Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Mehler--Heine formulas and zeros of discrete Charlier and Meixner Sobolev-type polynomials

Anier Soria Lorente

asoria@uloyola.es

Universidad Loyola Andalucía

Junior Michel

Universidad Loyola Andalucía

Research Article

Keywords: Discrete Sobolev orthogonal polynomials, Charlier and Meixner polynomials, Mehler–Heine formulas, Zeros, Asymptotic analysis.

Posted Date: November 13th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11447970/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

Mehler–Heine formulas and zeros of discrete Charlier and Meixner Sobolev-type polynomials

Anier Soria-Lorente [1], Junior Michel [1]

[1]Department of Quantitative Methods, Loyola University,
Avda. de las Universidades, 2. Dos Hermanas, Seville,
41704, Andalusia, Spain.
asoria@uloyola.es, jmichel@uloyola.es

Abstract

We consider the monic polynomials orthogonal with respect to the discrete Sobolev-type inner product $\langle f, g\rangle_\lambda = \sum_{x\geq 0} f(x)g(x)\rho(x) + \lambda\, \Delta^j f(\alpha)\, \Delta^j g(\alpha)$, where ρ is the Charlier or the Meixner weight, $\lambda > 0$, $j \geq 1$, and the mass point $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$, a negative real number which is not an integer, lies outside the support of ρ. Starting from the connection formula with the classical family, we establish Mehler–Heine type formulas for both Sobolev-type families. Since the mass point is exterior, the Sobolev perturbation is not asymptotically negligible with respect to the classical setting the scaling gains a factor n and the limit function acquires the factor $z - \alpha$, whereas it is independent of both the Sobolev mass parameter λ and the order j of the forward difference. As a consequence, for large degree the Sobolev-type polynomials have exactly one zero near the mass point, which is real and simple, and exactly one zero near each fixed nonnegative integer. We also deduce the outer relative asymptotics with respect to the classical families. Numerical experiments in high-precision arithmetic, with the computed zeros certified by exact rational sign changes, illustrate the results.

Keywords. Discrete Sobolev orthogonal polynomials, Charlier and Meixner polynomials, Mehler–Heine formulas, Zeros, Asymptotic analysis.
2020 Mathematics Subject Classification: Primary 33C45; Secondary 41A60, 26C10.
Corresponding author: Anier Soria-Lorente

1 Introduction

Classical orthogonal polynomials possess a remarkable collection of structural properties, such as three-term recurrence relations, difference or differential equations, Christoffel–Darboux kernels, generating functions and asymptotic expansions, which provide complementary descriptions of the same polynomial system [14, 15, 23].

Within the Askey scheme of hypergeometric orthogonal polynomials, the Charlier and Meixner families occupy a central position as the canonical discrete analogues of the Hermite and Laguerre families. They are closely related to the Poisson and negative binomial probability distributions, respectively, and their classical theory includes second-order difference equations, reproducing kernels, and Mehler–Heine asymptotic formulas [10, 14, 15, 23]. In this paper we restrict ourselves to these two classical families supported on $\mathbb{N}_0$; the families with finite support (Krawtchouk and Hahn) require a different asymptotic scaling and are not considered here.

One of the most important generalizations of classical orthogonality is provided by Sobolev-type orthogonal polynomials, where the underlying inner product is modified by incorporating

derivatives or, in the discrete setting, finite differences evaluated at prescribed points. See [16, 26, 25] for surveys. Replacing derivatives by forward or backward difference operators leads to families whose analytical behaviour is considerably more intricate than in the classical setting, connection formulas become substantially more involved, recurrence relations increase in order, and the asymptotic behaviour may change qualitatively.

1.1 Related Work

The theory of Sobolev orthogonal polynomials originated from the study of orthogonality with respect to inner products involving derivatives evaluated together with classical measures; see Marcellán and Ronveaux [18], Alfaro, Marcellán, Rezola and Ronveaux [1], and the surveys [16, 26] and Koekoek [25].

The discrete counterpart of Sobolev-type orthogonality was introduced by Bavinck [4, 5, 6], who replaced derivatives by finite difference operators evaluated at a point, obtained the corresponding connection formulas, and constructed an infinite-order difference operator having the Sobolev-type Charlier polynomials as eigenfunctions. Bavinck and Koekoek [7] studied the analogous problem for Sobolev-type Meixner polynomials. Analytic properties of Sobolev-type Charlier polynomials, including properties of their zeros, were obtained by Huertas and Soria-Lorente [13], whereas Costas-Santos, Soria-Lorente and Vilaire [9] developed a unified algebraic framework for Sobolev-type Meixner polynomials associated with arbitrary-order forward and backward differences, including connection formulas, ladder operators, $(2j+3)$-term recurrence relations, and a Mehler–Heine formula in the boundary case $\alpha = 0$. More recently, Neha and Swaminathan [22] analysed the zeros and the asymptotic behaviour of a class of Sobolev-type Meixner polynomials with a first-order difference.

A different, non-diagonal discrete Sobolev setting arises when the nonstandard part of the inner product is itself a discrete measure supported on $\mathbb{N}_0$ rather than a point evaluation. For the Meixner case this setting was investigated by Área, Godoy and Marcellán [2] and by Área, Godoy, Marcellán and Moreno-Balcázar [3], who obtained ratio and Plancherel–Rotach asymptotics; a generating function was obtained by Moreno-Balcázar, Pérez and Piñar [21], and a Mehler–Heine formula together with the asymptotic behaviour of the zeros was established by Moreno-Balcázar [20]; see also [19] for the use of Mehler–Heine formulas in the asymptotics of the smallest zeros. Related holonomic equations and asymptotic results for neighbouring families can be found in [24, 12, 11]. Although these inner products are not of Sobolev type in the sense considered here, the techniques developed in those works are closely related to ours.

Table 1 summarises the works on zeros and asymptotics that are closest to the present one, with the features that distinguish them: the inner product, the order of the difference operator, the position of the point mass, and the type of result.

1.2 Our Contributions

In the present paper we study the monic polynomials orthogonal with respect to the Sobolev-type inner product (13), with the Charlier or Meixner weight, a forward difference of arbitrary order $j \geq 1$, and a mass point $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$ outside the support. In this setting the connection formula contains the degree-dependent denominator

$$\delta_n = 1 + \lambda\, K_{n-1}^{(j,j)}(\alpha, \alpha),$$

which, unlike the boundary case $\alpha = 0$, grows at least geometrically with n (Lemma 3.3). In [9] the Meixner case was considered with forward or backward differences at $\alpha \leq 0$, and a Mehler–Heine formula was given for the evaluation point $\alpha = 0$; for forward differences the

Table 1: Closest related works on discrete Sobolev(-type) polynomials, compared with the present paper. "Diagonal" refers to a point-evaluation term $\lambda Tf(c)\,Tg(c)$ as in (13); "non-diagonal" to a second measure supported on $\mathbb{N}_0$.

Reference	Weight	Sobolev term	Results on zeros and asymptotics
[4, 5, 7]	general, Charlier, Meixner	diagonal, Δ (order 1) at a point	connection formulas, location of zeros, difference operators; fixed n
[13]	Charlier	diagonal, Δ (order 1) at c with $c+1<0$	zeros as functions of λ: interlacing, monotonicity, limits and rate $O(1/\lambda)$ as $\lambda\to\infty$; fixed n
[9]	Meixner	diagonal, Δ^j or ∇^j at $\alpha\le 0$	connection formula, ladder operators, $(2j+3)$-term recurrence; Mehler–Heine formula for $\alpha=0$
[22]	Meixner	diagonal, Δ-type term at a	five-term recurrence, zeros as functions of a and λ, Plancherel–Rotach asymptotics (as described in its published abstract)
[3, 20]	Meixner	non-diagonal, Δ	ratio and Plancherel–Rotach asymptotics; Mehler–Heine formula and asymptotics of zeros
[11]	generalized Charlier	non-diagonal, Δ	asymptotic expansions
[17]	Laguerre	diagonal, derivatives of higher order at 0	monotonicity and asymptotics of zeros in the mass parameter
This paper	Charlier, Meixner	diagonal, Δ^j, any $j\ge 1$, at $\alpha\in\mathbb{R}_-\setminus\mathbb{Z}_-$	Mehler–Heine formulas with scaling n and factor $z-\alpha$; local zeros; outer relative asymptotics

kernel $K_{n-1}^{(j,j)}(0,0)$ then converges [9, Corollary 1]. Since $\nabla^j f(0)=\Delta^j f(-j)$, backward differences evaluated at 0 correspond to forward differences evaluated at a negative integer, which is outside the hypotheses of the present paper. As a consequence, the Sobolev perturbation is not asymptotically negligible under the classical scaling.

Our main result is a Mehler–Heine type formula for both Sobolev-type families (Proposition 3.10 and Corollaries 3.11 and 3.12). Unlike the boundary case $\alpha=0$ for forward differences [9], where the Sobolev perturbation leaves the classical limit unchanged, the scaling gains a factor n and the limit function acquires the factor $z-\alpha$, while it remains independent of both the Sobolev parameter λ and the order j of the forward difference operator. By Hurwitz's theorem, we deduce that for large degree the Sobolev-type polynomials have exactly one zero near the mass point α, which is real and simple, and exactly one zero near each fixed nonnegative integer (Corollary 3.13). We also obtain the outer relative asymptotics with respect to the classical families (Corollary 3.14).

The paper is organized as follows. Section 2 introduces the classical Charlier and Meixner families together with the discrete Sobolev inner product and the corresponding connection formula. Section 3 establishes the Mehler–Heine formulas and their consequences for the zeros, and illustrates them numerically. Finally, Section 4 summarizes the main conclusions.

2 Mathematical Background

In this section we recall the main definitions and results on the Charlier and Meixner families of orthogonal polynomials, and we introduce the discrete Sobolev-type Charlier and Meixner

polynomials that are the object of this paper. Following the unified style of [9, 15], we present the results in a compact form, valid for both families upon particularising the two columns of Table 2. Throughout, Δ and ∇ denote the forward and backward difference operators

$$\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1), \tag{1}$$

applied recursively by

$$\Delta^n f(x) = \Delta[\Delta^{n-1} f(x)], \tag{2}$$

and a subscript Δ_x or Δ_y indicates the variable acted upon. The forward operator satisfies the product rule

$$\Delta[f(x)g(x)] = f(x)\Delta g(x) + g(x+1)\Delta f(x). \tag{3}$$

Throughout this paper, $[\cdot]_k$ denotes the falling factorial of order k, defined by

$$[x]_k = \begin{cases} 1, & k = 0, \\ \displaystyle\prod_{j=0}^{k-1} (x-j), & k \geq 1. \end{cases}$$

Moreover,

$$(x)_k = (-1)^k [-x]_k = \frac{\Gamma(x+k)}{\Gamma(x)},$$

denotes the Pochhammer symbol of order k, and Γ stands for the Gamma function. Let $\mathbb{P}$ denote the linear space of polynomials with real coefficients.

Fix one of the two columns of Table 2, that is, either the monic Charlier polynomials $C_n^{(\mu)}(x)$ with $\mu > 0$ or the monic Meixner polynomials $M_n^{(\gamma,\mu)}(x)$ with $\gamma > 0$ and $0 < \mu < 1$. We denote by P_n the polynomial of degree n of the corresponding generic monic family $\{P_n\}_{n\geq 0}$, and refer to [14, 15, 23] for the background.

Table 2: Monic Charlier $C_n^{(\mu)}(x)$ and Meixner $M_n^{(\gamma,\mu)}(x)$ polynomials and their data.

