staller maximising the number of moves) the length of the game is $\gamma_{tg}(H|A)$ if Dominator starts and $\gamma'_{tg}(H|A)$ if Staller starts; $\gamma_{tg}(H) = \gamma_{tg}(H|\emptyset)$ and $\gamma'_{tg}(H) = \gamma'_{tg}(H|\emptyset)$.

A selected vertex v satisfies $N_H(v) \subseteq N_H(S)$ afterwards, so no vertex is selected twice; the game is finite and ends with a TD-set, whence $\gamma_{tg}(H), \gamma'_{tg}(H) \ge \gamma_t(H) \ge 2$. Legality and the end of the game depend only on the set $A \cup N_H(S_i)$ of totally dominated vertices, which together with the player to move is the *position* of the game. Below, "dominated" always means "totally dominated".

For a hypergraph F and a set B of edges declared to be hit, the *transversal game* on $F|B$ is played by Edge-hitter and Staller, who alternately select vertices that hit at least one edge not hit before, until every edge is hit; $\tau_g(F|B)$ and $\tau'_g(F|B)$ are its lengths under optimal play when Edge-hitter, respectively Staller, starts [4].

Proposition 2.2. *Let H be isolate-free and $A \subseteq V(H)$.*

(i) *The total domination game on $H|A$ coincides with the total domination game on $[H]_2|A$.*

(ii) *Let $\mathrm{ONH}^*(H)$ be the multi-hypergraph on $V(H)$ with one edge $E_u = N_H(u)$ for every $u \in V(H)$. The total domination game on $H|A$ coincides with the transversal game on $\mathrm{ONH}^*(H)|\{E_u : u \in A\}$. In particular $\gamma_{tg}(H) = \tau_g(\mathrm{ONH}(H))$ and $\gamma'_{tg}(H) = \tau'_g(\mathrm{ONH}(H))$.*

Proof. (i) Legality and the end of the game are expressed through the sets $N_H(v) = N_{[H]_2}(v)$ only. (ii) A vertex u is dominated by S if and only if $S \cap N_H(u) \neq \emptyset$, i.e. if and only if S hits E_u. Hence a move dominates a new vertex if and only if it hits a new edge, and the game ends if and only if all edges are hit. For $A = \emptyset$ repeated edges are irrelevant in the transversal game. □

Part (ii) is the hypergraph form of the observation of [4], and part (i) the total analogue of [7, Proposition 2.1]. Both are used to transfer known results; the following describes exactly which graphs arise.

Proposition 2.3. *Let $k \ge 2$. A graph is the 2-section of an isolate-free k-uniform hypergraph if and only if it has no isolated vertices and every edge lies in a clique of order k. Consequently the maxima of γ_{tg} and of γ'_{tg} over isolate-free k-uniform hypergraphs on n vertices equal the maxima over the class $\mathcal{G}_{n,k}$ of such graphs on n vertices.*

Proof. If $G = [H]_2$ and $uv \in E(G)$, then some edge of H contains u, v and spans a K_k in G. Conversely, if G has the stated property, the hypergraph whose edges are the vertex sets of the k-cliques of G is k-uniform, isolate-free and has 2-section G. The last statement follows from Proposition 2.2(i). □

For $k \geq 3$ every graph in $\mathcal{G}_{n,k}$ has minimum degree at least $k-1 \geq 2$. The following two facts were proved for graphs in [10]; by Proposition 2.2(i) they hold verbatim for hypergraphs. The first is known as the Total Continuation Principle.

Lemma 2.4. *If H is isolate-free and $B \subseteq A \subseteq V(H)$, then $\gamma_{tg}(H|A) \leq \gamma_{tg}(H|B)$ and $\gamma'_{tg}(H|A) \leq \gamma'_{tg}(H|B)$.*

Theorem 2.5. *If H is isolate-free, then $|\gamma_{tg}(H) - \gamma'_{tg}(H)| \leq 1$, $\gamma_t(H) \leq \gamma_{tg}(H) \leq 2\gamma_t(H) - 1$ and $\gamma_t(H) \leq \gamma'_{tg}(H) \leq 2\gamma_t(H)$.*

For a graph G and $k \geq 2$, the *k-th power* $G^{(k)}$ [14] is obtained by enlarging every edge uv to $\{u,v\} \cup P_{uv}$, where the sets P_{uv} of $k-2$ new vertices are pairwise disjoint. Vertices of G are *spine vertices*, the others *private vertices*. For a private $x \in P_{uv}$ we have $N(x) = (\{u,v\} \cup P_{uv}) \setminus \{x\}$, and for a spine vertex v, $N(v)$ is the union of the edges through v, minus v. A *vertex cover* of G is a set $C \subseteq V(G)$ meeting every edge; $\tau(G)$ is the smallest size of a vertex cover, and we put $\varepsilon(G) = 1$ if some vertex cover of size $\tau(G)$ contains both ends of an edge of G, and $\varepsilon(G) = 0$ otherwise. The *loose path* $P_m^{(k)}$ and the *loose cycle* $C_m^{(k)}$ are the k-th powers of the path and of the cycle with m edges; we write $v_0, \ldots, v_m$ for the spine vertices of $P_m^{(k)}$ and $e_i \supseteq \{v_{i-1}, v_i\}$ for its edges (indices modulo m for $C_m^{(k)}$). The k-th power of the star $K_{1,m}$ is the *hyperstar* $S_m^{(k)}$.

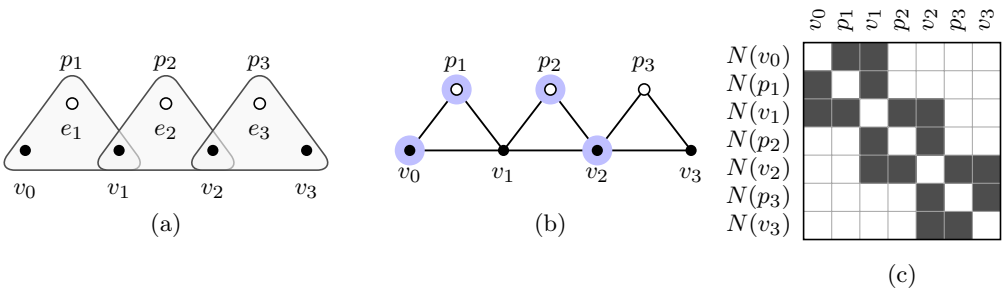


Fig. 1 Three descriptions of the same game, for the 3-uniform loose path $P_3^{(3)}$. (a) The hypergraph; black: spine vertices, white: private vertices. (b) Its 2-section (Proposition 2.2(i)); the open neighbourhood of v_1 is shaded. (c) The incidence matrix of $\mathrm{ONH}(P_3^{(3)})$: row u is the edge $N(u)$ of the transversal game, and a move v (a column) is legal exactly when it hits a row that has not yet been hit (Proposition 2.2(ii))

3 Sharp asymptotics and the threshold for disjoint edges

This section proves Theorem B. The upper bound is a first-moment estimate for γ_t combined with Theorem 2.5; the lower bound and the threshold both follow from a deterministic statement about set systems, the free-index lemma (Theorem 3.4). All logarithms are natural.

Lemma 3.1. *If H is an isolate-free hypergraph with $\delta_2 = \delta_2(H) \geq 1$, then $\gamma_t(H) \leq n(1 + \ln \delta_2)/\delta_2$.*

Proof. Put $p = \ln \delta_2/\delta_2 \in [0, 1/e]$ and let S contain every vertex independently with probability p. Let $T = \{v : N_H(v) \cap S = \emptyset\}$ and choose $u_v \in N_H(v)$ for every $v \in T$. Then $D = S \cup \{u_v : v \in T\}$ is a TD-set, and since $\mathbb{P}(v \in T) = (1-p)^{|N_H(v)|} \leq e^{-p\delta_2} = 1/\delta_2$,

$$\mathbb{E}|D| \leq np + n/\delta_2 = n(1 + \ln \delta_2)/\delta_2.$$

Some outcome satisfies $|D| \leq \mathbb{E}|D|$. □

Theorem 3.2. *For every $k \geq 2$, with $\delta_2 = \delta_2(H)$,*

$$\gamma_{tg}(H) \leq 2n\,\frac{1 + \ln \delta_2}{\delta_2} - 1 \leq 2n\,\frac{1 + \ln(k-1)}{k-1} - 1,$$
$$\gamma'_{tg}(H) \leq 2n\,\frac{1 + \ln \delta_2}{\delta_2} \leq 2n\,\frac{1 + \ln(k-1)}{k-1}.$$

If H is linear with minimum degree d, then $\delta_2 \geq (k-1)d$ may be used.

Proof. Combine Theorem 2.5 with Lemma 3.1; the function $x \mapsto (1 + \ln x)/x$ is non-increasing for $x \geq 1$ and $\delta_2 \geq k-1$. For linear H, $|N_H(v)| = (k-1)d_H(v)$. □

The parameter δ_2 is the minimum degree of the 2-section, not a codegree. The linear case is the special case $\Delta_{\mathrm{co}}(H) = 1$ of the following estimate. It is in terms of the minimum degree $\delta(H)$ and the maximum codegree $\Delta_{\mathrm{co}}(H) = \max_{u \neq v} |\{e \in E(H) : u, v \in e\}|$.

Corollary 3.3. *Let H be an isolate-free k-uniform hypergraph on n vertices and put $x = \max\{1, (k-1)\delta(H)/\Delta_{\mathrm{co}}(H)\}$. Then*

$$\gamma_{tg}(H) \le 2n\,\frac{1+\ln x}{x} - 1 \qquad \text{and} \qquad \gamma'_{tg}(H) \le 2n\,\frac{1+\ln x}{x}.$$

In particular $\gamma_{tg}(H)/n \to 0$ along every sequence of hypergraphs with $(k-1)\delta(H)/\Delta_{\mathrm{co}}(H) \to \infty$, for instance along linear d-regular k-uniform hypergraphs with $(k-1)d \to \infty$.

Proof. Fix v and count the pairs (u, e) with $e \in E(H)$, $v \in e$ and $u \in e \setminus \{v\}$. There are exactly $(k-1)d_H(v) \ge (k-1)\delta(H)$ such pairs, and every $u \in N_H(v)$ occurs in at most $\Delta_{\mathrm{co}}(H)$ of them. Hence $|N_H(v)| \ge (k-1)\delta(H)/\Delta_{\mathrm{co}}(H)$, and since also $|N_H(v)| \ge 1$, we get $\delta_2(H) \ge x$. The bounds follow as in the proof of Theorem 3.2, because $y \mapsto (1+\ln y)/y$ is non-increasing for $y \ge 1$. □

Let A be a finite set and let $\mathcal{F} = (e_1, \dots, e_q)$ be a family of non-empty subsets of A (repetitions allowed) with $\bigcup_i e_i = A$. Let $H(\mathcal{F})$ be the hypergraph with vertex set $A \cup \{x_1, y_1, \dots, x_q, y_q\}$, where the x_i, y_i are $2q$ new vertices, and with edges

$$f_i = e_i \cup \{x_i, y_i\} \qquad (1 \le i \le q).$$

$H(\mathcal{F})$ is isolate-free, it has $|A| + 2q$ vertices, and it is $(r+2)$-uniform if all e_i have size r. For $t \ge 0$ let

$$c_{\mathcal{F}}(t) = \max\big\{\, |\{i : e_i \cap X \neq \emptyset\}| \;:\; X \subseteq A,\ |X| \le t \,\big\}$$

be the largest number of members of $\mathcal{F}$ met by t points; thus $c_{\mathcal{F}}(t) \le q - j - 1$ means that every t-subset of A is disjoint from at least $j+1$ members of $\mathcal{F}$.

We refer to the following result as the *free-index lemma*.

Theorem 3.4. *If $j \ge 0$ and $c_{\mathcal{F}}(j) + j \le q - 1$, then $\gamma_{tg}(H(\mathcal{F})) \ge 2j+1$ and $\gamma'_{tg}(H(\mathcal{F})) \ge 2j+1$. If $c_{\mathcal{F}}(j) + j \le q - 2$, then moreover $\gamma'_{tg}(H(\mathcal{F})) \ge 2j+2$.*

Proof. Write $c = c_{\mathcal{F}}$. Since every e_i is non-empty, $c(t+1) \ge \min\{q, c(t)+1\}$: if a set X of size at most t meets $c(t) < q$ members, add to X a point of a member it misses. Iterating, $c(t+p) \ge \min\{q, c(t)+p\}$ for all $t, p \ge 0$.

Let Q be the set of vertices played so far. Call an index i *free* if $Q \cap f_i = \emptyset$. Since $N(y_i) = f_i \setminus \{y_i\}$ and $N(x_i) = f_i \setminus \{x_i\}$, a free index i has the following two properties: y_i is not dominated, so x_i is a legal move; and x_i is not dominated, so the game is not over.

Staller's strategy. Whenever it is her turn and a free index exists, she plays x_i for some free index i; otherwise she plays any legal move.

Counting non-free indices. Suppose Dominator has made d moves, of which those in A form the set X and $p = d - |X|$ are private vertices. Suppose Staller has made σ moves, all of the form x_i; in particular none of her moves lies in A, so X is the set of all played vertices in A. An index i is non-free only if $e_i \cap X \neq \emptyset$ or a private vertex of f_i has been played. At most $c(|X|)$ indices of the first kind exist, at most p are made non-free by Dominator's private moves and at most σ by Staller's moves, and the first two kinds together number at most q. Hence the number of non-free indices is at most

$$\min\{q,\ c(|X|) + p\} + \sigma \;\le\; c(d) + \sigma.$$

D-game. We show by induction on $i \in \{1, \dots, j\}$ that Staller's i-th move is some x_ℓ and that the game is not over after it. Before her i-th move the game is not over: for $i = 1$ because $\gamma_t \ge 2$, and for $i > 1$ because a free index remained after her $(i-1)$-st move, so that Dominator's i-th move cannot end the game. At that moment Dominator has made i moves and Staller $i-1$ moves of the form x_ℓ, so at most $c(i) + i - 1 \le c(j) + j - 1 \le q - 2$ indices are non-free and a free index exists. After her move at most $c(i) + i \le c(j) + j \le q - 1$ indices are non-free, so a free index remains and the game is not over. Thus the game lasts at least $2j+1$ moves.

S-game. Before Staller's i-th move, $1 \le i \le j+1$, both players have made $i-1$ moves, so at most $c(i-1) + i - 1 \le c(j) + j \le q - 1$ indices are non-free. Hence a free index exists before each of Staller's first $j+1$ moves; in particular the game is not over before her $(j+1)$-st move, which is the $(2j+1)$-st move of the game. If $c(j) + j \le q - 2$, then after this move at most $c(j) + j + 1 \le q - 1$ indices are non-free and the game continues. □

The hypothesis concerns only the set system $\mathcal{F}$, and it is the analogue of the transversal property used in [7]. What is specific to the total game is the pair x_i, y_i: a single private vertex x_i does not dominate itself, and its legality would depend on the vertices of e_i, which Dominator can dominate cheaply.

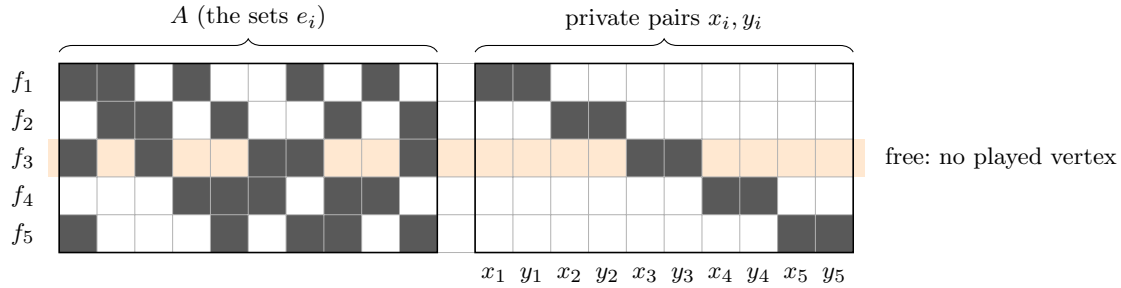


Fig. 2 The hypergraph $H(\mathcal{F})$ of Theorem 3.4, drawn as an incidence diagram: row i is the edge $f_i = e_i \cup \{x_i, y_i\}$. While the shaded row i contains no played vertex (index i is *free*), y_i is not dominated, so Staller's move x_i is legal and the game cannot be over

The following lemma is the standard random construction of [7], written out with explicit estimates.

Lemma 3.5. *Let r be sufficiently large, $N = \lceil r \ln r \rceil$, $A = [N]$ and $s = \lfloor \ln^2 r - 10 \ln r \ln\ln r \rfloor$. Let $e_1, \dots, e_r$ be independent, uniformly random r-subsets of A. With probability tending to 1 as $r \to \infty$, every $X \subseteq A$ with $|X| \le s$ is disjoint from at least $\frac{1}{2e^3} \ln^{10} r$ of the sets e_i.*

Proof. All inequalities below hold for sufficiently large r. It suffices to consider $|X| = s$. Fix such an X. For a uniformly random r-subset e of A,

$$p := \mathbb{P}(e \cap X = \emptyset) = \prod_{i=0}^{r-1} \Big(1 - \frac{s}{N-i}\Big) \ge \Big(1 - \frac{s}{N-r}\Big)^r .$$

Put $x = s/(N-r)$. As $N - r \ge r(\ln r - 1)$, we have $x \le \ln^2 r/(r(\ln r - 1)) \le \frac{1}{2}$, and $\ln(1-x) \ge -x - x^2$ for $0 \le x \le \frac{1}{2}$; hence $p \ge \exp(-rx - rx^2)$. Now

$$\begin{aligned} rx \le \frac{s}{\ln r - 1} &\le \frac{\ln r\,(\ln r - 10 \ln\ln r)}{\ln r - 1} \\ &= \Big(1 + \frac{1}{\ln r - 1}\Big)(\ln r - 10 \ln\ln r) \le \ln r - 10 \ln\ln r + 2 \end{aligned}$$

and $rx^2 \le \ln^4 r/(r(\ln r - 1)^2) \le 1$, so $p \ge e^{-3} \ln^{10} r / r$. The number Z_X of indices with $e_i \cap X = \emptyset$ has distribution $\mathrm{Bin}(r, p)$ with mean $\mu = rp \ge e^{-3} \ln^{10} r$, and by the Chernoff bound [1, Theorem A.1.13], $\mathbb{P}(Z_X < \mu/2) \le e^{-\mu/8}$. There are $\binom{N}{s} \le N^s \le e^{2 \ln^3 r}$ sets X (as $N \le r^2$ and $s \le \ln^2 r$), so the probability that some X has $Z_X < \mu/2$ is at most $\exp(2 \ln^3 r - e^{-3} \ln^{10} r/8) \to 0$. □

Theorem 3.6. *For every sufficiently large k there is an isolate-free k-uniform hypergraph H_k with $|V(H_k)| \le \lceil (k-2)\ln(k-2) \rceil + 2(k-2)$, $\gamma_{tg}(H_k) \ge 2s+1$ and $\gamma'_{tg}(H_k) \ge 2s+2$, where $s = \lfloor \ln^2(k-2) - 10 \ln(k-2) \ln\ln(k-2) \rfloor$.*

Proof. Let $r = k-2$, choose $e_1, \dots, e_r$ as in Lemma 3.5 with the stated property, and let $\mathcal{F} = (e_1, \dots, e_r)$ on $A^* = \bigcup_i e_i$. Every $X \subseteq A^*$ with $|X| \le s$ misses at least $\frac{1}{2e^3} \ln^{10} r \ge s + 2$ members, so $c_{\mathcal{F}}(s) + s \le r - 2$, and Theorem 3.4 applies with $j = s$ to $H_k = H(\mathcal{F})$, which is k-uniform with $|A^*| + 2r \le N + 2r$ vertices. □

Theorem 3.7. $\Gamma(k) = (2 + o(1))\dfrac{\ln k}{k}$ *and* $\Gamma'(k) = (2 + o(1))\dfrac{\ln k}{k}$. *More precisely,* $\Gamma(k), \Gamma'(k) \le 2(1 + \ln(k-1))/(k-1)$ *for every $k \ge 2$, and* $\Gamma(k), \Gamma'(k) \ge (2 - o(1)) \ln k / k$.

Proof. The upper bounds are Theorem 3.2, and $(1 + \ln(k-1))/(k-1) = (1 + o(1)) \ln k / k$. For the lower bounds, Theorem 3.6 gives $s = (1 - o(1)) \ln^2 r$ and $|V(H_k)| = (1 + o(1)) r \ln r$ with $r = k - 2$, so $\gamma_{tg}(H_k)/|V(H_k)| \ge (2 - o(1)) \ln r / r = (2 - o(1)) \ln k / k$, and the same for γ'_{tg}. □

We do not claim $\Gamma(k) \le \Gamma'(k)$: there are hypergraphs with $\gamma'_{tg} < \gamma_{tg}$ (Theorem 6.8).

A disjoint union of t edges of size k has $\gamma_{tg} = \gamma'_{tg} = 2t = 2n/k$ (Proposition 6.1), so $\Gamma(k), \Gamma'(k) \geq 2/k$, and by Theorem 3.7 this is not the truth for large k. The error terms of Lemma 3.5 are effective only for astronomically large k; the free-index lemma with $j \in \{1, 2\}$ shows that the threshold is at most 52.

For integers $1 \leq s \leq a$ let $q = \lceil 2a/s \rceil$ and consider, in $\mathbb{Z}_a$, the cyclic intervals

$$B_t = \{ts, ts+1, \ldots, ts+s-1\} \pmod{a}, \qquad 0 \leq t \leq q-1,$$

and the family $\mathcal{I}(a, s) = (\mathbb{Z}_a \setminus B_t)_{0 \leq t < q}$ of $(a-s)$-subsets of $\mathbb{Z}_a$.

Lemma 3.8. *If $q \geq 4$, every element of $\mathbb{Z}_a$ lies in at least two and at most three of the sets B_t. Consequently $\bigcup \mathcal{I}(a, s) = \mathbb{Z}_a$ and every element of $\mathbb{Z}_a$ is missed by at least two members of $\mathcal{I}(a, s)$.*

Proof. The sets B_t are the residues of the consecutive integer intervals $[ts, ts+s-1]$, which partition $\{0, 1, \ldots, qs-1\}$. Since $s \leq a$, no interval contains two congruent integers, so each B_t has s distinct elements, and the number of sets B_t containing a residue x equals the number of integers in $[0, qs-1]$ congruent to x. Every residue $x \in \{0, \ldots, a-1\}$ is represented by x and $x+a$, both smaller than $2a \leq qs$, so it lies in at least two sets B_t; and by at most three integers in $[0, qs-1]$, since $qs < 2a + s \leq 3a$. As $q \geq 4$, every element misses some B_t, i.e. lies in some member of $\mathcal{I}(a, s)$. □

Theorem 3.9. *The following hold.*

(i) *For every $k \geq 73$ there is an isolate-free k-uniform hypergraph H with n vertices such that $\gamma_{tg}(H) \geq 3$, $\gamma'_{tg}(H) \geq 3$ and $3k > 2n$.*

(ii) *For every k with $52 \leq k \leq 72$ there is an isolate-free k-uniform hypergraph H with n vertices such that $\gamma_{tg}(H) \geq 5$, $\gamma'_{tg}(H) \geq 5$ and $5k > 2n$.*

In particular $\Gamma(k) > 2/k$ and $\Gamma'(k) > 2/k$ for every $k \geq 52$.

Proof. (i) Let $s \leq a$ with $a - s = k - 2$ and $q = \lceil 2a/s \rceil \geq 4$, and let $H = H(\mathcal{I}(a, s))$. It is k-uniform and isolate-free, with $n = a + 2q$ vertices. By Lemma 3.8, $c_{\mathcal{I}}(1) \leq q - 2$, so Theorem 3.4 with $j = 1$ gives $\gamma_{tg}(H), \gamma'_{tg}(H) \geq 3$. It remains to choose s with $3k > 2a + 4q$. For $k \geq 78$ take $s = 16$: then $a = k + 14$, $q = \lceil (k+14)/8 \rceil \leq (k+21)/8$ and $2a + 4q \leq (5k + 77)/2 < 3k$. For $73 \leq k \leq 77$ take (s, a, q) from Table 1; the inequality $3k > 2n$ is checked there.

(ii) For each such k, Table 7 in Section A lists an integer p and base sets $D_1, \ldots, D_t \subseteq \mathbb{Z}_p$; let $S = \sum_i |D_i|$. Let $A = \{1, \ldots, t\} \times \mathbb{Z}_p$ and, for $c \in \mathbb{Z}_p$, $B_c = \{(i, x) \in A : \ c \in D_i + x\}$; since $c \in D_i + x$ holds for exactly $|D_i|$ values of x, $|B_c| = S$ for every c. Let $\mathcal{F} = (A \setminus B_c)_{c \in \mathbb{Z}_p}$, a family of $q = p$ sets of size $r = tp - S$. Two points $(i, x) \neq (i', x')$ lie together in $|(D_i + x) \cap (D_{i'} + x')|$ sets B_c, and a direct check of the $\binom{tp}{2}$ pairs (`free_index.py`) shows that this number is always at least 3. Hence every 2-subset of A is missed by at least three members of $\mathcal{F}$, i.e. $c_{\mathcal{F}}(2) + 2 \leq q - 1$; and every point (i, x) lies in exactly $p - |D_i| \geq 1$ members of $\mathcal{F}$, so $\bigcup \mathcal{F} = A$. By Theorem 3.4 with $j = 2$, $H(\mathcal{F})$ is k-uniform with $k = r + 2$, $\gamma_{tg}, \gamma'_{tg} \geq 5$ and $n = tp + 2p$ vertices; the parameters and the inequality $5k > 2n$ are listed in Table 1. □

Remark 3.10. For $k = 73$ we also verified $\gamma_{tg}(H) \geq 3$ directly, without Theorem 3.4, by testing the characterisation of $\gamma_{tg} = 2$ in Proposition 6.2 on all 109 vertices. The method cannot give $k \leq 72$ with $j = 1$: then every point of A lies in at most $q - 2$ of the q sets e_i, so $|A|(q-2) \geq q(k-2)$ and $n = |A| + 2q \geq \lceil q(k-2)/(q-2) \rceil + 2q$, and $3k > 2n$ has no integer solution with $k \leq 72$; the families $\mathcal{I}(a, s)$ attain this bound. For $j \geq 2$ the complements $A \setminus e_i$ must cover every j-subset of A at least $j + 1$ times. Inserting the Schönheim lower bound for the number of blocks of such coverings into the condition $(2j+1)k > 2n$ and searching all $|A| < 300$ and $j \leq 5$ gives no admissible parameters with $k < 34$. More precisely, the smallest admissible k is 41 for $j = 2$, 36 for $j = 3$ and 34 for $j = 4$ (`schoenheim.py`, recomputed independently). We have not determined the exact limit of the method; a simulated-annealing search for $j = 2$ families with $k < 52$ found none. Whether $\Gamma(k) > 2/k$ for some $k \leq 51$, and in particular for $k \in \{3, 4, 5\}$, remains open.

k	j	$\lvert A\rvert$	q	n	$2n/k$	family
73	1	87	11	109	2.986	$\mathcal{I}(87,16)$
74	1	88	11	110	2.973	$\mathcal{I}(88,16)$
75	1	88	12	112	2.987	$\mathcal{I}(88,15)$
76	1	89	12	113	2.974	$\mathcal{I}(89,15)$
77	1	90	12	114	2.961	$\mathcal{I}(90,15)$
52	2	90	18	126	4.846	$\mathbb{Z}_{18}$, $t=5$
53	2	88	22	132	4.981	$\mathbb{Z}_{22}$, $t=4$
54	2	88	22	132	4.889	$\mathbb{Z}_{22}$, $t=4$
55	2	102	17	136	4.945	$\mathbb{Z}_{17}$, $t=6$
56	2	102	17	136	4.857	$\mathbb{Z}_{17}$, $t=6$
57	2	100	20	140	4.912	$\mathbb{Z}_{20}$, $t=5$
58	2	112	16	144	4.966	$\mathbb{Z}_{16}$, $t=7$
59	2	87	29	145	4.915	$\mathbb{Z}_{29}$, $t=3$
60	2	105	21	147	4.900	$\mathbb{Z}_{21}$, $t=5$
61	2	114	19	152	4.984	$\mathbb{Z}_{19}$, $t=6$
62	2	114	19	152	4.903	$\mathbb{Z}_{19}$, $t=6$
63	2	104	26	156	4.952	$\mathbb{Z}_{26}$, $t=4$
64	2	104	26	156	4.875	$\mathbb{Z}_{26}$, $t=4$
65	2	126	18	162	4.985	$\mathbb{Z}_{18}$, $t=7$
66	2	108	27	162	4.909	$\mathbb{Z}_{27}$, $t=4$
67	2	115	23	161	4.806	$\mathbb{Z}_{23}$, $t=5$
68	2	143	13	169	4.971	$\mathbb{Z}_{13}$, $t=11$
69	2	120	24	168	4.870	$\mathbb{Z}_{24}$, $t=5$
70	2	112	28	168	4.800	$\mathbb{Z}_{28}$, $t=4$
71	2	105	35	175	4.930	$\mathbb{Z}_{35}$, $t=3$
72	2	125	25	175	4.861	$\mathbb{Z}_{25}$, $t=5$

Table 1 Explicit hypergraphs of Theorem 3.9. Top: the cyclic-interval families $\mathcal{I}(a,s)$ with the fewest vertices for $73 \le k \le 77$ (for $k \ge 78$, $s=16$ works). Bottom: the cyclic families with $j=2$ (t base sets in $\mathbb{Z}_p$, $q=p$; listed in Table 7), found by randomised search (`cyc.py`, `cyc2.py`) and verified exhaustively (`free_index.py`); in all cases $3k > 2n$, respectively $5k > 2n$.

4 Power hypergraphs

Throughout this section G is a graph without isolated vertices, with m edges and maximum degree Δ, and $k \ge 3$. For an edge uv of G we write $\widehat{uv} = \{u,v\} \cup P_{uv}$ for the corresponding edge of $G^{(k)}$. Recall that $N(x) = \widehat{uv} \setminus \{x\}$ for $x \in P_{uv}$, and that two vertices of $G^{(k)}$ are adjacent if and only if they lie in a common edge $\widehat{uv}$.

A *star* of G is a non-empty set of edges with a common end, its *centre*; a single edge is a star with either end as centre. A *star decomposition* of G is a partition of $E(G)$ into stars, called *blocks*, with a centre fixed for each block. If C is a vertex cover, assigning every edge to an end in C gives a star decomposition with at most $|C|$ blocks; conversely, the centres of a star decomposition form a vertex cover. Hence the least number of blocks of a star decomposition is $\tau(G)$.

For a block β let $V(\beta) = \bigcup_{uv\in\beta} \widehat{uv}$. Every vertex of $G^{(k)}$ lies in some $V(\beta)$. A private vertex of an edge of β (a *private vertex of* β) lies in no other $V(\beta')$, whereas spine vertices may lie in several. Call β *settled* if every vertex of $V(\beta)$ is dominated; the game is over exactly when all blocks are settled.