$P_n(x)$	$C_n^{(\mu)}(x)$, $\mu > 0$	$M_n^{(\gamma,\mu)}(x)$, $\gamma > 0$, $0 < \mu < 1$
$\rho(x)$	$\dfrac{e^{-\mu}\mu^x}{\Gamma(x+1)}$	$\dfrac{\mu^x \Gamma(\gamma+x)}{\Gamma(\gamma)\Gamma(x+1)}$
$\|P_n\|^2$	$n!\,\mu^n$	$\dfrac{n!\,(\gamma)_n\,\mu^n}{(1-\mu)^{\gamma+2n}}$
α_n	$n+\mu$	$\dfrac{n(1+\mu)+\mu\gamma}{1-\mu}$
β_n	$n\mu$	$\dfrac{n\mu(n-1+\gamma)}{(\mu-1)^2}$

Proposition 2.1. *Let $\{P_n\}_{n\geq 0}$ be the sequence of monic classical discrete polynomials of degree n. The following statements hold.*

1. *The sequence $\{P_n\}_{n\geq 0}$ consists of monic polynomials orthogonal with respect to the inner product defined on the space of polynomials $\mathbb{P}$,*

$$\langle f, g\rangle = \sum_{x\geq 0} f(x)g(x)\rho(x),$$

where $\rho(x)$ is defined by the corresponding column of the second row of Table 2.

2. *Squared norm. For every $n \in \mathbb{N}_0$, the quantity $\|P_n\|^2 = \langle P_n, P_n \rangle$ is given in Table 2.*

3. *The three-term recurrence relation*

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad n \geq 0, \tag{4}$$

holds, with initial conditions $P_{-1} \equiv 0$ and $P_0 \equiv 1$, where the recurrence coefficients α_n and β_n are given in the corresponding columns of Table 2.

4. *Forward shift operator. For $k \in \mathbb{N}_0$ we denote by $\{P_m^{[k]}\}_{m\geq 0}$ the monic classical family obtained from $\{P_m\}_{m\geq 0}$ after k forward differences, namely $P_m^{[k]} = C_m^{(\mu)}$ in the Charlier case and $P_m^{[k]} = M_m^{(\gamma+k,\mu)}$ in the Meixner case. For every $n \geq 1$ and $0 \leq k \leq n$, the k-th forward difference lowers the degree by k according to*

$$\Delta^k P_n(x) = [n]_k \, P_{n-k}^{[k]}(x), \quad 0 \leq k \leq n. \tag{5}$$

In each particular case this reads as follows. For the monic Charlier polynomials $C_n^{(\mu)}$ with $\mu > 0$,

$$\Delta^k C_n^{(\mu)}(x) = [n]_k \, C_{n-k}^{(\mu)}(x), \quad 0 \leq k \leq n, \tag{6}$$

whereas for the monic Meixner polynomials $M_n^{(\gamma,\mu)}$ with $\gamma > 0$, $0 < \mu < 1$, the lowered family carries a shifted parameter,

$$\Delta^k M_n^{(\gamma,\mu)}(x) = [n]_k \, M_{n-k}^{(\gamma+k,\mu)}(x), \quad 0 \leq k \leq n. \tag{7}$$

5. *Mehler–Heine asymptotics. In the monic normalisation of Table 2, Dominici's formulas [10] read, for the monic Charlier polynomials $C_n^{(\mu)}$ with $\mu > 0$,*

$$\lim_{n\to\infty} \frac{(-1)^n}{\Gamma(n-z)} \, C_n^{(\mu)}(z) = \varphi_C(z), \quad \varphi_C(z) = \frac{e^{\mu}}{\Gamma(-z)}, \tag{8}$$

and, for the monic Meixner polynomials $M_n^{(\gamma,\mu)}$ with $\gamma > 0$, $0 < \mu < 1$,

$$\lim_{n\to\infty} \frac{(\mu-1)^n}{\Gamma(n-z)} \, M_n^{(\gamma,\mu)}(z) = \varphi_M(z), \quad \varphi_M(z) = \frac{1}{(1-\mu)^{\gamma+z}\,\Gamma(-z)}, \tag{9}$$

both uniformly on compact subsets of $\mathbb{C}$. These are the monic forms of [10, Eqs. (23),(35)]. Writing φ for either limit function and κ_n for the corresponding sign/scale, both relations take the common form

$$\lim_{n\to\infty} \frac{1}{\kappa_n \, \Gamma(n-z)} \, P_n(z) = \varphi(z), \quad \kappa_n = \begin{cases} (-1)^n, & \textit{Charlier}, \\ (\mu-1)^{-n}, & \textit{Meixner}. \end{cases} \tag{10}$$

Both limit functions are entire multiples of $1/\Gamma(-z)$; hence their only zeros are the simple zeros of $1/\Gamma(-z)$ at $z = 0, 1, 2, \ldots$. Since $P^{[k]}$ is again a monic Charlier family, or a monic Meixner family with the admissible parameter $\gamma + k > 0$, it satisfies (10) with the same κ_n and with limit function $\varphi^{[k]} = \varphi_C$ (Charlier), respectively φ_M with γ replaced by $\gamma + k$ (Meixner). In particular, $\varphi^{[k]}(w) \neq 0$ for $w \in \mathbb{C} \setminus \mathbb{N}_0$.

Let K_n denote the n-th reproducing kernel of $\{P_n\}_{n\geq 0}$,

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\|P_k\|^2} = \frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{(x-y)\,\|P_n\|^2}, \tag{11}$$

where the second expression is the Christoffel–Darboux formula. For $(i,j) \in \mathbb{N}_0^2$ we write

$$K_n^{(i,j)}(x,y) = \Delta_x^i\big(\Delta_y^j K_n(x,y)\big) = \sum_{k=0}^{n} \frac{\Delta^i P_k(x)\,\Delta^j P_k(y)}{\|P_k\|^2}. \tag{12}$$

We now introduce the discrete Sobolev-type polynomials of order j, $\{\mathbb{S}_n(x)\}_{n\geq 0}$, orthogonal with respect to the Sobolev-type inner product

$$\langle f,g\rangle_\lambda = \sum_{x\geq 0} f(x)g(x)\rho(x) + \lambda\,\Delta^j f(\alpha)\,\Delta^j g(\alpha), \tag{13}$$

where $\lambda \in \mathbb{R}^+$ and $j \in \mathbb{N}$ with $j \geq 1$ are fixed, ρ is the weight of the chosen column of Table 2, and the mass point satisfies

$$\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-, \quad \mathbb{R}_- = (-\infty, 0), \quad \mathbb{Z}_- = \{-1,-2,-3,\dots\}. \tag{14}$$

This is a standing hypothesis throughout the paper. It delimits the problem studied here: the mass point is exterior to the support $\mathbb{N}_0$ of ρ, and the nodes $\alpha, \alpha+1, \dots, \alpha+j$ of $\Delta^j f(\alpha)$ are not integers. In the proofs below the hypothesis enters only through the inequality $\alpha < 0$. As before, P_n denotes the underlying classical monic family and all notation of Section 2 is retained.

The link between the Sobolev-type polynomials $\mathbb{S}_n$ and the classical polynomials P_n is the following connection formula, which underlies all subsequent constructions.

Proposition 2.2. *Let $\{\mathbb{S}_n\}_{n\geq 0}$ be the monic Sobolev-type family for the inner product* (13). *Then, for $n \geq 1$,*

$$\mathbb{S}_n(x) = P_n(x) - \lambda\,\frac{\Delta^j P_n(\alpha)}{1+\lambda\,K_{n-1}^{(j,j)}(\alpha,\alpha)}\,K_{n-1}^{(0,j)}(x,\alpha). \tag{15}$$

Proof. This connection formula is well known; see [4, 5, 7] for $j=1$ and [9] for the Meixner case with $j \geq 1$. For completeness, we recall that the argument of [18, Section 2] applies verbatim: expanding $\mathbb{S}_n$ in the basis $\{P_k\}_{k\leq n}$ and imposing orthogonality with respect to (13), all Fourier coefficients but the one carrying the mass term vanish, and the reproducing property of the kernel (11) yields (15). □

For convenience, we introduce the abbreviated notation

$$\mathbb{S}_n(x) = P_n(x) - \lambda\,\Omega_n\,K_{n-1}^{(0,j)}(x,\alpha), \tag{16}$$

where

$$\Omega_n = \frac{\Delta^j P_n(\alpha)}{1+\lambda\,K_{n-1}^{(j,j)}(\alpha,\alpha)} = \frac{\Delta^j P_n(\alpha)}{\delta_n}. \tag{17}$$

3 Mehler–Heine type formulas

We establish Mehler–Heine type formulas for the Sobolev-type families $\{\mathbb{S}_n\}_{n\geq 0}$. Since the mass point α lies outside the support $[0,+\infty)$, the Sobolev perturbation is not asymptotically negligible under the classical scaling: it cancels the leading term, shifts the Mehler–Heine scale by one factor of n, and produces a new zero of the limit function at $z=\alpha$. The resulting limit is independent of λ and of j. Throughout this section, $\widetilde{P} = P^{[j]}$ and $\widetilde{\varphi} = \varphi^{[j]}$ denote the family obtained after j forward differences and its Mehler–Heine limit (Proposition 2.1), so that $\Delta^j P_k(\alpha) = [k]_j \widetilde{P}_{k-j}(\alpha)$ by (5).

Lemma 3.1. *Let φ and κ_n be as in* (10)*, and set $c_\mu = 1$ for Charlier, $c_\mu = 1-\mu$ for Meixner. Let $k \in \mathbb{N}_0$ be fixed and let $K \subset \mathbb{C} \setminus \mathbb{N}_0$ be compact. Then there exists $n_K \in \mathbb{N}$ such that $P^{[k]}_{n-1}(w) \neq 0$ for all $w \in K$ and $n \geq n_K$, and*

$$\frac{P^{[k]}_n(w)}{P^{[k]}_{n-1}(w)} = -\frac{n-1-w}{c_\mu}\big(1+\varepsilon_n(w)\big), \quad \sup_{w\in K}|\varepsilon_n(w)| = O(n^{-2}), \tag{18}$$

and $\kappa_n/\kappa_{n-1} = -1/c_\mu$.

The first-order form of (18), with $\varepsilon_n = o(1)$, is the classical outer ratio asymptotics and follows at once from (10). What is needed below is the second-order information $\varepsilon_n = O(n^{-2})$, i.e., the absence of a term of order n^{-1}, because in Lemma 3.2 a term of order n^{-1} is extracted from the difference of two quantities of order one.

Proof. It suffices to consider $k = 0$, since $P^{[k]}$ is again a monic Charlier or Meixner family, with the same κ_n and c_μ (Proposition 2.1); in the Meixner case all the formulas below hold with γ replaced by $\gamma + k$. From (10), $\kappa_n/\kappa_{n-1} = -1$ for Charlier and $\kappa_n/\kappa_{n-1} = (\mu-1)^{-1}$ for Meixner; since $c_\mu = 1$ and $c_\mu = 1-\mu$ respectively, both are equal to $-c_\mu^{-1}$, which yields the last identity in (18).

Since φ is continuous and does not vanish on K, there is $m_K > 0$ with $|\varphi| \geq m_K$ on K, and by the uniform convergence in (10) there is n_K, with $n_K > 2 + \max_{w\in K}|w|$, such that $|P_n(w)| \geq \frac{m_K}{2}|\kappa_n\Gamma(n-w)| > 0$ for $w \in K$ and $n \geq n_K - 1$. Set $r_n(w) = P_n(w)/P_{n-1}(w)$ and $s_n(w) = -(n-1-w)/c_\mu$. Using the functional equation $\Gamma(n-w) = (n-1-w)\Gamma(n-1-w)$ and $\kappa_n/\kappa_{n-1} = -c_\mu^{-1}$,

$$1+\varepsilon_n(w) = \frac{r_n(w)}{s_n(w)} = \frac{P_n(w)/(\kappa_n\Gamma(n-w))}{P_{n-1}(w)/(\kappa_{n-1}\Gamma(n-1-w))} \longrightarrow \frac{\varphi(w)}{\varphi(w)} = 1,$$

uniformly on K; hence $\sup_K|\varepsilon_n| \to 0$.