Lemma 4.1. *Let β be a block with centre c and let Q be the set of played vertices. If $c \in Q$ and Q contains a vertex of $V(\beta) \setminus \{c\}$, then β is settled.*

Proof. The centre lies in every edge of β, so it dominates $V(\beta) \setminus \{c\}$. A played vertex $z \in V(\beta) \setminus \{c\}$ lies in an edge of β together with c, so it dominates c. □

Theorem 4.2. *Let $k \ge 3$ and let C be a vertex cover of G. Then*

$$\gamma_{tg}(G^{(k)}) \le 2|C| \qquad \text{and} \qquad \gamma'_{tg}(G^{(k)}) \le 2|C|.$$

If C contains two adjacent vertices, then moreover $\gamma_{tg}(G^{(k)}) \le 2|C| - 1$. Consequently

$$\gamma_{tg}(G^{(k)}) \le 2\tau(G) - \varepsilon(G) \qquad \text{and} \qquad \gamma'_{tg}(G^{(k)}) \le 2\tau(G).$$

Proof. Write $H = G^{(k)}$. Choose a map $\kappa : E(G) \to C$ with $\kappa(uv) \in \{u, v\}$ for every $uv \in E(G)$; it exists because C is a vertex cover. For $c \in C$ put $\beta_c = \kappa^{-1}(c)$, and let $\mathcal{B} = \{\beta_c : \ c \in C, \ \beta_c \neq \emptyset\}$ be the set of *blocks*; c is the *centre* of β_c. Then $\mathcal{B}$ is a star decomposition of G and

$$|\mathcal{B}| \leq |C|. \tag{1}$$

For $\beta \in \mathcal{B}$ let $P(\beta) = \bigcup_{uv \in \beta} P_{uv}$. For a play $z_1, \dots, z_L$ and $0 \leq j \leq L$ let

$$Q_j = \{z_1, \dots, z_j\}, \qquad D_j = \bigcup_{i=1}^{j} N(z_i),$$

and call β *settled* after move j if $V(\beta) \subseteq D_j$. Since $D_0 \subseteq D_1 \subseteq \cdots \subseteq D_L$, a settled block stays settled.

Step 1: two facts on blocks.

(F1) $\bigcup_{\beta \in \mathcal{B}} V(\beta) = V(H)$, and for $x \in P_{uv} \subseteq P(\beta)$,

$$x \notin V(\beta') \ \ (\beta' \in \mathcal{B} \setminus \{\beta\}) \qquad \text{and} \qquad N(x) = \widehat{uv} \setminus \{x\} \subseteq V(\beta).$$

Indeed, every edge of G lies in exactly one block, and x lies in no edge of H other than $\widehat{uv}$.

(F2) The game is over after move j if and only if every block is settled after move j. This follows from (F1), since $D_j = V(H)$ if and only if $V(\beta) \subseteq D_j$ for every β.

Step 2: bookkeeping. To every move z_j we assign a block $\beta(z_j)$ with

$$z_j \in V(\beta(z_j)) \qquad \text{and} \qquad V(\beta(z_j)) \not\subseteq D_{j-1}. \tag{2}$$

Such a block exists. The move z_j is legal, so there is $w \in N(z_j) \setminus D_{j-1}$; then $z_j, w \in \widehat{uv}$ for some $uv \in E(G)$, and the block $\beta \ni uv$ satisfies $z_j, w \in V(\beta)$ and $w \notin D_{j-1}$. If several blocks satisfy (2), a move of Staller may be assigned to any of them, and the argument below holds for every such choice; moves of Dominator are assigned as stated in his rules. Let $a_j(\beta)$ be the number of moves among $z_1, \dots, z_j$ assigned to β, and $a(\beta) = a_L(\beta)$. Then

$$L = \sum_{\beta \in \mathcal{B}} a(\beta). \tag{3}$$

By (F1) and (2):

(A1) a move $z_j \in P(\beta)$ is assigned to β;

(A2) if β is settled after move j, then $a_{j'}(\beta) = a_j(\beta)$ for all $j' \geq j$.

A block β is *pending* after move j if it is unsettled, $a_j(\beta) = 1$, and the move assigned to β is its centre.

Step 3 (Claim 1: settling moves). Let β be unsettled after move j, with centre c, and let $V(\beta) \cap Q_j \neq \emptyset$. Then there is an unplayed $y \in V(\beta)$ that is a legal move and satisfies $V(\beta) \subseteq D_j \cup N(y)$; namely $y = c$ if $c \notin Q_j$, and any $y \in P(\beta)$ if $c \in Q_j$. If $c \notin Q_j$, then $Q_j \cap (V(\beta) \setminus \{c\}) \neq \emptyset$, and after $y = c$ both c and a vertex of $V(\beta) \setminus \{c\}$ are played; by Lemma 4.1, β is settled. If $c \in Q_j$, then $Q_j \cap (V(\beta) \setminus \{c\}) = \emptyset$, since otherwise β would already be settled by Lemma 4.1. As $k \geq 3$, $P(\beta) \neq \emptyset$; every $y \in P(\beta)$ is unplayed, and after y Lemma 4.1 applies. In both cases

$$V(\beta) \not\subseteq D_j \qquad \text{and} \qquad V(\beta) \subseteq D_j \cup N(y),$$

so $N(y) \not\subseteq D_j$ and y is legal.

Step 4 (Claim 2). If β is pending after move $j-1$ and z_j is assigned to β, then β is settled after move j. The centre c of β is played, and $c \neq z_j$, as a played vertex is never legal again. Since $z_j \in V(\beta) \setminus \{c\}$, Lemma 4.1 applies.

Step 5: the basic strategy. Dominator maintains the invariant

> after each of his moves, and at the start of the game, every unsettled block β satisfies $a(\beta) = 0$, except at most one block, which is pending. (*)

Let $x = z_j$ be Staller's last move, with the game not over, or let Dominator make the first move of the D-game (then there is no x, and (D1) and (D2) do not apply). Dominator plays z_{j+1} by the first applicable rule.

(D1) *$\beta(x)$ is unsettled after move j.* By Step 4, $\beta(x)$ was not pending after move $j-1$; by (2) it was unsettled; so $a_{j-1}(\beta(x)) = 0$ by $(*)$. Since $x \in V(\beta(x)) \cap Q_j$, Step 3 gives a legal settling move, which Dominator plays and assigns to $\beta(x)$. Then

$$a_{j+1}(\beta(x)) = 0 + 1 + 1 = 2 \qquad \text{and} \qquad \beta(x) \text{ is settled.}$$

(D2) *Otherwise, a pending block π exists.* Its only assigned move is its centre c. The block π has been unsettled since the start, so by (A1) every played vertex of $P(\pi)$ is assigned to π; hence $P(\pi) \cap Q_j = \emptyset$. Dominator plays some $y \in P(\pi)$. By Lemma 4.1, π is settled after y and was unsettled before, so y is legal. He assigns y to π, and $a_{j+1}(\pi) = 2$.

(D3) *Otherwise, some unsettled block β satisfies $V(\beta) \cap Q_j \neq \emptyset$.* No block is pending, and $\beta \neq \beta(x)$ because $\beta(x)$ is settled. Hence $a_j(\beta) = a_{j-1}(\beta) = 0$ by $(*)$. Dominator plays the settling move of Step 3 and assigns it to β; then $a_{j+1}(\beta) = 1$ and β is settled.

(D4) *Otherwise.* By (F2) some block β is unsettled, and $V(\beta) \cap Q_j = \emptyset$. Dominator plays its centre c. For $x' \in P(\beta) \neq \emptyset$ we have $N(x') \subseteq V(\beta)$ by (F1), so

$$x' \in N(c) \qquad \text{and} \qquad x' \notin \bigcup_{z \in Q_j} N(z) = D_j,$$

because no vertex of $V(\beta)$ has been played. Hence c is legal. He assigns c to β; now $a_{j+1}(\beta) = 1$, and β is settled or pending.

$(*)$ *is preserved.* In each rule $\beta(x)$ is settled after move $j+1$. In (D1)–(D3) the block that receives Dominator's move is settled afterwards. Only (D4) can create a pending block, and (D4) applies only if no block is pending. Every other unsettled block has the same value of a as after Dominator's previous move, namely 0.

Step 6: the budget. Let $\beta \in \mathcal{B}$. By (A2), assignments to β occur only while β is unsettled. Combining $(*)$, Step 4 and the rules, the possible sequences of assignments to β are:

- a move of Staller with $a(\beta) = 0$ before it, after which β is settled, or else (D1) assigns a second move and settles β;
- a move of Dominator by (D3), after which β is settled;
- a move of Dominator by (D4), after which β is settled or pending, and a pending block receives at most one further move (by Step 4 or by (D2)), after which it is settled.

Hence $a(\beta) \leq 2$ for every $\beta \in \mathcal{B}$, and by (3) and (1),

$$L = \sum_{\beta \in \mathcal{B}} a(\beta) \leq 2|\mathcal{B}| \leq 2|C|$$

in both games. Thus $\gamma_{tg}(H) \leq 2|C|$ and $\gamma'_{tg}(H) \leq 2|C|$.

Step 7: the refined strategy in the D-game. Let $c_1, c_2 \in C$ with $c_1c_2 \in E(G)$, and choose κ as above. Put $T = \beta_{c_1} \cup \beta_{c_2}$ and treat T as one block, with blocks β_c, $c \in C \setminus \{c_1, c_2\}$, as before; there are at most $|C| - 2$ of them. Dominator opens with $z_1 = c_1$, which is legal because $N(c_1) \neq \emptyset = D_0$, and assigns it to T.

(T1) *For every $j \geq 1$, $V(T) \setminus D_j \subseteq N(c_2)$.* Let $w \in V(T) \setminus D_j$, and let $\widehat{e}$ be an edge with $e \in T$ and $w \in \widehat{e}$. If $c_1 \in \widehat{e}$ and $w \neq c_1$, then $w \in N(c_1) \subseteq D_j$, which is false. If $w = c_1$, then $w \in \widehat{c_1c_2} \setminus \{c_2\} \subseteq N(c_2)$. Otherwise $c_1 \notin \widehat{e}$, so $c_2 \in \widehat{e}$ because $\kappa(e) \in \{c_1, c_2\}$; moreover $w \neq c_2$, because $c_2 \in N(c_1) \subseteq D_j$. Hence $w \in \widehat{e} \setminus \{c_2\} \subseteq N(c_2)$.

(T2) *If T is unsettled after move $j \geq 1$, then $c_2 \notin Q_j$, and c_2 is a legal move that settles T.* By (T1), $V(T) \setminus D_j \subseteq N(c_2)$. If $c_2 \in Q_j$, then $N(c_2) \subseteq D_j$ and T is settled. Otherwise $\emptyset \neq V(T) \setminus D_j \subseteq N(c_2)$, so c_2 is legal and $V(T) \subseteq D_j \cup N(c_2)$.

Since $c_1 \notin N(c_1)$, T is unsettled after z_1; we regard T as pending while it is unsettled. Dominator follows the basic strategy with two changes:

- (D0), applied first: if x is assigned to T and T is unsettled after x, he plays c_2;
- in (D2), if the pending block is T, he plays c_2.

By (T2) both moves are legal and settle T; they are assigned to T. Rule (D1) never concerns T, because (D0) takes precedence. While T is unsettled it is pending, so (D2) precedes (D3) and (D4); no other block becomes pending, and $(*)$ is preserved as in Step 5. The moves assigned to T are c_1, at most one move of Staller (answered at once by (D0)), and at most one move c_2. Therefore $a(T) \leq 3$, every other block has

$a(\beta) \le 2$ by Step 6, and

$$L = a(T) + \sum_{\beta \neq T} a(\beta) \le 3 + 2\big(|C| - 2\big) = 2|C| - 1.$$

So $\gamma_{tg}(H) \le 2|C| - 1$.

Step 8: consequences. Apply Step 6 to a minimum vertex cover, and, if $\varepsilon(G) = 1$, apply Step 7 to a minimum vertex cover containing both ends of an edge. This gives $\gamma_{tg}(H) \le 2\tau(G) - \varepsilon(G)$ and $\gamma'_{tg}(H) \le 2\tau(G)$. □

With blocks of at most two edges, the basic strategy gives the bound $m + \mathrm{odd}(G)$; by the following lemma this bound is implied by Theorem 4.2.

Lemma 4.3. *Every connected graph with $m \ge 1$ edges has a partition of its edge set into $\lfloor m/2 \rfloor$ cherries (pairs of edges with a common end) and, if m is odd, one further edge. Consequently $\tau(G) \le (m + \mathrm{odd}(G))/2$ for every graph G without isolated vertices.*

Proof. Let $\mathcal{T}$ be a spanning tree rooted at ρ, and assign every edge not in $\mathcal{T}$ to one of its ends, arbitrarily. Process the vertices $u \neq \rho$ in order of non-increasing depth. When u is processed, let $L(u)$ consist of the edges assigned to u and the edges from u to its children that have not been used yet. If $|L(u)|$ is odd, add the edge from u to its parent. Split the resulting even set into pairs; each pair consists of two edges at u. Finally split the set $L(\rho)$, defined in the same way, into pairs and, if $|L(\rho)|$ is odd, one single edge. Every edge is used exactly once: a tree edge from u to its parent is used at u or at its parent, and every other edge is used at the end to which it is assigned. All blocks except possibly the last have two edges, so the last block is a single edge exactly when m is odd. Apply this to every component. The centres of the cherries, together with one end of every single edge, form a vertex cover with $(m + \mathrm{odd}(G))/2$ vertices. □

Theorem 4.2 can be much stronger than the bound $m + \mathrm{odd}(G)$. For the hyperstar $S_m^{(k)}$ it gives the exact value 2, and for $K_n^{(k)}$ ($n \ge 3$) it gives $\gamma_{tg} \le 2n - 3$ instead of roughly $n^2/2$. On the 280 power hypergraphs $G^{(k)}$ with G connected, $|V(G)| \le 6$, $k \in \{3, 4\}$ and at most 30 vertices, the bound $2\tau(G) - \varepsilon(G)$ for γ_{tg} is attained in 200 cases and the bound $2\tau(G)$ for γ'_{tg} in 182 cases; it is strictly smaller than $m + \mathrm{odd}(G)$ in 270 cases (Section 7).

The following lower bound uses two private vertices in every edge, hence $k \ge 4$.

Theorem 4.4. *If $\Delta \ge 2$ and $k \ge 4$, then*

$$\gamma_{tg}(G^{(k)}) \ge \frac{2(m+1)}{\Delta} - 1 \qquad \text{and} \qquad \gamma'_{tg}(G^{(k)}) \ge \frac{2m}{\Delta}.$$

Proof. Write $H = G^{(k)}$, and for $\widehat{e} = \widehat{uv} \in E(H)$ write $P(\widehat{e}) = P_{uv}$, so that $|P(\widehat{e})| = k - 2 \ge 2$. Recall that

$$N(x) = \widehat{e} \setminus \{x\} \quad \text{for } x \in P(\widehat{e}), \qquad d_H(v) = d_G(v) \le \Delta \quad \text{for } v \in V(G), \qquad d_H(x) = 1 \quad \text{for } x \in P(\widehat{e}). \tag{4}$$

Fix a play $z_1, z_2, \ldots, z_L$, and for $0 \le j \le L$ put

$$D_j = \bigcup_{i=1}^{j} N(z_i), \qquad \mathcal{T}_j = \big\{\widehat{e} \in E(H) : \ \widehat{e} \cap \{z_1, \ldots, z_j\} \neq \emptyset\big\}, \qquad \theta_j = |\mathcal{T}_j| - |\mathcal{T}_{j-1}| \quad (j \ge 1).$$

An edge in $\mathcal{T}_j$ is called *touched* after move j. Thus θ_j is the number of edges that contain z_j and none of $z_1, \ldots, z_{j-1}$; in particular

$$0 \le \theta_j \le d_H(z_j) \le \Delta. \tag{5}$$

Step 1: $\sum_{j=1}^{L} \theta_j = m$. Let $\widehat{e} \in E(H)$ and $x \in P(\widehat{e})$. Since $D_L = V(H)$, there is an $i \le L$ with $x \in N(z_i)$; then $z_i \neq x$ and z_i lies in an edge with x, and by (4) this edge is $\widehat{e}$. Hence $\widehat{e} \in \mathcal{T}_L$, so $\mathcal{T}_L = E(H)$, and

$$\sum_{j=1}^{L} \theta_j = \sum_{j=1}^{L} \big(|\mathcal{T}_j| - |\mathcal{T}_{j-1}|\big) = |\mathcal{T}_L| - |\mathcal{T}_0| = m - 0 = m.$$

Step 2 (Claim 1). If $\theta_j = d_H(z_j)$, then $z_j \notin D_j$; in particular $j < L$. Suppose $z_j \in N(z_i)$ for some $i \le j$. Then $i \neq j$, since $z_j \notin N(z_j)$, and z_i lies in an edge $\widehat{e} \ni z_j$. As $i < j$, we get $\widehat{e} \in \mathcal{T}_{j-1}$. But $\theta_j = d_H(z_j)$ means that no edge containing z_j lies in $\mathcal{T}_{j-1}$, a contradiction. Hence $z_j \notin D_j$, so $D_j \neq V(H)$ and the game continues.

Step 3 (Claim 2). Let $j < L$ and let z_{j+1} be a move of Staller.

(a) *If some $\widehat{e} \in \mathcal{T}_j$ contains a vertex $w \notin D_j$, then Staller has a legal move $x \in P(\widehat{e})$, and every such move has $\theta_{j+1} = 0$.* Let $z_i \in \widehat{e}$ with $i \le j$. If $z_i \neq w$, then $w \in \widehat{e} \setminus \{z_i\} \subseteq N(z_i) \subseteq D_j$, which is false. Hence

$$\widehat{e} \cap \{z_1, \dots, z_j\} = \{w\}.$$

Since $|P(\widehat{e})| \ge 2$, there is $x \in P(\widehat{e}) \setminus \{w\}$. Then x is unplayed and, by (4), $w \in \widehat{e} \setminus \{x\} = N(x)$, so $N(x) \not\subseteq D_j$ and x is legal. The only edge containing x is $\widehat{e} \in \mathcal{T}_j$, so $\theta_{j+1} = 0$.

(b) *Otherwise Staller has a legal move with $\theta_{j+1} = 1$.* Choose $w \notin D_j$. Since G has no isolated vertices, w lies in some $\widehat{e} \in E(H)$, and $\widehat{e} \notin \mathcal{T}_j$ by the case assumption. Choose $x \in P(\widehat{e}) \setminus \{w\}$. Then x is unplayed, $w \in N(x)$ and $w \notin D_j$, so x is legal; and $\theta_{j+1} = 1$, because $\widehat{e}$ is the only edge containing x.

Staller's strategy is to play a move as in (a) whenever (a) applies, and a move as in (b) otherwise. Consequently every move z_{j+1} of Staller satisfies

$$\theta_{j+1} \le 1, \qquad \text{and} \qquad \theta_{j+1} = 0 \ \text{ if some edge of } \mathcal{T}_j \text{ contains a vertex of } V(H) \setminus D_j. \tag{6}$$

Step 4: the ledger. Let z_j be a move of Dominator.

(i) If $j < L$, then

$$\theta_j + \theta_{j+1} \le \Delta. \tag{7}$$

Indeed, if $\theta_j = d_H(z_j)$, then $\theta_j \ge 1$, so some edge of $\mathcal{T}_j$ contains z_j, and $z_j \notin D_j$ by Step 2. By (6), $\theta_{j+1} = 0$, and

$$\theta_j + \theta_{j+1} = d_H(z_j) + 0 \le \Delta.$$

If $\theta_j \le d_H(z_j) - 1$, then by (5) and (6),

$$\theta_j + \theta_{j+1} \le \big(d_H(z_j) - 1\big) + 1 \le \Delta.$$

(ii) If $j = L$, then $\theta_L \neq d_H(z_L)$ by Step 2, and therefore

$$\theta_L \le d_H(z_L) - 1 \le \Delta - 1. \tag{8}$$

The D-game. Dominator makes the moves z_j with j odd. If L is even, Step 1 and (7) give

$$m = \sum_{t=1}^{L/2} \big(\theta_{2t-1} + \theta_{2t}\big) \le \Delta \cdot \frac{L}{2}, \qquad \text{so} \qquad L \ge \frac{2m}{\Delta} = \frac{2(m+1)}{\Delta} - 1 + \Big(1 - \frac{2}{\Delta}\Big) \ge \frac{2(m+1)}{\Delta} - 1,$$

since $\Delta \ge 2$. If L is odd, Step 1, (7) and (8) give

$$m = \sum_{t=1}^{(L-1)/2} \big(\theta_{2t-1} + \theta_{2t}\big) + \theta_L \le \Delta \cdot \frac{L-1}{2} + \Delta - 1 = \Delta \cdot \frac{L+1}{2} - 1, \qquad \text{so} \qquad L \ge \frac{2(m+1)}{\Delta} - 1.$$

The S-game. Staller makes the moves z_j with j odd, and $\theta_1 \le 1$ by (6). If L is odd,

$$m = \theta_1 + \sum_{t=1}^{(L-1)/2} \big(\theta_{2t} + \theta_{2t+1}\big) \le 1 + \Delta \cdot \frac{L-1}{2}, \qquad \text{so} \qquad L \ge 1 + \frac{2(m-1)}{\Delta} = \frac{2m}{\Delta} + \Big(1 - \frac{2}{\Delta}\Big) \ge \frac{2m}{\Delta}.$$

If L is even,

$$m = \theta_1 + \sum_{t=1}^{(L-2)/2} \big(\theta_{2t} + \theta_{2t+1}\big) + \theta_L \le 1 + \Delta \cdot \frac{L-2}{2} + (\Delta - 1) = \Delta \cdot \frac{L}{2}, \qquad \text{so} \qquad L \ge \frac{2m}{\Delta}.$$

In both games these inequalities hold for every play in which Staller follows her strategy, whatever Dominator does. Hence

$$\gamma_{tg}(G^{(k)}) \ge \frac{2(m+1)}{\Delta} - 1 \qquad \text{and} \qquad \gamma'_{tg}(G^{(k)}) \ge \frac{2m}{\Delta}. \qquad \square$$

Remark 4.5. For $k = 3$ every edge has only one private vertex, and Claim 2(a) in the proof of Theorem 4.4 fails. So does the D-game bound: for $G = K_{3,3}$ and for the prism $C_3 \square K_2$ ($m = 9$, $\Delta = 3$) we have $\gamma_{tg}(G^{(3)}) = 5 < 17/3$ and $\gamma'_{tg}(G^{(3)}) = 6$. The S-game bound $\gamma'_{tg}(G^{(3)}) \geq 2m/\Delta$ holds for all graphs G with $\Delta \geq 2$ and at most 6 vertices, by exact computation; whether it holds in general is open.

Corollary 4.6. *Let $k \geq 3$, $m \geq 1$ for paths and $m \geq 3$ for cycles.*

(i) *$\gamma_{tg}(P_m^{(k)}) \leq m$ for $m \geq 2$ and $\gamma_{tg}(C_m^{(k)}) \leq m$. The values $\gamma'_{tg}(P_m^{(k)})$ and $\gamma'_{tg}(C_m^{(k)})$ are at most m if m is even and at most $m+1$ if m is odd.*

(ii) *If $k \geq 4$ and $m \geq 2$, then*

$$\gamma_{tg}(P_m^{(k)}) = \gamma_{tg}(C_m^{(k)}) = m.$$

Moreover, $\gamma'_{tg}(P_m^{(k)})$ and $\gamma'_{tg}(C_m^{(k)})$ equal m if m is even and lie in $\{m, m+1\}$ if m is odd.

Proof. (i) The path with m edges has vertices $v_0, \ldots, v_m$ and $\tau = \lceil m/2 \rceil$: the $\lceil m/2 \rceil$ edges $v_0v_1, v_2v_3, \ldots$ are pairwise disjoint, so every vertex cover has at least $\lceil m/2 \rceil$ vertices, and the covers below attain this. If m is even, $\{v_1, v_3, \ldots, v_{m-1}\}$ is a minimum vertex cover. If $m \geq 3$ is odd, the minimum vertex cover $\{v_1, v_3, \ldots, v_{m-2}, v_{m-1}\}$ contains the edge $v_{m-2}v_{m-1}$, so $\varepsilon = 1$. The cycle with m edges has $\tau = \lceil m/2 \rceil$. For odd m every vertex cover of size $(m+1)/2$ contains an edge: the complement of a vertex cover is independent, so a vertex cover of size $(m+1)/2$ containing no edge would split the odd cycle into two independent sets, which is impossible. Apply Theorem 4.2.

(ii) The upper bounds are (i). For the lower bounds apply Theorem 4.4 with $\Delta = 2$: $\gamma_{tg} \geq (m+1) - 1 = m$ and $\gamma'_{tg} \geq m$. □

Corollary 4.6 gives, for $k \geq 4$, exact values of γ_{tg} for all loose paths and loose cycles. By comparison, the bound $\gamma_t(P_m^{(k)}) \leq 2\lceil m/3 \rceil$ together with Theorem 2.5 only gives $4\lceil m/3 \rceil - 1$. The odd case of the S-game remains open; the computations suggest the answer (Table 2).

Corollary 4.7. *Let $k \geq 4$ and let G be a d-regular bipartite graph with $d \geq 1$. Then*

$$\gamma_{tg}(G^{(k)}) = \gamma'_{tg}(G^{(k)}) = |V(G)|,$$

whereas $G^{(k)}$ has $|V(G)|\,(1 + (k-2)d/2)$ vertices. If G is d-regular with $d \geq 2$ but not bipartite, then $|V(G)| \leq \gamma_{tg}(G^{(k)}) \leq 2\tau(G) - 1$ and $|V(G)| \leq \gamma'_{tg}(G^{(k)}) \leq 2\tau(G)$.

Proof. Let $n_0 = |V(G)|$, so $m = dn_0/2$. For $d = 1$, $G^{(k)}$ is a disjoint union of $n_0/2$ edges and the claim is Proposition 6.1. Let $d \geq 2$. By Theorem 4.4, $\gamma'_{tg} \geq 2m/d = n_0$ and $\gamma_{tg} \geq 2(m+1)/d - 1 = n_0 - 1 + 2/d > n_0 - 1$, so $\gamma_{tg} \geq n_0$ since it is an integer. If G is bipartite with colour classes X and Y, then $d|X| = m = d|Y|$, so the colour class X is a vertex cover of size $n_0/2$, so $\gamma_{tg}, \gamma'_{tg} \leq n_0$ by Theorem 4.2. If G is not bipartite, no vertex cover is independent, because the complement of a vertex cover is always independent. Hence $\varepsilon(G) = 1$, and Theorem 4.2 gives the stated upper bounds. □

For example, for $k \geq 4$ we get $\gamma_{tg} = \gamma'_{tg} = 2^d$ for the k-th power of the hypercube Q_d, and $\gamma_{tg} = \gamma'_{tg} = 2d$ for the k-th power of $K_{d,d}$.

Observation 4.8. *For all computed instances and odd $m \geq 3$:*

$$\gamma_{tg}(P_m^{(k)}) = \gamma_{tg}(C_m^{(k)}) = m, \qquad \gamma'_{tg}(P_m^{(k)}) = m+1,$$

$$\gamma'_{tg}(C_m^{(3)}) = m, \qquad \gamma'_{tg}(C_m^{(k)}) = m+1 \quad \text{for } k \geq 4.$$

For even m and $k = 3$ all four values equal m. The computed instances are paths with $k \in \{3, 4\}$, $m \leq 12$ and with $k = 5$, $m \leq 9$, and cycles with $k \in \{3, 4\}$, $m \leq 12$, with $k = 5$, $m \leq 10$ and with $k = 6$, $m \leq 6$.

Conjecture 4.9. *For every odd $m \geq 3$, $\gamma'_{tg}(P_m^{(k)}) = m+1$ for every $k \geq 3$, and $\gamma'_{tg}(C_m^{(k)}) = m+1$ for every $k \geq 4$. For $k = 3$, $\gamma_{tg}(P_m^{(3)}) = \gamma_{tg}(C_m^{(3)}) = m$ for all $m \geq 2$, and $\gamma'_{tg}(P_m^{(3)}) = \gamma'_{tg}(C_m^{(3)}) = m$ for even m.*

By Corollary 4.6(i), Conjecture 4.9 only asks for lower bounds. The difference between $k = 3$ and $k \geq 4$ for $\gamma'_{tg}(C_m^{(k)})$ is consistent with the role of the second private vertex in Claim 2 of Theorem 4.4.

The dual H^* of a hypergraph H has vertex set $E(H)$ and, for every $v \in V(H)$, the edge $\{e \in E(H) : v \in e\}$. Edges of size one are discarded, since they create no adjacencies and do not affect the game. Duals of uniform hypergraphs need not be uniform.

Proposition 4.10. *The game total domination number is not monotone under duality in either direction.*

(i) *For $k \geq 4$ and $m \geq 3$, $\gamma_{tg}\big((C_m^{(k)})^*\big) \leq m - \frac{m-1}{3}$ and $\gamma'_{tg}\big((C_m^{(k)})^*\big) < m$, whereas $\gamma_{tg}(C_m^{(k)}) = m$ and $\gamma'_{tg}(C_m^{(k)}) \geq m$.*

	$P_m^{(k)}$			$C_m^{(k)}$			
m	$k=3$	$k=4$	$k=5$	$k=3$	$k=4$	$k=5$	$k=6$
1	2, 2	2, 2	2, 2	–	–	–	–
2	2, 2	2, 2	2, 2	–	–	–	–
3	3, 4	3, 4	3, 4	3, 3	3, 4	3, 4	3, 4
4	4, 4	4, 4	4, 4	4, 4	4, 4	4, 4	4, 4
5	5, 6	5, 6	5, 6	5, 5	5, 6	5, 6	5, 6
6	6, 6	6, 6	6, 6	6, 6	6, 6	6, 6	6, 6
7	7, 8	7, 8	7, 8	7, 7	7, 8	7, 8	–
8	8, 8	8, 8	8, 8	8, 8	8, 8	8, 8	–
9	9, 10	9, 10	9, 10	9, 9	9, 10	9, 10	–
10	10, 10	10, 10	–	10, 10	10, 10	10, 10	–
11	11, 12	11, 12	–	11, 11	11, 12	–	–
12	12, 12	12, 12	–	12, 12	12, 12	–	–

Table 2 Exact values $(\gamma_{tg}, \gamma'_{tg})$ for loose paths $P_m^{(k)}$ and loose cycles $C_m^{(k)}$ ("–": not computed or undefined). For $k \geq 4$ the values of γ_{tg} $(m \geq 2)$, and those of γ'_{tg} for even m, are determined by Corollary 4.6; all other entries are computational.

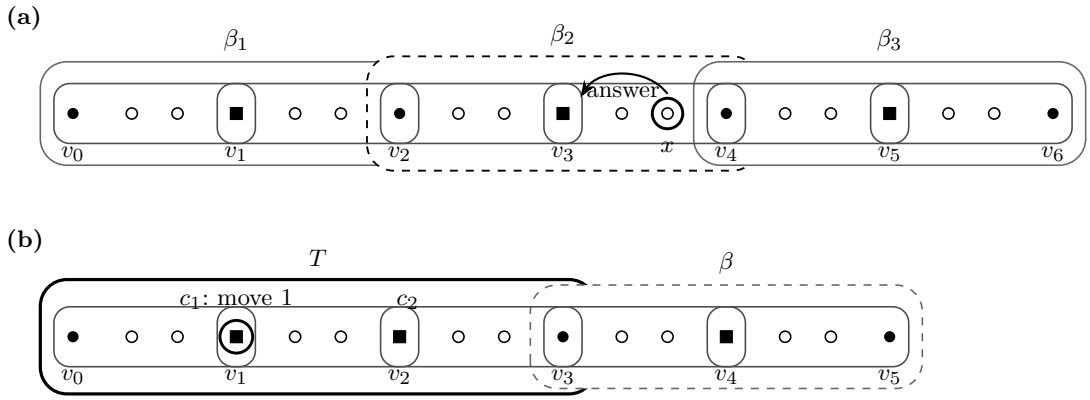


Fig. 3 The strategies of Theorem 4.2 for $k = 4$; squares mark the vertex cover. (a) $P_6^{(4)}$ with the vertex cover $\{v_1, v_3, v_5\}$ and its three blocks. When Staller plays in an unsettled block, Dominator settles it with one move (Lemma 4.1); here Staller plays $x \in V(\beta_2)$ and Dominator answers with the centre v_3. Every block receives at most two moves, so the game ends after at most 6 moves; by Theorem 4.4 Staller can force 6. (b) $P_5^{(4)}$ with the vertex cover $\{v_1, v_2, v_4\}$, which contains the edge v_1v_2. Dominator opens with $c_1 = v_1$. The block T (thick outline) then needs only one further move, $c_2 = v_2$, so the game ends after at most $5 = 2\tau - 1$ moves; by Theorem 4.4 this is optimal

(ii) *The complete* 3*-uniform hypergraph* $H = K_8^{(3)}$ *satisfies* $\gamma_{tg}(H) = 2 < 3 \leq \gamma_{tg}(H^*)$*, and* H^* *is* 21*-uniform.*
(iii) $\gamma_{tg}\big((S_m^{(k)})^*\big) = \gamma_{tg}(S_m^{(k)}) = 2$ *for* $m \geq 2$.