To obtain the rate, we insert $r_n = s_n(1+\varepsilon_n)$ into the three-term recurrence relation (4), which gives $r_n = w - \alpha_{n-1} - \beta_{n-1}/r_{n-1}$. A direct computation with the recurrence coefficients of Table 2 shows that

$$\varepsilon_n(w) = a_n(w) + \theta_n(w)\,\frac{\varepsilon_{n-1}(w)}{1+\varepsilon_{n-1}(w)}, \tag{19}$$

where, in the Charlier case,

$$a_n(w) = -\frac{\mu(1+w)}{(n-1-w)(n-2-w)}, \quad \theta_n(w) = \frac{(n-1)\mu}{(n-1-w)(n-2-w)},$$

and, in the Meixner case,

$$a_n(w) = -\frac{\mu(1+w)(\gamma+w)}{(n-1-w)(n-2-w)}, \quad \theta_n(w) = \frac{\mu(n-1)(n-2+\gamma)}{(n-1-w)(n-2-w)}.$$

Indeed, in both cases

$$s_n\varepsilon_n = \Big(w - \alpha_{n-1} - s_n - \frac{\beta_{n-1}}{s_{n-1}}\Big) + \frac{\beta_{n-1}}{s_{n-1}}\,\frac{\varepsilon_{n-1}}{1+\varepsilon_{n-1}},$$

and the first bracket equals $\mu(1+w)/(n-2-w)$ for Charlier and $\mu(1+w)(\gamma+w)/((1-\mu)(n-2-w))$ for Meixner. Consequently $\sup_K|a_n| \leq C_K n^{-2}$, and $\theta_n \to 1-c_\mu$ uniformly on K, where $1-c_\mu = 0$ (Charlier) or $1-c_\mu = \mu$ (Meixner). Fix $\theta \in (1-c_\mu, 1)$. Since $\varepsilon_n \to 0$ uniformly on K, there is

$n_1 \geq n_K$ such that $|\theta_n/(1+\varepsilon_{n-1})| \leq \theta$ on K for $n > n_1$, and (19) gives $e_n \leq C_K n^{-2} + \theta\, e_{n-1}$ for $e_n = \sup_K |\varepsilon_n|$ and $n > n_1$. Iterating,

$$\begin{aligned} e_n &\leq \theta^{\,n-n_1} e_{n_1} + C_K \sum_{m=n_1+1}^{n} \theta^{\,n-m} m^{-2} \\ &\leq \theta^{\,n-n_1} e_{n_1} + C_K \Big(\theta^{n/2} \sum_{m \geq 1} m^{-2} + \frac{4}{n^2(1-\theta)} \Big), \end{aligned}$$

where we split the sum at $m = n/2$. Hence $e_n = O(n^{-2})$. □

Lemma 3.2. *Let $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$, $j \geq 1$, and set*

$$\mathcal{N}_n(z) = \Delta^j P_{n-1}(\alpha)\, P_n(z) - \Delta^j P_n(\alpha)\, P_{n-1}(z). \tag{20}$$

Then, uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{N}_0$,

$$\frac{n\,\kappa_n^{-1}}{\Gamma(n-z)} \, \frac{\mathcal{N}_n(z)}{\Delta^j P_{n-1}(\alpha)} \xrightarrow[n\to\infty]{} -(z-\alpha)\,\varphi(z). \tag{21}$$

In particular, the left-hand side of (21) is bounded on each such compact set for n large enough.

Proof. Fix a compact set $K \subset \mathbb{C} \setminus \mathbb{N}_0$. By the forward-shift identity (5), $\Delta^j P_s(\alpha) = [s]_j \widetilde{P}_{s-j}(\alpha)$, and $\alpha \in \mathbb{C} \setminus \mathbb{N}_0$, so Lemma 3.1 applied to $\widetilde{P} = P^{[j]}$ at the point α gives

$$\begin{aligned} \varrho_n := \frac{\Delta^j P_n(\alpha)}{\Delta^j P_{n-1}(\alpha)} &= \frac{[n]_j}{[n-1]_j} \, \frac{\widetilde{P}_{n-j}(\alpha)}{\widetilde{P}_{n-1-j}(\alpha)} \\ &= -\frac{1}{c_\mu} \, \frac{n}{n-j} \, (n-1-j-\alpha)\big(1 + O(n^{-2})\big). \end{aligned}$$

Lemma 3.1 applied to P on K gives $r_n(z) = P_n(z)/P_{n-1}(z) = -(n-1-z)c_\mu^{-1}(1+O(n^{-2}))$ uniformly on K. Hence

$$\frac{\mathcal{N}_n(z)}{\Delta^j P_{n-1}(\alpha)} = P_n(z) - \varrho_n\, P_{n-1}(z) = P_n(z)\Big[1 - \frac{\varrho_n}{r_n(z)}\Big],$$

with

$$\frac{\varrho_n}{r_n(z)} = \frac{n(n-1-j-\alpha)}{(n-j)(n-1-z)}\big(1 + O(n^{-2})\big).$$

After straightforward calculations,

$$\frac{n(n-1-j-\alpha)}{(n-j)(n-1-z)} = 1 + \frac{z-\alpha}{n} + O(n^{-2}),$$

uniformly on K, so that $n\big[1 - \varrho_n/r_n(z)\big] = -(z-\alpha) + O(n^{-1})$. Since $P_n(z)/(\kappa_n \Gamma(n-z)) \to \varphi(z)$ uniformly on K by (10), we obtain (21). The second-order precision of Lemma 3.1 is essential here: a relative error of order n^{-1} in either quotient would modify the coefficient of n^{-1}. □

Lemma 3.3. *Let $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$ and $j \geq 1$. Then $K_{n-1}^{(j,j)}(\alpha,\alpha) \to \infty$ as $n \to \infty$. More precisely, there exist $q > 1$ and $n_0 \in \mathbb{N}$, depending only on α, j and the parameters of the family, such that*

$$K_{n-1}^{(j,j)}(\alpha,\alpha) \geq q^n, \quad n \geq n_0. \tag{22}$$

In particular, for each fixed $p > 0$ one has $n^{-p}\, K_{n-1}^{(j,j)}(\alpha,\alpha) \to \infty$ as $n \to \infty$.

Proof. By the definition (12) of the iterated kernel and the forward shift operator (5), the differences $\Delta^j P_k(\alpha) = [k]_j \widetilde{P}_{k-j}(\alpha)$ vanish for $k < j$, so that

$$K_{n-1}^{(j,j)}(\alpha,\alpha) = \sum_{k=j}^{n-1} t_k, \quad t_k = \frac{\big(\Delta^j P_k(\alpha)\big)^2}{\|P_k\|^2} = \frac{[k]_j^2\, \widetilde{P}_{k-j}(\alpha)^2}{\|P_k\|^2} > 0. \tag{23}$$

We first establish that the series diverges. Since the summands are positive, it suffices to show $t_k \to \infty$, which we do through the ratio t_{k+1}/t_k. The Mehler–Heine limit (10) for the descended family gives

$$\widetilde{P}_{k-j}(\alpha) = \kappa_{k-j}\, \Gamma(k-j-\alpha)\, \widetilde{\varphi}(\alpha)\big(1+o(1)\big), \tag{24}$$

where $\widetilde{\varphi}(\alpha) \neq 0$ precisely because $\alpha < 0$ avoids the nonnegative integers, the only zeros of $\widetilde{\varphi}$; this is where the hypothesis $\alpha < 0$ enters. Squaring (24) and forming the ratio of consecutive terms, the three structural factors of t_k evolve as follows. The falling factorial satisfies

$$\frac{[k+1]_j}{[k]_j} = \frac{k+1}{k+1-j} \to 1,$$

so its square contributes a factor tending to 1. The Gamma factor evolves through the functional equation

$$\frac{\Gamma(k+1-j-\alpha)}{\Gamma(k-j-\alpha)} = k-j-\alpha,$$

contributing $(k-j-\alpha)^2$ after squaring. The remaining factors, the scale κ_{k-j}^2 and the squared norm $\|P_k\|^2$, are read from (10) and Table 2, respectively. Therefore,

$$\frac{\|P_{k+1}\|^2}{\|P_k\|^2} = \begin{cases} (k+1)\mu, & \text{for Charlier,} \\ \dfrac{(k+1)(k+\gamma)\mu}{(1-\mu)^2}, & \text{for Meixner,} \end{cases}$$

while

$$\frac{\kappa_{k+1-j}^2}{\kappa_{k-j}^2} = \begin{cases} 1, & \text{for Charlier,} \\ (1-\mu)^{-2}, & \text{for Meixner.} \end{cases}$$

Collecting these,

$$\frac{t_{k+1}}{t_k} = \frac{\kappa_{k+1-j}^2}{\kappa_{k-j}^2}\,(k-j-\alpha)^2\,\frac{\|P_k\|^2}{\|P_{k+1}\|^2}\big(1+o(1)\big) = \begin{cases} \dfrac{(k-j-\alpha)^2}{(k+1)\mu}\big(1+o(1)\big), & \text{Charlier,} \\ \dfrac{(k-j-\alpha)^2}{(k+1)(k+\gamma)\mu}\big(1+o(1)\big), & \text{Meixner.} \end{cases}$$

In the Meixner case the factor $(1-\mu)^{-2}$ from the scale cancels the $(1-\mu)^{-2}$ in the norm ratio. Consequently

$$\lim_{k\to\infty} \frac{t_{k+1}}{t_k} = \begin{cases} +\infty, & \text{Charlier,} \\ \dfrac{1}{\mu}, & \text{Meixner.} \end{cases} \tag{25}$$

We now draw from (25) the geometric growth of the summands. Denote by $L \in (1, +\infty]$ the limit in (25), which exceeds 1 in both families: it is $+\infty$ for Charlier, and equals $\mu^{-1} > 1$ for

Meixner because $0 < \mu < 1$. Choose any q with $1 < q < L$, for instance $q = 2$ in the Charlier case and

$$q = \frac{1 + \dfrac{1}{\mu}}{2},$$

in the Meixner case. By the definition of the limit, applied with the threshold q, there exists $k_0 \geq j$ such that

$$\frac{t_{k+1}}{t_k} \geq q, \quad k \geq k_0. \tag{26}$$

Iterating (26), that is, multiplying the inequalities for $k = k_0, k_0 + 1, \ldots, m - 1$ and telescoping the left-hand side, we obtain

$$t_m \geq t_{k_0}\, q^{\,m-k_0}, \quad m \geq k_0. \tag{27}$$

This is the precise sense in which the terms increase at least geometrically: they dominate the geometric sequence $t_{k_0} q^{-k_0} \cdot q^m$ of ratio $q > 1$. Three consequences follow at once. First, since $t_{k_0} > 0$ and $q > 1$, the right-hand side of (27) tends to $+\infty$, hence $t_m \to \infty$. Second, the terms of the series (23) therefore do not tend to 0, so the series diverges. Third, and quantitatively, summing (27) over m gives

$$K_{n-1}^{(j,j)}(\alpha, \alpha) = \sum_{k=j}^{n-1} t_k \geq \sum_{k=k_0}^{n-1} t_{k_0}\, q^{\,k-k_0} = t_{k_0}\, \frac{q^{\,n-k_0} - 1}{q - 1} \longrightarrow \infty,$$

so that $K_{n-1}^{(j,j)}(\alpha, \alpha) \to \infty$, as asserted.

It remains to sharpen this into the geometric lower bound (22). Since the kernel is a sum of positive terms, in particular

$$K_{n-1}^{(j,j)}(\alpha, \alpha) \geq t_{n-1}, \tag{28}$$

and (27) with $m = n - 1$ gives $t_{n-1} \geq c_0\, q^n$, where

$$c_0 = t_{k_0}\, q^{-k_0-1} > 0.$$

Fix any $q' \in (1, q)$. Since $c_0\, (\frac{q}{q'})^n \to \infty$, there is $n_0 \geq k_0$ with $c_0\, q^n \geq (q')^n$ for all $n \geq n_0$; combined with (28), this yields

$$K_{n-1}^{(j,j)}(\alpha, \alpha) \geq t_{n-1} \geq (q')^n, \quad n \geq n_0,$$

which is (22) with q' in the role of q. Finally, fix $p > 0$; since $q' > 1$ one has

$$\frac{(q')^n}{n^p} \to \infty,$$

so $n^{-p} K_{n-1}^{(j,j)}(\alpha, \alpha) \to \infty$. □

We shall also need the following two-sided estimate at the mass point.

Lemma 3.4. *Let $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$ and $j \geq 1$. There exist constants $0 < \ell_0 \leq \ell_1$, depending only on α, j and the parameters of the family, such that, for all $s \geq 0$,*

$$\ell_0\, |\kappa_s|\, \Gamma(s - \alpha) \leq |\widetilde{P}_s(\alpha)| \leq \ell_1\, |\kappa_s|\, \Gamma(s - \alpha). \tag{29}$$

Proof. Taking absolute values in the Mehler–Heine formula (10) for $\widetilde{P}$, we obtain $|\widetilde{P}_s(\alpha)|/(|\kappa_s|\Gamma(s-\alpha)) \to |\widetilde{\varphi}(\alpha)|$. Since $\alpha < 0$, the argument $-\alpha$ is positive and is not a pole of Γ, so $\widetilde{\varphi}(\alpha) \neq 0$. Hence there exist constants $0 < a \leq b$ and an index s_0 with $a \leq |\widetilde{P}_s(\alpha)|/(|\kappa_s|\Gamma(s-\alpha)) \leq b$ for $s \geq s_0$. Since the orthogonality measure of $\widetilde{P}$ is positive and supported on $\mathbb{N}_0$, all zeros of $\widetilde{P}_s$ are real, simple, and lie in $(0,\infty)$ [14]; hence $\widetilde{P}_s(\alpha) \neq 0$ for every s, and the finitely many indices $s < s_0$ are absorbed by taking the minimum and the maximum with the corresponding finite set of positive numbers. □

Lemma 3.5. *Let $\{\mathbb{S}_n\}_{n\geq 0}$ be the Sobolev-type family of (13) with fixed $j \geq 1$, $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$ and $\lambda > 0$. For $j \leq k \leq n-1$ define the positive weights and confluent brackets*

$$Q_{n,k}(z) = P_n(z) - \Delta^j P_n(\alpha) R_k(z), \quad R_k(z) = \frac{P_k(z)}{\Delta^j P_k(\alpha)}. \tag{30}$$

Then, for every $z \in \mathbb{C}$ and every $n > j$,

$$\mathbb{S}_n(z) = \delta_n^{-1} \left(P_n(z) + \lambda \sum_{k=j}^{n-1} t_k \, Q_{n,k}(z) \right), \tag{31}$$

and consequently, for every $z \in \mathbb{C}$ and $n > j$,

$$\frac{\kappa_n^{-1} n}{\Gamma(n-z)} \mathbb{S}_n(z) = \vartheta_n \frac{\displaystyle\sum_{i=0}^{n-1-j} \tau_{n,i} \frac{\kappa_n^{-1} n}{\Gamma(n-z)} Q_{n,\,n-1-i}(z)}{\displaystyle\sum_{i=0}^{n-1-j} \tau_{n,i}} + \eta_n(z), \quad \tau_{n,i} = \frac{t_{n-1-i}}{t_{n-1}}, \tag{32}$$

where $\vartheta_n = \lambda K_{n-1}^{(j,j)}(\alpha,\alpha)/\delta_n \to 1$ and $\eta_n(z) = n\,\kappa_n^{-1} P_n(z)/(\delta_n \Gamma(n-z)) \to 0$ uniformly on compact subsets of $\mathbb{C}$.

Proof. Since $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$ lies outside the convex hull of the support of the orthogonality measure, whose interior contains the zeros of $\{P_k\}$, one has $\Delta^j P_k(\alpha) \neq 0$ for every $k \geq j$; hence the weights and brackets (30) are well defined and $t_k > 0$.

We first establish the exact identity (31). Using the definitions of the kernels (12) and (23), we obtain the exact identity

$$\begin{aligned} P_n(z)\, K_{n-1}^{(j,j)}(\alpha,\alpha) - \Delta^j P_n(\alpha)\, K_{n-1}^{(0,j)}(z,\alpha) &= \sum_{k=j}^{n-1} \frac{\left(\Delta^j P_k(\alpha)\right)^2}{\|P_k\|^2} \Big(P_n(z) - \Delta^j P_n(\alpha) R_k(z) \Big) \\ &= \sum_{k=j}^{n-1} t_k \, Q_{n,k}(z). \end{aligned}$$

From which it follows that

$$P_n(z)(1 + \lambda\, K_{n-1}^{(j,j)}(\alpha,\alpha)) - \lambda\, \Delta^j P_n(\alpha)\, K_{n-1}^{(0,j)}(z,\alpha) = P_n(z) + \lambda \sum_{k=j}^{n-1} t_k \, Q_{n,k}(z).$$

Dividing the above identity by δ_n and invoking (16), we deduce (31), valid for every $z \in \mathbb{C}$ and $n > j$.

We now derive (32). Fix a compact set $K \subset \mathbb{C}$ and multiply (31) by

$$\frac{\kappa_n^{-1} n}{\Gamma(n-z)}, \tag{33}$$

we treat the two resulting terms separately. For the first, the Mehler–Heine limit (10) gives

$$\frac{P_n(z)}{\kappa_n \Gamma(n-z)} \to \varphi(z),$$

uniformly on K. Then, from Lemma 3.3, we have

$$\frac{n}{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)} \cdot \frac{P_n(z)}{\kappa_n \Gamma(n-z)} \longrightarrow 0, \tag{34}$$

uniformly on K, since $P_n(z)/(\kappa_n\Gamma(n-z)) \to \varphi(z)$ uniformly on K, so that this quotient is bounded on K, while $n/\delta_n \to 0$.

For the second term, since Lemma 3.3 gives $K_{n-1}^{(j,j)}(\alpha,\alpha) \to \infty$, the prefactor satisfies

$$\frac{\lambda\, K_{n-1}^{(j,j)}(\alpha,\alpha)}{1+\lambda\, K_{n-1}^{(j,j)}(\alpha,\alpha)} = \frac{1}{1+\big(\lambda\, K_{n-1}^{(j,j)}(\alpha,\alpha)\big)^{-1}} = 1+o(1), \quad n\to\infty, \tag{35}$$

On the other hand, multiplying and dividing the second term of (31) by $K_{n-1}^{(j,j)}(\alpha,\alpha)$, we obtain

$$\frac{\lambda}{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)} \sum_{k=j}^{n-1} t_k\, Q_{n,k}(z) = \frac{\lambda\, K_{n-1}^{(j,j)}(\alpha,\alpha)}{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)} \cdot \frac{\displaystyle\sum_{k=j}^{n-1} t_k\, Q_{n,k}(z)}{\displaystyle\sum_{k=j}^{n-1} t_k}, \tag{36}$$

whose first factor is $1+o(1)$ by (35) and whose second factor is a weighted average of the brackets $Q_{n,k}$ with the positive weights t_k.

It remains to reindex this weighted average from the top of the sum, where the weights are largest. Setting $i = n-1-k$, so that $k = n-1-i$ runs over $0 \le i \le n-1-j$ as k runs from j to $n-1$, and dividing numerator and denominator by the top weight $t_{n-1} > 0$, the normalised weights $\tau_{n,i}$ appear, with $\tau_{n,0} = 1$, and

$$\frac{\displaystyle\sum_{k=j}^{n-1} t_k\, Q_{n,k}(z)}{\displaystyle\sum_{k=j}^{n-1} t_k} = \frac{\displaystyle\sum_{i=0}^{n-1-j} t_{n-1-i}\, Q_{n,\,n-1-i}(z)}{\displaystyle\sum_{i=0}^{n-1-j} t_{n-1-i}} = \frac{\displaystyle\sum_{i=0}^{n-1-j} \tau_{n,i}\, Q_{n,\,n-1-i}(z)}{\displaystyle\sum_{i=0}^{n-1-j} \tau_{n,i}},$$

the division by t_{n-1} leaving the quotient unchanged. Multiplying by (33), which enters each numerator term through $Q_{n,\,n-1-i}$, and combining with (36), we arrive

$$\frac{\kappa_n^{-1} n}{\Gamma(n-z)} \cdot \frac{\lambda}{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)} \sum_{k=j}^{n-1} t_k\, Q_{n,k}(z) = \vartheta_n \frac{\displaystyle\sum_{i=0}^{n-1-j} \tau_{n,i}\, \frac{\kappa_n^{-1} n}{\Gamma(n-z)}\, Q_{n,\,n-1-i}(z)}{\displaystyle\sum_{i=0}^{n-1-j} \tau_{n,i}}.$$

Since the first term of (31), rescaled, is $\eta_n(z)$, which tends to 0 uniformly on K by (34), and $\vartheta_n \to 1$ by (35), we obtain (32). □

Lemma 3.6. *Let $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$ and $j \geq 1$. For each fixed integer $l \geq 0$, setting $m = n - l$,*

$$\frac{\Delta^j P_n(\alpha)}{\Delta^j P_m(\alpha)} = \Big(-\frac{1}{c_\mu}\Big)^l n^{\,l}\big(1 + o(1)\big), \quad n \to \infty. \tag{37}$$

Proof. For $l = 0$ both sides equal 1 and there is nothing to prove, so assume $l \geq 1$. By the forward shift operator (5), $\Delta^j P_s(\alpha) = [s]_j \, \widetilde{P}_{s-j}(\alpha)$ for every $s \geq j$, where $\{\widetilde{P}_\cdot\}$ is the classical family descended j times; this family obeys (10), and its limit function does not vanish at α since $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$ avoids the nonnegative integers. Writing the quotient as a telescoping product over the l unit steps from $m = n - l$ to n,

$$\frac{\Delta^j P_n(\alpha)}{\Delta^j P_{n-l}(\alpha)} = \prod_{r=1}^{l} \frac{\Delta^j P_{n-r+1}(\alpha)}{\Delta^j P_{n-r}(\alpha)}, \tag{38}$$

a product of l consecutive-index ratios, each of which we now evaluate.

Fix $r \in \{1, \dots, l\}$ and set $s = n - r + 1$. Applying the forward shift operator to numerator and denominator, we deduce

$$\frac{\Delta^j P_s(\alpha)}{\Delta^j P_{s-1}(\alpha)} = \frac{[s]_j}{[s-1]_j} \cdot \frac{\widetilde{P}_{s-j}(\alpha)}{\widetilde{P}_{s-1-j}(\alpha)} = \frac{s}{s-j} \cdot \frac{\widetilde{P}_{s-j}(\alpha)}{\widetilde{P}_{s-1-j}(\alpha)}. \tag{39}$$

The consecutive-index ratio (18), applied to $\{\widetilde{P}_\cdot\}$ at the index $s - j$ and the point α, gives

$$\frac{\widetilde{P}_{s-j}(\alpha)}{\widetilde{P}_{s-1-j}(\alpha)} = -\frac{s - 1 - j - \alpha}{c_\mu}\big(1 + O(n^{-1})\big).$$

Substituting into (39) and using $s(s-j)^{-1} \to 1$ together with $(s - 1 - j - \alpha)(s - 1 - \alpha)^{-1} \to 1$, both uniformly for $r \leq l$ with l fixed, we infer

$$\frac{\Delta^j P_s(\alpha)}{\Delta^j P_{s-1}(\alpha)} = -\frac{s - 1 - \alpha}{c_\mu}\big(1 + o(1)\big) = -\frac{n - r - \alpha}{c_\mu}\big(1 + o(1)\big), \tag{40}$$

the last equality on recalling $s - 1 = n - r$.