Proof. (i) In $(C_m^{(k)})^*$ the spine vertex shared by e_i and e_{i+1} gives the edge $\{e_i, e_{i+1}\}$, and each private vertex gives an edge of size one, which is discarded. The dual is therefore the cycle C_m. By Proposition 2.2(i) and [8], $\gamma_{tg}((C_m^{(k)})^*) = \gamma_{tg}(C_m) \leq \lfloor (2m+1)/3 \rfloor \leq m - (m-1)/3$. By Theorem 2.5, $\gamma'_{tg}(C_m) \leq \lfloor (2m+1)/3 \rfloor + 1 < m$ for $m \geq 5$. Also $\gamma'_{tg}(C_3) = \gamma'_{tg}(C_4) = 2$: any two vertices of C_3 dominate it, and in C_4 the only legal replies to a first move v are the two neighbours of v, each of which dominates the two remaining vertices. The values for $C_m^{(k)}$ are Corollary 4.6.

(ii) $\gamma_{tg}(K_8^{(3)}) = 2$ by Corollary 6.3. The vertices of H^* are the 56 triples of [8], its edges are the 8 sets of 21 triples through a fixed point, and two triples are adjacent in H^* if and only if they intersect. Suppose $\{S_1, S_2\}$ were a TD-set of H^*. Then S_1 and S_2 dominate each other, so $S_1 \cap S_2 \neq \emptyset$ and $|S_1 \cup S_2| \leq 5$. Some triple is then disjoint from $S_1 \cup S_2$, and it is not dominated. Hence $\gamma_{tg}(H^*) \geq \gamma_t(H^*) \geq 3$.

(iii) In $(S_m^{(k)})^*$ the centre of the star gives an edge containing all m vertices, so the 2-section is K_m and the value is 2. Also $\gamma_{tg}(S_m^{(k)}) = 2$ by Corollary 6.3. □

Thus neither $\gamma_{tg}(H^*) \leq \gamma_{tg}(H)$ nor $\gamma_{tg}(H^*) \geq \gamma_{tg}(H)$ holds in general. Adjacency in H^* records pairs of intersecting edges of H, whereas the game on H is governed by the neighbourhoods $N_H(v)$. The duality that is compatible with the game is the self-duality of $\mathrm{ONH}^*(H)$: since $u \in N(v)$ if and only if $v \in N(u)$, the dual of $\mathrm{ONH}^*(H)$ is isomorphic to $\mathrm{ONH}^*(H)$.

5 Upper bounds for fixed uniformity

For fixed k the asymptotic bound of Theorem 3.2 is weak, and the best upper bounds that we know of are transfers of graph and transversal results. We record them for hypergraphs and check their hypotheses; apart from the shrinking lemma, the degeneracy lemma and their consequences, the results of this section are known. Throughout, H is an isolate-free k-uniform hypergraph on n vertices; Table 3 summarises the bounds for each k.

For $k \geq 3$ the 2-section $[H]_2$ has minimum degree at least $k-1 \geq 2$ (Proposition 2.3). Hence Proposition 2.2(i) and the graph theorems of [16, 4, 11] give the following.

Theorem 5.1. *Let H be an isolate-free k-uniform hypergraph on n vertices, $k \geq 3$. Then the following hold.*
(i) $\gamma_{tg}(H) \leq \frac{3}{4}n$ *[16, Theorem 1.2] and* $\gamma_{tg}(H) \leq \frac{5}{7}(n+1)$ *[16, Theorem 1.3];*
(ii) $\gamma_{tg}(H) < \frac{8}{11}n$ *[4];*
(iii) $\gamma'_{tg}(H) \leq \frac{4n+2}{5}$ *[11] and* $\gamma'_{tg}(H) \leq \gamma_{tg}(H)+1 \leq \frac{5}{7}(n+1)+1$.

The bound $5(n+1)/7$ is Theorem 1.3 of the arXiv version of [16]; for $k=3$ it should be compared with the conjectured $2n/3$ (Conjecture 7.3).

Bujtás, Henning, Tuza and Yeo [6] denote by $\mathcal{H}_k$ the class of k-uniform hypergraphs without isolated vertices, isolated edges and multiple edges, and proved the following bound [6, Theorem 3].

Theorem 5.2. *If $k \in \{2,\dots,6\}$ and $H \in \mathcal{H}_k$ has n vertices, then $\gamma_t(H) \leq 2n/(k+1)$.*

To use Theorem 5.2 for larger k we shrink edges without enlarging neighbourhoods.

Lemma 5.3. *Let $2 \leq k \leq k'$ and let H' be an isolate-free k'-uniform hypergraph which, if $k = k'$, has no isolated edges. Then there is $H \in \mathcal{H}_k$ on $V(H')$ with $N_H(v) \subseteq N_{H'}(v)$ for all v; in particular $\gamma_t(H') \leq \gamma_t(H)$.*

Proof. For $k = k'$ take $H = H'$. Let $k < k'$, and write every edge of H' as $e' = \{u_1,\dots,u_{k'}\}$, in an arbitrary order. Let $\mathcal{F}(e')$ consist of the *windows* $\{u_j,\dots,u_{j+k-1}\}$ for

$$j \in \{1+t(k-1) :\ t \geq 0,\ 1+t(k-1) \leq k'-k+1\} \cup \{k'-k+1\}.$$

Consecutive starting indices differ by at most $k-1$, so consecutive windows intersect; the first window contains u_1, the last contains $u_{k'}$, so the windows cover e'; and there are at least two windows. Let $E(H) = \bigcup_{e'} \mathcal{F}(e')$. Every vertex lies in a window, every window meets another window of the same family, and adjacent vertices of H lie in a common window $f \subseteq e'$, hence are adjacent in H'. Thus $H \in \mathcal{H}_k$, $N_H(v) \subseteq N_{H'}(v)$, and every TD-set of H is a TD-set of H'. □

Theorem 5.4. *Let H be an isolate-free k-uniform hypergraph on n vertices.*
(i) *If $k \in \{5,6\}$, then $\gamma_t(H) \leq n/3$, $\gamma_{tg}(H) \leq \frac{2}{3}n-1$ and $\gamma'_{tg}(H) \leq \frac{2}{3}n$; for $k=5$ this requires that H has no isolated edges.*
(ii) *If $k \geq 6$ and H has no isolated edges, or if $k \geq 7$, then $\gamma_t(H) \leq 2n/7$, $\gamma_{tg}(H) \leq \frac{4}{7}n-1$ and $\gamma'_{tg}(H) \leq \frac{4}{7}n$.*

Proof. (i) For $k=5$ apply Theorem 5.2 to H; for $k=6$ apply Lemma 5.3 with target uniformity $5<6$ and then Theorem 5.2. (ii) For $k=6$ without isolated edges apply Theorem 5.2 to H; for $k \geq 7$ apply Lemma 5.3 with target uniformity $6<k$. The game bounds follow from Theorem 2.5. □

The assumption on isolated edges cannot be dropped for $k=5$: a disjoint union of t edges has $\gamma_t = 2t = 2n/5 > n/3$. The bound of Theorem 3.2 is smaller than $4n/7$ exactly when $k \geq 14$.

If $[H]_2$ is r-regular, then $\mathrm{ONH}(H)$ is r-uniform with n vertices and at most n edges, and the transversal game bounds of [5] apply; we use the following one [5, Theorem 6].

Theorem 5.5. *If F is a 4-uniform hypergraph with n_F vertices and m_F edges, then $\tau_g(F) \leq \frac{71}{252}(n_F+m_F)$.*

Theorem 5.6. *If $[H]_2$ is 4-regular, then $\gamma_{tg}(H) \leq \frac{71}{126}n$ and $\gamma'_{tg}(H) \leq \frac{71}{126}n+1$. In particular this holds for every linear 2-regular 3-uniform hypergraph, and $\gamma_{tg}(L(G)) \leq \frac{71}{84}|V(G)|$ for the line graph of every cubic graph G.*

Proof. By Proposition 2.2(ii) and Theorem 5.5, $\gamma_{tg}(H) = \tau_g(\mathrm{ONH}(H)) \leq \frac{71}{252}(n+n)$; the S-game bound follows from Theorem 2.5. A linear 2-regular 3-uniform hypergraph has $|N(v)| = 4$ for all v. For a cubic graph G the hypergraph on $E(G)$ whose edges are the stars $\{e : v \in e\}$, $v \in V(G)$, is linear, 3-uniform and 2-regular with 2-section $L(G)$ and $|E(G)| = \frac{3}{2}|V(G)|$ vertices. □

Theorem 5.6 is of interest for $k=3$ only, and the analogous use of the 3-uniform bound $\tau_g(F) \leq \frac{5}{16}(n_F+m_F)$ of [5, Theorem 5] for cubic 2-sections is void. This follows from the next observation with $r=3$, respectively with $r=4$ and $k \geq 4$; recall that every edge of the 2-section of a k-uniform hypergraph lies in a clique of order k.

Lemma 5.7. *Let $r \geq 3$. If G is r-regular and every edge of G lies in a clique of order r, then every component of G is K_{r+1}.*

Proof. Let $v \in V(G)$ and suppose two neighbours a, b of v are non-adjacent. A clique of order r through va consists of v, a and $r-2$ vertices of $N(v) \setminus \{a\}$; it avoids b, so it is $\{v, a\} \cup (N(v) \setminus \{a, b\})$. Thus a is adjacent to all of $N(v) \setminus \{a, b\}$, and so is b. Moreover, every $c \in N(v) \setminus \{a, b\}$ (such c exists as $r \geq 3$) is adjacent to all of $N(v) \setminus \{a, b, c\}$, since c lies in the clique through va. Hence the r neighbours of c are exactly the vertices of $(\{v\} \cup N(v)) \setminus \{c\}$. The vertex a has $r-1$ neighbours in $\{v\} \cup N(v)$ and is not adjacent to b, so it has a neighbour $w \notin \{v\} \cup N(v)$. A clique of order r through aw contains $r-2 \geq 1$ common neighbours of a and w. These lie in $N(a) \setminus \{w\} = \{v\} \cup (N(v) \setminus \{a, b\})$; but w is adjacent neither to v nor to any such c. This is a contradiction. Hence $\{v\} \cup N(v)$ is a clique of order $r+1$ for every v, and by regularity it is a component. □

For K_{r+1}, and hence for disjoint unions of copies of it, $\gamma_{tg} = \gamma'_{tg} = 2$ per component (Proposition 6.1). For all 3-uniform hypergraphs with 4-regular 2-section that we computed, the ratio γ_{tg}/n is at most 0.467 (Section 7).

k	upper bound on γ_{tg}/n	source	lower bound on $\Gamma(k)$	data ($n \leq 11$)
3	$5/7$	Theorem 5.1	$2/3$	$2/3$
4	$5/7$	Theorem 5.1	$1/2$	$1/2$
5	$2/3^{\mathrm{a}}$	Theorem 5.4	$2/5$	$2/5$
6	$4/7^{\mathrm{a}}$, $2/3$	Theorem 5.4	$1/3$	–
$7 \leq k \leq 13$	$4/7$	Theorem 5.4	$2/k$	–
$14 \leq k \leq 51$	$2\frac{1+\ln(k-1)}{k-1}$	Theorem 3.2	$2/k$	–
$52 \leq k \leq 72$	$2\frac{1+\ln(k-1)}{k-1}$	Theorem 3.2	$5/n_k > 2/k$	–
$k \geq 73$	$2\frac{1+\ln(k-1)}{k-1}$	Theorem 3.2	$3/n_k > 2/k$	–
$k \to \infty$	$(2+o(1))\frac{\ln k}{k}$	Theorem 3.7	$(2-o(1))\frac{\ln k}{k}$	–

Table 3 Bounds on $\gamma_{tg}(H)/n$ for isolate-free k-uniform hypergraphs. "Upper" is the best bound proved or transferred here (constants up to $O(1/n)$); "lower" is the best ratio realised; "data" is the maximum over all hypergraphs with $n \leq 11$ (Observation 7.2). Bounds marked [a] assume that there are no isolated edges; without this assumption $5(n+1)/7$ ($k = 5$), respectively $2n/3$ ($k = 6$), apply. n_k is the number of vertices of the hypergraph used for k in the proof of Theorem 3.9 (Table 1; $\mathcal{I}(k+14, 16)$ for $k \geq 78$).

6 Small values, the grid hypergraph, and the two games

Proposition 6.1. *Let H be the vertex-disjoint union of $H_1, \ldots, H_t$, where in each H_i all maximal sequences of legal moves have the same length ℓ_i. Then $\gamma_{tg}(H) = \gamma'_{tg}(H) = \sum_i \ell_i$. In particular, a disjoint union of t copies of $K_q^{(k)}$ ($q \geq k$) has $\gamma_{tg} = \gamma'_{tg} = 2t$, and for all $n \geq k \geq 2$ there is an isolate-free k-uniform hypergraph on n vertices with $\gamma_{tg} = \gamma'_{tg} = 2\lfloor n/k \rfloor$.*

Proof. Neighbourhoods lie inside components, so a move in H_i is legal in H if and only if it is legal for the moves already made in H_i, and H is dominated if and only if every H_i is. Every maximal legal sequence in H therefore restricts to maximal legal sequences in the H_i, whatever the players do. In $K_q^{(k)}$ the first move dominates all vertices but itself and every legal second move ends the game. For $n = ak + b$, $0 \leq b < k$, take $a-1$ copies of $K_k^{(k)}$ and one copy of $K_{k+b}^{(k)}$. □

Proposition 6.2. *Let H be isolate-free with vertex set V.*
(i) *$\gamma_{tg}(H) = 2$ if and only if some vertex v satisfies $N(v) \cup N(w) = V$ for every w with $N(w) \not\subseteq N(v)$.*
(ii) *$\gamma'_{tg}(H) = 2$ if and only if for every vertex w some vertex v satisfies $N(v) \cup N(w) = V$.*

Proof. (i) After a first move v, the vertex v is undominated, and the legal replies are exactly the w with $N(w) \not\subseteq N(v)$ (a neighbour of v is one of them). The game ends after the reply, whatever it is, if and only if each such w satisfies $N(v) \cup N(w) = V$. (ii) Every vertex is a legal first move, after which the game is not over; Dominator can end the game with his reply v if and only if $N(v) \cup N(w) = V$, and such a v is legal because it dominates $w \notin N(w)$. Hence $\gamma'_{tg}(H) = 2$ if every w has such a v; conversely, if some w has none, Staller opens with w and the game lasts at least three moves. □

Corollary 6.3. *If an isolate-free hypergraph has a vertex adjacent to all other vertices, then $\gamma_{tg} = \gamma'_{tg} = 2$. This applies to complete k-uniform hypergraphs, to Steiner systems $S(2, k, v)$ (in particular finite projective*

and affine planes and Steiner triple systems), to sunflowers with non-empty kernel (in particular the hyperstars $S_m^{(k)}$), and to all hypergraphs whose edges share a common vertex.