Multiplying the l relations (40) for $r = 1, \dots, l$ in (38), the l factors $-c_\mu^{-1}$ combine into $(-c_\mu^{-1})^l$ and the l error factors $1 + o(1)$ into a single $1 + o(1)$, since l is fixed. This gives $\Delta^j P_n(\alpha)/\Delta^j P_{n-l}(\alpha) = (-c_\mu^{-1})^l \prod_{r=1}^{l}(n - r - \alpha)\big(1 + o(1)\big)$. Moreover, each factor of the product satisfies $(n - r - \alpha)\, n^{-1} \to 1$ as $n \to \infty$ with $r \leq l$ fixed, so

$$\prod_{r=1}^{l}(n - r - \alpha) = n^{\,l} \prod_{r=1}^{l} \frac{n - r - \alpha}{n} = n^{\,l}\big(1 + o(1)\big),$$

the product of finitely many factors each tending to 1. Substituting gives (37). □

Lemma 3.7. *Let $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$, $j \geq 1$, and let $Q_{n,k}$ and R_k be as in* (30)*. Then, for each fixed integer $i \geq 0$, we have*

$$\frac{\kappa_n^{-1} n}{\Gamma(n - z)}\, Q_{n,\,n-1-i}(z) \xrightarrow[n\to\infty]{} -(i + 1)(z - \alpha)\, \varphi(z), \tag{41}$$

uniformly on compact subsets $K \subset \mathbb{C} \setminus \mathbb{N}_0$.

Proof. From (30) the bracket reads $Q_{n,m}(z) = P_n(z) - \Delta^j P_n(\alpha) R_m(z)$, so its dependence on the lower index m is carried entirely by R_m. Hence the difference of two brackets with consecutive lower indices is

$$Q_{n,m}(z) - Q_{n,m+1}(z) = \Delta^j P_n(\alpha)\big(R_{m+1}(z) - R_m(z)\big), \tag{42}$$

the terms $P_n(z)$ cancelling. Summing (42) over m from $n-1-i$ to $n-1$, the left-hand side telescopes to $Q_{n,\,n-1-i}(z) - Q_{n,n}(z)$; since $Q_{n,n}(z) = P_n(z) - \Delta^j P_n(\alpha) R_n(z) = P_n(z) - P_n(z) = 0$, we obtain

$$Q_{n,\,n-1-i}(z) = \Delta^j P_n(\alpha) \sum_{l=0}^{i} \big(R_{n-l}(z) - R_{n-1-l}(z)\big), \tag{43}$$

a sum of exactly $i+1$ terms, indexed by $l = 0, 1, \dots, i$.

We next show that each summand of (43) is a delay-zero bracket. Fix l and set $m := n - l$. Placing the two fractions of $R_m - R_{m-1}$ over the common denominator $\Delta^j P_m(\alpha)\Delta^j P_{m-1}(\alpha)$, we get

$$\begin{aligned}
\Delta^j P_n(\alpha)\big(R_m(z) - R_{m-1}(z)\big) &= \Delta^j P_n(\alpha)\Big(\frac{P_m(z)}{\Delta^j P_m(\alpha)} - \frac{P_{m-1}(z)}{\Delta^j P_{m-1}(\alpha)}\Big) \\
&= \frac{\Delta^j P_n(\alpha)}{\Delta^j P_m(\alpha)} \cdot \frac{\Delta^j P_{m-1}(\alpha)\, P_m(z) - \Delta^j P_m(\alpha)\, P_{m-1}(z)}{\Delta^j P_{m-1}(\alpha)} \\
&= \frac{\Delta^j P_n(\alpha)}{\Delta^j P_m(\alpha)} \cdot \frac{\mathcal{N}_m(z)}{\Delta^j P_{m-1}(\alpha)},
\end{aligned} \tag{44}$$

where $\mathcal{N}_m(z)$ is precisely the bracket (20) at top index $m = n - l$.

It remains to rescale (44) by (33) and let $n \to \infty$ with l fixed, so that $m = n - l \to \infty$ as well. We write

$$\frac{\kappa_n^{-1} n}{\Gamma(n-z)} \cdot \frac{\Delta^j P_n(\alpha)}{\Delta^j P_m(\alpha)} \cdot \frac{\mathcal{N}_m(z)}{\Delta^j P_{m-1}(\alpha)} = A_{n,l}(z)\, B_m(z), \tag{45}$$

where

$$A_{n,l}(z) = \frac{n}{m}\,\frac{\kappa_m}{\kappa_n}\,\frac{\Gamma(m-z)}{\Gamma(n-z)}\,\frac{\Delta^j P_n(\alpha)}{\Delta^j P_m(\alpha)}, \quad B_m(z) = \frac{\kappa_m^{-1} m}{\Gamma(m-z)}\,\frac{\mathcal{N}_m(z)}{\Delta^j P_{m-1}(\alpha)}.$$

The factor B_m is governed by Lemma 3.2, which gives, uniformly on K,

$$\frac{\kappa_m^{-1} m}{\Gamma(m-z)}\,\frac{\mathcal{N}_m(z)}{\Delta^j P_{m-1}(\alpha)} \xrightarrow[n\to\infty]{} -(z-\alpha)\,\varphi(z), \tag{46}$$

while the prefactor is supplied by Lemma 3.6, which gives

$$\frac{\Delta^j P_n(\alpha)}{\Delta^j P_m(\alpha)} = \Big(-\frac{1}{c_\mu}\Big)^l n^{\,l}\big(1 + o(1)\big). \tag{47}$$

The scale and Gamma quotients follow from (18) iterated l times and from the functional equation of Γ,

$$\frac{\kappa_m}{\kappa_n} = (-c_\mu)^{\,l}, \quad \frac{\Gamma(m-z)}{\Gamma(n-z)} = \prod_{r=1}^{l} \frac{1}{n-r-z} = n^{-l}\big(1 + o(1)\big), \tag{48}$$

uniformly on K. In $A_{n,l}$, the four factors carrying l-dependence, namely $(-c_\mu^{-1})^l$ and n^l from (47), together with $(-c_\mu)^l$ and n^{-l} from (48), cancel in pairs, and $n/m \to 1$; hence $A_{n,l} \to 1$ uniformly on K. Consequently,

$$\frac{\kappa_n^{-1} n}{\Gamma(n-z)} \cdot \frac{\Delta^j P_n(\alpha)}{\Delta^j P_m(\alpha)} \cdot \frac{\mathcal{N}_m(z)}{\Delta^j P_{m-1}(\alpha)} \xrightarrow[n\to\infty]{} -(z-\alpha)\,\varphi(z), \tag{49}$$

uniformly on K. What survives is the l-independent value $-(z-\alpha)\varphi(z)$, so every one of the $i+1$ summands in (43) contributes the same limit.

Summing the $i+1$ equal contributions (49) over $l = 0, 1, \ldots, i$, a finite sum whose terms converge uniformly on K, yields (41). □

Lemma 3.8 (Asymptotics and summability of the weights)**.** *Throughout, we adopt the convention $0^0 = 1$; thus, in the Charlier case ($c_\mu = 1$), the limits below are 1 for $i = 0$ and 0 for $i \geq 1$. Let t_k be as in (23) and $\tau_{n,i}$ as given by (32). Then, for each fixed $i \geq 0$,*

$$\tau_{n,i} \xrightarrow[n\to\infty]{} (1-c_\mu)^{\,i}. \tag{50}$$

Consequently, the two series

$$\sum_{i\geq 0}(1-c_\mu)^{\,i} = \frac{1}{c_\mu}, \quad \sum_{i\geq 0}(i+1)(1-c_\mu)^{\,i} = \frac{1}{c_\mu^2}, \tag{51}$$

converge.

Proof. Clearly, taking reciprocals in (25), we arrive at

$$\frac{t_k}{t_{k+1}} \xrightarrow[k\to\infty]{} 1-c_\mu, \tag{52}$$

the limit being 0 for Charlier, where $c_\mu = 1$, and μ for Meixner, where $1-c_\mu = \mu$; in the Charlier case this is the statement that the reciprocal of a sequence diverging to $+\infty$ tends to 0.

To establish (50), fix $i \geq 0$ and telescope the quotient $\tau_{n,i}$ over the i unit steps from the index $n-1-i$ up to $n-1$,

$$\tau_{n,i} = \frac{t_{n-1-i}}{t_{n-1}} = \prod_{l=0}^{i-1} \frac{t_{n-2-l}}{t_{n-1-l}}, \tag{53}$$

each factor being of the form (52) at the index $k = n-2-l$. As $n \to \infty$ with i fixed, every one of these i indices tends to infinity, so each factor tends to $1-c_\mu$ by (52); the product of finitely many convergent factors converges to the product of the limits, which gives (50).

Finally, since $x = 1-c_\mu$ satisfies $0 \leq x < 1$ in both families, the geometric series and its termwise derivative converge, and

$$\sum_{i\geq 0} x^{\,i} = \frac{1}{1-x}, \quad \sum_{i\geq 0}(i+1)x^{\,i} = \frac{d}{dx}\sum_{i\geq 0} x^{\,i+1} = \frac{1}{(1-x)^2},$$

the termwise differentiation being legitimate inside the disc of convergence. Substituting $x = 1-c_\mu$, so that $1-x = c_\mu$, gives (51). □

Lemmas 3.7 and 3.8 describe the terms of the weighted average (32) for each *fixed* i. Since the number of terms grows with n, passing to the limit in the whole sum requires a bound that is uniform in n and summable in i. Write

$$X_{n,i}(z) = \frac{\kappa_n^{-1} n}{\Gamma(n-z)}\, Q_{n,\,n-1-i}(z), \quad 0 \leq i \leq n-1-j. \tag{54}$$

Lemma 3.9. *Let $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$, $j \geq 1$, $\lambda > 0$, and let $K \subset \mathbb{C} \setminus \mathbb{N}_0$ be compact. There exist $q > 1$, $p \geq 0$, $C > 0$ and $N_0 \in \mathbb{N}$ such that, for all $z \in K$, $n \geq N_0$ and $0 \leq i \leq n-1-j$,*

$$\tau_{n,i} \leq C\, q^{-i}, \quad \tau_{n,i}\,|X_{n,i}(z)| \leq C\,(i+1)^p\, q^{-i}. \tag{55}$$

Proof. Considere $q > 1$ and k_0 be as in (26). If $n-1-i \geq k_0$, telescoping as in (53) gives $\tau_{n,i} \leq q^{-i}$. If $n-1-i < k_0$, then by (27) $\tau_{n,i} = t_{n-1-i}/t_{n-1} \leq T_0\, q^{k_0} t_{k_0}^{-1} q^{-(n-1)} \leq T_0\, q^{k_0} t_{k_0}^{-1} q^{-i}$, where $T_0 = \max_{j \leq k \leq k_0} t_k$.

Let $x_K = \max_{z\in K} \Re z$ and $a = \max\{0, x_K - j - \alpha\}$, and fix an integer $N_0 \geq \max\{2j,\, k_0+1,\, 2(|x_K|+1)\}$ so large that the left-hand side of (21) is bounded on K by a constant C_1 for $m \geq N_0$ (Lemma 3.2) and that $|P_n(z)/(\kappa_n \Gamma(n-z))| \leq C_1$ on K for $n \geq N_0$ (by (10)). For $N_0 \leq m \leq n$ and $z \in K$ we have $|s-z| \geq s - x_K > 0$ for $s \geq m$, hence

$$\Big|\frac{\Gamma(m-z)}{\Gamma(n-z)}\Big| = \prod_{s=m}^{n-1} \frac{1}{|s-z|} \leq \prod_{s=m}^{n-1} \frac{1}{s-x_K}, \tag{56}$$

and

$$\prod_{s=m}^{n-1} \frac{s-j-\alpha}{s-x_K} \leq \exp\left(\sum_{s=m}^{n-1} \frac{a}{s-x_K}\right) \leq \Big(\frac{2n}{m}\Big)^a, \tag{57}$$

where we used $1+u \leq e^u$, we have

$$\sum_{s=m}^{n-1} (s-x_K)^{-1} \leq \log \frac{n-1-x_K}{m-1-x_K} \quad \text{and} \quad \frac{n-1-x_K}{m-1-x_K} \leq \frac{2n}{m}.$$

The last inequality follows from $m-1-x_K \geq m/2$ and $n-1-x_K \leq n$ if $x_K \geq -1$, and from

$$\frac{n-1-x_K}{m-1-x_K} \leq \frac{n}{m},$$

since $-1-x_K > 0$ and $m \leq n$ if $x_K < -1$. Moreover, by (5) and Lemma 3.4, and since $\kappa_{n-j}/\kappa_{m-j} = \kappa_n/\kappa_m$,

$$\Big|\frac{\Delta^j P_n(\alpha)}{\Delta^j P_m(\alpha)}\Big| \leq \frac{\ell_1}{\ell_0} \frac{[n]_j}{[m]_j} \Big|\frac{\kappa_n}{\kappa_m}\Big| \frac{\Gamma(n-j-\alpha)}{\Gamma(m-j-\alpha)}, \quad 2j \leq m \leq n, \tag{58}$$

with $[n]_j/[m]_j \leq (2n/m)^j$.