Proof. Let u be adjacent to all other vertices, so $N(u) = V \setminus \{u\}$. In Proposition 6.2(i) take $v = u$: every $w \neq u$ is adjacent to u, so $N(u) \cup N(w) = V$. For (ii), Dominator answers $w \neq u$ with u, and $w = u$ with any neighbour of u. In a Steiner system any two vertices lie in a common edge; in the other cases a kernel vertex, respectively the common vertex, is adjacent to all vertices. □

Corollary 6.4. *The complete k-partite k-uniform hypergraph $K^{(k)}_{n_1,\dots,n_k}$ ($n_i \geq 1$) satisfies $\gamma_{tg} = \gamma'_{tg} = 2$.*

Proof. For v in part V_i we have $N(v) = V \setminus V_i$. If $v \in V_1$ and $N(w) \not\subseteq N(v)$, then $w \in V_j$ with $j \neq 1$, and $N(v) \cup N(w) = V$. In the S-game Dominator answers in a different part. (This is the hypergraph form of the statement for complete multipartite graphs [10].) □

Proposition 6.5. *If H is an isolate-free intersecting k-uniform hypergraph, then every edge is a TD-set; hence $\gamma_t(H) \leq k$, $\gamma_{tg}(H) \leq 2k-1$ and $\gamma'_{tg}(H) \leq 2k$.*

Proof. Let $e \in E(H)$. A vertex $w \notin e$ lies in an edge f, and any $u \in f \cap e$ is a neighbour of w in e. A vertex $w \in e$ is adjacent to the other vertices of e. Apply Theorem 2.5. □

The hypergraph $H_{k,1}$ of [7, Construction 1] has vertex set $[k] \times [k]$; its edges are the k columns $C_j = \{(i,j) : i \in [k]\}$ and the first $k-1$ rows $R_i = \{(i,j) : j \in [k]\}$, $i \leq k-1$ (Fig. 4). It is k-uniform with $n = k^2$, and Bujtás et al. proved $\gamma_g(H_{k,1}) = k + \lfloor (k-1)/2 \rfloor$ [7, Construction 1]. We call $b_j = (k,j)$ the *bottom* vertices; $N(b_j) = C_j \setminus \{b_j\}$, and $N((i,j)) = (C_j \cup R_i) \setminus \{(i,j)\}$ for $i < k$.

Theorem 6.6. *For every $k \geq 2$, $\gamma_t(H_{k,1}) = k$, $\gamma_{tg}(H_{k,1}) = 2k-1$ and $2k-1 \leq \gamma'_{tg}(H_{k,1}) \leq 2k$.*

Proof. Total domination number. Every TD-set contains a vertex of $C_j \setminus \{b_j\}$ for each j, and the columns are disjoint, so $\gamma_t \geq k$. The row R_1 is a TD-set: (i,j) with $i \neq 1$ is adjacent to $(1,j)$, and $(1,j)$ to $(1,j')$, $j' \neq j$. Hence $\gamma_t = k$, and $\gamma_{tg} \leq 2k-1$, $\gamma'_{tg} \leq 2k$ by Theorem 2.5. It remains to show that Staller can force $2k-1$ moves in both games.

Terminology. Let Q be the set of played vertices. Column j is *served* if Q meets $C_j \setminus \{b_j\}$; a move *serves* column j if column j was not served before and is served afterwards. Let $R = \{i \leq k-1 : Q \cap R_i \neq \emptyset\}$ and $C = \{j : Q \cap C_j \neq \emptyset\}$. We use the following facts.

(F1) b_j is dominated if and only if column j is served; a move serves at most one column, and a bottom vertex serves none.

(F2) A non-bottom vertex (i,j) is dominated if and only if $Q \setminus \{(i,j)\}$ meets $R_i \cup C_j$.

(F3) If $i \notin R$ and $j \notin C$, then (i,j) is undominated. Consequently, if all non-bottom vertices are dominated, then $R = [k-1]$ or $C = [k]$.

(F4) If all non-bottom vertices are dominated, every later legal move serves exactly one column (the only undominated vertices are bottom vertices of unserved columns, and a vertex dominating b_j lies in $C_j \setminus \{b_j\}$).

(F5) If a non-bottom vertex (i,j) is undominated, then b_j is unplayed, legal, and serves no column.

(F2)–(F5) follow from the description of the neighbourhoods; in (F5), a played b_j would dominate (i,j).

Staller's strategy. While some non-bottom vertex is undominated, Staller plays by the first applicable rule:

(S1) if Dominator's last move is a non-bottom vertex (i,j) that is undominated, play b_j;

(S2) if there are $i \in [k-1] \setminus R$, a served column j' and a column $x \notin C$, play (i,j');

(S3) if a played non-bottom vertex (i,j) is undominated, play b_j;

(S4) play b_x for a column $x \notin C$ containing an undominated non-bottom vertex.

The strategy is well defined and Staller never serves. Let $u = (i,j)$ be an undominated non-bottom vertex. If u is played and (S1), (S2) do not apply, (S3) applies, and b_j is legal by (F5). If u is unplayed, then no vertex of $R_i \cup C_j$ is played, so $i \notin R$ and $j \notin C$; if some column is served, (S2) applies with $x = j$, and the move (i,j') is unplayed, lies in a served column, and is legal because it dominates u; otherwise (S4) applies with $x = j$. The moves of (S1), (S3), (S4) are bottom vertices and the move of (S2) lies in a served column, so Staller never serves. Moreover: (S1) and (S3) change neither R nor C (the column j already belongs to C); (S2) adds one row to R and nothing to C; (S4) adds one column to C and nothing to R. Finally, (S4) is applied only when no column is served: if (S3) does not apply, every undominated non-bottom vertex is unplayed, and the argument above shows that (S2) applies as soon as some column is served. Since every non-bottom move lies in a column that is served afterwards, at that moment all previous moves of Dominator are bottom vertices.

Counting. Let T be the number of moves played at the first moment at which all non-bottom vertices are dominated, and let d_T, s_T be the numbers of moves of Dominator and Staller among them. Call a move of Dominator *wasted* if it serves no column, and let w_T be the number of wasted moves among the first T. Up

to time T Staller never serves, so exactly $d_T - w_T$ columns are served at time T, and by (F4) the game lasts exactly

$$L = T + \big(k - (d_T - w_T)\big) = k + s_T + w_T$$

moves. We show $s_T + w_T \geq k - 1$, which gives $L \geq 2k - 1$ in both games.

Classify Dominator's first T moves z, according to the sets R and C just before z, as type α ($z = (i, j)$ non-bottom with $i \notin R$, $j \notin C$), β ($i \notin R$, $j \in C$), γ ($i \in R$, $j \notin C$), δ ($i \in R$, $j \in C$) or ε (a bottom vertex). After a move of type α the played vertex is undominated, so T has not yet been reached and Staller answers by (S1) within the first T moves. A column that belongs to C but is not served contains only its bottom vertex as a played vertex; such columns arise only from moves of type ε and from moves (S4), one column each, so at most $\#\varepsilon + \#\mathrm{S4}$ moves of type β or δ serve a column, and all other moves of these types are wasted. All moves of type ε are wasted. Write $\#\mathrm{S}i$ for the number of Staller's moves of rule (Si) among the first T.

Case $R = [k-1]$ at time T. Rows enter R only through moves of types α and β and moves (S2). Hence

$$\begin{aligned} k - 1 &\leq \#\alpha + \#\beta + \#\mathrm{S2} \\ &\leq \#\alpha + \#\mathrm{S2} + \#\mathrm{S4} + \big(\#\varepsilon + \#\{\text{wasted moves of type } \beta\}\big) \\ &\leq s_T + w_T, \end{aligned}$$

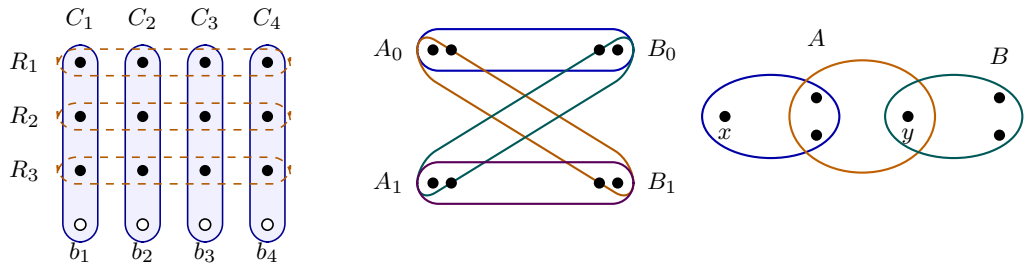
because the answers (S1) to the moves of type α, the moves (S2) and the moves (S4) are distinct moves of Staller.

Case $C = [k]$ at time T. Columns enter C only through moves of types α, γ, ε and moves (S4). A move of type ε adds a column only if it is played in a column not yet in C, so counting all such moves gives an upper bound. Every move of type γ except possibly the T-th move is followed, within the first T moves, by a move of Staller, which is not of type (S1) (the played vertex is dominated through its row) and not of type (S4) (its column is served). Every move (S4) follows a move of type ε or is Staller's opening move in the S-game. Hence the answers to the moves of types α and γ and the moves (S4) are distinct, and

$$k \leq \#\alpha + \#\gamma + \#\varepsilon + \#\mathrm{S4} \leq (s_T + 1) + w_T.$$

In both cases $s_T + w_T \geq k - 1$, so $L \geq 2k - 1$. □

The theorem shows that for every k the upper bound $\gamma_{tg} \leq 2\gamma_t - 1$ of Theorem 2.5 is attained by a k-uniform hypergraph with $\gamma_t = k$. Computations give $\gamma'_{tg}(H_{k,1}) = 2k - 1$ for $k \leq 6$, and we expect this to hold for every k; for the full $k \times k$ grid (all rows) the values are $(\gamma_{tg}, \gamma'_{tg}) = (4, 4), (5, 6), (7, 8)$ for $k = 3, 4, 5$.



(a) $H_{4,1}$ (b) $B_{2,2}$ (c) F_3

Fig. 4 Three hypergraphs with exactly known game values. (a) The grid $H_{4,1}$: columns (vertical) and the first three rows (horizontal); the bottom vertices (white) lie only in their column; $\gamma_{tg} = 7 = 2\gamma_t - 1$ (Theorem 6.6). (b) $B_{2,2}$, $k = 4$, with edges $A_i \cup B_j$: $\gamma'_{tg} = 2 < 3 = \gamma_{tg}$ (Theorem 6.8). (c) F_3, with edges $A \cup \{x\}$, $A \cup \{y\}$, $B \cup \{y\}$: $\gamma'_{tg} = 4 = \gamma_{tg} + 1 = 2\gamma_t$ (Theorem 6.7)

By Theorem 2.5, $|\gamma_{tg} - \gamma'_{tg}| \leq 1$, $\gamma_{tg} \leq 2\gamma_t - 1$ and $\gamma'_{tg} \leq 2\gamma_t$. Each of the three possible values of $\gamma'_{tg} - \gamma_{tg}$ occurs, and both upper bounds are attained, for every $k \geq 4$ (Theorems 6.7 and 6.8 and Proposition 6.1), and also for $k = 3$ (Theorem 6.7 and Proposition 6.1 and Example 6.9).

Theorem 6.7. *For $k \geq 3$ let A, B be disjoint $(k-1)$-sets and x, y two further vertices, and let F_k have the edges $A \cup \{x\}$, $A \cup \{y\}$ and $B \cup \{y\}$. Then $\gamma_t(F_k) = 2$, $\gamma_{tg}(F_k) = 3 = 2\gamma_t(F_k) - 1$ and $\gamma'_{tg}(F_k) = 4 = 2\gamma_t(F_k)$.*

Proof. We have $N(x) = A$, $N(y) = A \cup B$, $N(a) = (A \setminus \{a\}) \cup \{x, y\}$ for $a \in A$ and $N(b) = (B \setminus \{b\}) \cup \{y\}$ for $b \in B$; note $|A| = |B| \geq 2$. For $a \in A$, $N(y) \cup N(a) = V$, so $\gamma_t = 2$.

$\gamma_{tg} \leq 3$. Dominator plays y; then only x and y are undominated. Staller's legal replies are the vertices of A, which end the game, and of B, after which Dominator ends the game with any vertex of A.

$\gamma_{tg} \geq 3$. We exhibit, for every first move, a reply of Staller after which the game is not over. After y: Staller plays $b \in B$, and x is undominated. After $a \in A$: Staller plays $b \in B$, and a is undominated. After x: Staller plays $b \in B$ (legal, it dominates y), and x is undominated. After $b \in B$: Staller plays $b' \in B \setminus \{b\}$, and x is undominated.

$\gamma'_{tg} = 4$. The upper bound is Theorem 2.5. Staller opens with x, dominating A. If Dominator replies y, Staller plays $b \in B$ (legal, it dominates y) and x remains undominated. If he replies $a \in A$, then B is undominated and Staller plays $b \in B$, after which b is undominated. If he replies $b \in B$, Staller plays $b' \in B \setminus \{b\}$ and x remains undominated. In each case at least four moves are played. □

Theorem 6.8. *Let $a, b \geq 2$, $k = a + b$, let A_0, A_1, B_0, B_1 be disjoint sets with $|A_0| = |A_1| = a$ and $|B_0| = |B_1| = b$, and let $B_{a,b}$ be the k-uniform hypergraph with the four edges $A_i \cup B_j$. Then $\gamma_t(B_{a,b}) = \gamma'_{tg}(B_{a,b}) = 2$ and $\gamma_{tg}(B_{a,b}) = 3$.*

Proof. Let $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$. For $u \in A_i$, $N(u) = (A_i \setminus \{u\}) \cup B$, and for $w \in B_j$, $N(w) = (B_j \setminus \{w\}) \cup A$, so $N(u) \cup N(w) = V$ whenever $u \in A$, $w \in B$. Hence $\gamma_t = 2$ and, by Proposition 6.2(ii), $\gamma'_{tg} = 2$. For γ_{tg}, by symmetry, and since the roles of A and B are interchangeable in the following argument (as $a, b \geq 2$), we may assume that Dominator starts with $u \in A_0$; then u and A_1 are undominated. Staller replies $u' \in A_0 \setminus \{u\}$, which dominates u and leaves A_1 undominated; so $\gamma_{tg} \geq 3$. Conversely, after u and any reply of Staller, a reply in B ends the game, and after a reply in A Dominator plays a vertex of B, which dominates A (in particular A_1) and ends the game because $B \subseteq N(u)$. □

For $k = 3$ the construction of Theorem 6.8 is not available. The exhaustive computation of Section 7 shows that every isolate-free 3-uniform hypergraph on at most 6 vertices has $\gamma'_{tg} \geq \gamma_{tg}$, and that there are exactly 7 classes (2-sections) on 7 vertices with $\gamma'_{tg} = \gamma_{tg} - 1$. One of them is small enough to check by hand.

Example 6.9. Let H be the 3-uniform hypergraph on $\{0, \ldots, 6\}$ with the edges $\{0,3,4\}$, $\{0,5,6\}$, $\{1,3,5\}$, $\{1,5,6\}$, $\{2,4,6\}$, $\{2,5,6\}$. Then $\gamma_t(H) = 2$, $\gamma_{tg}(H) = 3$ and $\gamma'_{tg}(H) = 2$.

The open neighbourhoods are

$$N(0) = \{3,4,5,6\}, \quad N(1) = \{3,5,6\}, \quad N(2) = \{4,5,6\},$$
$$N(3) = \{0,1,4,5\}, \quad N(4) = \{0,2,3,6\},$$
$$N(5) = \{0,1,2,3,6\}, \quad N(6) = \{0,1,2,4,5\}.$$

Since $N(5) \cup N(6) = V$, we have $\gamma_t = 2$.

The S-game. For each w the following v satisfies $N(v) \cup N(w) = V$: $v = 5$ for $w \in \{0,2,3,6\}$ and $v = 6$ for $w \in \{1,4,5\}$. Hence $\gamma'_{tg} = 2$ by Proposition 6.2(ii).

The D-game. For each first move v we give a legal reply w with $N(v) \cup N(w) \neq V$, so $\gamma_{tg} \geq 3$ by Proposition 6.2(i):

first move v	0	1	2	3	4	5	6
reply w	3	0	0	1	2	1	2
a vertex still undominated	2	0	0	2	1	4	3

Each reply is legal because it dominates a vertex not in $N(v)$; for example, after $v = 5$ the reply 1 dominates the vertex 5. Finally $\gamma_{tg} \leq 2\gamma_t - 1 = 3$ by Theorem 2.5.