Case $n-1-i \geq N_0$. By (43) and (44), $X_{n,i} = \sum_{l=0}^{i} T_{n,l}$ with $m = n-l$ and

$$T_{n,l}(z) = \frac{\kappa_n^{-1} n}{\Gamma(n-z)} \frac{\Delta^j P_n(\alpha)}{\Delta^j P_m(\alpha)} \frac{\mathcal{N}_m(z)}{\Delta^j P_{m-1}(\alpha)}.$$

Since $m \geq n-i \geq N_0$, the bound C_1 for the left-hand side of (21) at index m, together with (56), (57) and (58), gives

$$|T_{n,l}(z)| \leq C_1 \frac{\ell_1}{\ell_0} \frac{n}{m} \Big(\frac{2n}{m}\Big)^j \Big|\frac{\Gamma(m-z)}{\Gamma(n-z)}\Big| \frac{\Gamma(n-j-\alpha)}{\Gamma(m-j-\alpha)} \leq C_1 \frac{\ell_1}{\ell_0} \Big(\frac{2n}{m}\Big)^{1+j+a}.$$

If $i \leq n/2$, then $n/m \leq 2$; if $i > n/2$, then $n/m \leq n < 2(i+1)$. Hence $|X_{n,i}(z)| \leq C(i+1)^{2+j+a}$.

Case $n-1-i < N_0$. Here $m := n-1-i \in \{j, \ldots, N_0-1\}$ and $n \leq N_0(i+1)$. From (30),

$$|X_{n,i}(z)| \leq n \Big|\frac{P_n(z)}{\kappa_n \Gamma(n-z)}\Big| + n \frac{|\Delta^j P_n(\alpha)|}{|\kappa_n|\,|\Gamma(n-z)|} \max_{j \leq m < N_0} \max_{z\in K} \Big|\frac{P_m(z)}{\Delta^j P_m(\alpha)}\Big|.$$

The first term is at most $C_1 n$. For the second, Lemma 3.4 gives $|\Delta^j P_n(\alpha)| \leq \ell_1\, [n]_j |\kappa_{n-j}| \Gamma(n-j-\alpha)$, while $|\Gamma(n-z)| \geq g_K\, \Gamma(n-x_K)/\Gamma(N_0-x_K)$ by (56) with $m = N_0$, where $g_K = \min_{z\in K} |\Gamma(N_0-z)| > 0$. Using (57), the second term is at most $C\, n^{1+j+a}$. Hence $|X_{n,i}(z)| \leq C(i+1)^{1+j+a}$.

Combining both cases with the bound for $\tau_{n,i}$ gives (55) with $p = 2+j+a$. □

Proposition 3.10. *Let $\{\mathbb{S}_n\}_{n\geq 0}$ be the Sobolev-type family of* (13) *with fixed $j \geq 1$, $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$, $\lambda > 0$. Then,*

$$\lim_{n\to\infty} \frac{\kappa_n^{-1}\, n}{\Gamma(n-z)}\, \mathbb{S}_n(z) = -\,\frac{(z-\alpha)\,\varphi(z)}{c_\mu}, \tag{59}$$

uniformly on compact subsets of $\mathbb{C}$. In particular, the limit is independent of λ and of j.

Proof. Fix a compact set $K \subset \mathbb{C} \setminus \mathbb{N}_0$. The restriction will be removed at the end using Vitali's theorem. Consider first the numerator of (32), namely $\sum_{i=0}^{n-1-j} \tau_{n,i} X_{n,i}(z)$, with $X_{n,i}$ as in (54). For each fixed i, its general term converges uniformly on K. Indeed, by Lemma 3.7 the rescaled bracket tends to $-(i+1)(z-\alpha)\varphi(z)$, while by (50) the weight tends to $(1-c_\mu)^{\,i}$, so that

$$\tau_{n,i}\, X_{n,i}(z) \xrightarrow[n\to\infty]{} -(i+1)(z-\alpha)\,\varphi(z)\,(1-c_\mu)^{\,i}. \tag{60}$$

By Lemma 3.9, $|\tau_{n,i} X_{n,i}(z)| \leq C(i+1)^p q^{-i}$ for all $z \in K$, $n \geq N_0$ and $0 \leq i \leq n-1-j$, and $\sum_{i\geq 0}(i+1)^p q^{-i} < \infty$. Hence Tannery's theorem (the dominated convergence theorem for series, whose proof gives uniform convergence when the termwise limits are uniform and the dominating sequence does not depend on z) yields, uniformly on K,

$$\sum_{i=0}^{n-1-j} \tau_{n,i}\, X_{n,i}(z) \xrightarrow[n\to\infty]{} -(z-\alpha)\,\varphi(z) \sum_{i\geq 0} (i+1)(1-c_\mu)^{\,i} = -\,\frac{(z-\alpha)\,\varphi(z)}{c_\mu^2}, \tag{61}$$

the last equality being the second series of (51).

In the same way, by (50), the first bound in (55) and the first series of (51),

$$\sum_{i=0}^{n-1-j} \tau_{n,i} \xrightarrow[n\to\infty]{} \sum_{i\geq 0} (1-c_\mu)^{\,i} = \frac{1}{c_\mu}. \tag{62}$$

Since this limit is a nonzero constant, the quotient in (32) converges to the quotient of the limits (61) and (62); hence, since $\vartheta_n \to 1$ and $\eta_n \to 0$ uniformly on K by Lemma 3.5, we obtain, uniformly on K,

$$\frac{\kappa_n^{-1} n}{\Gamma(n-z)}\, \mathbb{S}_n(z) \xrightarrow[n\to\infty]{} \frac{-(z-\alpha)\varphi(z)/c_\mu^2}{1/c_\mu} = -\,\frac{(z-\alpha)\,\varphi(z)}{c_\mu}. \tag{63}$$

The mass λ disappeared in the passage to (32), and the order j enters only through the ratios (47) defining the brackets $Q_{n,k}$, whose limit in Lemma 3.7 is independent of j; the limit therefore depends on neither, as asserted.

It remains to remove the restriction $z \notin \mathbb{N}_0$. Define

$$F_n(z) = \frac{\kappa_n^{-1}\, n}{\Gamma(n-z)}\, \mathbb{S}_n(z).$$

Since Γ^{-1} is an entire function, each F_n is entire.

Fix $p \in \mathbb{N}_0$. The circle

$$C_p = \{z \in \mathbb{C} : |z-p| = 1/2\},$$

contains no integer, so it is a compact subset of $\mathbb{C} \setminus \mathbb{N}_0$. By (63), the sequence $\{F_n\}$ converges uniformly on C_p. Hence there exist $M_p > 0$ and $n_p \in \mathbb{N}$ such that $|F_n(z)| \leq M_p$ for $z \in C_p$ and $n \geq n_p$. Since each F_n is entire, the Maximum Modulus Principle implies that the same bound holds throughout the closed disc $D_p = \{z \in \mathbb{C} : |z-p| \leq 1/2\}$. Now let $L \subset \mathbb{C}$ be compact. Then L is covered by the interiors of finitely many discs D_p together with the compact set

$L \setminus \bigcup_p \operatorname{int} D_p$, which is contained in $\mathbb{C} \setminus \mathbb{N}_0$; on the latter $\{F_n\}$ is uniformly bounded for large n by (63). Therefore $\{F_n\}$ is locally bounded on $\mathbb{C}$. Since F_n converges on the connected open set $\mathbb{C} \setminus \mathbb{N}_0$, Vitali's theorem [8] implies that $\{F_n\}$ converges locally uniformly on $\mathbb{C}$ to an entire function, which necessarily coincides with the right-hand side of (63). Hence (59) holds uniformly on compact subsets of $\mathbb{C}$, which completes the proof. □

The following two corollaries follow by inserting the family-specific data φ, κ_n, c_μ into Proposition 3.10.

Corollary 3.11. *Let $\{\mathbb{S}_n\}_{n\geq 0}$ be the monic Charlier-Sobolev type orthogonal polynomials of (13), with $\mu > 0$, $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$, $\lambda > 0$, $j \geq 1$. Then, uniformly on compact subsets of $\mathbb{C}$,*

$$\lim_{n\to\infty} \frac{(-1)^n\, n}{\Gamma(n-z)}\, \mathbb{S}_n(z) = -(z-\alpha)\, \frac{e^{\mu}}{\Gamma(-z)}. \tag{64}$$

The limit is independent of λ and j.

Proof. Here $\varphi = \varphi_C$, $\kappa_n = (-1)^n$ and $c_\mu = 1$; substituting into (59) gives (64). □

Corollary 3.12. *Let $\{\mathbb{S}_n\}_{n\geq 0}$ be the monic Meixner-Sobolev type orthogonal polynomials of (13), with $\gamma > 0$, $0 < \mu < 1$, $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$, $\lambda > 0$, $j \geq 1$. Then, uniformly on compact subsets of $\mathbb{C}$,*

$$\lim_{n\to\infty} \frac{(\mu-1)^n\, n}{\Gamma(n-z)}\, \mathbb{S}_n(z) = -\,\frac{z-\alpha}{(1-\mu)^{\gamma+z+1}\,\Gamma(-z)}. \tag{65}$$

The limit is independent of λ and j.

Proof. Here $\varphi = \varphi_M$, $\kappa_n = (\mu-1)^{-n}$ and $c_\mu = 1-\mu$; substituting into (59), the factor $1/(1-\mu)$ produces $-(z-\alpha)\varphi_M(z)/(1-\mu)$, which is (65). □

As a consequence of the Mehler–Heine formulas we obtain the following local information on the zeros of $\mathbb{S}_n$.

Corollary 3.13. *Let $\{\mathbb{S}_n\}_{n\geq 0}$ be the Charlier or Meixner Sobolev-type family of (13), with $j \geq 1$, $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$ and $\lambda > 0$, and let $0 < r < \min\{1/2, |\alpha|\}$. Then:*

1. *there exists $n_0 = n_0(r)$ such that, for every $n \geq n_0$, $\mathbb{S}_n$ has exactly one zero in the disc $|z-\alpha| < r$, and this zero is real and simple; denoting it by $x_{n,\alpha}$, we have $x_{n,\alpha} \to \alpha$;*
2. *for each fixed $k \in \mathbb{N}_0$ there exists $n_0 = n_0(r,k)$ such that, for every $n \geq n_0$, $\mathbb{S}_n$ has exactly one zero in the disc $|z-k| < r$, and this zero is real and simple.*

Proof. Let F_n be as in the proof of Proposition 3.10 and $F(z) = -(z-\alpha)\varphi(z)/c_\mu$. By Proposition 3.10, $F_n \to F$ uniformly on compact subsets of $\mathbb{C}$. The zeros of F are $z = \alpha$ and the nonnegative integers, all of them simple, because φ is a nonvanishing entire multiple of $1/\Gamma(-z)$ and $\alpha \notin \mathbb{N}_0$. Let $\zeta \in \{\alpha\} \cup \mathbb{N}_0$ be fixed. By the choice of r, the closed disc $|z-\zeta| \leq r$ contains no other zero of F, and F does not vanish on its boundary. By Hurwitz's theorem [8], for n large F_n has exactly one zero, counted with multiplicity, in $|z-\zeta| < r$. For $n > |\zeta| + r$ the factor $1/\Gamma(n-z)$ has no zeros in this disc (its zeros are $z = n, n+1, \dots$), so the same holds for $\mathbb{S}_n$. Finally, $\mathbb{S}_n$ has real coefficients, because P_n, Ω_n and $K_{n-1}^{(0,j)}(\cdot,\alpha)$ are real for real α; hence its nonreal zeros occur in conjugate pairs. Since the disc is symmetric with respect to the real axis and contains exactly one zero, this zero is real, and it is simple. As r can be taken arbitrarily small, $x_{n,\alpha} \to \alpha$. □

Corollary 3.13 is a local statement: it describes the zeros of $\mathbb{S}_n$ in fixed bounded regions and does not, by itself, determine the location of all n zeros. In particular, for n large the zero $x_{n,\alpha}$ lies in $(-\infty, 0)$, outside the convex hull of the support of ρ.