7 Computational results

The computations verify every concrete value claimed in the paper, test the strategies used in the proofs against all opponent strategies on small instances, and provide the evidence behind the conjectures. None of the theorems depends on a computation, except Theorem 3.9(ii), whose families are verified by a finite check. Table 4 lists the experiments.

The software consists of nauty 2.8.8 (`geng`) for isomorph-free generation and Python with NetworkX and matplotlib. There are two independent implementations of the solver, one in Python and one in C. Software versions and the command for every table and figure are recorded in Online Resource 1, which contains all code, data and verification logs.

Use of large language models. Large language models (Anthropic) were used to assist with the computer experiments, with checking computations and proofs, and with editing parts of the text. All proofs were checked by the authors, every computation reported here was reproduced with the programs in Online Resource 1, and the authors take full responsibility for the content.

object	parameters	method	size	role
all 2-sections in $\mathcal{G}_{n,k}$	$k \in \{3,4,5\}$, $k \le n \le 11$	`geng` + exact DP / DFS	289,443,894 classes	evidence for Conjecture 7.3, Observation 7.2
loose paths, loose cycles	$k \le 6$, $m \le 12$	DFS with symmetry reduction	65 instances	Observation 4.8, Conjecture 4.9; checks Corollary 4.6
$G^{(k)}$, G connected	$\lvert V(G)\rvert \le 6$, $k \in \{3,4\}$	exact values	280 instances	checks Theorems 4.2 and 4.4
Dominator strategies of Theorem 4.2	$\lvert V(G)\rvert \le 5$, every min. cover; paths/cycles	all Staller strategies	142+32 instances	checks the proof
Staller strategy of Theorem 6.6	$2 \le k \le 5$	all Dominator strategies	8 games	checks the proof
$H(\mathcal{F})$, random $\mathcal{F}$	$\lvert V(H(\mathcal{F}))\rvert \le 16$	exact values	437 systems	checks Theorem 3.4
threshold families	$52 \le k \le 72$	pair co-occurrence count	21 families	proof of Theorem 3.9(ii)
tight paths/cycles, 4-regular 2-sections	$n \le 18$	exact values	184 instances	context for Theorem 5.6
random search	$k \le 6$, $11 \le n \le 15$	local search	51 runs	evidence for Observation 7.2

Table 4 Computational experiments, the mathematical objects studied, and the statements they test or support. $\mathcal{G}_{n,k}$ is the class of 2-sections defined in Section 7.

A position of the game is the pair (D, P) of the set D of dominated vertices and the player P to move (Definition 2.1); played vertices need not be stored, because a played vertex is never legal again. With $\mathrm{val}(D, P)$ the number of remaining moves under optimal play,

$$\mathrm{val}(D,P) = \begin{cases} 0, & D = V(H), \\ 1 + \min\{\mathrm{val}(D \cup N(v), \mathrm{St}) : N(v) \not\subseteq D\}, & P = \mathrm{Dom}, \\ 1 + \max\{\mathrm{val}(D \cup N(v), \mathrm{Dom}) : N(v) \not\subseteq D\}, & P = \mathrm{St}, \end{cases} \tag{9}$$

and $\gamma_{tg}(H) = \mathrm{val}(\emptyset, \mathrm{Dom})$, $\gamma'_{tg}(H) = \mathrm{val}(\emptyset, \mathrm{St})$. We store D as an n-bit integer and memoise val (Algorithm 1). There are at most 2^{n+1} positions, each evaluated with $O(n^2)$ word operations; the pruning below reduces the branching substantially.

Lemma 7.1. *In (9) the minimum may be restricted to moves v for which $N(v) \setminus D$ is inclusion-maximal among the sets $N(u) \setminus D$, and the maximum to moves for which it is inclusion-minimal; moves with equal sets $N(v) \setminus D$ may be identified.*

Proof. If $N(u) \setminus D \subseteq N(v) \setminus D$, then $D \cup N(u) \subseteq D \cup N(v)$, and Lemma 2.4 (with the same player to move) gives $\mathrm{val}(D \cup N(v), P) \le \mathrm{val}(D \cup N(u), P)$. The value depends on v only through $D \cup N(v)$. □

For power hypergraphs the $k-2$ private vertices of an edge are interchangeable by automorphisms, and we replace every position by the canonical one in which, within each such block, the dominated vertices come first. Since automorphisms preserve val, the values are unchanged; this makes loose paths and cycles with up to 40 vertices feasible.

Algorithm 1 Exact values $(\gamma_{tg}(H), \gamma'_{tg}(H))$ of an isolate-free hypergraph

Require: neighbourhood bit masks $N(v)$, $v \in V(H)$
1: **function** VAL(D, P) ▷ memoised on (D, P); D = dominated set, P = player to move
2: **if** $D = V(H)$ **then return** 0
3: **end if**
4: $\mathcal{C} \leftarrow \{N(v) \setminus D : v \in V(H)\} \setminus \{\emptyset\}$ ▷ new sets of the legal moves
5: **if** $P =$ Dominator **then**
6: $\mathcal{C} \leftarrow$ inclusion-maximal members of $\mathcal{C}$ ▷ Lemma 7.1
7: **return** $1 + \min_{S \in \mathcal{C}}$ VAL$(D \cup S,$ Staller)
8: **else**
9: $\mathcal{C} \leftarrow$ inclusion-minimal members of $\mathcal{C}$
10: **return** $1 + \max_{S \in \mathcal{C}}$ VAL$(D \cup S,$ Dominator)
11: **end if**
12: **end function**
13: **return** $\big($VAL$(\emptyset, \mathrm{Dominator}),$ VAL$(\emptyset, \mathrm{Staller})\big)$

All computations were carried out twice, with implementations that share no code.

(A) The original pipeline (Python, `tdgame.py`; Algorithm 1 with the pruning of Lemma 7.1). It was validated against a second Python implementation that uses the set of played vertices as state and no pruning, on 635 random graphs with 3 to 8 vertices.

(B) An independent re-implementation that shares no code with (A):
- a C plug-in for `geng` (`tdg_geng.c`) that tests the clique condition and computes γ_t by a subset recursion;
- two game solvers inside the plug-in: a backward recursion over *all* 2^n sets D without any pruning, and a memoised forward search with the pruning of Lemma 7.1;
- a stand-alone C solver (`tdsolve_stdin.c`);
- a Python solver with the symmetry reduction for power hypergraphs (`tdsolve.py`).

The checks were the following.

(a) For the cycles C_n, $3 \le n \le 20$, both pipelines reproduce the formula $\gamma_{tg}(C_n) = \lfloor (2n+1)/3 \rfloor - [n \equiv 4 \pmod 6]$ of Dorbec and Henning [8]. Complete multipartite graphs and complete hypergraphs give $\gamma_{tg} = \gamma'_{tg} = 2$, and disjoint unions of complete hypergraphs give twice the number of components (Proposition 6.1).

(b) Every concrete exact value proved in this paper was recomputed:
- F_k for $3 \le k \le 7$ and $B_{a,b}$ for $a, b \in \{2, 3\}$;
- Example 6.9;
- $\gamma_t(H_{k,1}) = k$ and $\gamma_{tg}(H_{k,1}) = \gamma'_{tg}(H_{k,1}) = 2k-1$ for $k \le 5$, and $\gamma_{tg}(H_{6,1}) = \gamma'_{tg}(H_{6,1}) = 11$ by a further independent solver;
- the values 5 and 6 in Remark 4.5;
- all entries of Table 6, and the entries of Table 2 with $m \le 9$ ($k \in \{3, 4\}$), $m \le 7$ ($k = 5$) and $m \le 5$ ($k = 6$).

Proposition 6.2 agrees with the solver on every class of the exhaustive enumeration.

(c) Theorem 3.4 was compared with exact values on 287 random set systems with $|V(H(\mathcal{F}))| \le 16$ (original) and on 150 further ones with $|V(H(\mathcal{F}))| \le 15$ (independent). The families of Theorem 3.9 were re-verified from the printed Table 7. Theorems 4.2 and 4.4 were compared with exact values on all power hypergraphs $G^{(k)}$, $k \in \{3, 4\}$, of connected graphs G with at most 6 vertices, and on random graphs (`explore_power.py`, `test_power_upper.py`). No violation occurred.

(d) *Strategy checks.* Dominator's strategies in the proof of Theorem 4.2 (basic and refined) were played against *every* strategy of Staller and every admissible choice of the assignment $\beta(x)$, with the legality of each prescribed move and each budget asserted. The instances were:
- $P_m^{(k)}$ and $C_m^{(k)}$ with $k = 3$, $m \le 9$ and $k = 4$, $m \le 7$;
- every connected graph with at most 5 vertices, with every minimum vertex cover, for $k \in \{3, 4\}$.

Staller's strategy in the proof of Theorem 6.6 was played against every strategy of Dominator on $H_{k,1}$, $2 \le k \le 5$, in both games (`gridstrat.py`); it always forced $2k-1$ moves.

By Proposition 2.3 the values of $(\gamma_{tg}, \gamma'_{tg})$ over isolate-free k-uniform hypergraphs on n vertices are the values over the class $\mathcal{G}_{n,k}$ of graphs without isolated vertices in which every edge lies in a K_k. Different hypergraphs with the same 2-section have the same values, so we enumerate the graphs of $\mathcal{G}_{n,k}$ up to isomorphism. We generated all graphs with minimum degree at least $k-1$ with `geng` from nauty 2.8.8 and kept those in which every edge lies in a K_k (Fig. 5). Table 5 summarises the results.

Note what is counted. For fixed (k, n) every isomorphism class of $\mathcal{G}_{n,k}$ is counted once. A graph in $\mathcal{G}_{n,4}$ also lies in $\mathcal{G}_{n,3}$, so the number 3,150,526 for $n \le 10$ is a sum over the 21 pairs (k, n) and not a number of distinct graphs. The number of distinct graphs is 3,019,582, the $k = 3$ total. Disconnected graphs are included and isolated vertices are excluded.

The two pipelines agree completely for $n \le 10$:
- the same 3,150,526 classes (graph6 codes);
- the same triple $(\gamma_t, \gamma_{tg}, \gamma'_{tg})$ for every class;
- hence identical statistics in all eleven columns of Table 5.

The unpruned and the pruned C solvers also agree on every class. The rows with $n = 11$ were computed with pipeline (B) only. On four cores (`run_exhaustive.py`), the largest cases took the following times:
- $k = 3$, $n = 10$: 35 s (9,808,968 graphs from `geng`, 2,954,334 classes);
- $k = 4$, $n = 11$: 270 s;
- $k = 3$, $n = 11$: 80 min (277,512,114 classes);
- every other case: under two minutes.

Observation 7.2. *For $k \in \{3, 4, 5\}$ and $k \le n \le 11$:*

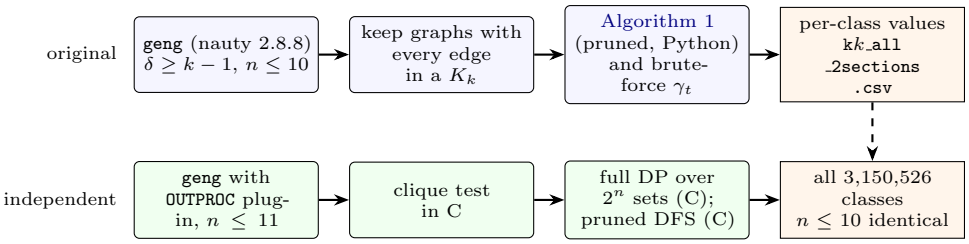


Fig. 5 The computational pipelines. Top: the original implementation. Bottom: the independent re-implementation, which extends the range to $n = 11$. Both produce identical values on all classes with $n \leq 10$. `README.md` in Online Resource 1 lists the commands

k	n	classes	conn.	$\max \gamma_{tg}$	$\max \gamma'_{tg}$	$\max \gamma_{tg}^{\text{conn}}$	$\max \gamma_t$	$\gamma_{tg}{=}2\gamma_t{-}1$	$\gamma'_{tg}{=}\gamma_{tg}{+}1$	$\gamma'_{tg}{=}\gamma_{tg}{-}1$
3	3	1	1	2	2	2	2	0	0	0
3	4	2	2	2	2	2	2	0	0	0
3	5	7	7	2	2	2	2	0	0	0
3	6	32	31	4	4	3	4	5	1	0
3	7	220	218	4	4	3	4	81	25	7
3	8	2,654	2,644	4	4	4	4	1,618	591	121
3	9	62,332	62,286	6	6	5	6	48,528	22,370	2,316
3	10	2,954,334	2,954,024	6	6	5	6	2,545,599	1,478,311	60,925
3	11	277,512,114	277,508,807	6	6	6	6	244,524,961	174,126,890	2,448,353
4	4	1	1	2	2	2	2	0	0	0
4	5	2	2	2	2	2	2	0	0	0
4	6	7	7	2	2	2	2	0	0	0
4	7	32	32	2	2	2	2	0	0	0
4	8	260	259	4	4	3	4	30	4	5
4	9	3,809	3,807	4	4	3	4	1,048	182	160
4	10	122,456	122,446	4	4	4	4	58,226	13,119	7,505
4	11	8,636,991	8,636,945	4	4	4	4	5,627,267	1,668,750	475,905
5	5	1	1	2	2	2	2	0	0	0
5	6	2	2	2	2	2	2	0	0	0
5	7	7	7	2	2	2	2	0	0	0
5	8	32	32	2	2	2	2	0	0	0
5	9	261	261	3	3	3	2	1	0	0
5	10	4,074	4,073	4	4	3	4	224	16	37
5	11	144,263	144,261	4	4	3	4	20,618	1,749	4,029
total, $n \leq 10$: 3,150,526		total, $n \leq 11$: 289,443,894								

Table 5 Exhaustive results over isomorphism classes of 2-sections of isolate-free k-uniform hypergraphs on n vertices. "conn." counts connected classes; $\max \gamma_{tg}^{\text{conn}}$ is the maximum over connected classes. The last three columns count the classes attaining $\gamma_{tg} = 2\gamma_t - 1$ and the two possible differences between the games. The rows with $n = 11$ were computed by pipeline (B) only.

(a) $\max \gamma_{tg} = \max \gamma'_{tg} \leq 2n/k$*; for* $k \in \{3, 4\}$ *both maxima equal* $2\lfloor n/k \rfloor$ *and are attained by disjoint unions of complete hypergraphs;*

(b) for $k = 5$ *the maximum is* $2\lfloor n/5 \rfloor$ *except at* $n = 9$*, where it is* $3 > 2\lfloor 9/5 \rfloor$*, attained by a unique class, which is connected and has* $\gamma_t = 2$*,* $\gamma_{tg} = \gamma'_{tg} = 3$*;*

(c) for $k = 3$ *and* $n = 3, \ldots, 11$*, the maximum of* γ_{tg} *over connected hypergraphs is* 2, 2, 2, 3, 3, 4, 5, 5, 6*, respectively;*

(d) both $\gamma'_{tg} = \gamma_{tg} + 1$ *and* $\gamma'_{tg} = \gamma_{tg} - 1$ *occur: for* $k = 3$ *from* $n = 6$*, respectively* $n = 7$*, on; for* $k = 4$ *from* $n = 8$ *on; and for* $k = 5$ *from* $n = 10$ *on;*

(e) $\gamma_{tg} = 2\gamma_t - 1$ *holds for* 2,545,599 *of the* 2,954,334 *classes with* $(k, n) = (3, 10)$*; all but* 180 *of them have* $\gamma_t = 2$*.*

The range $n \leq 11$ is small, and Observation 7.2 is evidence, not proof. Theorem 3.9 shows that the ratio $2/k$ is exceeded for larger k, by hypergraphs with more than 100 vertices. For $k = 3$, however, the data support the following conjecture, which would improve Theorem 5.1 and is sharp by Proposition 6.1.

Conjecture 7.3. *Every isolate-free* 3*-uniform hypergraph* H *on* n *vertices satisfies* $\gamma_{tg}(H) \leq 2n/3$ *and* $\gamma'_{tg}(H) \leq 2n/3$*. Equivalently,* $\gamma_{tg}(G), \gamma'_{tg}(G) \leq 2|V(G)|/3$ *for every graph* G *without isolated vertices in which every edge lies in a triangle.*

Table 6 lists exact values for structured families, and Fig. 6 shows one member of each of six of them; the values for loose paths and cycles are in Table 2. All entries of Table 6 were obtained by both pipelines. The entries of Table 2 with $m \leq 9$ for $k \in \{3, 4\}$, with $m \leq 7$ for $k = 5$, and with $m \leq 5$ for $k = 6$ were also obtained by both; the remaining, larger entries were obtained by the original pipeline only.

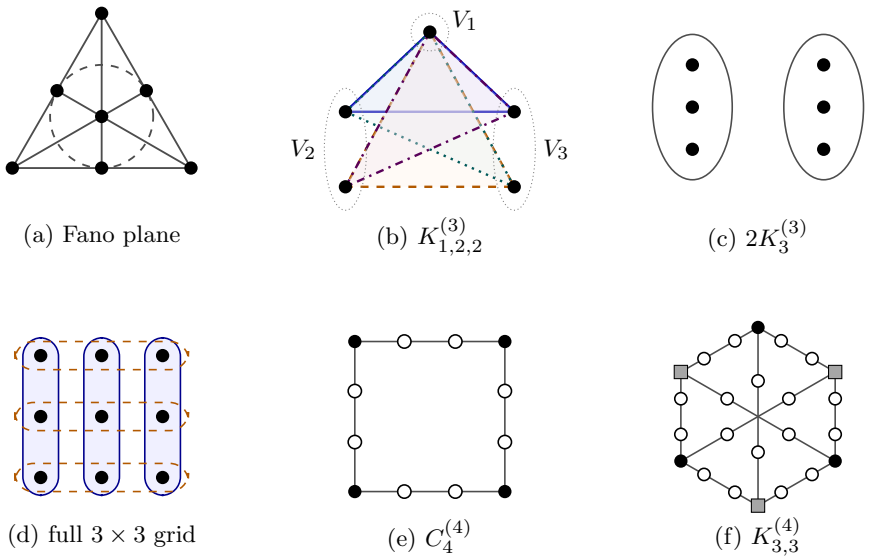
A randomised local search (`random_search.py`, seed 20261002; 51 runs with $k \in \{3,4,5,6\}$ and $11 \le n \le 15$) maximised $\max(\gamma_{tg}, \gamma'_{tg})/n$. It found no hypergraph with ratio above $2/k$; for $k = 3$ the best ratio was exactly $2/3$.

For 3-uniform hypergraphs with 4-regular 2-section we computed two families: the line graphs of all 112 connected cubic graphs on at most 12 vertices, and the squares C_n^2 of cycles, $7 \le n \le 18$. The largest ratio was $7/15 \approx 0.467$, well below $71/126 \approx 0.563$ (Theorem 5.6).

Tight paths and tight cycles (consecutive k-sets on a line or a circle) have values growing roughly like $n/2$ for $k = 3$ and $n/3$ for $k = 4$ in the computed range; we have not found exact formulas.

family	computed parameters	γ_t	γ_{tg}	γ'_{tg}	proof
$K_q^{(k)}$	$k \in \{3,4\}$, $k \le q \le k+3$	2	2	2	Corollary 6.3
complete k-partite	$k \in \{3,4\}$, parts of size $1,2,3$	2	2	2	Corollary 6.4
Fano plane, STS(9)	$k = 3$	2	2	2	Corollary 6.3
t disjoint edges	$k \in \{3,4,5\}$, $t \le 3$	$2t$	$2t$	$2t$	Proposition 6.1
F_k	$3 \le k \le 7$	2	3	4	Theorem 6.7
$B_{a,b}$	$a, b \in \{2,3\}$	2	3	2	Theorem 6.8
$H_{k,1}$	$k = 3,4,5$	k	$2k-1$	$2k-1$	Theorem 6.6 (for γ_{tg})
full $k \times k$ grid	$k = 3,4,5$	–	$4,5,7$	$4,6,8$	–
$P_m^{(k)}, C_m^{(k)}$, m even	see Table 2		m	m	Corollary 4.6 ($k \ge 4$)
$P_m^{(k)}, C_m^{(k)}$, m odd	see Table 2		m	$m+1$[a]	Corollary 4.6 (γ_{tg}, $k \ge 4$)
$G^{(k)}$, G d-regular bipartite	$K_{3,3}^{(4)}$, $C_{2r}^{(4)}$		$\lvert V(G)\rvert$	$\lvert V(G)\rvert$	Corollary 4.7 ($k \ge 4$)

Table 6 Exact values for structured families (all computed by both pipelines). The last column gives the statement that proves the value for all parameters, where available. [a]Except $\gamma'_{tg}(C_m^{(3)}) = m$; computational (Observation 4.8).



(a) Fano plane (b) $K_{1,2,2}^{(3)}$ (c) $2K_3^{(3)}$

(d) full 3×3 grid (e) $C_4^{(4)}$ (f) $K_{3,3}^{(4)}$

Fig. 6 Members of the structured families of Table 6, with their exact values $(\gamma_t, \gamma_{tg}, \gamma'_{tg})$. (a) The Fano plane, $(2,2,2)$ (Corollary 6.3); its seven edges are the three sides, the three medians and the dashed circle. (b) The complete 3-partite 3-uniform hypergraph with parts V_1, V_2, V_3 of sizes $1,2,2$, whose four edges are drawn as triangles with different line patterns, $(2,2,2)$ (Corollary 6.4). (c) Two disjoint edges, $k = 3$, $(4,4,4)$ (Proposition 6.1). (d) The full 3×3 grid, whose edges are the three columns (solid) and the three rows (dashed), $(3,4,4)$; computed. (e) The loose cycle $C_4^{(4)}$: every edge of C_4 is drawn as a segment carrying its two private vertices (white), $(3,4,4)$ (Corollaries 4.6 and 4.7). (f) The power $K_{3,3}^{(4)}$, drawn in the same way, with the two colour classes of $K_{3,3}$ as circles and squares; $\gamma_{tg} = \gamma'_{tg} = 6 = |V(K_{3,3})|$ (Corollary 4.7)

8 Conclusion

We studied the total domination game on uniform hypergraphs. We proved that the largest ratios $\Gamma(k)$ and $\Gamma'(k)$ of the game total domination numbers to the order over k-uniform hypergraphs are both $(2+o(1))\ln k/k$ (Theorem 3.7); the lower bound rests on the free-index lemma, in which two private vertices per edge keep Staller's moves legal. The same lemma, applied to explicit cyclic families, shows that the ratio $2/k$ of disjoint edges is exceeded for every $k \ge 52$ (Theorem 3.9). For power hypergraphs we showed that $\gamma_{tg}(G^{(k)})$ lies

between $2(m+1)/\Delta - 1$ and $2\tau(G) - \varepsilon(G)$ and that $\gamma'_{tg}(G^{(k)})$ lies between $2m/\Delta$ and $2\tau(G)$, where the upper bounds hold for $k \geq 3$ and the lower bounds for $k \geq 4$; this determines γ_{tg} for all loose paths and loose cycles, γ'_{tg} for those with an even number of edges, and both values for the powers of all regular bipartite graphs (Corollaries 4.6 and 4.7). We further determined $\gamma_{tg}(H_{k,1}) = 2\gamma_t(H_{k,1}) - 1 = 2k - 1$ for the grid hypergraphs (Theorem 6.6), showed that each of the three possible values of $\gamma'_{tg} - \gamma_{tg}$ occurs and both upper bounds of Theorem 2.5 are attained in every uniformity $k \geq 3$, characterised the value 2 of both games, and collected the best upper bounds for fixed k that follow from the graph and transversal results. Finally, an exact computation over all isolate-free k-uniform hypergraphs with $k \in \{3, 4, 5\}$ and at most 11 vertices found no hypergraph exceeding the ratio $2/k$, which supports Conjecture 7.3 for $k = 3$, while the computed values of loose paths and cycles with an odd number of edges support Conjecture 4.9.

A The cyclic families in the proof of the threshold theorem

For each k in the second part of Table 1 we list the base sets $D_1, \ldots, D_t \subseteq \mathbb{Z}_p$. The ground set is $A = [t] \times \mathbb{Z}_p$, the complements are $B_c = \{(i, x) : c \in D_i + x\}$, $c \in \mathbb{Z}_p$, and the hypergraph is $H(\mathcal{F})$ with $\mathcal{F} = (A \setminus B_c)_{c \in \mathbb{Z}_p}$; it has $n = tp + 2p$ vertices and uniformity $k = tp - \sum_i |D_i| + 2$. For $k \in \{53, 55, 60, 61, 63, 64, 69\}$ the base sets have two different sizes; repeated base sets (for instance for $k = 52$) are intended and allowed. The property used in the proof of Theorem 3.9(ii) is that $|(D_i + x) \cap (D_{i'} + x')| \geq 3$ for all $(i, x) \neq (i', x')$; it is verified by `free_index.py`.

k	base sets $D_1, \ldots, D_t \subseteq \mathbb{Z}_p$
52 ($p = 18$)	$\{0,2,4,5,7,13,14,17\}$, $\{1,7,8,11,12,14,16,17\}$, $\{3,4,7,8,10,12,13,15\}$, $\{4,5,8,9,11,13,14,16\}$, $\{4,5,8,9,11,13,14,16\}$
53 ($p = 22$)	$\{0,1,2,3,4,7,9,13,15,20\}$, $\{0,4,7,8,10,15,17,20,21\}$, $\{1,2,3,7,10,11,13,18,20\}$, $\{1,3,6,7,8,12,15,16,18\}$
54 ($p = 22$)	$\{1,2,4,5,6,9,11,13,17\}$, $\{1,3,7,13,14,16,17,18,21\}$, $\{2,4,6,10,16,17,19,20,21\}$, $\{3,9,10,12,13,14,17,19,21\}$
55 ($p = 17$)	$\{0,2,3,6,8,13,15,16\}$, $\{0,3,5,9,10,11,14,16\}$, $\{1,3,4,5,7,8,11,13\}$, $\{1,3,4,6,8,11,13,14,15\}$, $\{2,4,5,6,8,9,12,14\}$, $\{3,5,6,7,9,10,13,15\}$
56 ($p = 17$)	$\{0,1,3,6,7,8,9,13\}$, $\{0,4,7,9,10,11,12,13\}$, $\{1,2,3,4,8,12,13,15\}$, $\{1,4,6,7,8,9,10,14\}$, $\{2,4,5,6,7,8,12,16\}$, $\{3,7,8,10,13,14,15,16\}$
57 ($p = 20$)	$\{0,3,8,9,10,11,13,14,15\}$, $\{1,2,3,4,6,7,8,13,16\}$, $\{1,2,8,10,11,15,16,17,18\}$, $\{1,2,8,10,11,15,16,17,18\}$, $\{3,5,6,7,9,11,16,18,19\}$
58 ($p = 16$)	$\{0,1,2,5,6,8,9,14\}$, $\{0,2,3,4,5,8,9,11\}$, $\{0,2,3,5,6,11,13,15\}$, $\{0,2,3,8,10,12,13,15\}$, $\{1,2,6,7,9,10,11,15\}$, $\{3,5,6,7,10,11,13,14\}$, $\{4,6,7,8,9,12,13,15\}$
59 ($p = 29$)	$\{1,2,3,5,10,14,15,22,25,28\}$, $\{1,3,4,5,7,12,16,17,24,27\}$, $\{3,4,11,14,17,19,20,21,23,28\}$
60 ($p = 21$)	$\{0,1,3,7,9,12,13,14,15,18\}$, $\{0,2,6,11,15,16,17,18,19,20\}$, $\{0,4,5,7,9,10,12,16,20\}$, $\{0,4,5,9,10,12,14,15,17\}$, $\{1,3,5,8,12,15,16,17,18\}$
61 ($p = 19$)	$\{0,2,4,5,7,8,15,16,18\}$, $\{0,4,6,8,9,11,12,17,18\}$, $\{1,2,5,7,8,9,11,13,17,18\}$, $\{2,4,6,7,9,10,11,16,17\}$, $\{3,4,5,9,11,13,14,16,17\}$, $\{5,8,9,10,12,13,15,17,18\}$
62 ($p = 19$)	$\{0,4,6,8,13,14,15,16,18\}$, $\{0,5,7,10,11,13,15,16,17\}$, $\{1,2,3,5,8,9,11,15,16\}$, $\{2,3,5,7,8,9,14,15,17\}$, $\{3,5,7,10,11,13,14,17,18\}$, $\{4,6,7,9,10,12,13,15,16\}$
63 ($p = 26$)	$\{0,1,3,7,8,12,14,19,22,24,25\}$, $\{0,3,7,9,10,12,13,14,18,20,22\}$, $\{1,6,7,8,10,11,16,17,19,20\}$, $\{2,4,8,10,15,16,20,21,22,24,25\}$
64 ($p = 26$)	$\{0,2,6,7,12,14,19,21,23,24,25\}$, $\{0,3,9,11,13,15,16,21,22,25\}$, $\{1,2,7,8,10,11,12,15,21,25\}$, $\{3,6,9,12,14,15,17,18,19,20,25\}$
65 ($p = 18$)	$\{0,1,3,5,6,7,15,16,17\}$, $\{0,1,4,5,12,13,14,16,17\}$, $\{0,1,4,8,9,11,12,15,17\}$, $\{1,2,4,5,9,10,12,14,16\}$, $\{1,2,5,8,11,13,14,16,17\}$, $\{1,3,5,6,8,11,12,13,15\}$, $\{2,4,5,7,8,10,11,14,17\}$
66 ($p = 27$)	$\{0,2,5,13,14,16,17,21,22,23,24\}$, $\{1,2,3,6,11,13,14,16,22,23,26\}$, $\{2,5,6,9,11,12,13,14,16,22,26\}$, $\{3,4,5,6,12,13,14,17,18,20,26\}$
67 ($p = 23$)	$\{2,3,5,6,7,9,12,13,18,20\}$, $\{2,4,5,6,11,13,17,19,21,22\}$, $\{3,4,8,9,16,17,18,19,20,21\}$, $\{4,8,9,11,12,14,15,18,20,22\}$, $\{5,6,7,8,9,10,15,16,19,21\}$
68 ($p = 13$)	$\{0,1,2,3,4,6,9\}$, $\{0,1,2,4,7,11,12\}$, $\{0,1,3,5,8,9,12\}$, $\{0,2,5,9,10,11,12\}$, $\{1,2,3,4,5,7,10\}$, $\{1,2,4,8,9,10,12\}$, $\{1,4,5,8,9,10,12\}$, $\{1,5,6,7,8,9,11\}$, $\{2,3,4,6,8,11,12\}$, $\{2,6,7,8,9,10,12\}$, $\{3,4,5,6,7,9,12\}$
69 ($p = 24$)	$\{0,2,3,4,8,10,13,17,20,21,22\}$, $\{0,2,4,8,10,11,13,14,19,20,22\}$, $\{0,2,5,10,12,13,14,18,19,20,22\}$, $\{0,3,5,8,10,11,12,16,17,21\}$, $\{6,7,9,11,12,13,17,18,19,22\}$
70 ($p = 28$)	$\{0,2,6,7,11,14,15,18,19,20,21\}$, $\{0,4,9,10,12,14,17,21,22,23,24\}$, $\{1,2,3,5,8,12,14,15,17,22,25\}$, $\{3,4,6,11,12,14,15,16,17,18,22\}$
71 ($p = 35$)	$\{0,3,5,6,12,17,21,22,25,27,28,29\}$, $\{2,3,7,9,11,12,15,17,18,26,30,34\}$, $\{4,9,13,14,17,19,20,21,27,30,32,33\}$
72 ($p = 25$)	$\{0,1,2,3,7,8,12,15,17,20,24\}$, $\{1,2,4,7,8,9,11,14,15,18,20\}$, $\{1,2,5,7,10,11,12,15,19,21,22\}$, $\{4,6,9,10,11,13,15,17,20,21,23\}$, $\{4,6,11,12,15,16,17,19,21,23,24\}$

Table 7 Base sets of the cyclic families with $j = 2$.

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