The Mehler–Heine formulas also yield the outer relative asymptotics of the Sobolev-type polynomials with respect to the classical ones. For a first-order difference, the asymptotic behaviour of Sobolev-type Meixner polynomials was studied in [22]; see also [3] for ratio asymptotics in the non-diagonal Meixner–Sobolev case.

Corollary 3.14. *Under the hypotheses of Proposition 3.10, for every compact set $K \subset \mathbb{C} \setminus \mathbb{N}_0$ there exists $n_K \in \mathbb{N}$ such that P_n has no zeros on K for $n \geq n_K$, and*

$$\lim_{n\to\infty} n \frac{\mathbb{S}_n(z)}{P_n(z)} = -\frac{z-\alpha}{c_\mu}, \tag{66}$$

uniformly on K. If $K \subset \mathbb{C} \setminus [0, +\infty)$, one can take $n_K = 1$.

Proof. Let $K \subset \mathbb{C} \setminus \mathbb{N}_0$ be compact. Since φ is continuous and has no zeros on K, there is $m_K > 0$ with $|\varphi| \geq m_K$ on K. By (10), $\kappa_n^{-1} P_n(z)/\Gamma(n-z) \to \varphi(z)$ uniformly on K, hence there is n_K such that $|\kappa_n^{-1} P_n(z)/\Gamma(n-z)| \geq m_K/2$ on K for $n \geq n_K$; in particular, P_n has no zeros on K for $n \geq n_K$. For such n,

$$n \frac{\mathbb{S}_n(z)}{P_n(z)} = \frac{F_n(z)}{\kappa_n^{-1} P_n(z)/\Gamma(n-z)},$$

where the numerator tends to $-(z-\alpha)\varphi(z)/c_\mu$ by Proposition 3.10 and the denominator tends to $\varphi(z)$, both uniformly on K; this gives (66). Finally, the zeros of P_n lie in $(0, \infty)$, so P_n does not vanish on $\mathbb{C} \setminus [0, +\infty)$ for any n. □

3.1 Numerical illustration of the Mehler–Heine formulas

Figure 1 illustrates the Mehler–Heine formulas through the normalised sequences appearing on their left-hand sides, which remove the dominant growth of the monic polynomials.

For the Charlier–Sobolev family, the plotted sequence is $\mathcal{F}_n^C(x) = (-1)^n n \mathbb{S}_n(x)/\Gamma(n-x)$, which, according to Corollary 3.11, converges to $\mathcal{F}^C(x) = -(x-\alpha)e^\mu/\Gamma(-x)$; for the Meixner–Sobolev family it is $\mathcal{F}_n^M(x) = (\mu-1)^n n \mathbb{S}_n(x)/\Gamma(n-x)$, which, by Corollary 3.12, converges to $\mathcal{F}^M(x) = -(x-\alpha)/((1-\mu)^{\gamma+x+1}\Gamma(-x))$. Both are the functions F_n and F of the proof of Proposition 3.10.

The computations were performed with $\alpha = -5/2$, $\mu = 5$, $j = 10$ and $\lambda = 100$ in the Charlier case, and with $\alpha = -5/2$, $\gamma = 2$, $\mu = 1/2$, $j = 10$ and $\lambda = 100$ in the Meixner case. The classical polynomials were generated by the three-term recurrence relation (4), the differences $\Delta^j P_k(\alpha)$ by the forward-shift identity (5) (with the shifted parameter $\gamma + j$ in the Meixner case, and checked against direct finite differences), and $\mathbb{S}_n$ by the connection formula (15). All computations were carried out in arbitrary-precision arithmetic (Python, `mpmath`) with 60 and with 120 significant digits; at every grid point the two runs differ by less than $10^{-56} \max\{1, |\mathcal{F}_n(x)|\}$. High precision is necessary, since near the nonnegative integers the values of P_n are strongly affected by cancellation in the recurrence. In addition, the connection formula and the identity (31) were verified in exact rational arithmetic against the monic Gram–Schmidt orthogonalisation with respect to (13), computed from the exact factorial moments of the weight (both families, $j = 1, 2, 3$, $\alpha \in \{-5/2, -3/2, -1/2\}$, $n \leq 7$; and the parameters above with $n \leq 14$).

Figure 1 shows, for each family, $\mathcal{F}_n(x)$ for $n = 50, 200, 800$ together with the limit function on the interval $[-4, 2]$, which contains the exterior mass point $\alpha = -5/2$, where the new zero of the limit function appears, and the zeros $0, 1, 2$ inherited from $1/\Gamma(-z)$. Each panel has its

own linear vertical scale, and the insets magnify the neighbourhood of α. The curves approach the limit as n increases. On the plotted interval, the maximal deviation $\max_x |\mathcal{F}_n(x) - \mathcal{F}(x)|$, computed on a uniform grid of 481 points (and hence only an approximation of the continuous maximum), decreases from 254 ($n = 50$) to 43.0 ($n = 200$) and 9.90 ($n = 800$) in the Charlier case, and from 36.1 to 6.11 and 1.41 in the Meixner case. The deviation is largest between the zeros 1 and 2, where $|\mathcal{F}|$ is largest.

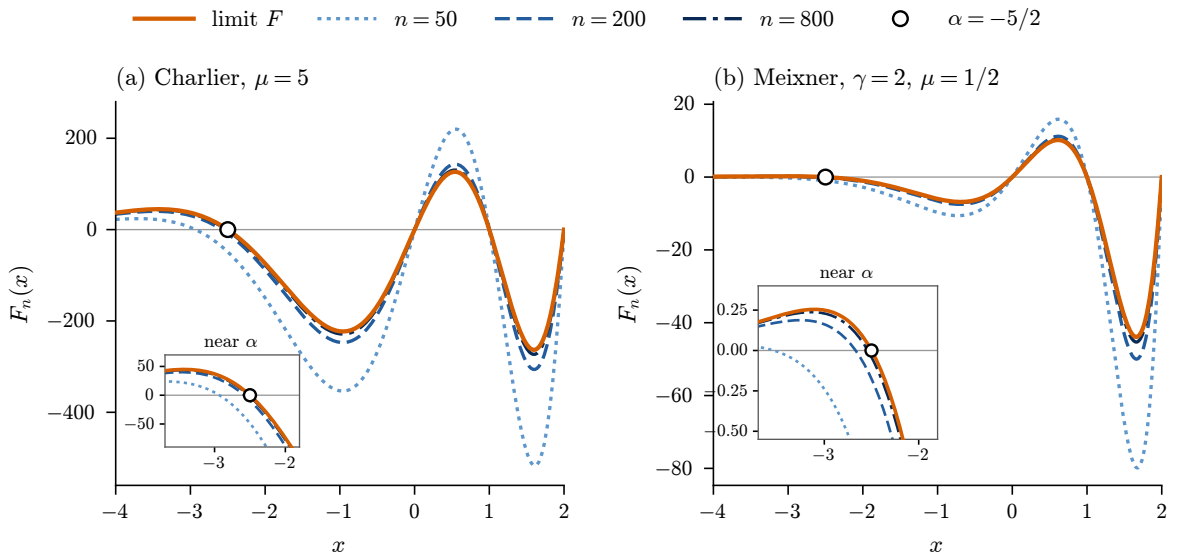


Figure 1: Convergence of the scaled Sobolev-type polynomials to the Mehler–Heine limits on $[-4, 2]$: (a) $\mathcal{F}_n^C(x)$ ($\alpha = -5/2$, $\mu = 5$, $j = 10$, $\lambda = 100$) and (b) $\mathcal{F}_n^M(x)$ ($\alpha = -5/2$, $\gamma = 2$, $\mu = 1/2$, $j = 10$, $\lambda = 100$), for $n = 50$ (dotted), $n = 200$ (dashed) and $n = 800$ (dash-dotted), together with the limits $\mathcal{F}^C$ and $\mathcal{F}^M$ (solid). The open circle marks the mass point α, where the limit has the additional zero $z = \alpha$; the insets magnify its neighbourhood. Each panel has its own linear vertical scale.

Table 3 lists, for each n, the zero $x_{n,\alpha}$ of $\mathbb{S}_n$ found in $[-12, -0.05]$; on a uniform grid of step $1/40$ in this interval exactly one sign change of $\mathbb{S}_n$ was detected. This does not exclude other zeros on the negative real axis, and no global statement is intended. The distance $|x_{n,\alpha} - \alpha|$ decreases as n increases, which illustrates the convergence $x_{n,\alpha} \to \alpha$ of Corollary 3.13(1). In the computed cases the zero lies to the left of $\alpha = -5/2$; this is only an observation from the experiment.

Each zero was approximated by Newton's method, after bisection from the sign change closest to α on the grid above, doubling the working precision from 60 digits until the offset $d = x_{n,\alpha} - \alpha$ agreed to 10 significant digits between two consecutive precisions; 120 digits were sufficient in all cases. Since the recurrence coefficients of Table 2 are integers for the parameters used, $\mathbb{S}_n$ can be evaluated exactly at the decimal points $\alpha + d(1 \pm 10^{-6})$, and opposite signs were obtained in all 12 cases of Table 3. Hence each offset is certified to a relative accuracy of 10^{-6}; the digits shown come from the high-precision computation. The computations do not certify that the zero found is the zero $x_{n,\alpha}$ of Corollary 3.13, whose threshold n_0 is not explicit, and they give no information on the other zeros of $\mathbb{S}_n$. The parameters, exact endpoints and signs, the execution logs and the scripts that compute the data of the table and the figures are part of the supplementary material.

Finally, Figure 2 illustrates the outer relative asymptotics of Corollary 3.14 on the interval $[-5, -0.25]$, where P_n has no zeros. The quotients $n\mathbb{S}_n(x)/P_n(x)$ were computed

Table 3: The zero $x_{n,\alpha}$ near the mass point and its distance to α in the main experiment ($\alpha = -5/2$, $j = 10$, $\lambda = 100$), illustrating the convergence $x_{n,\alpha} \to \alpha$ of Corollary 3.13(1). The digits come from the high-precision computation; a zero of $\mathbb{S}_n$ within relative distance 10^{-6} of each offset $x_{n,\alpha} - \alpha$ is certified in exact arithmetic.

n	$x_{n,\alpha}$	$\lvert x_{n,\alpha} - \alpha \rvert$
Charlier, $\mu = 5$		
50	-2.9303709735	0.4303709735
100	-2.6766269210	0.1766269210
200	-2.5810975854	0.0810975854
400	-2.5389629412	0.0389629412
800	-2.5191085088	0.0191085088
1600	-2.5094637504	0.0094637504
Meixner, $\gamma = 2$, $\mu = 1/2$		
50	-3.5347935223	1.0347935223
100	-2.8813763763	0.3813763763
200	-2.6680033103	0.1680033103
400	-2.5792537048	0.0792537048
800	-2.5385349045	0.0385349045
1600	-2.5190052966	0.0190052966

with 60 and with 120 significant digits, and the two computations differ by less than $10^{-56} \max\{1, |n\mathbb{S}_n(x)/P_n(x)|\}$ at every grid point.

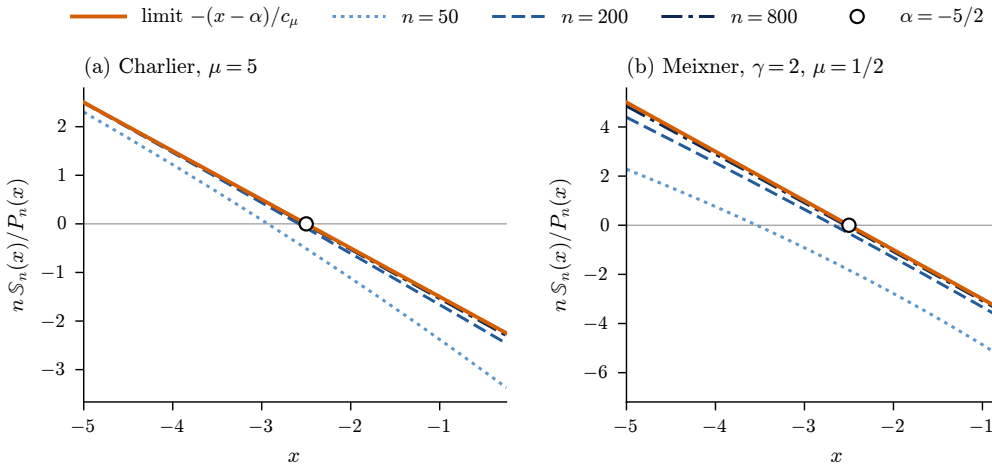


Figure 2: Outer relative asymptotics (Corollary 3.14): $n\mathbb{S}_n(x)/P_n(x)$ for $n = 50$ (dotted), 200 (dashed), 800 (dash-dotted) and the limit $-(x-\alpha)/c_\mu$ (solid), on $[-5, -0.25]$, for the parameters of Figure 1: (a) Charlier, (b) Meixner. The open circle marks $\alpha = -5/2$.

4 Conclusions

In this paper we have studied the asymptotic behaviour of the discrete Sobolev-type Charlier and Meixner orthogonal polynomials associated with arbitrary-order forward differences evaluated

at an exterior mass point $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$. Starting from the connection formula relating the Sobolev-type and classical families, we have established Mehler–Heine type formulas for both families.

Unlike the case of forward differences at $\alpha = 0$ considered in [9], where the Sobolev perturbation leaves the classical limit unchanged, the exterior mass modifies the leading asymptotic behaviour, the scaling gains a factor n and the limit function acquires the factor $(z - \alpha)$. The underlying mechanism is the at least geometric growth of the kernel $K_{n-1}^{(j,j)}(\alpha, \alpha)$, which makes the Sobolev correction comparable to the classical term. Although the Sobolev-type polynomials depend explicitly on the mass parameter λ and on the order j of the forward difference operator, both parameters disappear from the Mehler–Heine limits; the only remaining trace of the Sobolev perturbation is the factor $(z - \alpha)$, which records the location of the exterior mass.

As a consequence, for large degree exactly one zero of the Sobolev-type polynomial lies near the exterior mass point, and it is real and simple, while exactly one zero lies near each fixed nonnegative integer, as in the classical case. These are local statements; a description of the location of all the zeros is not provided by the Mehler–Heine formulas alone and remains an open problem. We have also obtained the outer relative asymptotics with respect to the classical families.

The numerical experiments, carried out in high-precision arithmetic, illustrate the convergence of the scaled Sobolev-type polynomials towards the predicted limiting functions, the approach of one zero to the exterior mass point; the computed zeros were certified by exact rational sign changes.

Several questions remain open, among them the global behaviour of the zeros, the possible determination of rates of convergence of the local zeros, and the extension of the present approach to other families of the Askey scheme, to several mass points, or to other asymptotic regimes, for instance by means of the Riemann–Hilbert method.

Acknowledgements

The authors would like to thank the Department of Quantitative Methods at Universidad Loyola Andalusia for providing an excellent research environment and institutional support during the development of this work.

Declarations

Ethical Approval

Not applicable.

Consent to Participate

Not applicable.

Consent to Publish

Not applicable.

Data Availability Statement

The scripts and data used to produce Figures 1 and 2 and Table 3, together with the exact certification records of the zeros, the exact-arithmetic checks and the execution logs, are provided

as a self-contained supplementary archive submitted with the manuscript.

Code and data availability

All formulas and numerical data used in this study are contained in the manuscript or generated by the executable programs of the public repository `https://github.com/asorial/Mehler--Heine-formulas-and-zeros.git`. The repository contains the complete Python/SymPy/Matplotlib program producing the figures, the exact-arithmetic verification of the identity, exact parameter values and dependency versions, and machine-readable verification results.

Author Contributions

Conceptualization, A.S.-L.; methodology, A.S.-L. and J.-M.; software, A.S.-L. and J.-M.; validation, A.S.-L. and J.-M.; formal analysis, A.S.-L. and J.-M.; investigation, A.S.-L. and J.-M.; resources, A.S.-L.; data curation, A.S.-L. and J.-M.; writing–original draft preparation, A.S.-L.; writing–review and editing, A.S.-L. and J.-M.; visualization, A.S.-L.; supervision, A.S.-L.; project administration, A.S.-L. Both authors contributed equally to the research reported in this work. All authors have read and approved the final version of the manuscript.

Generative AI Disclosure

During the preparation of this manuscript, the authors used Claude (Anthropic) for English-language editing and assistance with Python scripts for numerical verification and figure generation. All mathematical ideas, original results, proofs, interpretations, and conclusions were independently developed and verified by the authors, who take full responsibility for the content of the manuscript.

Funding

This research received no external funding.

Competing Interests

The authors declare that they have no competing interests.

References

[1] M. Alfaro, F. Marcellán, M. L. Rezola, and A. Ronveaux. On orthogonal polynomials of Sobolev type: algebraic properties and zeros. *SIAM J. Math. Anal.*, 23(3):737–757, 1992. doi:10.1137/0523038.

[2] I. Área, E. Godoy, and F. Marcellán. Inner products involving differences: the Meixner–Sobolev polynomials. *J. Difference Equ. Appl.*, 6(1):1–31, 2000. doi:10.1080/10236190008808211.

[3] I. Área, E. Godoy, F. Marcellán, and J. J. Moreno-Balcázar. Ratio and Plancherel–Rotach asymptotics for Meixner–Sobolev orthogonal polynomials. *J. Comput. Appl. Math.*, 116(1):63–75, 2000. doi:10.1016/S0377-0427(99)00281-2.

[4] H. Bavinck. On polynomials orthogonal with respect to an inner product involving differences. *J. Comput. Appl. Math.*, 57:17–27, 1995. doi:10.1016/0377-0427(93)E0231-A.

[5] H. Bavinck. On polynomials orthogonal with respect to an inner product involving differences (the general case). *Appl. Anal.*, 59:233–240, 1995. doi:10.1080/00036819508840402.

[6] H. Bavinck. A difference operator of infinite order with the Sobolev-type Charlier polynomials as eigenfunctions. *Indag. Math. (N.S.)*, 7(3):281–291, 1996. doi:10.1016/0019-3577(96)83721-9.

[7] H. Bavinck and R. Koekoek. Difference operators with Sobolev type Meixner polynomials as eigenfunctions. *Computers & Mathematics with Applications*, 36(10–12):163–177, 1998. https://doi.org/10.1016/S0898-1221(98)80018-7.

[8] J. B. Conway. *Functions of One Complex Variable I*. Second Edition. Graduate Texts in Mathematics, Vol. 11. Springer, New York, 1978. doi:10.1007/978-1-4612-6313-5.

[9] R. S. Costas-Santos, A. Soria-Lorente, and J.-M. Vilaire. On polynomials orthogonal with respect to an inner product involving higher-order differences: the Meixner case. *Mathematics*, 10(11):1952, 2022. doi:10.3390/math10111952.

[10] D. Dominici. Mehler–Heine type formulas for Charlier and Meixner polynomials. *Ramanujan J.*, 39(2):271–289, 2016. doi:10.1007/s11139-014-9665-5.

[11] D. Dominici and J. J. Moreno-Balcázar. Asymptotic analysis of a family of Sobolev orthogonal polynomials related to the generalized Charlier polynomials. *J. Approx. Theory*, 293:105918, 2023. doi:10.1016/j.jat.2023.105918.

[12] G. Filipuk, J. F. Mañas-Mañas, J. J. Moreno-Balcázar, and C. Rodríguez-Perales. Second–order difference equation for quasi–orthogonal polynomials related to Hahn difference operator. *J. Math. Anal. Appl.*, 557(1):130268, 2026. doi:10.1016/j.jmaa.2025.130268.

[13] E. J. Huertas and A. Soria-Lorente. New analytic properties of nonstandard Sobolev-type Charlier orthogonal polynomials. *Numer. Algorithms*, 82(1):41–68, 2019. doi:10.1007/s11075-018-0593-0.

[14] M. E. H. Ismail. *Classical and Quantum Orthogonal Polynomials in One Variable*. Encyclopedia of Mathematics and its Applications, Vol. 98. Cambridge University Press, Cambridge, 2013. doi:10.1017/CBO9781107325982.

[15] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric Orthogonal Polynomials and Their q-Analogues*. Springer Monographs in Mathematics. Springer, Berlin, 2010. doi:10.1007/978-3-642-05014-5.

[16] F. Marcellán and Y. Xu. On Sobolev orthogonal polynomials. *Expo. Math.*, 33:308–352, 2015. doi:10.1016/j.exmath.2014.10.002.

[17] F. Marcellán and F. R. Rafaeli. Monotonicity and asymptotics of zeros of Laguerre–Sobolev-type orthogonal polynomials of higher order derivatives. *Proc. Amer. Math. Soc.*, 139(11):3929–3936, 2011. doi:10.1090/S0002-9939-2011-10806-2.

[18] F. Marcellán and A. Ronveaux. On a class of polynomials orthogonal with respect to a discrete Sobolev inner product. *Indag. Math. (N.S.)*, 1(4):451–464, 1990. doi:10.1016/0019-3577(90)90013-D.

[19] J. J. Moreno-Balcázar. Smallest zeros of some types of orthogonal polynomials: asymptotics. *J. Comput. Appl. Math.*, 179:289–301, 2005. doi:10.1016/j.cam.2004.09.045.

[20] J. J. Moreno-Balcázar. Δ-Meixner–Sobolev orthogonal polynomials: Mehler–Heine type formula and zeros. *J. Comput. Appl. Math.*, 284:228–234, 2015. doi:10.1016/j.cam.2014.11.018.

[21] J. J. Moreno-Balcázar, T. E. Pérez, and M. A. Piñar. A generating function for nonstandard orthogonal polynomials involving differences: the Meixner case. *Ramanujan J.*, 25:21–35, 2011. doi:10.1007/s11139-010-9254-1.

[22] N. Neha and A. Swaminathan. Zeros and asymptotics of a certain class of Sobolev-type Meixner polynomials. *Ukr. Mat. Zh.*, 78(7–8):609–610, 2026 (abstract). English translation: *Ukrainian Math. J.*, 2026, doi:10.1007/s11253-026-02632-2. doi:10.3842/umzh.v78i7-8.9553.

[23] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov. *Classical Orthogonal Polynomials of a Discrete Variable*. Springer Series in Computational Physics. Springer, Berlin, 1991. doi:10.1007/978-3-642-74748-9.

[24] M. N. Rebocho. On the second-order holonomic equation for Sobolev-type orthogonal polynomials. *Appl. Anal.*, 101(1):314–336, 2022. doi:10.1080/00036811.2020.1742881.

[25] R. Koekoek. *Generalizations of the Classical Laguerre Polynomials and Some q-Analogues*. Ph.D. thesis, Delft University of Technology, 1990. https://resolver.tudelft.nl/uuid:5545d761-01b1-4f0f-8e55-3956da27d878.

[26] F. Marcellán, M. Alfaro, and M. L. Rezola. Orthogonal polynomials on Sobolev spaces: old and new directions. *J. Comput. Appl. Math.*, 48(1–2):113–131, 1993. doi:10.1016/0377-0427(93)90318-6.